Lectures on Minimal Models And Birational Transformations of Two Dimensional Schemes

By

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Preface

These lectures contain an exposition of fundamental concepts and results of the theory of birational transformations and minimal models for schemes of dimension two. In the case of surfaces over an algebraically closed field of characteristic zero, these results were obtained by old Italian geometers. In the case of fields of arbitrary characteristic, they are due to Zariski (cf. his book on minimal models). Later Neron observed the importance of obtaining such results in the case of schemes of dimension 2; for certain questions of number theory. In particular, he proved the existence of absolute minimal models for two dimensional schemes over rings of integers of global fields in the case where the genus of the generic fibre is 1.

The first aim of these lectures was to give a proof of the corresponding result in the case of arbitrary genus $g \geq 1$. This proof is quite short and is contained in Lecture 7. (Curiously the proof is much simpler for schemes over a one dimensional base than for surfaces over a field). However, in the proof, one has to use certain results on the geometry of two dimensional schemes that were proved only for surfaces. The correctness of these results in the general situation is more or less obvious but as the proofs were never written down, I preferred to give an exposition of the subject in the required generality starting from the beginning. This is why the lectures grew so Long.

For general properties of schemes used in the lectures, the reader is referred to "Elements" of Grothendieck. Of course, it does not mean that the knowledge of the whole treatise is assumed. For example, a general survey given at the beginning of Mumford's "Lectures on curves on an algebraic surface" would be sufficient (with isolated exceptions).

The notes of these lectures were taken and the manuscript was prepared by *C.P.* Ramanujam. I want to thank him here for the splendid job he has done. He has not only corrected several mistakes but also complemented proofs of many results that were only stated in the oral exposition. To mention some of them, he has written the proofs of the Castelnuovo theorem in lecture 6, of the "chain condition" in lecture 7, the example of Nagata of a non-projective surface in lecture 6, the proof of Zariski's theorem in lecture 6.

I also thank A. Bialynicki-Birual and J. Manin who have read the manuscript and made many useful remarks.

I.R. Shafarevich

Contents

Lecture 1

Minimal models

Summary of results. The subject of these lectures is the theory of birational transformations of two dimensional algebraic varieties, or more generally two dimensional reduced schemes over a base, scheme. We first recall the definition of a birational map. Let X and Y be two preschemes over a base scheme B and f and g B-morphisms of open dense sets U and V respectively of X into Y. We shall say that f and g are equivalent if they coincide, as morphisms of B- preschemes, on an open dense subset W of $U \cap V$. This is clearly an equivalence relation, and equivalence classes are called B-*rational maps* (or *rational maps over* B) of X/B into Y/B. A rational map of X/B into Y/B is called *birational* if it admits a representative which is a B-isomorphism of an open dense subset of X on to an open dense subset of Y, and X and Y are then said to be birationally equivalent over B. **1**

For obvious reasons, it is in general not possible to 'compose' two rational maps, and consequently, we cannot from a category whose morphisms are rational maps. However, if X and Y are irreducible B-preschems and φ a rational map of X/B into Y/B such that the image of some representative of φ is dense in Y and ψ a rational map of Y/B into a third B-prescheme Z/B, it is possible to define uniquely the 'composite map' $\psi \circ \varphi$. Thus, when X and Y are irreducible, a rational map φ of **2**
X/B into Y/B birational if and only if there is a rational map ψ of Y/B into X/B such that $\psi \circ \varphi$ is defined and is the identity map of X/B (that

is, contains the identity morphism of X/B as a member) and $\varphi \circ \psi$ is the identity map of Y/B.

We are mainly interested in the case when the base scheme B is a noetherian integral scheme of dimension ≤ 1 which is regular. We recall that a prescheme is *regular* if the rings of its points are regular. (For the definitions of the other terms used, see EGA, I). Thus B can be either $(a) B = \operatorname{Spec} k$, k a field, or (b) an irreducible noetherian scheme such that for every close point $b \in B$, the local ring $\mathscr{O}_b$ is a discrete (rank 1) valuation ring.

First consider the case when X is an irreducible and reduced B-scheme of finite type and of dimension 1. Let $R(B)$ and $R(X)$ denote the fields of rational functions (that is rational maps into Spec $\mathbb{Z}[T]$) of B and X respectively. When we are in the case (a) where B-Spec k, it is well-known [EGA II, 7.4] that there is an irreducible, regular, proper (and even projective) k-scheme X' and a birational map φ of X' into X over k. Further, if ψ is a rational map of X into a k-proper scheme Y/k, $\psi \circ \varphi$ extends to (or has as a representative) a *morphism* of X' into Y over k. [EGA II, 7.4]: In particular, if X is itself k-proper, φ is a morphism of X' into X. It follows that X' is uniquely determined upto a k-isomorphism, by the conditions that it be regular, k- proper and bira-
3 tional with X. The local rings of the closed points of X' can be canonically identified with the discrete valuation rings in $R(X)$ containing k and having $R(X)$ for quotient field. When we are in case (b), it can happen that the image of X under the structural morphism is a single point b_o of B. In this case, since X is reduced, we can factor this morphism as $X \to \operatorname{Spec} k(b_o) \to B$ where $k(b_o)$ is the field of residues at b_o and the second morphism is the canonical morphism of Spec $k(b_o)$ into B. Thus we are essentially in case (a). However, if the image of X in B does not reduce to a single point, since X is irreducible and B is one-dimensional, the image of X must be dense in B. By composition therefore, we get a monomorphism of $R(B)$ into $R(X)$, and $R(X)$ may be considered as a finite algebraic extension of $R(B)$. Let us assume that for any affine open subset B' of B, the integral closure of its ring $\Gamma(B', \mathscr{O}_B)$ (considered as a subring of $R(B)$) in the field $R(X)$ is a finite $\Gamma(B', \mathscr{O}_B)$-module. (This is a mild assumption, and is fulfilled, for example, if $R(X)$ is separable

over $R(B)$ or if B is an algebraic scheme). Let X' be the normalisation of B in the field R is an field $R(X)$ (EGA, II, 6.3). Then X' is irreducible, regular, and proper over B and there is a birational map φ of X'/B into X/B. As in the earlier case, for any rational map ψ of X/B into a scheme Y proper B, $\psi \circ \varphi$ extends to a B-morphism of X' into Y, φ is itself a morphism if X/B is proper, and X' is uniquely determined upto a B-isomorphism by the conditions that it be irreducible, regular, B-proper **4**
and birationally equivalency to X/B. The local rings of closed points of X' are precisely the discrete valuation rings having $R(X)$ for quotient field and dominating the local ring of some closed point of B.

Thus, when $\dim X = 1$ (and X and B satisfy the conditions above), the picture is very simple. Among the B-schemes birationally equivalent to X/B, there is a unique one (upto a B-isomorphism)- X'/B which is regular and B-proper. If further X/B is proper, we have a birational morphism $X' \to X$ over B. The local rings of closed points of X' are discrete valuation rings with quotient field $R(X)$ which dominate the local ring of some closed point of B.

The situation is far from being this simple, even when $\dim X = 2$, as is shown by the following two examples. We use the usual language of algebraic varieties.

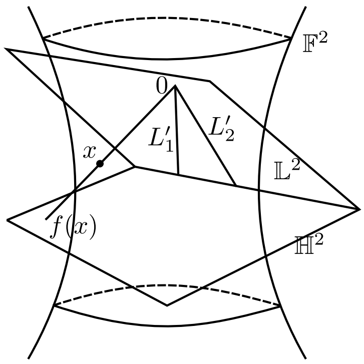
Example 1. Let $X = \mathbb{P}^2(k)$ be the two dimensional projective space over k with homogeneous co-ordinates (x_0, x_1, x_2). Then X is regular and k-proper. Consider the rational map $(x) \to (y)$ of X. into itself determined by the equations

$$y_0 = x_1x_2,\ y_1 = x_0x_2,\ y_2 = x_0x_1$$

and defined on the open subset of X which is the complement of the three points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. One verifies by substitution that the square of this rational map is the identity map of X onto itself, so that in particular, this map is birational.

It is in fact a biregular isomorphism of the complement of the lines **5**
$x_0 = 0$, $x_1 = 0$ and $x_2 = 0$ onto itself. But the entire line $x_0 = 0$ is mapped onto the single point $(1, 0, 0)$ and similarly the lines $x_1 = 0$ and $x_2 = 0$ are mapped onto the points $(0, 1, 0)$ and $(0, 0, 1)$ respectively. Thus, this rational map cannot extend to an automorphism of X.

Example 2. Assume k algebraically closed, and Let $\mathbb{F}^2$ be a non - degenerate quadratic surface in $\mathbb{P}^3$. Then $\mathbb{F}^2$ is regular and k-proper.



We set up a birational correspondence between $\mathbb{F}^2$ and the projective plane $\mathbb{P}^2$ by a method familiar under the name of stereographic projection in function theory. Let 0 be any (closed) point on $\mathbb{F}^2$ and $\mathbb{H}^2$ a plane in $\mathbb{P}^3$ not containing 0. For any point $x \in \mathbb{F}^2 - \{0\}$, define $f(x)$ to be the point of $\mathbb{H}^2$ where the line $0x$ meets $\mathbb{H}^2$. (fig. 1) f is then a morphism of $\mathbb{F}^2 - \{0\}$ into $\mathbb{H}^2$.

f cannot be extended to a morphism of $\mathbb{F}^2$ into $\mathbb{H}^2$, as is shown by a simple calculation. ($f(x)$ tends to different limits as x approaches 0 from different directions in $\mathbb{F}^2$, when k is the filed $\mathbb{C}$ of complex numbers).
6 Let $\mathbb{L}^2$ be the tangent plane to $\mathbb{F}^2$ at 0, so that $\mathbb{L}^2$ intersects $\mathbb{F}^2$ along two straight lines $\mathbb{L}_1'$ and $\mathbb{L}_2'$ of $\mathbb{F}^2$. Then f maps $\mathbb{L}_i'$ onto the points $\mathbb{L}_i \cap \mathbb{H}^2$, and f is an isomorphism of the open subset$\mathbb{F}^2 - (\mathbb{L}_1' \cup \mathbb{L}_2')$ of $\mathbb{F}^2$ onto the open subset $\mathbb{H}^2 - (\mathbb{H}^2 \cap \mathbb{L}^2)$ of $\mathbb{H}^2$. In particular, f gives a birational equivalence of $\mathbb{F}^2$ and $\mathbb{H}^2 \simeq \mathbb{P}^2$.

In this example not only is f not biregular, but there is *no* biregular isomorphism of $\mathbb{F}^2$ and $\mathbb{P}^2$. When $k = \mathbb{C}$, this follows from the fact that the second Betti-number of $\mathbb{P}^2$ is 1, while that of $\mathbb{F}^2$ is 2.

Our chief concern in these lectures is with the case when $\dim X = 2$. *For this preliminary discussion, we assume for the sake of simplicity that $B = Spec\ k$ where k is algebraically closed.* We assume that X is regular and k-proper. With the one dimensional case in mind, we may ask if we can pick out from among the k-schemes which are proper over k and birationally equivalent to X a unique 'model' which has some 'nice' properties. The above examples show that, unlike the one dimensional case, being regular, k-proper and birational with X does not ensure uniqueness. Again in analogy with the one dimensional case, we may look for an X' which is maximal in the birationality class of X, in the sense that if Y is any other irreducible k-proper scheme birationally equivalent to X and $Y \xrightarrow{f} X'$ is a birational k- morphism, f is an isomorphism. However, such a maximal model does not exits, because of a process of construction, known as dilatation or 'blowing up a point'. We shall prove its existence later, but we shall now describe its essential features. **7**

Let X be an irreducible 2 dimensional regular algebraic k-scheme, and x a closed point of X. The dilatation of X at x is then an irreducible regular k-scheme Y together with a proper morphism $f : Y \to X$ which is birational, f induces an isomorphism of $Y - f^{-1}(x)$ into $X - \{x\}$. The fibre $f^{-1}(x)$ is an irreducible curve C in Y. These properties suffice to fix Y uniquely (upto an X-isomorphism) in terms of X and x. One can deduce that C must be isomorphic with the projective line $\mathbb{P}'$. Let D be a regular curve on X through x. The inverse of f on $X - \{x\}$, defines by restriction a morphism g of $D - \{x\}$ into Y, and since Y is X-proper and D is regular at x, g extends to a morphism of D into Y. (In the case when k is the complex field $\mathbb{C}$, $g(x)$ must be the limit of $g(y)$ as y tends to x from $D - \{x\}$). It can be shown that $g(x)$ depends only on the tangent line to D at x. Thus to every 'direction' at x (that is, a one-dimensional subspace of the tangent space to X at x), we obtain a point of the fibre $f^{-1}(x) = C$. This correspondence can be shown to be bijective, and since the set of directions at x can be identified with $\mathbb{P}'$, we have a further heuristic confirmation of the fact that $C \simeq \mathbb{P}'$,

The possibility of building such a dilatation associated to any regular algebraic k-scheme and any closed on it shows that we cannot find a "maximal model" in the birationality class of X, in the sense described in the previous paragraph. And for the same reason, we see that there is **8**
no model whose local whose rings at closed points are all the regular local rings of dimension 2 of the field $R(X)$ containing k and having $R(X)$ for quotient field. One might ask if it is not possible to form some of

projective limit of all regular k-proper models of $R(X)$, the maps of this projective system being the birational morphisms. This could be done, and has also proved to be useful (cf. [26] §17 and [17]), but unfortunately the resulting set does not carry a structure of scheme.

Is this process of dilatation reversible? In other words, given Y irreducible, two dimensional, regular, and k-proper, and an irreducible curve C on Y, does there exist an X and a point $x \in X$ such that Y is obtained by dilating (or blowing up) x on X and C is the fibre $f^{-1}(x)$? If this is possible, we also say that C on Y can be contracted or blown down to a point. Clearly, a necessary condition is that C be isomorphic to $\mathbb{P}'$, Further, let U be an affine open neighbourhood of x on X, and V the inverse image of U in Y. If C' is any irreducible closed curve on Y contained in V, the image of C' in X is proper, affine and irreducible, and hence must be a point. Since the morphism is one- one on $Y - C, C'$ must coincide with C. Thus, any irreducible closed curve on Y which is 'sufficiently near' to C must coincide with C. This shows that such curves on any surface are quite exceptional, and we see for instance that on a complete homogeneous surface of any algebraic group, no curve can be
9 contracted to a point. A more precise criterion for the contractibility of C on Y is given by a theorem of Castelnuovo, which states that under the above assumptions on Y and C, C is contractible to a point if and only if C is isomorphic to $\mathbb{P}'$ and the intersection number $(C).(C)$ is -1.

These considerations suggest that given X/k which is regular, proper and two dimensional, we should look for an X'. having the same properties, birationally equivalent to X and *minimal* for these properties, in the sense that any birational *morphism* $X' \to Y$, where Y is again regular and k-proper, is an isomorphism. An X' satisfying the above properties is called a *relatively minimal model* in the birationality class of X or for the function field $R(X)/k$. If such a relatively minimal model for $R(X)/k$ is unique upto a k-isomorphism, it is called a *minimal* model.

We are now in a position to be able to state the principal theorems that we shall prove.

We shall show that there always exist relatively minimal models (for given X/k or equivalently,) given extension K/k of finite type and transcendence degree two.) But these are not always unique, as shown by

example 2 above. (Both $\mathbb{P}^2$ and $\mathbb{F}^2$ are homogeneous spaces, and homogeneous surfaces can be shown to be relatively minimal). However, it was discovered by Enriques that this is one of a very few examples in which uniqueness fails to hold. He found that if all relatively minimal models models are not isomorphic, or in other word, if a minimal model fails to exist, X must be birationally equivalent to a product $C\times\mathbb{P}'$ where **10**
C is a complete curve over k. In this case, the relatively minimal models are precisely the algebraic fibre spaces over C with the projective line $\mathbb{P}'$ as fibre. These are the so called ruled surfaces. (We mention that when $C = \mathbb{P}'$, there is one exceptional $\mathbb{P}'$ bundle on C which is not relatively minimal, but there is a line on this bundle can which be contracted, and we obtain the projective plane $\mathbb{P}^2$ which is relatively minimal). The above theorem of Enriques is our second principal theorem.

Our first principal theorem says that any birational transformation of two dimensional, irreducible, regular and proper k-schemes can be decomposed into birational transformations of the simplest kind namely dilatations. More precisely, it states that given two irreducible regular proper surfaces X, Y over k and a birational transformations of X into Y, there exists a Z having the same properties and birational *morphisms* $Z \xrightarrow{f} X$ and $Z \xrightarrow{g} Y$ such that (i) $g\circ f^{-1}$ is the given birational equivalence of X and $Y, (ii)$ f and g can respectively be factored as

$$Z = X_n \overset{f_{n-1}}{\to} X_{n-1} \to \ldots \overset{f_o}{\to} X_o = X, Z = Y_m \overset{g_{m-1}}{\to} Y_{m-1} \to \ldots$$
$$\ldots \overset{g_o}{\to} Y_o = Y, f = f_o o f_1 o \ldots o f_{m-1}, g = go \ldots og_{m-1}$$

where each $f_i : X_i \to X_{i-1}$ and $g_j : Y_j \to Y_{j-1}$ is a dilatation.

Analogues of these theorems also hold in the 'arithmetical case' where B is of dimension 1 (case (b) above). We mention, however, **11**
that whereas the problem of classifying ruled surfaces over a curve C (or equivalently, the classification of vector bundles of rank two on C) has been treated by many authors, the analogous problem of classifying the relatively minimal models over space $\mathbb{Z}$ for example, has not been considered.

Our principal tool will be the intersection theory of divisors on a two-dimensional scheme. However, we meet here a difficulty. Namely, in the case of a regular surface X, we can define an intersection number. $(C_1.C_2)$ of two divisors in such a way that $(C.(f)) = 0$ if f is a function, only of X is complete. If X is a scheme of dimension 2, proper over Spec $\mathcal{O}$, where $\mathcal{O}$ is a Dedekind domain, the situation is different because spec $\mathcal{O}$ itself is not, in any sense, complete.

Fortunately, the fibres of $X \to \text{Spec}\, \mathcal{O}$ are complete and we can define $(C_1.C_2)$ with good properties when C_1 is contained in a fibre and C_2 is an arbitrary divisor. This incomplete intersection theory is sufficient for our purposes. Still it would be very interesting to find a general definition. This seems possible in the case where is the ring of integers in a number field. The intersection function should naturally involve the infinite primes of $\mathcal{O}$ since these serve to "compactify" Spec $\mathcal{O}$. When the general fibre of $X \to \text{Spec}\, \mathcal{O}$ is an elliptic curve, Tate has defined a function that has all the desired properties of the intersection number and Neron has proved for this function a product formula which can be considered as an expression of the global intersection number as a sum of local ones (See S. Lang "Les formes bilineaires de Neron et Tate", Seminaire Bourbaki No. 274, 1963/64).

Lecture 2

Blowing up a closed point of a two dimensional scheme

Let X be an equidimensional, noetherian, two dimensional regular prescheme, and x closed point of X. We shall then define another prescheme X' together with a morphism $\sigma : X' \to X$ such that (i) σ is proper, (ii) the fibre $\sigma^{-1}(x)$, considered as a scheme over the residue field $k(x) = k$ at x, isomorphic to the projective line $\mathbb{P}'(k)$ over k, and (iii) σ induces an isomorphism of $X' - \sigma^{-1}(x)$ onto $X - \{x\}$. The prescheme X' is said to be the *dilatation* of X at x, or is said to be obtained from X by blowing up the closed point x. 12

First assume that $X = \operatorname{Spec} A$, where A is a noetherian, equidimensional, regular domain of dimension 2, and suppose also that the maximal ideal $\mathcal{M}_x$ defining x is generated by two elements $u, v \in A$. Let R be the quotient field of A, $w' = \dfrac{v}{u} \in R$, and $B' = A[w']$ the subring of R generated by w' over A. We define Y' to be the scheme $\operatorname{Spec} B'$, and $\sigma' : Y' \to X$ to be the morphism induced by the inclusion $A \hookrightarrow B'$ of rings. To study Y' and σ', we need the following

Lemma. *If $f(T)$ is a polynomial in $A[T]$ such that $f(w') = 0$, $f(T) = (uT - v).g(T)$ with $g(T) \in A[T]$.*

Proof. This being trivial when deg. $f = 0$, it is sufficient to prove it for f of positive degree, assuming it to hold for lower degree polyno-

13 mials. Let $f(T) = a_o T^n + a_1 T^{n-1} + \cdots a_n$. Multiplying the equation $f(w') = f\left(\frac{v}{u}\right) = 0$ by u^n, we see tht u divides $a_o v^n$. In the local ring $A_{\mathcal{M}}, u$ is a prime element and v is not a multiple of u, since $A_{\mathcal{M}}$ is a regular local ring and (u, v) form a regular system of parameters. Hence $\frac{a_o}{u}$ belong to $A_{\mathcal{M}}$. On the other hand, if $\mathscr{Y}$ is a prime ideal of A distinct from $\mathcal{M}$ either u or v is invertible in $A_{\mathscr{Y}}$ since $(u, v) = \mathcal{M}$, so that $\frac{a}{u} = -\left(a_1.\frac{1}{v} + a_2.\frac{u}{v^2} + \cdots + a_n\frac{u^{n-1}}{v^n}\right)$ belongs to $A_{\mathscr{Y}}$ also. Since $\bigcap\limits_{\mathscr{Y} \in \text{ Spec } A} A_{\mathscr{Y}} = A, b = \frac{a_o}{u} \in A$. Thus, $f(t) = bT^{n-1}(uT - v) + g(T)$ where $g(T) \in A[T], \deg g < \deg f$ and $g(w') = 0$. The assertion follows from the induction hypothesis. □

It follows from the lemma that we can identify B' which the residue ring $A[W']/'_{(u.W'-v)}$, where W' is an indeterminate, such that W' corresponds to w'.

We have moreover,

$$\mathcal{M}B' = uB' + vB' = u.B' + u.w'B' = uB',$$

which shows that the inverse image in Y' of the closed set $V(u)$ of X defined by u coincides with the inverse image of the point $x = V(\mathcal{M})$. Further, the fibre $\sigma'^{-1}(X)$, considered as a scheme over the residue field $k(x) = k$, is given by

$$\sigma'^{-1}(X) = \text{Spec } k \underset{X}{\times} Y' = \text{Spec}\left(\frac{A}{\mathcal{M}} \otimes_A B'\right),$$

and we have the isomorphisms

$$\frac{A}{\mathcal{M}} \otimes_A B' = \frac{B'}{\mathcal{M}B'} \simeq \frac{B'}{uB'} = \frac{A[W']}{(uW' - v, u)} = \frac{a[W']}{(u, v)} \simeq k[W'],$$

14 where W' denotes an indeterminate over k, and W' is the image if $w' \in B'$ under the above isomorphism. In particular, $\sigma'^{-1}(x)$ is isomorphic to the affine line $\mathbb{A}'(k) = \text{Spec } k[W']$ over k. Now, let X_u and Y'_u denote the

open subsets of X and Y' respectively where $u \neq 0$. Then $Y'_u = \sigma'^{-1}(X_u)$, and since

$$Y'_u \simeq \operatorname{Spec} B'\left[\frac{1}{u}\right] \simeq \operatorname{Spec} A\left[\frac{v}{u}, \frac{1}{u}\right] = \operatorname{Spec} A\left[\frac{1}{u}\right] \simeq X_u,$$

σ' induces an isomorphism of Y'_u onto X_u.

We have thus proved that $\sigma'^{-1}(V(u)) = \sigma'^{-1}(x) = \operatorname{Spec} k[W'] \simeq \mathbb{A}'(k)$, where W' is the 'restriction' of w' to the fibre $\sigma'^{-1}(x)$, and W' is transcendental over k; and that σ' induces an isomorphism of $\sigma'^{-1}(X_u) = Y'_u$ onto X_u.

Interchanging u and v, we see that if we put $w'' = \dfrac{u}{v} \in R, B'' = A[w''] \subset R, Y'' = \operatorname{Spec} B''$ and $\sigma'' : Y'' \to X$ is the morphism induced by the inclusion $A \hookrightarrow B''$ of rings then $\sigma''^{-1}(V(v)) = \sigma''^{-1}(x) = \operatorname{Spec} k[W''] \simeq \mathbb{A}'(k)$, where W'', the restriction of w'' to $\sigma''^{-1}(x)$, is transcendental over k, and σ'' induces an isomorphism of $\sigma''^{-1}(X_v) = Y''_v$ onto X_v.

We shall now obtain the required scheme X' and the morphism $\sigma : X' \to X$ by 'patching up' Y' and Y''. Since $Y'_{w'} \simeq \operatorname{Spec} B'\left[\dfrac{1}{w'}\right] \simeq \operatorname{Spec} A\left[w', \dfrac{1}{w'}\right] = \operatorname{Spec} A[w', w'']$, and similarly $Y''_{w''} \simeq \operatorname{Spec} A[w', w'']$, we have an X-isomorphism χ of $Y'_{w'}$ onto $Y''_{w''}$. Therefore there exists a prescheme X' covered by two affine open subsets Z' and Z'' and isomorphisms $\tau' : Z' \to Y'$ and $\tau'' : Z'' \to Y''$ such that $\tau'(Z' \cap Z'') =$ **15**
$Y'_{w'} \subset Y', \tau''(Z' \cap Z'') = Y''_{w''} \subset Y''$ and $\tau'' \circ \tau'^{-1}$ defined on $Y'_{w'}$ is the isomorphism χ of $Y'_{w'}$ onto $Y''_{w''}$. Since χ is a morphism of X-schemes, we obtain a morphism σ of X' onto X such that $\sigma \circ \sigma'^{-1} = \sigma'$, $\sigma o \sigma''^{-1} = \sigma''; X'$ is by definition of finite type over X, and X' is a scheme, since $Z' \cap Z'' \simeq \operatorname{Spec} A[w', w'']$ is affine and its co-ordinate ring $A[w', w'']$ generated by the restrictions to $Z' \cap Z''$ of the co-ordinate rings $A[w']$ and $A[w'']$ of Z' and Z'' respectively. (EGA, I, 5.5.6).

Now, $\sigma'^{-1}(x) \cap Y'_{w'}$ is isomorphic to $\operatorname{Spec} k\left[W', \dfrac{1}{W'}\right]$ since it is the set of points of $\sigma'^{-1}(x) \simeq \operatorname{Spec} k[W']$ where W' (which is the restriction to $\sigma'^{-1}(x)$ of w') is different from zero, and similarly $\sigma''^{-1}(x) \cap Y''{}_{w''} \simeq$

Spec $k\left[W'', \dfrac{1}{W''}\right]$. The restriction χ_1 of χ to $\sigma'^{-1}(x) \cap Y'_{w'}$ is an isomorphism of $\sim'^{-1}(x) \cap Y'_{w'}$ onto $\sigma''^{-1}(x) \cap Y''_{w''}$, which is clearly induced by the isomorphism of k-algebras $k\left[W', \dfrac{1}{W'}\right] \simeq k\left[W'', \dfrac{1}{W''}\right]$ determined by $W' \to \frac{1}{W''}$. It follows that $\sigma^{-1}(x)$, which is obtained by patching up $\sigma'^{-1}(x)$ and $\sigma''^{-1}(x)$ by the isomorphism χ_1, is isomorphic to the projective line $\mathbb{P}'(k)$ over k. Further, σ is an isomorphism of $X' - \sigma^{-1}(x)$ onto $X - \{x\}$, since σ' and σ'' are isomorphisms of $Y' - \sigma'^{-1}(x)$ and $Y'' - \sigma''^{-1}(x)$ respectively onto X_u and X_v respectively.

We assert that X' is a regular scheme. It is clearly sufficient to show
16 that Y' and Y'' are regular. If y' is a point of Y' and $y = \sigma'(y') \neq x$, $\mathcal{O}_y$ and $\mathcal{O}_{y'}$ are isomorphic, and y' is a simple point. Hence suppose y' is a closed point of the fibre $\sigma'^{-1}(x) =$ Spec $k[W']$. Then there is an irreducible polynomial $\overline{f}(W') \in k[W']$ which generates the ideal of y' on $\sigma'^{-1}(x)$. Let $f(w')$ be any lift of $\overline{f}(W')$ in $B' = A[w']$. Since u generates the ideal $\mathcal{M}B'$ defining the fibre $\sigma'^{-1}(x)$ in B', the maximal ideal of B' which defines the point y' on Y' is generated by u and $f(w')$. Since $\mathcal{O}_y$ is a two dimensional local ring it follows that it is regular. This proves our assertion.

To prove that X' is uniquely determined upto an X-isomorphism by X and the closed point x of X and does not depend on the choice of the parameters u, v, we adopt a more intrinsic definition of 'blowing up' due to Grothendieck. Let S be the graded subring

$$S = A + \mathcal{M}T + \mathcal{M}^2T^2 + \cdots$$

of the polynomial ring $A[T]$, and let Proj S be the projective $X-$ scheme determined by S(EGA, II, 2.3). We shall show that X' is isomorphic to Proj S as X- schemes. If U and V are independent variables over A, we have a surjective homomorphism of graded algebras $A[U, V] \to S$ determined by $U \to uT, V \to v.T$. It follows from the lemma proved above that the kernel of this homomorphism is the ideal generated in $A[U, V]$ by the element $uV - vU$, so that $\dfrac{A[U, V]}{(uV - vU)} \simeq S$. Proj S is
17 therefore covered by two affine open sets isomorphic respectively to

Spec $\dfrac{A[U]}{(vU-u)} \simeq \operatorname{Spec} B'' = Y''$ and Spec $\dfrac{A[V]}{(uV-v)} \simeq \operatorname{Spec} B' = Y'$. The intersection of these affine open sets corresponds precisely to the open subsets $Y'_{w'}$ and $Y''_{w''}$ of Y' and Y'' respectively, and the isomorphism of $Y'_{w'}$ and $Y''_{w''}$ so obtained is precisely χ. This shows that X' and Proj S are isomorphic over X. Since Proj S is uniquely determined by X and x, it follows that X' is uniquely determined upto an X-isomorphism by X and x, and does not depend on the regular system of parameters used. We also see that $X' \xrightarrow{\sigma} X$ is a proper morphism (EGA, II, 5.5.3).

Now consider the general case, when X is any noetherian (or even locally noetherian) everywhere two dimensional regular prescheme and x any closed point of X. Since the maximal ideal of x in the local ring $\mathscr{O}_x$ is generated by two elements, we can choose an affine open neighbourhood X_o of x in X such that $X_o = \operatorname{Spec} A_o$ where A_o is a noetherian regular domain of dimension 2 and the maximal ideal $\mathcal{M}_o$ defining x in A_o is generated by two elements. Let X_1 be the open set $X-\{x\}$ of X. Let $\{X'_o, \sigma_o\}$ be the dilatation of x_o with respect to x as defined above, and put $X'_1 = X_1$ and $\sigma_1 : X'_1 \to X_1$ the identity morphism. By identifying the open subset $X'_o - \sigma_o^{-1}(x)$ and $\sigma_1^{-1}(X_o - \{x\})$ of X'_o and X'_1 respectively by means of the composite isomorphism

$$X'_o - \sigma_o^{-1}(x) \xrightarrow{\sigma} X_o - \{x\} \xrightarrow{\sigma_1^{-1}} \sigma_1^{-1}(X_o - \{x\}),$$

we obtain a prescheme X' and a morphism $\sigma : X' \to X$. Then X' is again **18**
(locally) noetherian, regular and two dimensional, σ is proper (since being proper is a local property on the second space), and in particular also separated; $\sigma^{-1}(x)$ is isomorphic to the projective line $\mathbb{P}'(k)$ on k; and σ induces an isomorphism of $X' - \sigma^{-1}(x)$ on $X - \{x\}$. X' is called the *dilatation* of X at x, and is said to be obtained by blowing up the point x of X.

We shall now study the inverse images of divisors on X by the morphism $\sigma : X' \to X$. Since both X and X' are regular noetherian preschemes, it makes no difference whether we define a divisor as an element of the free abelian group generated by irreducible reduced closed subschemes of codimension one, or as a coherent sheaf of inversible fractionary ideals and we shall use the latter definition. Let D be a positive divisor on X whose support contains x. Then there exists an irreducible affine open neighbourhood $X_o = \operatorname{Spec} A_o$ of x such that the maximal ideal $\mathcal{M}_x$ is generated by two elements (u, v) of A_o and D is defined by a non-zero element $f \in A_o$. The unique integer l such that $f \in \mathcal{M}_x^l$, but $f \notin \mathcal{M}_x^{l+1}$, is called the *multiplicity* of D at x, and is clearly independent of the f chosen to represent D at x. (Since A_o is a regular ring, it is well known that the map $f \to l$ extends uniquely to a discrete valuation on the quotient field R of A_o so that we can define the multiplicity at x of an arbitrary, not necessarily positive, divisor.) We can then write

$$f = \sum_{i=0}^{l} a_i u^i v^{l-i} + g, a_i \in A_o, g \in \mathcal{M}_x^{l+1}$$

19 and not all a_i belonging to $\mathcal{M}_x$ Let $Y_o' = \operatorname{Spec} A_o[w]$ be the affine open subset of $\sigma^{-1}(X_o)$ considered earlier, where $w = \dfrac{v}{u}$. In the ring $A_o[w]$, we have

$$f = u^l \sum_{i=0}^{l} a_i \left(\frac{v}{u}\right)^{l-i} + u^{l+1} h = u^l \left(\sum_{i=0}^{l} a_i w^{l-i} + uh\right), h \in A_o[w].$$

Since $\dfrac{A_o[w]}{\mathcal{M}_x A_o[w]} = \dfrac{A_o[w]}{uA_o[w]} = k[W]$ where W, the image of w in $A_o[w]/ \mathcal{M}_x A_o[w]$, is transcendental over k, we see that $\sum\limits_{i=0}^{l} a_i w^{l-i} + uh \notin u.A_o[w]$ since the image $\overline{a_i}$ in $A_o/\mathcal{M}_x$ is different from 0. Thus if we put $L = \sigma^{-1}(x)$, the divisor $\sigma^*(D)$, which is defined in $\sigma^{-1}(X_o)$ by the element f, contains L with multiplicity exactly l.

Let τ be the restriction of σ to the open subset $X' - \sigma^{-1}(x)$ of X', so that τ is an isomorphism of this open subset onto $X - \{x\}$. Since X' is regular, there is a unique divisor on X' which shall denote by $\sigma'(D)$, which does not contain L as a component and which restricted to $X' - \sigma^{-1}(x)$ coincides with $\tau^*(D)$. One may describe $\sigma'(D)$ as the closure in X' of $\tau^*(D)$; $\sigma'(D)$ is called the *proper transform* of D by σ. We then have

$$\sigma^*(D) = \sigma'(D) + l.L$$

20 where l is the multiplicity of D at x.

Now, $\sigma'(D)$ is defined in $\sigma^{-1}(X_o)$ by the element $\sum_{i=0}^{l} a_i w^i + u.h$. Reading this element modulo $\mathcal{M}_x A_o[W] = uA_o[w]$, we see that the points of intersection of $\sigma'(D)$ with the fibre $\sigma^{-1}(x) \cap Y'_o = \text{Spec } k[W]$ are precisely the closed points of this fibre corresponding to the irreducible factors of the polynomial $\sum_{o}^{l} \bar{a}_i W^{l-i}$ in $k[W]$. In particular, if D is a 'curve' which is regular at $x, l = 1$ and D is defined by an f of the form $a_o v + a_1 u + g$, where $a_o, a_1 \in A_o, \bar{a}_o \neq 0$ in k, and $g \in \mathcal{M}_x^2$; and in this case, $\sigma'(D) = \overline{\tau^{-1}(D)}$ passes through the k-rational point $W = -\dfrac{\bar{a}_1}{\bar{a}_o}$ of $\sigma^{-1}(x) \cap Y'_o$ This point depends only on the image $\bar{f} = \bar{a}_o \bar{v} + \bar{a}_1 \bar{u}$ of f in the k-vector space $\mathcal{M}_x / _{\mathcal{M}_x^2}$ of differentials at x.

Let us say that two irreducible closed one-dimensional regular subschemes D_1 and D_2 (or, as we shall say more shortly, regular curves) containing x have *contact of order* $\geq l$ at x if there exits elements f_1, f_2 of $A_\circ$ defining D_1 and D_2 respectively in a possibly smaller neighbourhood of x, such that in the local ring $\mathcal{O}_x$ of x on X, we have $f_1 = f_2($ mod $\mathcal{M}_x^{l+1})$. We have then proved that $\sigma'(D_1)$ and $\sigma'(D_2)$ pass through the same point y of $\sigma^{-1}(x)$ if and only if they have contact of order ≥ 1. Suppose now that two curves D_1 and D_2 are regular at x and have order of contact $l > 0$ at x, and let their defining elements be chosen that **21**
$f_1 \equiv f_2(\mathcal{M}_x^{l+1})$. We may then write $f_1 = a_o v + a_1 u, a_i \in A_o$ with say $\bar{a}_o \neq 0$ in k, so that we have $f_2 = a_o v + a_1 ug, g \in \mathcal{M}_x^{l+1}$. In the ring $A_o[w]$ of Y'_o, the defining elements of the proper transforms $\sigma'(D_1)$ and $\sigma'(D_2)$ are respectively $a_o w + a_1$ and $a_o w + u^l h$, where $h \in A_o[w]$. It follows that $\sigma'(D_1)$ and $\sigma'(D_2)$ have order of contact $l - 1$ at the point $W = \dfrac{-\bar{a}_1}{\bar{a}_o}$ of the fibre $\sigma^{-1}(x) \cap Y'_o \simeq \text{Spec } k[W]$. Thus, the order of contact at a point of the fibre $\sigma^{-1}(x)$ of the proper transforms of two regular curves through x is one less than the order of contact of these curves at x.

Lecture 3

Generalities on rational maps-Zariski's main theorem

Let X be a *reduced* B−prescheme, and Y a B-*scheme*. Let φ be a rational map of X/B into Y/B. Then there is a unique pair (U, f) consisting of an open dense subprescheme U of X and a B-morphism f of U into Y such that $(i)(U, f)$ represents φ, and (ii) if g is a B-morphism of an open dense subscheme V of X into Y which represents φ, then $V \subset U$ and $f/U = g$ (EGA I, 7.2.1). When these assumptions are fulfilled, we call U the *(maximal open) set of definition* of φ, and we say that φ is *defined* or *regular* at the points of U. Also, we shall use the same symbol φ to mean the B-morphism f defined on U. **22**

Let X be a reduced B-scheme which is either locally noetherian or irreducible. Let Y be a B-proper scheme, and φ a rational map of X/B into Y/B. If x is any of X such that $\mathscr{O}_x$ is a discrete valuation ring, φ is defined at x.

Proof. When X is locally noetherian, since $\mathscr{O}_x$ is an integral domain, we can find a neighbourhood X' of x which is irreducible, and we may clearly restrict ourselves to this neighbourhood. Thus, it is sufficient to prove the theorem in the case when X is irreducible. □

If U is the set of definition of φ, and if π_1 and π_2 are the first and

second projections of $U \times_B Y$, the graph morphism $U \xrightarrow{\Gamma_\varphi} U \times_B Y$ defined
23 by the conditions that $\pi_1 \circ \Gamma_\varphi = I_U$, $\pi_2 \circ \Gamma_\varphi = \varphi$, is a closed immersion, since Y is separated over B. Let Γ be the unique reduced subprescheme of $X \times_B Y$ whose support is the closure in $X \times_B y$ of $\mathrm{Im}(\Gamma_\varphi)$, and let g be the restriction to Γ of the projection $X \times_B Y \to X$. Then Γ is irreducible, $\Gamma \cap (U \times_B Y) = \mathrm{Im}(\Gamma_\varphi)$ is an open subprescheme of Γ and the restriction of g to this open subscheme is an isomorphism onto U. Hence g is birational. Further g is proper, being the composite of the closed immersion $\Gamma \to X \times_B Y$ and the proper morphism $X \times_B Y \to X$. It follows that g is surjective.

Let z be a point of Γ such that $g(z) = x$. Choose an affine open neighbourhood B' of the image of x in B, an affine neighbourhood Γ' of x in X which is mapped into B^1, and an affine neighbourhood Γ' of z in in Γ such that $g(\Gamma') \subset X'$. If $\Gamma(B', \mathscr{O}_B), \Gamma(X', \mathscr{O}_X)$ and $\Gamma(\Gamma, \mathscr{O}_\Gamma)$ are respectively denoted by P, Q and R, Q and R are P-algebras, and R is finitely generated over Q. We have the comutative diagram of P-algebras

$$\begin{array}{ccccc} R & \hookrightarrow & \mathscr{O}_z & \hookrightarrow & R(\Gamma) \\ {\scriptstyle g^*}\uparrow & & \uparrow & & \uparrow \wr \\ Q & \hookrightarrow & \mathscr{O}_x & \hookrightarrow & R(X) \end{array}$$

Since a discrete valuation ring is a maximal proper subring of its quotient field, $\mathscr{O}_x \to \mathscr{O}_z$ is an isomorphism. Let $h_1, \ldots, h_n$ be generators of R over Q. Then we can find an $h \in Q$ which is invertible in $\mathscr{O}_x$ such that $h_i \in g * \left(Q\left[\frac{1}{h}\right]\right)$. It follows that g^* induces an isomorphism of X_h^1
24 onto $\Gamma^1_{g^*(h)}$. The composite morphism $X_h^1 \xrightarrow{g^{-1}} \Gamma \to Y$ clearly represents the rational map φ so that by the maximality of $U, X_h^1 \subset U$ and $x \in U$.

This proves the assertion.

In particular, it follows from this proposition that the complete regular model of a function field of one variable is unique.

We now state that celebrated "Main theorem" of Zariski (Z.M.T for short) without proof. This has several variants, but we shall base all our conclusions on the following formulation:

Theorem Zariski. *Let Y be a a locally noetherian prescheme and $f : X \to Y$ a birational morphism of finite type. Let Y be any (not necessarily closed) point Y, such that*

(i) *the local ring $\mathscr{O}_y$ of Y at y is normal(that is, an integrally closed integral domain), and*

(ii) *there is a point $x \in f^{-1}(y)$ which is isolated (that is, open) in $f^{-1}(y)$.*

Then there is an open neighbourhood U of x in X and an open neighbourhood V of y in Y such that f/U is an isomorphism of U onto V. If further f is separated, we have $U = f^{-1}(V)$.

The geometric meaning of this theorem is: if no subvariety of X of positive dimension going through x is blown down by f, f is regular at x.

For a proof, we refer the reader to (EGA, III, 4.4).

Some geometric examples of dilatations.

We shall for simplicity consider only (irreducible and reduced) varieties
over an algebraically closed field K, and we shall only consider points 25
which are rational over K.

Let φ be a rational map of a variety X into a projective space $\mathbb{P}^n$ (with homogeneous co-ordinates $(y_\circ, \ldots, y_n)$), and let x be a point of X where φ is defined. Then $\varphi(x)$ is contained in one of the 'standard' affine open sets of $\mathbb{P}^n$, say in $y_\circ \neq 0$. Hence there exist rational functions $f_1, \ldots, f_n$ on X, all of them regular in a neighbourhood of x, such that $\varphi(x^1) = (1, f_1(x^1), \ldots, f_n(x^1))$ for any x^1 in this neighbourhood. Conversely, any system $(f_\circ, f_1, \ldots, f_n)$ of rational functions on X with not all $f_i \equiv 0$, defines a rational map of X into $\mathbb{P}^n$ which is regular at atleast all points where the f_i are all regular and atleast one f_i does not vanish. Two such systems $(f_\circ, \ldots, f_n)$ and $(g_\circ, \ldots, g_n)$ define the same rational map if and only if there is a $h \in R(X)$ such that $g_i = hf_i$. The rational map defined by the system $(f_\circ, \ldots, f_n)$ is regular at x if and only if there is an $h \in R(X)$ such that hf_i are regular at x and $(hf_i)(x) \neq 0$ for atleast one i.

For any $\lambda = (\lambda_\circ, \ldots, \lambda_n) \in \mathbb{P}^n$, we shall denote by $H_\lambda (H = $ *Hyper-plane*) the divisor in $\mathbb{P}^n$ defined by $\sum_{i=0}^{n} \lambda_i y_i = 0$. If φ is a rational map of X into $\mathbb{P}^n$ determined by a system $(f_\circ, \ldots, f_n)$ of functions on X all regular and not all 0 at a point of x, the inverse image divisor $\varphi^*(H_\lambda)$ is defined in a neighbourhood of x by the equation $f_\lambda = \sum_0^n \lambda_i f_i = 0$.

We shall say φ is *biregular* ar x if it is regular at x, and induces an isomorphism of a neighborhood of x onto a We shall say that φ is *biregular* at x if it is regular at x, and induces an isomorphism of a
26 neighbourhood of x onto a locally closed subvariety of $\mathbb{P}^n$ (i.e.φ is a local immersion at x, in the sense of EGA. (Ch.I, 4.5.1). We shall now find conditions for this. Let φ be described by $(f_\circ, \ldots, f_n)$, where the f_i are all regular and at least one f_i does not vanish in a neighbourhood U of x on X. Suppose φ is an immersion on U. Then firstly, φ must be injective in U, so that given $x_1, x_2 \in U$ with $x_1 \neq x_2$, there is a hyperplane H_λ containing x_1, but not x_2. This may be expressed by the condition that

(A) *for any* $x_1, x_2 \in U$ *with* $x_1 \neq x_2$, *there is a* $\lambda = (\lambda_\circ, \ldots, \lambda_n)$ *such that* $f_\lambda(x_1) = 0$, $f_\lambda(x_2) \neq 0$.

Secondly, under the homomorphism $\mathscr{O}_{\varphi(x),\mathbb{P}^n} \to \mathscr{O}_{x,X}$, the image of the maximal ideal $\mathcal{M}_{\varphi(x)}$ must be the maximal ideal $\mathcal{M}_x$ at x. Since $\mathcal{M}_{\varphi(x)}$ is itself generated over $\mathscr{O}_{\varphi(x),\mathbb{P}^n}$ by the linear functions' $\dfrac{L_1(y)}{L_2(y)}$ where L_1 and L_2 are linear forms with $L_2(\varphi(x)) \neq 0, \mathcal{M}_x$ must be generated by the functions $L_1(f(x'))$ of $x' \in U$ for $L_2(\varphi(x'))$ are invertible at x. Thus we have the second condition.

(B) *As* $\lambda = (\lambda_\circ, \ldots, \lambda_n)$ *ranges through values such that* $f_\lambda(x) = 0$, *the functions* f_λ *generate the maximal ideal* $\mathcal{M}_x$, *or what is the same, their images* $d f_\lambda$ *in* $\dfrac{\mathcal{M}_x}{\mathcal{M}_x^2}$ *span this K-vector space.*

Speaking geometrically, (A) says that for distinct points x_1, x_2 of U,
27 there is a member of 'linear system' of divisors div (f_λ) which passes through x_1 but not x_2. When X is nonsingular, (B) says that the tangent spaces at x to those members of the linear system which pass through x

and are non-singular at x do not contain any one-dimensional subspace of the tangent space to X at x in common.

Conversely, if U is a neighbourhood of x and φ is given by functions $(f_\circ, \ldots, f_n)$ which are all regular on U and at least one vanishes nowhere in U, and if the conditions (A) and (B) above are fulfilled, *and* if further the closed subvariety $\overline{\varphi(U)}$ of $\mathbb{P}^n$ is *normal* at $\varphi(x), \varphi$ is biregular at x. In fact, because of (A), the morphism $\varphi : U \to \overline{\varphi(U)}$ must be radical (that is, $R(U)$ is a purely inseparable finite algebraic extension of $R(\overline{\varphi(U)})$ (EGA I, 3.5)). But (B) is evidently equivalent to saying that the mapping of the Zariski tangent space to U at x int that of $\overline{\varphi(U)}$ at $\varphi(x)$ is injective, and this implies that $R(U)$ is separable over $R(\overline{\varphi(U)})$. Hence $R(U) = R(\overline{\varphi(U)})$ and φ is birational. But now, by (A), x is the unique point of the fibre $\varphi^{-1}(x)$ (where φ is considered as restricted to U),and since $\varphi(x)$ is a normal point of $\overline{\varphi(U)}$, it follows by Z.M.T. that φ is an isomorphism of a nighbourhood of x onto a neighbourhood of $\varphi(x)$ in $\overline{\varphi(U)}$. This proves the converse assertion.

Suppose now that X is itself imbedded as a closed subvariety of a projective space $\mathbb{P}^m$, and let $k[x_\circ, \ldots, x_m]$ be its homogeneous coordinate ring. If $x \in X$, a rational function on X regular at x can **28**
be written in the form $\frac{f}{g}$, where f, g are homogeneous elements of $K[x_\circ, \ldots, x_m], g(x) \neq 0$ and $\deg f = \deg g$. Thus, in this case, a rational map regular at x can be described by a system of elements $(f_\circ, \ldots, f_n)$ homogeneous of the same degree in $K[x_\circ, \ldots, x_m]$ with at least one f_i such that $f_i(x) \neq 0$. The statement of condition (A) for biregularity remains unaltered, whereas in the condition (B), one replaces the f_λ (which are not functions on X) by $\frac{f_\lambda}{g}$ where g is *any* homogeneous element of $K[x_\circ, \ldots, x_m]$ with $g(x) \neq 0$ and deg. $g = \deg f_i$. The geometric statement of these conditions are again the same.

Let X be a non-singular surface, and $(f_\circ, \ldots, f_n)$ a system of regular functions on X, which have a point $x \in X$ for common zero (but no other common zero). The rational map $X \xrightarrow{\varphi} \mathbb{P}^n$ is then not defined at x. But suppose further that the condition (B) above is fulfilled at x. Let $X' \xrightarrow{\sigma} X$ be the dilatation of X at x. We shall show that $\varphi \circ \sigma$ is regular on X'. It is clearly sufficient to prove this at any point z of the fibre $\sigma^{-1}(x)$.

Let $u = 0$ be the defining equation of $\sigma^{-1}(x)$ in a neighbourhood of z in X'. We have seen that there is a one-one correspondence between the points of $\sigma^{-1}(x)$ and the set of one-dimensional subspaces of the
29 tangent space to X at x, such that any curve which is non- singular at x has for proper transform a curve on X', which passes through the unique point of $\sigma^{-1}(x)$ corresponding to the tangent line of the base curve at x. Because of condition (B), we can find an f_i such that $f_i = 0$ is non-singular at x and its proper transform does not pass through z. Now, the rational map $\varphi \circ \sigma : X' \to \mathbb{P}^n$ is defined in a neighbourhood of z in X' by the system of rational functions $\left(\dfrac{f_\circ \circ \sigma}{u}, \dfrac{f_1 \circ \sigma}{u}, \ldots, \dfrac{f_n \circ \sigma}{u}\right)$. All of these are regular at z, and since div $\left(\dfrac{f_n \circ \sigma}{u}\right) = \sigma^*(div f_i)$- div u = the proper transform $\sigma'(div(f_i))$, we see that f_i/u does not vanish at z. Hence $\varphi \circ \sigma$ is defined at z.

One can similarly deduce conditions for $\varphi \circ \sigma$ to be biregular at points of $\sigma^{-1}(x)$. Suppose in fact that the condition (A) above is fulfilled for $x_2 \neq x$, and assume that

$(B)'$ *for any curve C through x which is non-singular at x, there is a $\lambda = (\lambda_\circ, \ldots, \lambda_n)$ such that the divisor $f_\lambda = 0$ is non-singular at x and touches C at x, but does not have contact of order 2 with C at x.* (This condition is fulfilled if, the images of the f_i in $\dfrac{\mathcal{M}_x}{\mathcal{M}_x^3}$ generate this as a K-vector space, but is not equivalent to this).

Then $\varphi \circ \sigma$ is biregular on X' if $\overline{\varphi(X)}$ is normal. In fact, in view of what we have seen, the proper transforms of two curves which are non-singular and touch at x touch at a point of $\sigma^{-1}(x)$ if and only if the base curves have order of contact atleast two. By assumption (B'), given any point of $\sigma^{-1}(x)$, we can find $\lambda = (\lambda_\circ, \ldots, \lambda_n)$ and $\mu = (\mu_\circ, \ldots, \mu_n)$ such that the curves $f_\lambda = 0$ and $f_\mu = 0$ are non- singular and touch at x, but do
30 not have second order contact at x, and such that their proper transforms pass through z. Hence $\varphi \circ \sigma$ satisfies the conditions for biregularity.

One has similar results when X is a projective non-singular surface and f_i are homogeneous elements of the same degree in the homogeneous coordinate ring of X.

The above considerations suggest a method of imbedding the dilatation of a non-singular (and hence, by a theorem of Zariski, quasi-projective) surface in some projective space. One could take $(f_\circ, \ldots, f_n)$ to be a basis of the homogeneous elements of degree m in the homogeneous ring of X which vanish at a given point x, and hope to prove that m is large, the f_i satisfy the above conditions. This can indeed be done and it is even sufficient to take $m = 2$. We shall content ourselves with examining some particular cases.

i) Let $X = \mathbb{P}^n$, and take $(f_\circ, \ldots, f_N)$ to be the set of monomials of degree $m \geq 1$ in the homogeneous co-ordinates $(x_\circ, \ldots, x_n)$. The conditions for an imbedding are trivially fulfilled, and we get an imbedding of $\mathbb{P}^n$ in $\mathbb{P}^N (N = \binom{m+n}{m} - 1)$, called the *Veronese* imbedding. The hyperplane sections of the image are precisely the hypersurfaces in $\mathbb{P}^n$ of degree m, and this imbedding is useful for precisely this reason, that problems of intersection with hypersurfaces are reduced to problems of hyperplane sections in the image.

   Since the intersection of n general hyperplanes in $\mathbb{P}^N$ and the Veronesean variety is the intersection of n hypersurfaces of degree m in $\mathbb{P}^n$, by the theorem of Bezout, it consists of m^n points, and this is **31**
   the degree of the imbedded variety.

ii) Let $X = \mathbb{P}^2$, and x any point of $\mathbb{P}^2$. By applying a projective transformation, we can assume that $x = (0, 0, 1)$. Let $(f_\circ, \ldots, f_4)$ be the monomials of degree two which vanish at this point. One checks easily that these satisfy the conditions for projective imbedding of the blown up variety X' in $\mathbb{P}^4$ (A point $x' \neq x$ may be brought to (1,0,0) by a projective transformation which fixes (0,0,1), and this leaves the vector space $\sum_0^4 K f_i$ stable. Hence it is sufficient to check the conditions at (1, 0, 0) and (0, 0, 1)). A general hyperplane section of X' in $\mathbb{P}^4$ is the proper transform in X' of a general conic in $\mathbb{P}^2$ through x. Hence, the intersection of X' and a general linear variety of dimension 2must be the intersection of proper transforms of two general conics in $\mathbb{P}^2$ through x. Since two general conics through x are non-tangential at x and meet at three other points of $\mathbb{P}^2$, it follows that degree of X' in $\mathbb{P}^4$ is 3.

Suppose we try to get an imbedding of $\mathbb{P}^2$ blown up at two points in $\mathbb{P}^3$, by using the quadratic forms vanishing at these points. We do get a morphism of the blown up variety onto a quadratic surface in $\mathbb{P}^3$. However, since any conic must intersect a line at almost two points unless it contains it as a component, the whole of the line joining the two chosen points is mapped into a single point in $\mathbb{P}^3$. It can be checked directly that the morphism $X' \to \mathbb{P}^3$ so obtained is biregular in the complement of the proper transform of this line, but
32 contracts the proper transform of this line to a point on the quadric surface.

iii) Next we consider imbeddings by cubic forms. The dimension of the vector space of cubic forms is 10. If we impose the condition that they vanish at $k(\leq 10)$ points in 'general position' we get k homogeneous linear equations for the coefficients, and the dimension of the space of solutions is $10 - k$. Thus, to get an imbedding of $\mathbb{P}^2$ blown up at k points in $\mathbb{P}^{9-k}$, we must have $9-k \geq 2, k \leq 7$. Further for $k = 7$, we cannot get an isomorphism of X blown up at 7 points with $\mathbb{P}^2$ (blowing up a point increases 2nd betti-number by one, for instance when $K = \mathbb{C}$). Thus to get an imbedding of $\mathbb{P}^2$ blown up at k points by cubic forms, we must have $k \leq 6$.

Further, when $k \geq 3$, on three of these points must lie on a line. For, a cubic curve through 3 points on a line, if it also contains a fourth point of the line, must contain the entire line as a component, so that the corresponding rational map will contract the entire line through these three points to a single point. For, similar reasons, when $k = 6$, all the six points should not lie on a conic.

Suppose then that $k \leq 3$ and a set $P_1, \ldots, P_k$ of k points in $\mathbb{P}^2$ are given satisfying the above conditions, Let $(f_\circ, \ldots, f_n)$ be a basis of the vector space of cubic forms vanishing at these points.We shall verify that condition (A) above is fulfilled by this set of functions and $x_1 \neq P_i$ for any i. ($x_1 x_2$ are as in statement of condition (A)). By choosing further
33 points suitably if necessary, it is sufficient to verify this when $k = 6$, the 'worst possible' case. We may also clearly assume that $x_2 \neq P_i$ for any i. The dimension of the space of forms vanishing at $P_i(1 \leq i \leq 6)$ and

x_i is at least $10 - 7 = 3$. Suppose the line x_1x_2 does not contain any P_i. Choose distinct points y, z of this line distinct form x_1, x_2. Then there is a non-zero cubic form F vanishing at the P_i, x_1, x_2, y and z. Since a cubic form not vanishing identically on a line can have at most 3 zeros on this line, F must vanish identically on the line x_1x_2. Hence, $F = 0$ must have as its other component a conic through all the P_i. But this is impossible. Next suppose the line x_1x_2 contains just one P_i, say P_1. Choose a y on this line distinct from x_1, x_2 and P_1. The space of cubic forms vanishing at the P_i, x_1 and y has dimension 2,and all these forms vanish on the line x_1x_2. By dividing out by a defining linear form of this line, we get a vector space of quadratic forms of dimension ≥ 2, all of which vanish at the points $P_2, \ldots, P_6$. Hence,we can find a quadratic form of this space which is non-zero and vanishes also at P_i. This is a contradiction. The case when the line x_1, x_2 contains two points among the P_i is similarly dealt with.

We shall next verify the condition (B) for points x distinct from the P_i. Suppose all the forms F vanishing at the P_i and x vanish to the second order at x, or suppose all the curves $F = 0$ which are non-singular at x have fixed tangent line at x. In either case we can find a line L through x such that the restrictions to L of all these forms vanish to the second order at x. If L does not pass through any of the P_i, choose two points **34**
y, z on L distinct from x. The dimension of the space of cubic forms vanishing at the P_i, x, y and z is $\geq 10 - 6 - 3 = 1$, so that we can find an $F \neq 0$ and vanishing at these points. Since F vanishes to the second order at x and also at y and z when restricted to L, F vanishes identically on L. Thus, $F = 0$ has a component which is a conic passing through all the P_i. This is a contradiction. One argues similarly when L passes through one or two of the P_i

Finally, we verify condition $(B)'$, at the point P_1, say.We need a simple fact, namely that given four points of $\mathbb{P}^2$, on three of which are collinear, and given a non-zero vector at one of them, there is a conic through these points (which may be degenerate) which has the given tangent vector at the specified point. In fact, we may assume after a projective transformation that the points are (1, 0, 0), (0, 1, 0), (0, 0, 1) and (1, 1, 1), and conics through these points are given by $fx_1x_2 +$

$gx_2x_\circ + hx_\circ x_1 = 0$, and this has the tangent line $hx_1 + gx_2 = 0$ at (1, 0, 0). But this represents an arbitrary line through (1, 0, 0). This proves the assumption. Now let C be an arbitrary non- singular curve through P_1 and D the conic through P_1, P_2, P_3 and P_4 which is non-singular at P_1 and touches C at P_1. Suppose every cubic through $P_1, \ldots, P_6$ which is non-singular at P_1 and touches C at P_1 has second order contact with C at P_1. In particular, the degenerate cubic consisting of D and the line P_5P_6 has second order contact with C at P_1, so that we may assume that
35 $C = D$. Now, the space of all cubic forms which vanish at P_2, P_3, P_4, P_5 and P_6, and whose restriction to D vanishes to the second order at P_1 is of dimension 3. We know that D does not containing P_5 or P_6. Choose a point x on D distinct from the P_i, and a point y not on D or the line P_5P_6. Then there is a non-zero cubic form F vanishing at $P_2, \ldots, P_6$, x and y, and whose restriction to D vanishes to the second order at P_1. But this last condition means that the curve $F = 0$ is non-singular at P_1 and touches D at P_1. By assumption, it must therefore have contact of order 2 with D at P_1, or equivalently, its restriction to D vanishes to the 3rd order at P_1. Since F also vanishes at 4 other distinct points of D, F must vanish on D. The other component of $F = 0$ must therefore be the line P_5P_6. But F vanishes also at y, and y does not lie on either D or P_5P_6, which is a contradiction.

Hence, we have proved that for a set of $k \leq 6$ points of $\mathbb{P}^2$ such that no three lie on a line (if $k \geq 3$), and not all six lie on a conic (if $k = 6$), the variety X' got by blowing up $\mathbb{P}^2$ at these points can be imbedded biregularly in $\mathbb{P}^{9-k}$ by means of the cubic forms vanishing at the given points. Since hyperplane sections on X' are proper transforms of cubic curves on $\mathbb{P}^2$ through the k points, we see that the degree of X' for this imbedding is $9 - k$.

The resulting surfaces are called the *Del Pezzo* surfaces. It has been shown ([18] §11) that these are the only non-singular surfaces in $\mathbb{P}^n$ of degree n which are not contained in a linear subvariety.

36 In particular, all cubic surfaces in $\mathbb{P}^3$ which are non-singular are obtained by six blowings up from the plane $\mathbb{P}^2$. We shall now derive the notorious 27 lines on such a cubic surface.

A line in $\mathbb{P}^3$ is characterised by the fact that it is a curve which has

exactly one point of intersection, counting multiplicity, with a general hyperplane. Since the fibre in X' over a blown up point P_i has a unique point of intersection with the proper transform of a general cubic curve through the $P_j (1 \leq j \leq 6)$, each such fibre is a line on X'. Any other line on X' is the proper transform of an irreducible curve C on $\mathbb{P}^2$. Further, since a line on X' can be realised as a (proper) component of a hyperplane section of X' it follows that C must be a (proper) component of a cubic curve through the P_j in $\mathbb{P}^2$. Hence, C must be either a line or a non-degenerate conic. Since any point of intersection of C and a cubic curve through the P_j which is distinct from all the P_j corresponds to a point of intersection of the proper transform of C in X' and a hyperplane, it follows that C can intersect a general cubic through the P_j at most one point of $\mathbb{P}^2$ besides possibly the $P_j (1 \leq j \leq 6)$, counting multiplicity. Suppose C is a line on $\mathbb{P}^2$. Then C meets a general cubic through the P_j at 3 points, counting multiplicity. Since the cubic curves through the P_j 'separate directions' at the P_j as we have shown earlier, a general cubic through the P_j is not tangential to C at any point P_j contained in C; so that such a point $P_j$occurs with multiplicity 1 the intersection of **37**
C and a general cubic through the P_j. It follows that C must contain at least two points among the P_j, since other wise it would have at least two points of intersection, counting multiplicity, outside the $P_j's$, which is impossible.Similarly, if C is a conic, one sees that C must contain 5 of the 6 points P_j. Conversely, we see that any line through two of the points P_j or any conic through 5 of the points P_j has for proper transform in X' a curve of degree 1 on X', that is, a line on X'. Thus we get exactly $6 + 6 + \binom{6}{2} = 27$ lines on the cubic surface X', and the relations of incidence in this configuration of straight lines are also very clearly seen by means of this representation.

By means of cubic forms vanishing at 7 points of $\mathbb{P}^2$ in general position, in the above sense, we get a morphism of the surface obtained by blowing up $\mathbb{P}^2$ at these points onto $\mathbb{P}^2$. This morphism is in fact a (ramified) covering of $\mathbb{P}^2$ of order 2 described on the affine open set $x_\circ \neq 0$ of $\mathbb{P}^2$ by an equation $Z^2 = f_4(x_1, x_2)$, where f_4 is a polynomial of degree 4.

The surface obtained by blowing up $k \leq 7$ points in general position on $\mathbb{P}^2$ (together with the limiting cases where blowing up certain points of the fibres of earlier dilatations are also allowed), have been characterised as the only non-singular surfaces ion $\mathbb{P}^n$ all of whose hyperplane sections are of genus 1. ([4]).

38 *The resolution of singularities of a curve on a surface by dilatations.*

Theorem M.Noether. *Let X be a noetherian equidimensional regular prescheme of dimension 2, and C an irreducible, reduced, closed subscheme of dimension one in X. Assume that for any affine open subset U of C, the integral closure of $\Gamma(U, \mathcal{O}_C)$ in its quotient field is a module of finite type over $\Gamma(U, \mathcal{O}_C)$.*

Then the set of singular (i.e., non-regular) points of C is a finite set of closed points of C. Further, there is a prescheme $Y$and a morphism $\tau : Y \to X$ satisfying the following conditions:

(i) τ can be factorised as $Y = X_n \xrightarrow{\sigma_{n-1}} X_{n-1} \xrightarrow{\sigma_{n-2}} \cdots \xrightarrow{\sigma_\circ} X_\circ = X, \tau = \sigma_\circ \circ \sigma_1 o \ldots \circ \sigma_{n-1}$, where each $\sigma_i : X_{i+1} \to X_i$ is X_i isomorphic to the dilatation of X_i at a closed point x_i of X_i, whose image in X by $\sigma_\circ \circ \ldots \circ \sigma_{i-1}$ is a singular point of C;

(ii) The unique reduced, irreducible closed subscheme C' of Y such that $\tau(C') = C$ (which shall be referred to as the proper transform of C on Y) is non -singular.

Proof. Let U be any affine open set on $C, B = \Gamma(U, \mathcal{O}_C)$ and $\bar{B}$ the integral closure of B in its quotient field. Then for any closed point x of U defined by the maximal ideal $\mathcal{M}_x$ of B, the integral closure of $\mathcal{O}_{x,C} = B_{\mathcal{M}x}$ is precisely the localisation $(\bar{B})_{\mathcal{M}_x}$ of $\bar{B}$ with respect to the multiplicatively closed subset $B - \mathcal{M}_x$. Now, a one-dimensional noethe-
39 rian local domain is regular (i.e., is a discrete valuation ring) if and only if it is integrally closed. □

Thus, the set of singular points of C in U is precisely the set of points x in U such that $B_{\mathcal{M}_x} \neq (\bar{B})_{\mathcal{M}_x}$, or equivalently, the set of points x such that $(\bar{B}/B)_{\mathcal{M}_x} \neq 0$, that is, the support of the B-module $(\bar{B}/B)$. Since $\bar{B}$ is a finite B-module by assumption, this set is closed. This proves the first part of the theorem, since X is quasi-compact, and the singular set is closed and discrete.

Let us now prove the second part.For any $x \in C$, let $\mathscr{O}_{x,C}$ denote the local ring of x on C, and $\overline{\mathscr{O}}_{x,C}$ its integral closure in the quotient field. Since by assumption $\overline{\mathscr{O}}_{x,C}$ is an $\mathscr{O}_{x,C}$-module of finite type, and since there is a non-zero element a $\in \mathscr{O}_{x,C}$ such that a $\overline{\mathscr{O}}_{x,C} \subset \mathscr{O}_{x,C}$it follows that the $\mathscr{O}_{x,C}$-module $\overline{\mathscr{O}}_{x,C/\mathscr{O}_{x,C}}$ is of finite length. We denote the length of this module by $^{1}\mathscr{O}_{x,C}(\overline{\mathscr{O}}_{x,C/\mathscr{O}_{x,C}})$, and we put

$$N(C,X) = \sum_{x=C} l_{\mathscr{O}_{x,C}}(\overline{\mathscr{O}}_{x,C/\mathscr{O}_{x,C}})$$

Then $N(C,X)$ is a non-negative integer associated to C and X, and $N(C,X) = 0$ if and only if C is regular.

Suppose then that C is not regular, and let x be a singular point of C. Let X' be the dilatation of X at x, and C' the proper transform of C on X'. If we can show that $N(C',X') < N(C,X)$ the theorem clearly follows by induction on the integer $N(C,X)$.

We thus have to show that for the dilatation X' of X at a singular **40**
point x of $C, N(C',X') < N(C,X)$. Let $\sigma : X' \to X$ be the dilatation morphism. Since σ is an isomorphism of $X' - \sigma^{-1}(x)$ onto $X - \{x\}$, we have only to show that

$$\sum_{\substack{\sigma(y)=x \\ y\in C'}} l_{\mathscr{O}_{y,C'}}(\overline{\mathscr{O}}_{y,C'/\mathscr{O}_{y,C'}}) < l_{\mathscr{O}_{x,C}}(\overline{\mathscr{O}}_{x,C/\mathscr{O}_{x,C}})$$

Suppose we can prove that there is a ring $A \neq \mathscr{O}_{x,C}$ integral over $\mathscr{O}_{x,c}$ and contained in its quotient field, such that the $\mathscr{O}_{y,C'}$ for $y \in \sigma^{-1}(x) \cap C'$ are the distinct localisations of A at its maximal ideals. We would then have

$$\begin{aligned}
\sum_{\substack{\sigma(y)=x \\ y\in C'}} l_{\mathscr{O}_{y,C'}}(\overline{\mathscr{O}}_{y,C'/\mathscr{O}_{y,C'}}) &\leq \sum_{\substack{\sigma(y)=x \\ y\in C'}} [k(y):k(x)] l_{\mathscr{O}_{y,C'}}(\overline{\mathscr{O}}_{y,C'/\mathscr{O}_{y,C'}}) \\
&= \sum_{\substack{\sigma(y)=x \\ y\in C'}} l_{\mathscr{O}_{x,C}}(\overline{\mathscr{O}}_{y,C'/\mathscr{O}_{y,C'}}) = l_{\mathscr{O}_{x,C}}(\overline{\mathscr{O}}_{x,C}/A) \\
&< l_{\mathscr{O}_{x,C}}(\overline{\mathscr{O}}_{x,C/\mathscr{O}_{x,C}}),
\end{aligned}$$

as required.

Let τ be the restriction of σ to C', considered as a morphism into C. Since τ is proper with finite fibres, one could appeal to (EGA, III, 4.4.2) to conclude that τ is a finite morphism, from which the existence of an A integral over $\mathscr{O}_{x,C}$ such that the localisations of A at its maximal ideals
41 are the local rings $\mathscr{O}_{y,C'}$ of points y of $\tau^{-1}(x)$, follows. However, we shall prove this directly, without appeal to the above mentioned theorem. Let us put $A = \bigcap\limits_{y\in\tau^{-1}(x)} \mathscr{O}_{y,C'}$, the intersection being taken in the quotient field $R(C)$. Let $\bar{C}$ be the normalisation of C in $R(C)$ and $\pi : \bar{C} \to C$ the projection. Since C' is C-proper, by the statement proved at the beginning of the lecture, there is a morphism $\pi' : \bar{C} \to C'$ such that $\tau \circ \pi' = \pi$. It follows that $\bar{C}$ is the normalisation of C' also. Now, A is a subring of $\bigcap\limits_{\substack{\pi(z)=x\\ z\in\tau}} \mathscr{O}_{z,\bar{C}} = \bar{\mathscr{O}}_{x,C}$, so that A is integral over $\mathscr{O}_{x,C}$. It remains to be shown that each $\mathscr{O}_{y,C'}(y \in \tau^{-1}(x))$ is a localisation of A. Since $\bar{C}$is the normalisation of C', we can find an integer $n > o$ such that $\bigcap\limits_{\pi'(z)=y} \mathcal{M}^n_z, \bar{C} \subset \mathcal{M}_{y,C'}$ for all $y \in \tau^{-1}(x)$. Let $y_1, \ldots, y_n$ be the distinct points of $\tau^{-1}(x)$. By the theorem of independence of valuations (or by the Chinese remainder theorem, applied to the co-ordinate ring of an affine open set $\tau^{-1}(U)$ of $\bar{C}$, where U is an affine open neighbourhood of x on C), we can find a rational function φ on $\bar{C}$ such that $\varphi \equiv 1($ mod $\mathcal{M}^n_{z,\bar{C}}$ for all z with $\pi'(z) = y_1$ and $\varphi \equiv 0($ mod $\mathcal{M}^n_{z,\bar{C}}$ for all z with $\pi'(z) = \gamma_\xi(i > 1)$. It then follows that $\varphi \in A$ and φ vanishes at $y_2, \ldots, y_n$ but not at y_1. Hence, for any $f \in \mathscr{O}_{y_1,C'}$, we have $\varphi^m f \in \bigcap\limits_{i=1}^{n} \mathscr{O}_{y_i,C'} = A$ for m large. This proves that $\mathscr{O}_{y_1,C'}$ (and similarly also $\mathscr{O}_{y_2,C'}$,etc). is a localisation of A.

42 It only remains to show that (when x is a singular point) $A \neq \mathscr{O}_{x,C}$, or equivalently that $\tau : C' \to C$ is not an isomorphism. Suppose C is defined at x by an element f, so that we have $f \in \mathcal{M}^l_{x,X}$, $f \notin \mathcal{M}^{l+1}_{x,X}$ for some $l > 1$. If (u, v) generate $\mathcal{M}_{x,X}$, we can then write

$$f = a_\circ u^l + a_1 u^{l-1} v + - + a_l v^l, a_i \in \mathscr{O}_{x,X},$$

and not all a_i belonging to $\mathcal{M}_{x,X}$. Let $\bar{a}_i$ denote the canonical image of a^i in $k(x) = k$. Identifying $\sigma^{-1}(x)$ with $\mathbb{P}^1(k) = \text{proj } (k[W', W''])$ as in

Lecture 2, we have shown in that lecture that the points of intersection of C' and $\sigma^{-1}(x)$ are the "zeros" (i.e., correspond to the irreducible factors) of the binary from $\bar{a}_\circ W''^l + \bar{a}_1 W''^{l-1} W' + \cdots + \bar{a}_l W'^l$. If τ were an isomorphism, there can in particular be only one irreducible factor, and this factor should further be linear (since otherwise, the residue field $k(y)$ at $y \in \tau^{-1}(x)$ would be a non-trivial extension of $k(x)$). Hence, by changing (u, v) if necessary, we may assume that $f = v^l + g$, $g \in \mathcal{M}^{l+1}_{x,X}$. In this case, there is a unique point y in $\sigma^{-1}(x) \cap C'$, and $w' = \dfrac{v}{u}$ is regular and vanishes at this point. Let ω' be the image of w' in $\mathscr{O}_{y,C'}$ so that $\omega' \in \mathcal{M}_{y,C'}$. If τ were an isomorphism, we must have $\mathscr{O}_{x,C} = \mathscr{O}_{y,C'}$, so that we would have $v_C = \omega u_C$, where v_C and u_C are the images of v and u respectively, in $\mathscr{O}_{x,C}$. But then, it would follow that

$$\mathcal{M}_{x,C} = (u_C, v_C) = (u_C),$$

so that x would be a regular point of C. Contradiction.

The theorem is completely proved. **43**

Remarks. 1) The assumption made on C, that for any affine open subset U of C, the integral closure of $\Gamma(U, \mathscr{O}_C)$ is of finite type over $\Gamma(U, \mathscr{O}_C)$, is necessary not only for the closeness of the set of singular points, but is also essential even for the resolution of the singularity at a single point of C by dilatations. In fact, it is easily shown (from what has been established during the course of the above proof) that if x is a singular point of C which can be resolved by a finite sequence of dilatations, $\bar{\mathscr{O}}_{x,C}$ must be an $\mathscr{O}_{x,C}$ module of finite type.

Incidentally, we may note that for a one-dimensional local domain A, its integral closure $\bar{A}$ is a finite A-module if and only if the completion $\hat{A}$ of A has no nilpotent elements.(L.R. Chap. V, §33, Ex. 1).

This assumption is further fulfilled in all 'good' cases, for instance when X is of finite type over a field or over $\mathbb{Z}$. More generally,the suitable assumption to make on X is that it be a "Japanese prescheme" (EGA, O, §23), in order to ensure the above condition.

2) It is not necessary to assume that C is irreducible, but only that it is reduced. Indeed, we may first resolve all the singularities of all

the components of C by a finite sequence of dilatations. After this is done, one has only to separate two components of C which may
44 intersect at a point x of X, by further dilatations. Suppose C_1, C_2 are two components of C simple at x, and having order of contact l at x. We know that after a dilatation at x, the proper transforms C'_1, C'_2 of C_1, C_2 respectively have contact of order $l-1$ at a point of the fibre over x. Thus, after $l+1$ dilatations, we see that the proper transform of C_1 and C_2 do not intersect at any point lying over x.

Lecture 4

The elimination of indeterminacies of a rational map by dilatations

Let X and Y be locally noetheiran two dimensional regular preschemes **45**
and $f : X \to Y$ a morphism. We say that f is a *composite* of *dilatations* if there exist a finite set of preschemes $Y_k(0 \leq k \leq n)$ $X = Y_n, Y = Y_o$, and morphisms $f_k : Y_{k+1} \to Y_k(0 \leq k \leq n-1)$ such that $f = f_o o f_1 o \ldots o f_{n-1}$ and each (Y_{k+1}, f_k) is Y_k-isomorphic to the dilatation of Y_k *at some closed point of* Y_k.

We shall prove the following:

Theorem. (Elimination of indeterminacies). *Let X be a noetherian, two dimensional, regular prescheme over an arbitrary base prescheme B, Y a proper B-scheme and φ a B-rational map of X into Y. Then there exists a B-prescheme Z and B-morphisms $f : Z \to X$ and $\psi : Z \to Y$ such that f is a composite of dilatations and $\varphi \circ f$ is represented by ψ.*

Before we set out to prove this theorem, we shall deduce an immediate corollary. For any reduced, irreducible and B-proper scheme X, we denote by $\mathscr{B}(X/B)$ the class of all couples $(Y/B, \varphi)$ consisting of a reduced, irreducible and B-proper scheme Y and a birational map φ of Y/B into X/B. We introduce a partial ordering in this class by say-

ing that $(Y/B, \varphi)$ *dominates* $(Y'/B, \varphi')(Y \geq Y'$ in symbols, when B, φ and φ' are understood) if the B-rational map $\varphi'^{-1} \circ \varphi$ is represented by a B-morphism ψ of Y onto Y'. ψ is then automatically proper, since the composite $Y \xrightarrow{\psi} Y' \to B$ is proper and $Y' \to B$ is separated. We
46 shall show that $\mathscr{B}(X/B)$ is directed for this partial ordering. For $(Y/B, \varphi)$ and $(Y'/B, \varphi')$ belonging to this class, let U be the maximal open set of definition of $\varphi'^{-1} \circ \varphi = \chi$, and i_U the inclusion of U in Y. Let Γ be the unique reduced closed subscheme of $Y \times_B Y'$ having for support the closure in $Y \times_B Y'$ of the image of U by means of the immersion $U \xrightarrow{(i_U, \chi)} Y \times_B Y'$. If we denote by f and f' the restrictions to Γ of the projections of $Y \times_B Y'$ onto the first and second factors respectively, and if we put $\psi = \varphi \circ f = \varphi \circ f', \psi$ is birational, Γ is an irreducible, reduced B-proper scheme and (Γ, ψ) dominates both (Y, φ) and (Y', φ'). This proves our assertion. We now have the

Corollary. *Let X be an irreducible, noetherian, two dimensional, regular and B-proper scheme. The members (X', f) of $\mathfrak{B}(X/B)$, where $f : X' \to X$ is a composite of dilatations, form a cofinal system in $\mathfrak{B}(X/B)$, for the partial ordering introduced above. In other words, given a reduced and B-proper scheme Y and a B-birational map φ of Y into X, there is an (X', f) as above and a proper birational B-morphism $\psi : X' \to Y$ such that $\varphi \circ \psi = f$.*

Proof. Apply the theorem with φ replaced by φ^{-1}. □

It is convenient to prove a slightly more general version of the theorem, and for this purpose, we give a definition. Let X be a reduced B-prescheme and Y a B-scheme, and let φ be a rational map of X into Y whose maximal open set of definition is U. Denote by i_U the inclusion
47 of U in X, and let Γ be the reduced closed subscheme of $X \times_B Y$ whose support is the closure in $X \times_B Y$ of the image of U by the immersion $U \xrightarrow{(i_U, \varphi)} X \times_B Y$. Let π be the restriction to Γ of the first projection $X \times_B Y \to X$. We say that a point $x \in X$ is a *point of indeterminacy* of φ if (i) $x \in X - U$, and (ii) $\pi^{-1}(x) \neq \phi$. If we suppose further that Y is of finite type over B, a point $x \in X$ such that $\mathscr{O}_x$ is a discrete valuation ring cannot be an indeterminacy point of φ. Indeed, this is what we

have proved at the beginning of lecture 3. We note further that when Y is B-proper, $\pi(\Gamma) = X$ so that the points of indeterminacy are precisely the points of the set $X - U$. In view of this, the above theorem is an immediate consequence of the following

Theorem. *Let X be a noetherian two dimensional regular B-prescheme, and Y a B-scheme of finite type. Let φ be a rational map of X/B into Y/B. Then there is a prescheme Z and a morphism $f : Z \to X$ which is a composite of dilatations, such that $\varphi \circ f$ has no points of indeterminacy on Z.*

Proof. We shall denote by U the maximal open set of definition of φ, and by Γ the reduced closed subscheme of $X \times_B Y$ having for support the closure in $X \times_B Y$ of the image of the immersion $U \xrightarrow{(i_U, \varphi)} X \times_B Y$. Let π_1 be the restriction to Γ of $X \times_B Y \to X$.

Since X is regular, it is the disjoint union of its components which are finite in number, so that we may assume that X is further irreducible. □

We shall first make some simplifications. **48**

1) We may assume B and Y to be affine. In fact, since X is quasi-compact, its image in B is covered by a finite number of affine open sets $B_i (i = 1, \ldots, k)$, and the inverse image of each B_i in Y is a-finite union of affine open sets $Y_{ij} (j = 1, \ldots, n_i)$ of Y. Let X_i be the inverse image of B_i in X, and $X_{ij} = \varphi^{-1}(Y_{ij}) \subset X_i$. If a certain X_{ij} is void, $\Gamma \cap X \times_B Y_{ij}$ is also void, and we leave out this Y_{ij} and renumber the rest. Then we define φ_{ij} to be the rational map of X_i/B_i into Y_{ij}/B_i defined by the B_i morphism φ of X_{ij}.

   Suppose the theorem is true for B and Y affine. Then by blowing up X a finite number of times at points in or lying over X_1, we can eliminate all points of indeterminacy of the B-rational map φ_{11} of X_1 into Y_{11}. Suppose X' is the prescheme obtained from X in this way, X'_i the inverse images in X' of the X_i and φ'_{ij} the rational maps from X'_i into Y_{ij}. Now if $n_1 > 1$, again by blowing up X' a finite number of times at points over X_1 we can eliminate all

indeterminacies of φ'_{12} for X'_1 to Y_{12}. If X'' is the new prescheme we get over X and X''_{11} the inverse image of X'_1 the rational map $X''_{11} \to X'_1 \xrightarrow{\varphi'_{11}} Y_{11}$ does not have any indeterminacies either, since φ'_{11} does not and $X''_{11} \to X'_1$ is a morphism. Repeating this for all (i, j), we obtain an Z/X such that if Z_i is the inverse image of B_i, the rational maps of Z_i into the Y_{ij} do not have indeterminacies. If
49 ψ is the morphism of Z onto X and Γ_Z the closed subset of $Z \times_B Y$ defined with respect to $Z; Y$ and $\varphi \circ \psi$ in the same way as Γ was defined with respect to X, Y and φ, then, we have evidently $\Gamma_Z \subset \bigcup_{(i,j)} (Z_i X_{B_i} Y_{ij})$, and this shows that $\varphi \circ \psi$ has no indeterminacies.

This complete the proof of 1).

2) We may assume $B = \operatorname{Spec} \mathbb{Z}$ and $Y = \operatorname{Spec} \mathbb{Z}[T] = \mathbb{A}'(\mathbb{Z})$

Because of 1), we may assume that $B = \operatorname{Spec} A$, and $Y = \operatorname{Spec} C$, where C is an A-algebra of finite type. By expressing C as the quotient of a polynomial ring in a finite number of variables, we see that we can realise Y as a closed subscheme of a scheme $B \times_{\mathbb{Z}} \mathbb{A}^n(\mathbb{Z}) = \mathbb{A}^n(A) = \operatorname{Spec} A[T_1, T_2, \ldots, T_n]$. Let φ' be the composite of φ with the inclusion $Y \hookrightarrow \mathbb{A}^n(A)$, and suppose the theorem to be true for φ'. We then get a morphism $f : Z \to X$ which is a composite of dilatations such that $\varphi' \circ f$ has no indeterminacies. Let V be the set of definition of $\varphi' \circ f$, so that the morphism $V \xrightarrow{(i_V, \varphi' \circ f)} Z \times_A \mathbb{A}^n(A)$ is a closed immersion. The image of an open (hence dense) subset of V is contained in the closed subscheme $Z \times_A Y$, so that the image of V is itself contained in this closed subscheme. Since V is reduced, the above morphism factors as $V \to Z \times_A Y \to Z \times_A \mathbb{A}^n(A)$. Composing the first of these morphisms with the projection of $Z \times_A Y$ onto Y, we get a morphism of V into Y which considered as a rational map of Z into Y has no indeterminacies, since $V \to Z \times_A Y$ is a closed immersion.

50 Thus, we are reduced to proving the theorem in the case when $B = \operatorname{Spec} A$ and $Y = \mathbb{A}^n(A) \simeq A \times_{\mathbb{Z}} \mathbb{A}^n(\mathbb{Z})$. But now, for any B-prescheme Z, B- morphisms (resp. B-rational maps) of Z into

$\mathbb{A}^n(A)$ are in canonical one-one correspondence with Z-sections (resp. Z-rational sections) of $Z \times_A \mathbb{A}^n(A) \simeq Z \times_A (A \underset{\mathbb{Z}}{\times} \mathbb{A}^n(\mathbb{Z}))$ and the sets of definition of a rational map and the corresponding rational section coincide. This is by the very definition of the product. A B-rational map has no indeterminacies if and only if the corresponding Z-section is a closed immersion. But now, we have a canonical isomorphism of Z-schemes $Z \times_A (A \times_{\mathbb{Z}} \mathbb{A}^n(\mathbb{Z})) \simeq Z \underset{\mathbb{Z}}{\times} \mathbb{A}^n(\mathbb{Z})$ and applying the remarks made above with A replaced by $\mathbb{Z}$ we see that it is sufficient to prove the theorem in the case when $B = \operatorname{Spec} \mathbb{Z}$, $Y = \mathbb{A}^n(\mathbb{Z})$.

Again, let Z be an X-prescheme and χ a Z-rational section of $Z \underset{\mathbb{Z}}{\times} \mathbb{A}^n(\mathbb{Z}) \simeq (Z \underset{\mathbb{Z}}{\times} \mathbb{A}'(\mathbb{Z})) \times_Z (Z \underset{\mathbb{Z}}{\times} \mathbb{A}'(\mathbb{Z}) \times_Z ..$ (n times) with maximal open set of definition V. Let p_i be the i^{th} projection of $Z \underset{\mathbb{Z}}{\times} \mathbb{A}'(\mathbb{Z})) \times_Z \ldots \times_Z (Z \underset{\mathbb{Z}}{\times} \mathbb{A}'(\mathbb{Z}))$ and let $p_i \circ \chi$ have the maximal open set of definition V_i. We have then clearly $\bigcap_i V_i \subset V$. If each $p_i \circ \chi$, as a Z-morphism of V_i into $Z \underset{\mathbb{Z}}{\times} \mathbb{A}'(\mathbb{Z})$ is a closed immersion, so is the morphism

$$\cap V_i \simeq V_1 \times_Z V_2 \times_Z \times \cdots\cdots \times_Z V_n \to (Z \underset{\mathbb{Z}}{\times} \mathbb{A}'(\mathbb{Z})) \times_Z \cdots \times_Z (Z \underset{\mathbb{Z}}{\times} \mathbb{A}'(\mathbb{Z}))$$

This proves that if each $p_i \circ \chi$ were a closed immersion, $V = \cap V_i$ and χ is a closed immersion of V.

Hence, it is sufficient to prove the theorem when $B = \operatorname{Spec} \mathbb{Z}$ and **51**
$Y = \mathbb{A}^1(\mathbb{Z})$.

3) We are thus reduced to the case when φ is a rational function $f \in R(X)$. When $f = 0$, f is a morphism of X into $\mathbb{A}'(\mathbb{Z})$ and there is nothing to prove. Thus we may assume that $f \in R(X)$, $f \neq 0$.

We shall make use of the following

Lemma. *Write the divisor D of f on X, considered as a cycle of codimension one, as $D_1 - D_2$ where D_1 are positive integral linear combinations of irreducible closed subsets of dimension one, such that D_1 and*

D_2 have no common components. Denote by $|D_i|$ the support of D_i on X. Then the maximal open set of definition of f is $X - |D_2|$, and the points of indeterminacy are precisely the points of $|D_1| \cap |D_2|$.

Proof. Identifying the local ring $\mathscr{O}_x$ of a point $x \in X$ with a subring of $R(X)$, we see from the definitions that f is defined at x if and only if $f \in \mathscr{O}_x$. Writing A for $\mathscr{O}_x$, A is an integrally closed noetherian domain, and hence a well-known result, $A = \bigcap\limits_{\mathscr{G}} A_{\mathscr{G}}$ where $\mathscr{G}$ runs through the prime ideals in A of high one. It follows from this and the definition of D_2 that f is regular at x if and only if no component of D_2 contains x, that is, if and only if $x \notin |D_2|$. This proves the first assertion. □

Let $\underline{0}$ be the closed subset of $\mathbb{A}'(\mathbb{Z}) = \operatorname{Spec} \mathbb{Z}[T]$ defined by the ideal (T), so that $\underline{0} \simeq \operatorname{Spec} \dfrac{\mathbb{Z}[T]}{(T)} = \operatorname{Spec} \mathbb{Z}$. By the very definition
52 of D_1, $|D_1| - (|D_1| \cap |D_2|)$ is the inverse image of $\underline{0}$ for the morphism $X - |D_2| \xrightarrow{f} \mathbb{A}'(\mathbb{Z})$. Since $(|D_1| - (|D_1| \cap |D_2|)) \underset{\mathbb{Z}}{\times} \underline{0} \simeq (|D_1| - (|D_1| \cap |D_2|)) \underset{\mathbb{Z}}{\times} \operatorname{Spec} \mathbb{Z} \simeq |D_1| - (|D_1| \cap |D_2|)$, the image of $|D_1| - (|D_1| \cap |D_2|)$ for the immersion $X - |D_2| \xrightarrow{(1,f)} X \underset{\mathbb{Z}}{\times} \mathbb{A}'(\mathbb{Z})$ is the subscheme $(|D_1| - (|D_1| \cap |D_2|)) \underset{\mathbb{Z}}{\times} \underline{0}$ of $X \underset{\mathbb{Z}}{\times} \mathbb{A}'(\mathbb{Z})$. Since D_1 and D_2 have no common components, $|D_1| - (|D_1| \cap |D_2|)$ is dense in $|D_1|$, and it follows that Γ (defined at the beginning of the proof of the theorem), being a closed set, must contain $|D_1| \underset{\mathbb{Z}}{\times} \underline{0} (\simeq |D_1|)$. Hence, every point of $|D_1| \cap |D_2|$ is a point of indeterminacy.

It only-remains to be shown that no point of $|D_2| - (|D_1| \cap |D_2|)$ is a point of indeterminacy of f. Applying what we have proved to $\dfrac{1}{f}$ instead of f, we see that $\dfrac{1}{f}$ is defined as a morphism from $X - |D_1|$ into $\mathbb{A}^1(\mathbb{Z})$, and the inverse image of $\underline{0}$ is precisely $|D_2| - (|D_1| \cap |D_2|)$. Now, it is easy to deduce from definition that the image of the open dense subset

$$\Gamma \cap \{(X - |D_1| - |D_2|) \underset{\mathbb{Z}}{\times} \mathbb{A}^1(\mathbb{Z})\} \text{ of } \Gamma \cap \{(X - |D_1|) \underset{\mathbb{Z}}{\times} \mathbb{A}^1(\mathbb{Z})\}$$

under the morphism

$$(X - |D_1|) \underset{\mathbb{Z}}{\times} \mathbb{A}'(\mathbb{Z}) \xrightarrow{(\frac{1}{f},1)} \mathbb{A}'(\mathbb{Z}) \underset{\mathbb{Z}}{\times} \mathbb{A}'(\mathbb{Z}) = \text{Spec } \mathbb{Z}[T, T']$$

is contained in the closed subset H of $\mathbb{A}'(\mathbb{Z}) \times_{\mathbb{Z}} \mathbb{A}'(\mathbb{Z})$ defined by $TT' - 1$. Hence, the image of $\Gamma \cap \{(X - |D_1|) \underset{\mathbb{Z}}{\times} \mathbb{A}'(\mathbb{Z})\}$ is itself contained in this closed subset. But the projection of H onto the first factor $\mathbb{A}'(\mathbb{Z})$ of $\mathbb{A}'(\mathbb{Z}) \underset{\mathbb{Z}}{\times} \mathbb{A}'(\mathbb{Z})$ does not <u>meet</u> the set $\underline{0}$, since the image of T under the **53**
homomorphism $\mathbb{Z}[T] \to \mathbb{Z}[T, T']/(TT' - 1)$ generates (as an ideal) the second ring. Thus, the projection of $\Gamma \cap \{(X - |D_1|) \underset{\mathbb{Z}}{\times} \mathbb{A}'(\mathbb{Z})$ into $X - |D_1|$ does not meet $|D_2| - (|D_1| \cap |D_2|)$.

This completes the proof of the lemma.

We now return to the proof of the theorem under the assumptions (3). We give the proof here only under the assumption that X is Japanese. The proof in the general case will be given in Lecture 6. Because of the lemma and the assumption that X is noetherian, f has only a finite number of indeterminacy points. Because of the theorem of resolution of singularities of a one-dimensional closed subscheme at a finite number of points by dilatations, we can find a morphism $\tau : X' \to X$ which is a composite of dilatations, such that with the notations of lemma, the components of the proper transforms $\tau'(D_1)$ and $\tau'(D_2)$ are all regular at all points of $\tau^{-1}(x)$, where x is any indeterminacy of f on X. But now, if we write div $(f \circ \tau) = D'_1 - D'_2$ where D'_i are divisors ≥ 0 on X' with no common components, we have

$$D'_1 - D'_2 = div(f \circ \tau) = \tau^*(div\ f) = \tau^*(D_1) - \tau^*(D_2)$$

and $\tau^*(D_i)$ differ from $\tau'(D_i)$ only by a linear combination of components of fibres $\tau^{-1}(x)$ of τ. But the components of the fibres $\tau^{-1}(x)$ are isomorphic to $\mathbb{P}'(k')$, where k' is an extension of the field k, so that these are again regular.

Thus, we may assume that all components of D_1 and D_2 are regular **54**
at the points of indeterminacy. Further, let C_1, C_2 be two components of either D_1 or D_2 which meet at a point x of X, and suppose they are regular at x and have order of contact l at x. Thus we have seen that the

order of contact of their proper transforms C_1' and C_2' at any point in the dilatation of X at x is $l-1$ (with the interpretation that when $l = 0$, C_1' and C_2' do not intersect at any point of the fibre over x). Moreover, C_1' and the fibre over x have order of contact 0 (that is, intersect transversally). It follows from these observations, that we may actually assume that at any point of indeterminacy x of f on X, there is exactly one component of D_1 and one component of D_2, that these are both regular at x and that they have order of contact 0 at x.

Under this assumption, for any point of indeterminacy x of f on X, let $C_i (i = 1, 2)$ be the unique component of $D_i (i = 1, 2)$ which contains x,and let $r_i > 0$ be the multiplicities with which C_i occur in D_i. Put $n(x, f) = \max(r_1, r_2)$. Let $\sigma : X' \to X$ be the dilatation of X at x, and $L = \sigma^{-1}(x)$. Then the rational function $f \circ \sigma$ on X' also satisfies these assumptions. Further, if the only components of div $(f \circ \sigma)$ which pass through any point of $\sigma^{-1}(x)$ are the proper transforms C_1', C_2' of C_1, C_2 respectively and L, and these occur with multiplicities $r_1, -r_2$ and $r_1 - r_2$ respectively. C_1' and L intersect at a unique point x_1 of L, C_2'
55 and L intersect at another distinct unique point x_2 of L. If $r_1 > r_2$, $f \circ \sigma$ is regular at x_1 and $n(x_2, f \circ \sigma) = \max(r_2, r_1 - r_2) < r_1 = n(x, f)$, and similarly when $r_2 > r_1$ also, $n(x_1, f \circ \sigma) < n(x, f)$, and x_2 is a polar point of f. If $r_1 = r_2$, $f \circ \sigma$ has no indeterminacies at any point of $\sigma^{-1}(x)$. It trivially follows from these there is a morphism $\tau : Y \to X$ which is a composite of dilatations, such that τ is an isomorphism of $Y - \tau^{-1}(x)$ onto $X - \{x\}$ and $f \circ \tau$ has no indeterminacies on $\tau^{-1}(x)$. By repeating this procedure for each of the finite number of points of indeterminacy, we arrive at an Z and a morphism $f : Z \to X$ having the properties stated in the theorem.

This completes the proof of the theorem.

Remark. We shall later give a much simpler proof of case 3) (which is the really difficult case) of the theorem. But the proof given here is more straightforward.

Our next theorem gives the structure of any proper birational morphism of two dimensional noetherian regular preschemes.

Theorem. (of decomposition). *Let X and Y be two dimensional noethe-*

rian regular preschemes, and $f : X \to Y$ a proper birational morphism. Then f is a composite of dilatations.

Proof. We may clearly assume X and Y to be irreducible. □

Since f is proper, the indeterminacy set of f^{-1} is precisely the complement of the set of definition of f^{-1}, and is therefore a closed set. Since no points of Y such that $\mathscr{O}_y$ is a discrete valuation ring can belong to this set, and since Y is regular and noetherian, the indeterminacy set **56**
consists of a finite number of closed points of Y, and if Y' denotes the complement, $f|f^{-1}(Y') \to Y'$ is an isomorphism.

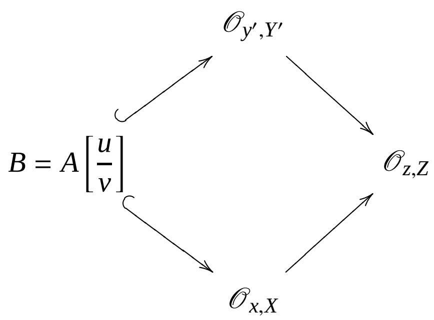
Thus, to prove the theorem, we may assume further that X, Y irreducible, and there is a single closed point y of Y such that $f : X - f^{-1}(y) \to Y - \{y\}$ is an isomorphism

Let $\sigma : Y' \to Y$ be the dilatation of Y at y, and $g = \sigma^{-1} \circ f$. We shall show that g is a morphism. Suppose we have done this. All components of the fibre $f^{-1}(y)$ cannot be mapped onto single points of Y' by g, since g is proper and hence surjective, and the fibre $L = \sigma^{-1}(y) \simeq \mathbb{P}'(k(y))$ contains an infinity of points. Thus, for any point $y' \in Y'$, the number of components of dimension 1 of $g^{-1}(y')$ is strictly less than the number of components of $f^{-1}(y)$. The theorem would then follow by induction, using *Z.M.T*.

Thus, it is sufficient to show that g is a morphism. Since X and Y' are birationally equivalent over Y, there is a unique closed reduced component Z of $X \times_Y Y'$ which dominates both X and Y'. Let p_1 and p_2 be the restrictions to Z of the projections of $X \times_Y Y'$ onto its first and second factors, so that p_1 and p_2 are birational proper morphisms. It is sufficient to show that p_1 is an isomorphism, since it would then follow that $g = \sigma^{-1} \circ f = p_2 \; o \; p_1^{-1}$ is a morphism, Suppose then that p_1 is not an isomorphism. It then follows by *Z.M.T* that there is a closed point x of X such that the fibre $p_1^{-1}(x)$ contains an irreducible one dimensional component C. Let z be the generic point of C, and $y' = p_2(z)$. Then **57**
y' cannot be a closed point of Y', since if it were, $(x) \times_Y (y')$ would be a finite set of closed points of $X \times_Y Y'$ and z would be contained in this set. Hence, $\mathscr{O}_{y',Y'}$ is a discrete valuation ring with is dominated by the local domain $\mathscr{O}_{z,Z}$ and their quotient fields (which are $R(Y')$ and

$R(Z)$ respectively) are isomorphic. It follows that the homomorphism $\mathscr{O}_{y',Y'} \to \mathscr{O}_{z,Z}$ is an isomorphism. Further, since $Y' - L$ and $X - f^{-1}(y)$ are Y-isomorphic, both are isomorphic to $Z - p_1^{-1}(f^{-1}(y)) = Z - p_2^{-1}(L)$. It follows that y' must be a point of L, and since $\mathscr{O}_{y',Y'}$ is a discrete valuation ring, y' must be the generic point of L.

Let u and v be a regular system of parameters of $\mathscr{O}_y$. Then we have seen that either one of $u \circ \sigma$ or $v \circ \sigma$ generates the maximal ideal of $\mathscr{O}_{y',Y'}$. Hence either one of $u \circ \sigma \circ p_2 = u \circ f \circ p_1$ and $v \circ \sigma \circ p_2 = v \circ f \circ p_1$ generates the maximal ideal of $\mathscr{O}_{z,Z}$. Since we have a local homomorphism $\mathscr{O}_{x,X} \to \mathscr{O}_{z,Z}$, it follows that none of the elements $u \circ f$, $v \circ f$ of $\mathscr{O}_{x,X}$ can be in $\mathcal{M}^2_{x,X}$. Now the fibre $f^{-1}(y)$ cannot contain any isolated closed points by *Z.M.T*, so that there is a component D of $f^{-1}(y)$ of dimension 1 which contains x. Let t be the element of $\mathscr{O}_{x,X}$ which defines D at x. Since $u \circ f$ and $v \circ f$ vanish on D, $u \circ f = tu'$, $v \circ f = tv'$ for some $u', v' \in \mathscr{O}_{x,X}$. Since $u \circ f$ and $v \circ f$ do not belong to $\mathcal{M}^2_x$, u' and v' must be invertible in $\mathscr{O}_{x,X}$, so that $\dfrac{u \circ f}{v \circ f} = \dfrac{u'}{v'}$ (and
58 similarly $\dfrac{v \circ f}{u \circ f}$) is defined at x. Let A be the co-ordinate ring of an affine neighbourhood of y in Y such that $Au + Av$ is the maximal ideal of A which defines y. Identifying all the above local rings with subrings of the field $R(Y)$, we have the diagram



From the very definition of the dilatation, $\mathscr{O}_{y',Y'} = \mathscr{O}_{z,Z}$ is the localisation of B with respect to the prime ideal $\mathcal{M}_{y',Y'} \cap B = \mathcal{M}_{z,Z} \cap B = \mathcal{M}_{x,X} \cap B$. Hence, $\mathscr{O}_{x,X}$ contains $\mathscr{O}_{z,Z}$, so that $\mathscr{O}_{x,X}$ and $\mathscr{O}_{z,Z}$ coincide and $\mathscr{O}_{x,X}$ is a discrete valuation ring. This contradicts the fact that x is a closed points, since X is two dimensional at x.

Thus, the theorem is proved.

Lecture 5

The behaviour of various groups associated to a scheme under birational transformations

Let X be a locally noetherian, everywhere two dimensional, regular prescheme, x a closed point of X and $\sigma : X' \to X$ the dilatation of X at x. Let $\mathscr{F}$ be a coherent sheaf of $\mathscr{O}_{X'}$-modules on X'. For any point $y \neq x$ on X, let V be an affine open neighbourhood of y on X not containing x. Then $\sigma^{-1}(V)$ is isomorphic to V, and hence is also affine, so that $H^p(\sigma^{-1}(V), \mathscr{F}) = 0$ for $p \geq 1$(EGA III). This proves that the higher direct images $R^p\sigma(\mathscr{F})$ for $p \geq 1$ are concentrated at the point x of X. Further, let $X_o = \operatorname{Spec} A_o$ be an affine open neighbourhood of x in X, such that the maximal ideal $\mathcal{M}_o$ of x in A_o is generated by two elements u, v of A_o. Then the open set $X'_o = \sigma^{-1}(X_o)$ of X' is covered by two affine open sets $Y'_o = \operatorname{Spec} A_o[w']$ and $Y''_o = \operatorname{Spec} A_o[w'']$, where $w' = \dfrac{v}{u}$ and $w'' = \dfrac{u}{v}$. Denoting this affine covering $\{Y'_o, Y''_o\}$ by $\mathscr{G}$, we have canonical isomorphisms 59

$$H^p(X'_o, \mathscr{F}) \xleftarrow{\sim} \check{H}^p(\mathscr{G}, \mathscr{F})$$

where $\check{H}^*$ denotes the Čech cohomology groups of this covering (EGA III, 1,2). But now, the Čech groups $\check{H}^p(\mathscr{G}, \mathscr{F})$ can be computed by using the complex of alternating cochains of $\mathscr{G}$ with values in $\mathscr{F}$ (EGA III, 1.4.1). Since $\mathscr{G}$ contains just two elements, any alternating p-cochain on $\mathscr{F}$ is 0 if $p \geq 2$. Hence, $H^p(X'_o, \mathscr{F}) \simeq \check{H}^p(\mathscr{G}, \mathscr{F}) = 0$ for $p \geq 2$, and $R^p\sigma(\mathscr{F}) = 0$ for $p \geq 2$.

60 Hence, in this case, the (convergent) spectral sequence of Leray

$$E_2^{p,q} = H^p(X, R^q\sigma(\mathscr{F})) \Longrightarrow H^n(X', \mathscr{F})$$

degenerates into an exact sequence

$$0 \to H'(X, R^o\sigma(\mathscr{F})) \xrightarrow{\sigma_1^*} H^1(X', \mathscr{F})^\alpha \to H^o(X, R'\sigma(\mathscr{F}))$$
$$\to H^2(X, R^o\sigma(\mathscr{F})) \xrightarrow{\sigma_2^*} H'(X', \mathscr{F}) \to 0.$$

Note that here, σ_1^* and σ_2^* are simply the canonical homomorphisms induced in cohomology by the morphism σ, and $\alpha : H'(X', \mathscr{F}) \to H^o(X, R'\sigma(\mathscr{F}))$ is the homomorphism of a presheaf into the sections of the associated sheaf.

Let us take for $\mathscr{F}$ the structure sheaf $\mathscr{O}_{X'}$, of X'. The homomorphism $\mathscr{O}_X \to R^o\sigma(\mathscr{O}_{X'})$ is clearly an isomorphism outside the point x. Let s be a section of $R^o\sigma(\mathscr{O}_{X'})$ in a neighbourhood U of x, so that s is a regular function in a neighbourhood of $\sigma^{-1}(x)$ on X'. But now, s defines a section s' of $\mathscr{O}_X$ in $U - \{x\}$. But we have seen earlier that any rational function on a regular (or even normal) prescheme Y, if it is regular at all points $y \in Y$ such that $\mathscr{O}_y$ is a discrete valuation ring, is regular on Y. Hence, s' extends to a section s_1 of $\mathscr{O}_x$ on U. Denoting the image of s_1 in $R^o\sigma(\mathscr{O}_{X'})$ by s_2, $s - s_2$ is a regular function in a neighbourhood of $\sigma^{-1}(x)$ which to 0 outside $\sigma^{-1}(x)$, so that $s = s_2$. Thus, $\mathscr{O}_X \to R^o\sigma(\mathscr{O}_{X'})$ is an isomorphism. We next show that $R'\sigma(\mathscr{O}_{X'}) = 0$. With notations as in the first paragraph, we have to show that $\check{H}'(\mathscr{G}, \mathscr{O}_{X'}) = 0$. An alternating
61 1-cocycle of the covering $\mathscr{G}$ is simply an element f of $\Gamma(Y'_o Y''_o, \mathscr{O}_{X'}) = A_o[w', w'']$, so that can write

$$f = \sum_O^n a_n w'^n + \sum_O^n b_n w''^n = f' - f'',$$

where $f' \in \Gamma(Y'_o, \mathscr{O}_{X'})$ and $f'' \in \Gamma(Y''_o, \mathscr{O}_{X'})$. This means that any such cocycle is a coboundary, and our assertion is proved.

We have thus the results

$R^o\sigma(\mathscr{O}_{X'}) \backsimeq \mathscr{O}_X$

$R^p\sigma(\mathscr{O}'_X) = 0, p \geq 1, H^p(X', \mathscr{O}'_X) \backsimeq H^p(X, \mathscr{O}_x), p \geq 0.$

In view of the theorems of domination and composition of the earlier lecture, we deduce that if X_1 and X_2 are two regular, noetherian, proper B-schemes (B being arbitrary) of dimension two, and φ a birational map of X_1 into X_2, φ induces isomorphisms

$$H^p(X_1, \mathscr{O}_{X_1}) \simeq H^p(X_2, \mathscr{O}_{X_2}),\ \ p \geq 0.$$

When in particular X is a proper scheme over $B = \operatorname{Spec} k$ where k is a field, the k-vector spaces $H^i(X, \mathscr{O}_X)$ are finite dimensional (EGA, III 3.2.4), and the integer $p_a(X) = \dim_k H'(X, \mathscr{O}_X)$ is called the *arithmetic genus* of X. We have thus proved the invariance of arithmetic genus under birational transformations. ($H^p(X, \mathscr{O})$ is invariant under birational isomorphisms.)

It is not known whether this is true for complete non-singular varieties of arbitrary dimension. However, when the base field is the field $\mathbb{C}$ of complex numbers and X and Y are assumed projective, we have from the theory of Kahler manifolds the equalities $\dim_{\mathbb{C}} H^p(X, \mathscr{O}_X) = \dim_{\mathbb{C}} H^o(X, \Omega^p_X), \dim_{\mathbb{C}} H^p(Y, \mathscr{O}_Y) = \dim_{\mathbb{C}} H^o(Y, \Omega^p_Y)$, where Ω^p_X and Ω^p_Y denote the sheaves of germs of holomorphic p-forms on X and Y respectively ; and by an argument similar to the one used for proving that $R^o\sigma(\mathscr{O}_{X'}) = \mathscr{O}_X$, we see that $R^o f(\Omega^p_X) = \Omega^p_Y$, so that $H^o(X, \Omega^p_X) \backsimeq H^o(Y, \Omega^p_Y)$. Hence we have in this case the equalities 62

$$\dim_{\mathbb{C}} H^p(X, \mathscr{O}_X) = \dim_{\mathbb{C}} H^p(Y, \mathscr{O}_Y), p \geq 0.$$

This naturally holds when X and Y are projective non-singular over any algebraically closed field of characteristic zero.

Again, let X be a regular, proper, two dimensional schemes over a field K, and $\sigma : X' \to X$ the dilatation of X at a closed point x of X. Let $\Omega'_X = \Omega'_{X/K}$ (resp. $\Omega'_{X'/K}$) be the sheaf of K-differentials of X(resp, X'). We have a canonical map $\Omega'_X \to R^o\sigma(\Omega'_{X'})$ which is clearly an isomorphism when restricted to $X - \{x\}$, and we shall

show that it is surjective. Let $X_o = \operatorname{Spec} A_o$ be an affine open neighbourhood of x satisfying the conditions mentioned in the first paragraph, $X'_o = \sigma^{-1}(X_o)$, and let $Y'_o = specA_o[w']$, $Y''_o = specA_o[w'']$
63 be the usual affine open sets of X'_o covering X'_o. We put $B' = A_o[w']$, $B'' = A_o[w'']$ and $C = B'.B'' = A_o[w', w'']$. For any commutative K-algebra $\wedge$, let us denote by $D_K(\wedge)$ the $\wedge$- module of differentials of $\wedge$ over K. An element $\omega \in \Gamma(X_o, R^o\sigma(\Omega'_X)) = \Gamma(X'_o, \Omega'_{X'})$ is represented by a pair of elements $\omega' \in D_K(B')$ and $\omega'' \in D_K(B'')$ such that the images of ω' and ω'' in $D_K(C)$ coincide. Now, ω' can be written as $\omega' = \sum b'_i da'_i + f'(w')dw'$ with $f' \in A_o[X], a'_i \in A_o, b'_i \in B'$ and similarly $\omega'' = \sum b''_j da''_j + f''(w'')dw''$ with $a''_j \in A_o, b''_j \in B''$ and $f'' \in A_o[X]$. Let us denote by $\overline{f'}$ and $\overline{f''}$ the polynomials over $k[X] = \dfrac{A}{\mathcal{M}}[X]$ got by reduction modulo $\mathcal{M}$ from f' and f''. Let l be the algebraic closure of k, and $\theta : \mathbb{P}'(l) \to \mathbb{P}'(k) \simeq \sigma^{-1}(x) \hookrightarrow X'$ the composite morphism. Then $\theta(\omega)$ is a regular differential form on $\mathbb{P}'(l)$, and hence is 0, as is well-known (and trivial to check). But $\theta^*(\omega)$ is represented on a standard affine open set of $\mathbb{P}'(\ell)$ by $\overline{f'}(X)dX$, so that $\overline{f'} \equiv 0$, and similarly $\overline{f''} \equiv 0$. Thus the coefficients of f' and f'' belong to $\mathcal{M}$. Since $\mathcal{M}B' = uB'$, we can write $f'(w')dw' = g'(w').udw' = g'(w')dv - g'(w')w'du$ with $g' \in B'[X]$. Thus, we may assume without loss of generality that $\omega' = \sum b'_i da'_i, \omega'' = \sum b''_j da''_j, a'_i, a''_j \in A_o, b'_i \in B'' b''_j \in B''$. Now, since $C = A_o[w', w'^{-1}]$ and $A_o[w']$ is the quotient of the polynomial ring $A_o[w]$ by the principal ideal $(uW - v)$, it follows from well - known facts concerning the behaviour of modules of differentials under passage to
64 quotient and formation of rings of fractions (EGA, O, § 20) that

$$D_K(C) \simeq \frac{C \otimes_{A_o} D_K(A_o) \oplus C.dw'}{C.(w'du + udw' - dv)}.$$

where $C.dw'$ denotes a free module of rank 1. It follows that the canonical homomorphism $C \otimes_{A_o} D_K(A_o) \to D_K(C)$ is an injection. Since $\sigma_*(\mathscr{O}_{X'}) = \mathscr{O}_X$, we have an exact sequence of A_o-modules

$$0 \to A_o \xrightarrow{\xi} B' \oplus B'' \xrightarrow{\eta} C \to 0$$

where $\xi(a) = (a, a)$ and $\eta(b', b'') = b' - b''$, and tensoring with $D_K(A_o)$, we obtain the exact sequence

$$D_K(A_o) \xrightarrow{\xi\otimes 1} B' \otimes_{A_o} D_K(A_o) \oplus B'' \otimes_{A_o} D_K(A_o) \xrightarrow{\eta\otimes 1} C \otimes_{A_o} D_K(A_o) \to 0$$

Since by assumption, (ω', ω'') belong to the kernal of $\eta \otimes 1$, it follows that $\omega = \sigma^*(\omega_1)$ where $\omega_1 \in D_K(A_o)$. In other words, the canonical homomorphism $\Omega'_X \to \sigma_*(\Omega'_{X'})$ is surjective. The kernel of this homomorphism has support at the point x, or is empty. It follows from the cohomology exact sequence that we have an isomorphism $H'(X, \Omega'_X) \simeq H'(X, R^o\sigma(\Omega'_{X'}))$.

Let us look back at the exact sequence obtained from the Leray spectral sequence with $\mathscr{F}$ replaced by the sheaf $\Omega'_{X'}$. We shall show that in this case, the homomorphism α is non-trivial. Let $\mathcal{U} = \{U_o, U_1, U_2\}$ be the covering of X given by $U_o = X' - \sigma^{-1}(x), U_1 = Y'_o, U_2 = Y''_o$, and let ξ be the cohomology class of $H'(X', \Omega'_{X'})$ given by the following alter- 65
nating cocycle on the covering $\mathcal{U}$: $\xi_{o1} = \dfrac{du}{u} \in \Gamma(U_o \cap U_1, \Omega'_{X'}), \xi_{o2} = \dfrac{dv}{v} \in \Gamma(U_o \cap U_2, \Omega'_{X'})$,

$$\xi_{12} = \frac{dw}{w} \in \Gamma(U_1, \cap U_2, \Omega'_{X'})$$

(This is the element of $H'(X', \Omega'_{X'})$ given by the divisor $\sigma^{-1}(x)$ of X'). We shall show that $\alpha(\xi) \neq 0$. As we have already remarked α is obtained by passage to the limit from the restriction maps in cohomology to neighbourhoods of the fibre $\sigma^{-1}(x)$. Identifying $\sigma^{-1}(x)$ with $\mathbb{P}'(k)$ as usual, we have a homomorphism $\Omega'_{X'} \to \Omega'_{\mathbb{P}'(k)}$, and hence a linear map

$$H^o(X, R'\sigma(\Omega'_{X'})) \xrightarrow{\beta} H'(\mathbb{P}'(k), \Omega'_{\mathbb{P}'(k)})$$

and it is sufficient to show that $\beta o \alpha(\xi) \neq 0$. If $V_1 = \operatorname{Spec} k[W]$ and $V_2 = \operatorname{Spec} k[W^{-1}]$ from the standard affine covering of $\mathbb{P}'(k), \eta = \beta o \alpha(\xi)$ is represented by the alternating 1-cocycle

$$\eta_{12} = \frac{dW}{W},$$

which is easily seen not to be a coboundary (represents the divisor class of degree -1 on $\mathbb{P}'(k)$).

Hence, when $\mathscr{F} = \Omega'_{X'}, \alpha$ is not trivial. Thus we obtain that the map $\sigma_1^* : H'(X, R^o)\sigma(\Omega'_{X'})) \to H'(X', \Omega'_{X'})$ is not surjective, and hence the inequality

$$\dim_K H'(X, \Omega'_X) = \dim_K H'(X, R^o\sigma(\Omega'_{X'})) < \dim_K H'(X', \Omega'_{X'}).$$

This will be useful to us later on.

66 We remark that more exact information can easily be obtained by an explicit computation of $R'\sigma(\Omega'_{X'})$ by using the affine covering $\mathscr{G} = \{Y'_{o'}, Y''_o\}$ of X'_o. We summarise the results, *when K is assumed algebraically closed*, both for Ω' and Ω^2, the sheaf of forms of degree 2.

$$R^o\sigma(\Omega'_{X'}) = \Omega'_X, \dim_K H^o(X, \Omega'_X) = \dim_K H^o(X', \Omega'_{X'})$$

$$R^o\sigma(\Omega'_{X'}) \simeq K = \frac{\mathscr{O}_x}{\mathcal{M}_x} \text{ at } x, \dim_K H'(X, \Omega'_X) = \dim_K H'(X', \Omega'_{X'}) - 1$$

$$R^o\sigma(\Omega'_{X'}) = 0 \text{ for } p \geq 2, \dim_K H^2(X, \Omega'_X) = \dim_K H^2(X', \Omega'_{X'})$$

$$R^o\sigma(\Omega^2_{X'}) = \Omega^2_X,$$

$$R^o\sigma(\Omega^2_{X'}) = 0, p \geq 1, \dim_K H^p(X', \Omega^2_X) = \dim_K H^p(X, \Omega^2_{X'}), p \geq 0.$$

The results for Ω^2 also follow from Serre duality, of course ([27]).

We shall next investigate the behaviour of the group $\vartheta(X)$ of divisors and the Picard group Pic (X) of classes of inversible sheaves on a noetherian, two dimensional, regular prescheme under a dilatation $\sigma : X' \to X$. Since σ induces an isomorphism of $X' - \sigma^{-1}(x)$ onto $X - \{x\}, \sigma^* : \vartheta(X) \to \vartheta(X')$ is clearly injective. Further, any divisor on X' is a sum of an element of $\sigma^*(\vartheta(X))$ and a multiple of the fibre $\sigma^{-1}(x)$, and no multiple of $\sigma^{-1}(x)$ belongs to $\sigma^*(\vartheta(X))$. Thus we have the isomorphism

$$\vartheta(X') \simeq \sigma^*(\vartheta(X)) \oplus \mathbb{Z}.\sigma^{-1}(x) \simeq \vartheta(X) \oplus \mathbb{Z}.$$

67 Further let $\vartheta_l(X)$ denote the subgroup of $\vartheta(X)$ consisting of principal divisors, so that we have a canonical isomorphism Pic $(X) \simeq \dfrac{\vartheta(X)}{\vartheta_l(X)}$.

Since $\vartheta_l(X') = \sigma^*(\vartheta_l(X))$, it follows that we have

$$Pic\ (X') \simeq Pic(X) \oplus \mathbb{Z}.$$

Suppose now that X is a non-singular surface over an algebraically closed field. In this case, algebraic equivalence of divisors is defined ([15], p. 55) and if $\vartheta_a(X)$ is the subgroup of divisors of X which are algebraically equivalent to 0, the quotient group $S(X) = \dfrac{\vartheta(X)}{\vartheta_a(X)}$ is called the *Neron-Severi group* of X. It is known ([16]) that $S(X)$ is a finitely generated abelian group. Now, we shall soon define for a pair of elements $\alpha, \beta \in \dfrac{\vartheta(X)}{\vartheta_l(X)}$ an intersection number $(\alpha.\beta) \in \mathbb{Z}$, when X is a complete variety. We shall also show that if $L = \sigma^{-1}(x)$ and (L) its class in $\vartheta(X')/\vartheta_l(X'), ((L).\sigma^*(\vartheta(X))) = 0$ and $(L).(L) = -1$. But it can be shown that $((L).\vartheta_a(X')) = 0$ (the intersection number is preserved under algebraic equivalence on an algebraic variety). It follows that $\vartheta_a(X') \subset \sigma^*(\vartheta(X))$. Since it can also be shown that if $\alpha \in \vartheta_a(X')$, the direct image $\sigma_*(\alpha) \in \vartheta_a(X)$, and since we have for $\alpha \in \sigma^*(\vartheta(X)), \alpha = \sigma^*\sigma_*(\alpha)$, we deduce that $\vartheta_a(X') = \sigma(\vartheta_a(X))$. We therefore have the following relationship between the Neron-Severi groups of X and X':

$$S(X') \simeq S(X) \oplus \mathbb{Z}.$$

A diversion

Several questions, especially of number theory, are connected with **68**
birational isomorphisms of surfaces X defined over a field K which is not necessarily algebraically closed. The principal problem is to classify under K-birational isomorphism those surfaces that become birationally isomorphic when one extends the base field to the algebraic closure $\overline{K}$ of K. We have constructed above some numbers that are attached to the surface and are birational invariants e.g. the arithmetic genus. But such invariants are unaltered by extension of the base field and so are not useful for our purpose. We have to construct finer invariants for this problem. The situation is simple in the case of curves, where birational isomorphism essentially coincides with isomorphism of schemes.

Let us consider the simplest possible example, namely that of a scheme X over a field K, such that the scheme $\bar{X} = XX_K\bar{X}$ over the algebraic closure $\bar{K}$ of K is isomorphic to the projective line $\mathbb{P}'(\bar{K})$ over $\bar{K}$. It is easily shown that such an X is isomorphic as a K-scheme to a conic $Q(X_o, X_1, X_2) = 0$ in $\mathbb{P}^2(K)$, where Q is a non-degenerate quadratic form with coefficients in K. In fact, from the isomorphism $XX_K\bar{K} \backsimeq \mathbb{P}'\bar{K}$, it follows that X is an integral scheme defined over K, such that its function field $R(K)$ is a regular extension of $K, R(\bar{X}) = R(X) \otimes_K \bar{K}$, and X is absolutely simple. In particular, the sheaf $\Omega_{\bar{X},\bar{K}}$ of differentials of $\bar{X}$ over $\bar{K}$ is got by base extension from the sheaf of K-differentials of X, that is, $\Omega_{\bar{X},\bar{K}} = \Omega_{X,K} \otimes_K \bar{K}$. Hence we have a canonical iso-
69 morphism $H^o(\bar{X}, \Omega^*_{\bar{X}}) \simeq H^o(X, \Omega^*_X) \otimes_K \bar{K}$. Further, $\Omega^*_{\bar{X}}$ is the unique invertible sheaf of degree 2 on $\bar{X} \simeq \mathbb{P}'(\bar{X})$, so that we can choose a basis $\sigma_o, \sigma_1, \sigma_2$ of $H^o(\bar{X}, \Omega^*_{\bar{X}})$ such that if t denotes the identity function of $\mathbb{P}'(\bar{K}), \dfrac{\sigma_1}{\sigma_o} = t, \dfrac{\sigma_2}{\sigma_o} = t^2$. The rational map of $\bar{X} \to \mathbb{P}^2(\bar{K})$ given by $x \mapsto (1, \dfrac{\sigma_1}{\sigma_o}(x), \dfrac{\sigma_2}{\sigma_o}(x))$ is therefore a closed immersion of $\bar{X}$ onto a non-degenerate conic in $\mathbb{P}^2(\bar{K})$ (defined by $X_oX_2 = X_1^2$). Hence, a similar assertion also holds for any other choice $\sigma^1_o, \sigma^1_1, \sigma^1_2$ of basis for $H^o(\bar{X}, \Omega^*_{\bar{X}})$. Now, we can choose such a basis σ'_i of $H^o(X, \Omega^*_X)$ over K, and its remains a basis of $H^o(\bar{X}, \Omega^*_{\bar{X}})$. But for such a choice of $'_i$, we have a K-*morphism* of X/K onto a conic in $\mathbb{P}^2(K)$ defined over K, which must of necessity be non-degenerate. This morphism is again a closed immersion, since it is so after a base extension. This proves that any X/K as above is isomorphic to a conic in $\mathbb{P}^2(K)$ defined over K. We naturally ask, when such conics C_1 and C_2 defined over K in $\mathbb{P}^2(K)$ are K-isomorphic. A K-isomorphism of C_1 and C_2 induces an isomorphism of the vector spaces $H^o(C_1, \Omega'^*_{C_1})$ and $H^o(C_2, \Omega'^*_{C_2})$. But by what we have said above, $\Omega^*_{C_i}$ is simply the restriction of the canonical sheaf $\mathscr{O}(1)$ of $\mathbb{P}^2(K)$ to C_i. This means that the isomorphism in question is actually
70 induced by a projective transformation over K of $\mathbb{P}^2(K)$. Thus, the problem of classifying all K-schemes X such that $\bar{X} = XX_K\bar{X} \backsimeq \mathbb{P}'(\bar{K})$ is equivalent ot the problem of finding all the classes of non-degenerate quadratic forms over K for the equivalence upto a constant factor de-

fined by the full linear group over K. This problem has been solved, for example, when K is a p-adic field (i.e., a complete inequicharacteristic discrete valuation ring with finite residue field) or when K is an algebraic number field. In either case (and more generally, when the characteristic of K is different from 2), any non-degenerate ternary quadratic form is equivalent to a form $Z^2 - aX^2 - bY^2, a, b \in K^*$. In the p-adic case, the forms $Z^2 - aX^2 - bY^2$ and $Z^2 - a'X^2 - b'Y^2$ are equivalent if and only if the Hilbert symbols $(a, b)_{\mathscr{G}}$ and $(a', b')_{\mathscr{G}}$ are equal ([10]). In the case of an algebraic number field, the forms are equivalent if and only if the forms have the same signature at all the real infinite primes (i.e., are equivalent at the infinite prime spots) and the forms are equivalent in all the $\mathscr{G}$-adic completions of K with respect to all the prime divisors $\mathscr{G}$ of K ([10]).

Finally, we mention that over any field K, such an X is isomorphic to $\mathbb{P}'(K)$ over K if and only if X contains a rational point over K. The necessity is clear, since $\mathbb{P}'(K)$ contains at least three rational points $(0, 1), (1, 0)$ and $(1, 1)$ over K. Suppose then that $x \in X$ is a rational point, and ϑ the inversible sheaf of ideals of $\mathscr{O}_X$ which defines the point x. The inversible sheaf $\vartheta^{-1} \otimes_K \bar{K}$ on $\bar{X} \simeq \mathbb{P}'(\bar{K})$ is then of 71
degree one, and hence admits two independent regular sections. Since $H^o(\bar{X}, \vartheta^{-1} \otimes_K \bar{K}) \simeq H^o(X, \vartheta^{-1}) \otimes_K \bar{K}$, it follows that there is an element f of $R_K(X)$ which has a simple pole at the point x of $\bar{X}$, and hence defines a K-isomorphism of X and $\mathbb{P}'(K)$. This proves our statement.

We proceed to discuss the analogous question for surfaces. We shall attach to a surface X over K (which is defined and absolutely simple over K) a group which is 'manageable' as we shall show by an example, and which is not necessarily the same for surfaces which become isomorphic over the algebraic closure $\bar{K}$ of K. We need some preliminary definitions.

Let X be scheme of finite type over a field K, and let K_s and $\bar{K}$ denote respectively the separable and algebraic closures of K. Let $\mathscr{G} = \mathscr{G}(K_S/K)$ be the Galois group of K_s over K with the Krull topology. Then $\mathscr{G}$ also acts as the groups of automorphisms of $\bar{K}$ over K. Let $\bar{X} = XX_K\bar{K}$ and $\pi : \bar{X} \to X$ the first projection. If x is any point of X, $k(x)$ the field of residues at x and $l_s(x)$ the largest separable alge-

braic extension of K in $k(x)$, it follows from EGA, I,(3.4.9), that the points of $\pi^{-1}(x)$ are canonically in one-one correspondence with the K-monomorphisms of $l_s(x)$ into $\bar{K}$(or K_s), and in particular, there are precisely $[l_s(x) : K]$ points in the fibre $\pi^{-1}(x)$. Points of the same fibre are said to be conjugate over K. Further, if Y is a an irreducible closed set of X with generic point y, the components of $\pi^{-1}(Y)$ are precisely the closures in $\bar{X}$ of the points of the fibre $\pi^{-1}(y)$, all these components have
72 the same dimension and are mapped onto Y by π. These components are again said to be conjugate over K, so that two irreducible closed sets of $\bar{X}$ are conjugate over K if and only if their generic points are conjugate over K.

If σ is any element of $\mathscr{G}, \sigma$ induces a K-isomorphism $\tilde{\sigma} : \operatorname{Spec} \bar{K} \to \operatorname{Spec} \bar{K}$, and hence also a K-isomorphism $\check{\sigma} = I_X \times \tilde{\sigma} : \bar{X} \to \bar{X}$. We have clearly $(\sigma\tau)\check{} = \check{\tau}\check{\sigma}$, so that we may consider $\mathscr{G}$ as acting (as an abstract group) on the right on $\bar{X}$. From what we have said above, it follows that $\mathscr{G}$ acts transitively on the fibres $\pi^{-1}(x)$, and hence also on any complete set of conjugate irreducible closed subsets. If $\bar{x}$ is any point of $\pi^{-1}(x)$, to which there corresponds a K-monomorphism $\theta : l_s(x) \to K_s$, the subgroup of $\mathscr{G}$ which fixes $\bar{x}$ is precisely the subgroup which fixes $\theta(l_s(x))$.

Let us now assume further that X is two dimensional and K-proper such that $\bar{X} = XX_K\bar{K}$ is irreducible and regular (that is, $R(K)$ is a regular extension of K, and X is absolutely regular, in particular regular). We can make $\mathscr{G}$ act on the left on the group $\vartheta(\bar{X})$ of divisors of $\bar{X}$ by defining for $\sigma \in \mathscr{G}$ and $D \in \vartheta(\bar{X})$, $\sigma D = \check{\sigma}^*(D)$. If we provide $\vartheta(\bar{X})$ with the discrete topology, it follows from our earlier remarks that $\mathscr{G}$ acts continuously on $\vartheta(\bar{X})$. Further, the subgroups $\vartheta_l(\bar{X})$ and $\vartheta_a(\bar{X})$ are seen to be stable for this action. Thus, $\mathscr{G}$ acts continuously on the factors $\vartheta(\bar{X})/\vartheta_l(\bar{X})$ and $S(\bar{X}) = \vartheta(\bar{X})/\vartheta_a(\bar{X})$ (both with discrete topology). In future, we shall denote $S(\bar{X})$ simply by $S(X)$. If Y is another K-scheme satisfying the above assumptions and $f : X \to Y$ a K-morphism, the
73 induced homomorphisms $\vartheta(\bar{Y}) \to \vartheta(\bar{X})$, $\dfrac{\vartheta(\bar{Y})}{\vartheta_l(\bar{Y})} \to \dfrac{\vartheta(\bar{X})}{\vartheta_l(\bar{X})}$ and $S(\bar{Y}) \xrightarrow{f^*} S(\bar{X})$ are $\mathscr{G}$-homomorphisms.

Now, let K be a perfect field, and X a regular two dimensional K-

proper scheme such that K is algebraically closed in $R(K)$. It follows automatically that $XX_K\bar{K}$ is regular and irreducible [EGA III, 4.3.5]. It is not difficult to show that there is a family $\{x_\alpha, \varphi_\alpha\}_{\alpha\in I}$ of couples consisting of regular K-proper schemes X_α and K-morphisms $\varphi_\alpha : X_\alpha \to X$ which are birational, indexed by a set I, such that if Y is any regular K-proper scheme and $\psi : Y \to X$ a birational K-morphism, there is a *unique* $\alpha \in I$ and an isomorphism $\psi : Y \to X_\alpha$ such that $\varphi_\alpha o \psi_\alpha = \psi$ (one can in fact take the X_α to be suitable collections of regular local rings of $R(X)$ containing K, that is, schemes in the sense of Chevalley). For $\alpha, \beta \in I$, let us define $\alpha \geq \beta$ if there is a K-morphism $\varphi^\alpha_\beta : X_\alpha \to X_\beta$ such that $\varphi_\beta o \varphi^\alpha_\beta = \varphi_\alpha$. From what we have seen in lecture 4, I is filtered for this partial ordering. Each $\bar{X}_\alpha = X_\alpha X_K \bar{X}$ is irreducible and K-regular (and it is for this that we assumed K perfect). Since for $\alpha \geq \beta \geq \gamma$ we have clearly $\varphi^\beta_\gamma o \varphi^\alpha_\beta = \varphi^\alpha_\gamma$, if we put

$$\bar{\varphi}^\alpha_\beta = \varphi^\alpha_\beta X_K \bar{K} : \bar{X}_\alpha \to \bar{X}_\beta, \text{ and } \psi^\beta_\alpha = (\bar{\varphi}^\alpha_\beta)^* : S(X_\beta) \to S(X_\alpha)),$$

$(S(X_\alpha), \psi^\beta_\alpha)$ form an inductive system of discrete $\mathscr{G}$ modules. We define

$$\gamma(X) = \varinjlim_\alpha S(X_\alpha),$$

so that $\gamma(X)$ is a discrete $\mathscr{G}$-module associated to X. This group can **74**
be interpreted as the Severi-group of the infinite dimensional object that was mentioned in lecture 1 (projective limit of all models of the filed $R(X)$) It is clear that $\gamma(X)$ is 'independent' of the choice of the family $\{X_\alpha, \varphi_\alpha\}_{\alpha\in I}$. Further, $\gamma(X)$ is a contravariant functor on the category of proper, K-regular absolutely irreducible schemes over K. We shall show that $\gamma(X)$ is a K-birational invariant, in the sense that if Y is again regular and K-proper and $\psi : Y \to X$ a K-birational map, ψ induces a canonical $\mathscr{G}$-isomorphism of $\gamma(X)$ and $\gamma(Y)$. Indeed, by the theorem of domination, we may assume that ψ is actually a morphism, so that we may take $Y = X_\alpha, \psi = \varphi_\alpha$ for some $\alpha \in I$. Since the set $I_\alpha = \{\beta \in I | \beta \geq \alpha\}$ is cofinal in I, we have canonical isomorphisms $\gamma(Y) \backsimeq \varinjlim_{\beta\in I_\alpha} S(X_\beta) \backsimeq \varinjlim_{\beta\in I} S(X_\beta) \backsimeq \gamma(X)$.

Now, $\gamma(X)$ itself is 'much too big' to be of any use by itself. However, it was pointed out by Manin that the cohomology group H'

$(\mathscr{G}, \gamma(\mathscr{X}))$ (in the sense of cohomology of profinite groups); (see [21]) is of manageable proportions. In fact, we shall show that if $\psi : S(X) \to \gamma(X)$ is the canonical homomorphism, ψ induces isomorphisms

$$H'(\mathscr{G}, S(X)) \to H'(\mathscr{G}, \gamma(X)).$$

By an elementary lemma on the cohomology of profinite groups we have that $\varinjlim_{\alpha} H'(\mathscr{G}, S(X_\alpha)) \simeq H'(\mathscr{G}, \gamma(X))$.

75 Thus, it sufficient to show that $\varphi_\alpha : X_\alpha \to X$ induces an isomorphism $H'(\mathscr{G}, S(X)) \tilde{\to} H'(\mathscr{G}, S(X_\alpha))$. Moreover, by the theorem of decomposition of birational morphisms, φ_α is a composite of dilatations, so that it is sufficient to show that if $\sigma : X' \to X$ is the dilatation of X at a closed point x of X, σ induces an isomorphism $H'(\mathscr{G}, S(X)) \tilde{\to} H'(\mathscr{G}, S(X'))$. Let $\bar{x}^{(1)}, \ldots \bar{x}^{(t)}$ be the complete set of conjugate points of $\bar{X}$ lying over x, so that $t = [k(x) : K]_s = [k(x) : K]$. Since $\bar{X}$ is regular, $\bar{\sigma} : \bar{X}' \to \bar{X}$ induces an isomorphism of $\bar{X}' - \bigcup_i \bar{\sigma}^{-1}(\bar{x}^{(i)})$ onto $\bar{X} - \{\bar{x}^{(-1)}, \ldots, \bar{x}^{(t)}\}$ and $\bar{\sigma}^{-1}(\bar{x}^{(i)}) \simeq \sigma^{-1}(x) X_{k(x)} \bar{K} \simeq \mathbb{P}'(\bar{K})$, we see that $\bar{X}'$ is the dilatation of $\bar{X}$ at the points $\bar{x}^{(1)}, \ldots, \bar{x}^{(t)}$. Let l be the image of the monomorphism of residue fields $k(x) \to k(x^{(1)}) = \bar{K}$, and $\mathscr{G}$ the subgroup of $\mathscr{J}$ which fixes l. By our remarks of the previous paragraph, we have a bijection $\mathscr{J}/\mathscr{G} \to \{\bar{x}^{(1)}, \ldots, \bar{x}^{(t)}\}$, given by $g\mathscr{G} \to g\bar{x}^{(1)}$, and this is compatible with the action of $\mathscr{J}$ on both the sets. Since $\bar{\sigma}$ again commutes with the action of $\mathscr{J}$ on $\bar{X}'$ and $\bar{X}$, we see that the set of divisors $\{\bar{\sigma}^{-1}(\bar{x}^{(1)})\}$ of $\bar{X}'$ is again stable for $\mathscr{J}$, and identifies itself as a $\mathscr{J}$-set with $\mathscr{J}/\mathscr{G}$. Now, we have proved earlier that $\bar{\sigma}^*$ is an injection of $S(\bar{X})$ into $S(\bar{X}')$, and $S(\bar{X}')$ decomposes as a direct sum $S(\bar{X}') \simeq \bar{\sigma}^*(S(\bar{X})) \oplus \sum_{i=1}^{t} \mathbb{Z}\bar{\sigma}^{-1}(\bar{x}^{(i)})$ and it follows that this is a decomposition of $\mathscr{J}$-modules. As a $\mathscr{J}$-module,
76 $\sum_{i=1}^{t} \mathbb{Z}\bar{\sigma}^{-}1(\bar{x}^{(i)}) = T$ is isomorphic to the $\mathbb{Z}$-free module $\mathbb{F}_{\mathbb{Z}}(\mathscr{J}/\mathscr{G})$ over $\mathscr{J}/\mathscr{G}$ considered as a left $\mathscr{J}$-module in the natural way. In order words this module is 'induced' from the trivial $\mathscr{G}$- module $\mathbb{Z}$. Hence, we have the isomorphisms $H^p(\mathscr{J}, T) \tilde{\to} H^p(\mathscr{G}, \mathbb{Z})$ ([21]), and since $H'(\mathscr{G}, \mathbb{Z}) = \varinjlim H'(H_\alpha, \mathbb{Z})$ where H_α are finite quotients of the profinite group $\mathscr{G}$, and $H'(H_\alpha, \mathbb{Z}) = \mathrm{Hom}(H_\alpha, \mathbb{Z}) = 0$, it follows that $H'(\mathscr{J}, T) = 0$. This proves our assertion that $H'(\mathscr{J}, S(X)) \simeq H'(\mathscr{J}, \gamma(X))$. Now, as we

have stated already, $S(X)$ is a finitely generated abelian group. Suppose further that $S(X)$ is torsion free. There is a normal subgroup $\mathscr{J}_\circ$ which is open and of finite index in $\mathscr{J}$ which fixes the generators of $S(X)$, and hence the whole of $S(X)$ ($\mathscr{J}_\circ$ can be taken as the subgroup which fixes a Galois extension L of K such that all the generators of $S(X)$ are defined over L'). Since $S(X)$ has no torsion by assumption $H'(\mathscr{J}_\circ, S(X)) = 0$, and it follows that

$$H'(\mathscr{J}, \gamma(X)) \xleftarrow{\sim} H'(\mathscr{J}, S(X)) \xleftarrow{\sim} H'(\mathscr{J}/\mathscr{J}_\circ, S(X)).$$

In particular, $H'(\mathscr{J}, \gamma(X))$ is a finite group which is 'Computable'.

We shall illustrate this by an example.

Let K be a perfect filed of characteristic $\neq 3$, and containing a primitive cube root ϱ of unity. Let X be the (absolutely irreducible and absolutely simple) projective surface X defined by the equation

$$x_1^3 + x_2^3 + x_3^3 = az_0^3, \tag{$*$}$$

where $a \neq 0$. We shall compute $H'(\mathscr{G}, \gamma(X)) = H'(\mathscr{G}, S(X))$. 77

Consider the set of six points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 1)$, $(1, \varrho, \varrho^2)$, $(1, \varrho^2, \varrho)$ of $\mathbb{P}^2$. Keeping in mind the fact that this set of points is invariant under permutations of co-ordinates, one verifies trivially that no three of these points lie on a line and all six do not lie on a conic. The cubic forms $U_1 = X_O(X_O X_1 - X_2^2), U_2 = X_O(X_O X_2 - X_1^2), U_3 = X_1(X_1 X_2 - X_O^2), U_4 = X_1(X_1 X_O - X_2^2)$ are linearly independent and vanish at all these points, and hence form a basis for all the cubic forms vanishing at these points. Further, these satisfy the equation

$$U_1 U_3 (U_1 + U_3) = U_2 U_4 (U_2 + U_4),$$

so that $\mathbb{P}^2$ blown up at the above six points is isomorphic to the cubic surface in $\mathbb{P}^3$ (with U_1, U_2, U_3 and U_4 as homogeneous co-ordinates) defined by the above equation. If we put $U_1 = x_1 + x_2, U_3 = \varrho(x_1 + \varrho x_2), U_2 = \sqrt[3]{a}\, x_o - x_3, U_4 = \varrho(\sqrt[3]{a}\, x_o - \varrho x_3)$, the above equation gets transformed into $(*)$. Thus, over $K(\sqrt[3]{a}), X$ becomes birationally isomorphic to $\mathbb{P}^2$. In particular, if a belongs to K^{*3}, that is, if a is a cube in K, X is birationally equivalent to $\mathbb{P}^2$ over K.

Now, choose a line L in $\mathbb{P}^2(\bar{K})$ defined over K. Any divisor of $\mathbb{P}^2(\bar{K})$ is linearly equivalent to a multiple of L. On the other hand, since the self intersection number of L is $+1$, no multiple of L is algebraically equivalent to 0.

78 Hence, $S(\mathbb{P}^2(\bar{K})) \backsimeq \mathbb{Z}$, and the action of $\mathcal{G}$ on $S(\mathbb{P}^2(\bar{K}))$ is trivial. It follows that *if* $a \in K^{*3}$,

$$H'(\mathcal{G}, \gamma(X)) = H'(\mathcal{G}, \gamma(\mathbb{P}^2)) = H'(\mathcal{G}, \mathbb{Z}) = 0.$$

Next, suppose that $a \notin K^{*3}$. The group $S(\bar{X})$ is a free group on seven generators. Six of these generators are the fibers of the blown up points in $\mathbb{P}^2$, and the seventh is the inverse image of a line in $\mathbb{P}^2$. But since this line may be chosen to pass through two of the blown up points, we see that the classes of the lines lying on $\bar{X}$ generate $S(\bar{X})$.

The 27 lines on $\bar{X}$ are easily written down by inspection. They are given by the equations

$$\left.\begin{matrix} x_i + \varrho^m x_j = 0 \\ x_k - \varrho^n \sqrt[3]{a} x_\circ = 0 \end{matrix}\right\} (i, j, k) \text{ a permutation of } (1, 2, 3), 0 \leq m, n < 3,$$

where $\sqrt[3]{a}$ denotes any fixed cube root of a. All these lines are defined over the field $K(\sqrt[3]{a})$ which is Galois over K, and which has for Galois group G a cyclic group of order 3 generated by an element g with $g(\sqrt[3]{a}) = \varrho \sqrt[3]{a}$. It follows that if l denotes the line defined by the equations

$x_i + \varrho^m x_j = 0, x_k - \varrho^n \sqrt[3]{a}\, x_o = 0$, then $l + gl + g^2 l$ is precisely the divisor cut out on $\bar{X}$ by the hyperplane $x_i + \varrho^m x_j = 0$.

We shall first establish that the group $S(\bar{X})^G$ of G-invariant elements 79 of $S(\bar{X})$ is isomorphic to $\mathbb{Z}$. Since $S(\bar{X})$ is torsion free, we have only to show that $S(\bar{X})^G$ is of rank 1. Let us put $S_Q(\bar{X}) = S(\bar{X}) \oplus_{\mathbb{Z}} \mathbb{Q}$, where $\mathbb{Q}$ denotes the rational filed, so that we have $S^G(\bar{X}) \oplus_{\mathbb{Z}} \mathbb{Q} \backsimeq (S_{\mathbb{Q}}(\bar{X}))^G$. If we put $N = 1 + g + g^2 \in \mathbb{Z}[G]$, we have clearly $S_{\mathbb{Q}}(\bar{X})^G = N.S_{\mathbb{Q}}(\bar{X})$, since for $\alpha \in S_{\mathbb{Q}}(\bar{X})^G$, we have $\alpha = \dfrac{1}{3} N\alpha$. Now, $S_{\mathbb{Q}}(\bar{X})$ is generated over $\mathbb{Q}$ by the canonical images of the above lines. If l is a line, we have shown above that $Nl = l + gl + g^2 l$ is a hyperplane section. Finally, the difference of two hyperplane of two hyperplane section is linearly equivalent to 0 on $\bar{X}$ (since any two hyperplanes are linearly equivalent in $\mathbb{P}^3$). In consequence of these remarks, it follows that the rank of $S(\bar{X})^G$= dimension over $\mathbb{Q}$ of $S_{\mathbb{Q}}(\bar{X})^G \leq 1$. Now, G admits precisely two irreducible representations over the rational filed, viz., the trivial one-dimensional representation and the two dimensional representation in the cyclotomic filed $\mathbb{Q}(\varrho)$. Since $S_{\mathbb{Q}}(\bar{X})$ is seven dimensional over $\mathbb{Q}$, it follows that we must necessarily have $S_{\mathbb{Q}}(\bar{X}) \simeq \mathbb{Q} \oplus \mathbb{Q}(\varrho)^3$ as $\mathbb{Q}[G]$-modules, because of the complete reducibility of $\mathbb{Q}$-representations of G. In particular, we must have $\dim_{\mathbb{Q}} S_{\mathbb{Q}}(\bar{X})^G = 1$, $S(\bar{X})^G = \mathbb{Z}$.

Now, for any finitely generated G-module M, the cohomology groups $H^p(G, M)(p \geq 1)$ are finite dimensional vector spaces over $\mathbb{Z}_3 = \mathbb{Z}/_{3\mathbb{Z}}$. Let us put $\chi(M) = \dim_{\mathbb{Z}_3} H^2(G, M) - \dim_{\mathbb{Z}_3} H'(G, M)$. (The rational number $3^{\chi(M)}$ is usually called the Herband quotient of M). It is λ **80**
then well-known ([21]) that (i) if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of finitely generated G-modules, $\chi(M) = \chi(M') + \chi(M'')$, and (ii) $\chi(M) = 0$ if M is finite. It easily follows from (i) and (ii), that if M_1 and M_2 are two finitely generated G-modules such that $M_1 \oplus_{\mathbb{Z}} \mathbb{Q}$ and $M_2 \oplus_{\mathbb{Z}} \mathbb{Q}$ are isomorphic $\mathbb{Q}[G]$- modules, $\chi(M_1) = \chi(M_2)$.

Applying this remark to $S(\bar{X})$, we deduce that

$$\chi(S(\bar{X})) = \chi(\mathbb{Z} \oplus \mathbb{Z}[\varrho]^3) = \chi(\mathbb{Z}) + 3\chi(\mathbb{Z}[\varrho]).$$

Now, the exact sequence of G-modules $0 \to \mathbb{Z}.N \to \mathbb{Z}[G] \to \mathbb{Z}[\varrho] \to 0$ yields that $\chi(\mathbb{Z}[\varrho]) + \chi(\mathbb{Z}) = \chi(\mathbb{Z}[G]) = 0$. Further, since $H'(G, \mathbb{Z}) = 0$ and $H^2(G\mathbb{Z}) \simeq \dfrac{\mathbb{Z}^G}{N\mathbb{Z}} \simeq \dfrac{\mathbb{Z}^G}{N\mathbb{Z}} \simeq \mathbb{Z}_3, \chi(\mathbb{Z}) = 1$. It follows that

$$\chi(S(\bar{X})) = \dim_{\mathbb{Z}_3} H^2(G, S(\bar{X})) - \dim_{\mathbb{X}_3} H'(G, S(\bar{X})) = -2,$$
$$H'(\mathscr{G}, \gamma(X)) = H'(G, S(\bar{X})) \neq 0.$$

In view of our earlier remarks, it follows that if $a \notin K^{*3}$ the K-surface X is not birationally equivalent to $\mathbb{P}^2(K)$ *over* K. It follows that if $a, b \in K^*$ with $K(\sqrt[3]{a}) \neq K(\sqrt[3]{b})$, and if X_a an X_b denoted the corresponding surfaces in $\mathbb{P}^3$, X_a and X_b are not birationally equivalent even over $K(\sqrt[3]{a})$, and a fortiori not equivalent over K. Now, by Kummer theory, $K(\sqrt[3]{a}) = K(\sqrt[3]{b})$ if and only if a and b generate the same subgroup in K^*/K^{*3}, that is, if and only if either $a = bc^3$ or $a = c^3/b$ for same

$c \in K^*$. In the first case, X_a and X_b are clearly even projectively isomor-
81 phic over K. However, we are unable to decide whether X_a and X_b are birationally equivalent over K or not when $a = c^3/b$ with $c \in K^*$.

Let us mention here another problem of geometric interest, where the difficulties are similar. A variety X over a filed K is said to be *rational* if it is birationally equivalent to a projective space over K, and *unirational* if there is a dominant k-rational map of a projective space onto X. Thus, X is rational if and only if its function field $R(X)$ is a purely transcendental extension of K, and X is unirational if and only if $R(X)$ is K-isomorphic to a subfield of a pure transcendental extension of K. It has been proved by Luroth that any unirational curve is rational. Castelnuovo ([3]) (in the classical case $K = \mathbb{C}$) and Zariski ([24], [25]) have proved that when K is algebraically closed and L an extension of K contained in pure transcendental extension $K(X, Y)$ of K such that $K(X, Y)$ is separable over L, then L is purely transcendental over K. It has been generally believed that the analogous statement in dimension three is false. Many examples of unirational three dimensional varieties which are thought not to be rational have been suggested, the most notable being the general cubic hypersurface in $\mathbb{P}^4$. But in no case has it been established that the suggested variety is not rational. The difficulty
82 in this case is again the lack of suitable invariants which distinguish between varieties which are rational and those which are merely unirational, the dimensions of sheaf cohomologies being insufficient for this purpose. Incidentally, there is a closer connection between the problem we discussed earlier and the present problem than is apparent. Indeed, the function filed R of a cubic hypersurface in $\mathbb{P}^4$ over a field K is of the form $R = K(X, Y, Z, T)$ with X, Y, Z and T connected by a single cubic polynomial relation $F_3(X, Y, Z, T) = 0$. Now R may also be considered as the field of functions of a cubic surface over $K(T)$. But now, since a general cubic in $\mathbb{P}^3$ over an algebraically closed filed is rational, as we have stated earlier, if L is the algebraic closure of $K(T)$, the composite field $L\ R$ is a field of rational functions over L in two variables. In other words, the field extension $R/K(T)$, becomes a rational function filed on extension of base from $K(T)$ to L. But the nature of the extension $R/K(T)$ itself is unknown. Of course the real difficulty lies in the

fact that the subfield $K(T)$ is not uniquely determined in $K(X, Y, Z, T)$.

Lecture 6

Intersection theory on two dimensional regular preschemes

Let X be a noetherian two dimensional regular prescheme. We shall say **83**
that two divisors D_1 and D_2 on X *intersect properly at a closed point* $x \in X$ if x is an isolated point of the intersection $|D_1| \cap |D_2|$ of their supports. When D_1 and D_2 intersect properly at a closed point x of X, we shall associate to D_1, D_2 and x an integer $(D_1, D_2)_x \in \mathbb{Z}$ called the *intersection multiplicity* of D_1 and D_2 at x. First assume that D_1 and D_2 are effective, that is, that they are linear combinations of irreducible divisors with non-negative integral coefficients, or equivalently, that they are defined at any point of X by functions which are regular at x. Suppose D_i is defined at x by an element f_i of the local ring $\mathscr{O}_x$ of x at X. Since x is an isolated point of $|D_1| \cap |D_2|$ by assumption, the ideal (f_1, f_2) in $\mathscr{O}_x$ is primary for the maximal ideal $\mathfrak{M}_x$ of $\mathscr{O}_x$. We then define

$$(D_1.D_2)_x = l_{\mathscr{O}x}(\mathscr{O}_x/(f_1, f_2)) \tag{1}$$

where $l_{\mathscr{O}x}(M)$ denotes the length over $\mathscr{O}_x$ of an Artinian module M. This definition is obviously independent of the choice of the f_i defining D_i, and we have evidently $(D_1.D_2)_x = (D_2.D_1)_x$. Suppose D_1' is another effective divisor intersecting D_2 properly at x. and let f_1' be a defining

84 element of D'_1 at x. Let B be the local ring $\mathscr{O}_x/f_2.\mathscr{O}_x$; then f_1 and f'_1 are not-zero-divisors in B, since f_1 and f'_1 do not belong to the prime ideals associated to f_2 in $\mathscr{O}_x$. We have therefore

$$\begin{aligned}(D_1 + D'_1, D_2) &= l_{\mathscr{O}_x}(\mathscr{O}_x/(f_1 f'_1, f_2)) = l_B(B/f_1 f'_1 B)\\ &= l_B(B/f_1 B) + l_B(f_1 B/f_1 f'_1 B)\\ &= l_B(B/f_1 B) + l_B(B/f'_1 B)\\ &= (D_1.D_2)_x + (D'_1, D_2)_x \qquad (2)\end{aligned}$$

If now D_1 and D_2 are arbitrary (not necessarily effective) divisors intersecting properly at x, we can write $D_i = D'_i - D''_i$ where D'_i, D''_i are effective and either one of D'_1, D''_1 intersects either one of D'_2, D''_2 properly at x, and we define

$$(D_1.D_2)_x = (D'_1.D'_2)_x + (D''_1.D''_2)_x - (D'_1.D''_2)_x - (D''_1.D'_2)_x.$$

It follows from what we have shown above that this integer is independent of the representation of D_i as $D'_i - D''_i$. Commutativity and biadditivity of intersection multiplicity continue to hold for arbitrary divisors (which intersect properly).

Now suppose X is a *B-scheme of finite type* fulfilling the condition stated at the beginning of the last paragraph, and let $\varphi : X \to B$ be the structural morphism. Let b be a point of B and D a divisor on X
85 such that $|D| \subset \varphi^{-1}(b)$ and $|D|$ considered as a reduced scheme is *proper over the residue residue field $k(b)$ at b*. Since any closed point x of X belonging to $\varphi^{-1}(b)$ is also a closed point of $\varphi^{-1}(b)$ is a scheme of finite type over $k(b)$, it follows from Hilbert's Nulletellensatz that $k(x)$ is a finite algebraic extension of $k(b)$. Let D' be any divisor on X which does not have any common component with D, so that D and D' intersect properly at any point of $|D| \cap |D'|$ and $|D| \cap |D'|$ is a finite set of closed points of X. We define *the intersection number* $(D.D')$ of D and D' (over b) as

$$(D.D') = \sum_{x \in |D| \cap |D'|} [k(x) : k(b)].(D.D')_x \qquad (3)$$

We shall show that if D'' is another divisor which is linearly equivalent to D' and has no common component with D, we have

$$(D.D') = (D.D'') \qquad (4)$$

We may clearly assume D to be an irreducible divisor, and X to be irreducible. We have to show that if $f \in R(X)$, $f \neq 0$ and div (f) does not contain D as a component, $(D.div(f)) = 0$. Let C be the reduced subscheme of X with $|D|$ for its underlying set, so that C is an integral scheme, proper over $k(b)$ and of dimension one (a *curve* over $k(b)$, for short). By assumption, the restriction g of f to C is defined as a rational function on C, and g is not identically 0.

We recall the definition of the *order* of a non-zero rational function **86**
g on a curve a field k at a closed point x of C. If g belongs to the local ring $\mathscr{O}_{x,C}$ of C at x, we define the order $o_x(g)$ of g at x to be the integer $l_k(\mathscr{O}_{x/g\mathscr{O}_x})$. Since for $g, g' \in \mathscr{O}_x$, both different from 0, we have

$$O_x(gg') = l_k(\mathscr{O}_{x/gg'\mathscr{O}_x}) = l_k\left(\frac{\mathscr{O}_x}{g\mathscr{O}_x}\right) + l_k\left(\frac{g\mathscr{O}_x}{gg'\mathscr{O}_x}\right) + l_k\left(\frac{\mathscr{O}_x}{g'\mathscr{O}_x}\right) = \mathscr{O}_x(g) + \mathscr{O}_x(g'),$$

we can extend the definition of O_x uniquely to the multiplicative group $R(C)^*$ of non-zero elements of $R(C)$ such that $O_x : R(C)^* \to \mathbb{Z}$ is a homomorphism. With this definition, it is clear that we have, for D and f, and g as in the previous paragraph,

$$(D.\text{div}(f)) = \sum_{x\in C} O_x(g)$$

Thus, if we can show that the sum of the orders of a non-zero element g of $R(C)$ over all closed points of a complete curve C over a field k is 0, the proof of (4) would be complete. We shall now prove this statement. Let C' be the normalisation of C in its field $R(C)$ of rational functions, and $\pi : C' \to C$ the associated morphism. We identify $R(C)$ and $R(C')$ by means of π. Let x be a closed point of C and $x'_1, \ldots, x'_n$ be the points of the fibre $\pi^{-1}(x)$. We shall show that

$$O_x(g) = \sum_{i=1}^{n} 0_{x'_i}(g)$$

We may assume that $g \in O_x$, because of the additivity of the order. **87**
Let $\overline{\mathscr{O}_x}$ be the integral closure of $\mathscr{O}_x$, so that $\overline{\mathscr{O}_x}$ is a semilocal ring and the local rings $\mathscr{O}_{x'_1,C'}$ are the localisations of $\overline{\mathscr{O}_x}$ at the distinct maximal

ideals of $\overline{\mathscr{O}_x}$. Further $\overline{\mathscr{O}_x}$ is an $\mathscr{O}_x$ -module of finite type. We have therefore

$$\sum_{i=1}^{n} o_{x_l(g)} = l_k\left(\frac{\overline{\mathscr{O}_x}}{g\overline{\mathscr{O}_x}}\right) = l_k\left(\frac{\overline{\mathscr{O}_x}}{g\mathscr{O}_x}\right) - l_k\left(\frac{g\overline{\mathscr{O}_x}}{g\mathscr{O}_x}\right)$$
$$= l_k\left(\frac{\overline{\mathscr{O}_x}}{g\mathscr{O}_x}\right) = l_k\left(\frac{\overline{\mathscr{O}_x}}{\mathscr{O}_x}\right) = l_k\left(\frac{\overline{\mathscr{O}_x}}{g\mathscr{O}_x}\right) = O_x(g).$$

Since we have $\sum\limits_{x\in C} O_x(g) = \sum\limits_{x\in C}\sum\limits_{\pi(x'_i)=x} O_{x'_l}(g) = \sum\limits_{x'\in C'} O_{x'}(C')$, it is sufficient to prove the equation $\sum\limits_{x\in C} O_x(g) = 0$ for a *regular* complete curve a field k. This is of course well-known, and we shall indeed deduce this form a more general lemma, which we postpone for the moment in order not to break the continuity of the argument.

It follows from (4) (X assumed noetherian, regular, two dimensional, and separated, and of finite type over a base prescheme B, D a divisor with $|D| \subset \varphi^{-1}(b)$ where b is a point of B and $|D|$ proper over $k(b)$), that $(D.D')$ depends only on the divisor class of D' (that is, on the image of D' in $\vartheta(X)/\vartheta_l(X)$). We shall show that in every divisor class, there is a divisor D' which has no common components with D. For this, one may clearly assume X to be irreducible. Let D'' be any divisor
88 on X, so that we can write $D'' = D''_1 + D''_2$, where all the components of D''_1 are components of D and D''_2 has no common components with D. Let $C_1, \ldots, C_r$ be the irreducible component of D, and $V_1, \ldots, V_r$ the corresponding discrete valuations of the functions field $R(X)$ of X, Since X is by assumption separated over B and C_i are contained in the same fibre $\varphi^{-1}(b)$, the valuations V_i are all distinct. If we write $D''_1 = \sum\limits_1^r n_i C_i$, it follows from the theorem of independence of valuations ([14]) that we can find an $f \in R(X)^*$, with $V_i(f) = -n_i$. Thus, if we put $D' = D'' + div(f)$, D' and D'' are linearly equivalent and D' has no common component with D. Thus, we have a unique homomorphism of $\dfrac{\vartheta(X)}{\vartheta_l(X)}$ into $\mathbb{Z}$ which for a divisor class (D') takes the value (D, D') if D' takes the value $(D.D')$ if D' has no common components with D. We shall denote the image of a divisor class ξ under this homomorphism again

by the same symbol $(D.\xi)$.

If the base B is Spec K where K is a field and X is proper over B, it follows that we have a symmetric from $\frac{\vartheta(X)}{\vartheta_l(X)} \times \frac{\vartheta(X)}{\vartheta_l(X)} \to \mathbb{Z}$ which is defined by the condition that if D and D' are divisors having no common components, the value $((D).(D'))$ of this bilinear form on the pair $((D),(D'))$ is the intersection number $(D.D')$. We shall again call this integer $((D).(D'))$ the intersection number of the divisor classes (D) and (D'). Assume further that K is algebraically closed. It can then be shown that if $D \in \vartheta_a(X)$ (that is, if D is algebraically equivalent to 0), $(D.D') = 0$ for any D'. Thus, the above pairing 'goes down' to a pairing of the Neron-Severi group $S(X) = \frac{\vartheta(X)}{\vartheta_a(X)}$. It has been shown 89
that for any element $\xi \in S(X)$, $(\xi.\eta) = 0$ for all $\eta \in S(X)$ if and only if ξ is a torsion element of $S(X)$ ([22]). Thus, if we put $S_Q(X) = Q \otimes_{\mathbb{Z}} S(X)$, we have a non-degenerate rational symmetric bilinear form on the finite dimensional vector space $S_Q(X)$. It is also known that the signature of this form is $(+, \underbrace{-, \ldots, -}_{r-1})$ where r is the dimension of $S_Q(X)$. We will neither prove nor use any of these results mentioned.

Next, suppose the base prescheme B is noetherian, regular and of dimension one and $\pi : X \to B$ is surjective. Any closed point $b \in B$ determines a divisors of B, which is defined by a generator of the maximal ideal $\mathfrak{M}_b$ of the local ring $\mathscr{O}_{b,B}$ at b and by 1 at all other points of B. We shall denote the inverse image of this divisor under the structural morphism $\pi : X \to B$ also by $\pi^{-1}(b)$, and it will always to be clear from the contest whether we mean by $\pi^{-1}(b)$ this divisor or the closed subscheme of X which is the fibre at b. We now have the following lemma, which when X is proper over an algebraically closed field, is a special case of the invariance of intersection number under algebraic equivalence stated int the earlier paragraph.

Lemma. *Let B noetherian irreducible one-dimensional regular prescheme and let $\pi : X \to B$ be a surjective proper morphism, where X is two dimensional and regular. If D is a divisor on X and b_1, b_2 are two closed points of B such that D intersects $\pi^{-1}(b_i)$ properly, we have* 90

$$(D.\pi^{-1}(b_1)) = (D, \pi^{-1}(b_2)). \tag{5}$$

Proof. We may clearly suppose that $B - Spec\,A$ Where A is a Dedekind domain, and also that D is irreducible. Let us denote the reduced subscheme of X having $|D|$ for support by the same symbol D, and let π_1 denote the restriction of π to D. Since $\pi_1(D)$ is a closed irreducible subset of B, it must either be a single closed point b of B or it must be the whole of B. In the former case, we have necessarily $b \neq b_i$, so that both sides of the above equation (5) become. In the latter case, it is easily seen that (Since both D and B are one dimensional and π_1 of finite type) $\pi_1^{-1}(b)$ is finite for any point bB. Since π_1 is also proper, it follows from a theorem of Chevalley [EGA III, 4.4.2] that D is isomorphic as a $B-$*scheme* to Spec C, where C is an A-alegbra which is an A-module of finite type. Since D is irreducible and reduced and of dimension one, C is a domain and C has no. A-torsion. A being a Dedekind domain, this implies that C is a projective A-module of rank r(=degree of quotient field of C over that of A) say. It follows that for any maximal ideal $\mathfrak{M}$ of A, we have

$$l_A\left(\frac{C}{MC}\right) = r.$$

But $l_A\left(\frac{C}{\mathfrak{M}C}\right) = r = l_{A/\mathfrak{M}}\left(\frac{C}{\mathfrak{M}C}\right) = r$ is clearly the intersection number $(\pi^{-1}(b).\ (D))$, where b is the point of B defined by $\mathfrak{M}$. This proves the lemma. □

91 We shall deduce that for a regular curve C over a field k and for an element $f \in R(C)^*$, we have $\sum_{X \in C} O_x(f) = 0$, as promised earlier. We take $X = C\ X_k \mathbb{A}'(k) = \operatorname{Spec} \mathscr{O}_C[T]$ and $B = \mathbb{A}'(k) = \operatorname{Spec} k[T]$ in the above lemma. Let D be the principal divisor on X defined by the rational function $g = Tf + (1-T)$ on X. Let 0 and 1 be the points of $\mathbb{A}'(k)$ defined by the maximal ideals (T). The fibres $\pi^{-1}(0)$ and $\pi^{-1}(1)$ are canonically isomorphic with C over k, and the restrictions of g to $\pi^{-1}(0)$ and $\pi^{-1}(1)$ go over into the rational functions 1 and f on C respectively by means of these isomorphisms. It follows that

$$\sum_{X \in C} O_x(f) = (\pi^{-1}(1).D) = (\pi^{-1}(0.)D) = \sum_{x \in C} O_x(1) = 0,$$

which was the assertion to be proved.

Let us return ot the case of a regular two dimensional prescheme X proper over a regular one dimensional noetherian prescheme B. By the lemma, for any divisor D on X, the intersection number $(D.\pi^{-1}(b))$ is independent of the closed point b of B. We shall therefore call this integer the intersection number of D with a fibre. Suppose now D itself has support $|D|$ contained in the fibre $\pi^{-1}(b)$. Then we assert that $(D.\pi^{-1}(b)) = 0$. Infact, let f be a rational function on B which is regular at b and which generates the maximal ideal $\mathfrak{M}_b$ of $\mathscr{O}_{b,B}$. It is then clear for the rational function $f \circ \pi$ on X, we have

$$\text{div}\,(f \circ \bar{x}) = \pi^{-1}(b) + C,$$

where

$$|C| \cap \pi^{-1} = \phi.$$

We have therefore 92

$$\begin{aligned}(D.\pi^{-1}(b)) &= (D.\pi^{-1}(b)) + (D.C)\\ &= (D.\,\text{div}\,(f \circ \pi)) = 0,\end{aligned}$$

which proves our assertion. Now, let ϑ_b be the subgroup of $\vartheta(X)$ consisting of those divisors with support contained in $\pi^{-1}(b)$. Then ϑ_b is a free abelian group on the irreducible components of $\pi^{-1}(b)$, and ϑ_b decomposes as a direct sum $\vartheta_b = \vartheta_1 \oplus \cdots \oplus \vartheta_r$, where each ϑ_i consists of those divisors with support contained in a connected component F_i of $\pi^{-1}(b)$. It is then cleat that ϑ_i and ϑ_j are orthogonal for the symmetric bilinear form $(,)$ for $i \neq j$. Let us write $\pi^b = D_1 + \cdots + D_r$ where D_i is an effective divisor in ϑ_i. Since $\pi^{-1}(b)$ is orthogonal to the whole of ϑ_b, it follows that each D_i is orthogonal to ϑ_i. We shall show that for each $C \in \vartheta_i, C \notin \mathbb{Z}D_i, (C^2) < 0$.

This follows immediately from a purely formal result:

Lemma. *Let V be a Q-vector space and $C_1, \ldots, C_n$ a basis of V. Suppose that in V a bilinear symmetric form $(,)$ is given such that:*

a) $(C_i, C_j) \geq 0$ *for* $i \neq j$.

b) There exists $m_1, \ldots, m_n, m_i > 0$*, such that* $(C_i, \vartheta_0) = 0$*,* $i = 1, 2, \ldots, n, \vartheta_o = \sum m_i C_i$.

c) *The set* $\{C_1, \ldots, C_n\}$ *is connected, that is, it cannot be divided into two parts in such a way that* $(C_i, C_j) = 0$ *if* C_i *and* C_j *are in different parts.*

93 Then $(\vartheta, \vartheta) \leq 0$, for any $\vartheta \in V$. Moreover, if $(\vartheta, \vartheta) = 0$ then $\vartheta = \alpha\vartheta_\circ, \alpha \in Q$.

Proof. First, we prove by induction on n that $(\vartheta, \vartheta) \leq 0$. The cases $n = 1$ and $n = 2$ are easy to verify.

Suppose the statement true for $n - 1 \geq 2$

1) If $(A, A) > 0$ where $A = \sum_{i=1}^{n} \alpha_i C_i$, $\alpha_i \in \mathbb{Q}$, then the vector A cannot have two coordinates equal to zero.

   If, for example, $\alpha_{n-1} = \alpha_n = 0$, then we take $C_1^* = C_1, C_2^* = C_2, \ldots, C_{n-2}^*$, $C_{n-2}^* = m_{n-1}C_{n-1} + m_n C_n$. Let V^* be the subspace of V spanned by $C_1^*, \ldots, C_{n-1}^*$ and $(,)^*$ be the restriction of the form $(,)$ to v^*. Then a), b), c) are satisfied for V^* and $C^*, \ldots, C_{n-1}^*$. Hence $(,)^*$ is semi-negative. But $A \in V^*$. Thus $(A, A) \leq 0$. Contradiction.

2) If $A = \alpha_1 C_1 + \cdots + \alpha_n C_n$, $(A, A) > 0$ and $\alpha_i = 0$ for some i then either all the remaining coordinates are positive or are all negative.

Assume that $\alpha_n = 0$, $\alpha_1 > 0, \ldots, \alpha_i > 0$, $\alpha_{i+1} < 0, \ldots, \alpha_{n-1} < 0$ where $1 < i < n-1$. Then $A = A_1 - A_2$ for $A_1 = \alpha_1 C_1 + \cdots + \alpha_i C_i$ and $A_2 = (-\alpha_{i+1})C_{i+1} + \cdots + (-\alpha_{n-1})C_{n-1}$. It follows form a) that $(A_1, A_2) \geq 0$. But $o < (A, A) = (A_1, A_1) + (A_2, A_2) - 2(A_1, A_2)$. Thus either $(A_1, A_1) > 0$ or $(A_2, A_2) > 0$. Since A_1, A_2 have at least two coordinates equal to zero (for A_1 the n^{th} and $(n-1)^{st}$; for A_2, the n^{th} and 1^{st}; this contradicts 1)

94 Let $(A, A) > 0$; we may assume that $\alpha_n = 0$; otherwise we replace A by $A - \dfrac{\alpha_n}{m_n}\vartheta_\circ$. Hence, it follows from 2) that we may assume that $\alpha_1 > 0, \ldots, \alpha_{n-1} > 0$. Since, $n - 1 \geq 2$, we may choose $i \neq j$, $1 \leq i < j \leq n-1$ such that $\dfrac{\alpha_i}{m_i} - \dfrac{\alpha_j}{m_j} \geq 0$. Let $B = A - \dfrac{\alpha_j}{m_j}\vartheta_o$. Then

$$(B, B) > 0$$

The i^{th} coordinate of $B = \alpha_i - \dfrac{\alpha_j}{m_j} m_i \geq 0$, the j^{th} coordinate of $B = \alpha_j - \alpha_{j/m_j} m_j = 0$, the n^{th} coordinate of $B = 0 - \dfrac{\alpha_j}{m_j} m_n < 0$. As $i \neq j$, this contradicts 2). Now we can prove the second statement.

Since $(\vartheta, \vartheta_\circ) = 0$ for all $\nu \in V$, the rank of the bilinear form $(,)$ is $\leq n - 1$. We have to prove that it equals $n - 1$. Let it be $\leq n - 2$ and let

$$W = \Big\{ \omega \in V : (\omega, \nu) = 0 \text{ for all } \vartheta \in V \Big\}.$$

Then on V/W we have negative definite form. Let $C_1, \ldots, C_e$ be such that they give a basis in V/W. Then $(\alpha_1 C_1 + \cdots + \alpha_e C_e)^2$ is < 0 except when all $\alpha_i = 0$. Put $C_{e+1} = L + D$ where $D \in W$ and L is a linear combination of $C_1, \ldots, C_e$. Let $L = L_1 - L_2$ where L_1 (resp. -L_2) contains all C_i that enter in L with positive (resp. negative) coefficients. Then $(C_{e+1}, L_i) \geq 0$ because of a). On the other hand, $(C_{e+1}, L_1) = (L_1, L_1) - (L_2, L_1) \leq 0$. This is possible only when $(L_1, L_1) = 0$ and $(L_1, L_2) = 0$. From this it follows that $L_1 = 0$ and so $C_{e+1} + L_2 \in W$. In particular, $(C_{e+1} + L_2, c_i)$ is zero for all C_i. Since we have assume that 95
dim $V/W \leq n - 2$, there have to be C_i that do not enter in $C_{e+1} + L_2$ and because of a) this gives a division of the set $C_1, \ldots, C_n$, into two parts with properties contradicting c). □

This proves that the restriction of the symmetric bilinear form ϑ_i has the signature $(0, \underbrace{, -, -, \ldots -}_{s_i - 1})$ where s_i is the number of irreducible components of the connected component F_i. On the group ϑ_b itself this form is negative semidefinite, and has null space of dimension equal to the number r of connected components of the fibre $\pi^{-1}(b)$.

We now consider the case when the base is also of dimension two. Let X, Y be noetherian integral schemes of dimension two with Y regular, and let $f : Y \to X$ be a proper surjective morphism. Let x be a closed point of X, and let $C_1, \ldots, C_s$ be the irreducible divisors contained in $f^{-1}(x)$. We shall show that the restriction of $(,)$ to the free group generated by the C_i is negative definite. For this purpose, we fix a non-zero rational function g on X which is regular at x and vanishes at

x such that on Y, $if\ div(g \circ f) = \sum_{i}^{s} m_i C_i + P$ where P does not contain any C_i as a component, we have $m_i > 0$, P. is effective in a neighbourhood of $f^{-1}(x)$ and P has non-empty proper intersection with each C_i.
96 (This is possible, since we can find for each i an irreducible divisor D_i distinct from all the C_j and intersecting C_i, and we have only to take g to be a function regular at x and vanishing on $f(\bigcup_{i=1}^{s} D_i)$. Then (C_i, C_j) and (C_i, P) are defined. Let V be a Q-vector space with basis $C_1, \ldots, C_s$, P. We define the function $(,)$ on V taking for $(C_i, C_j), i \neq j$, and (C_i, P) the values already defined. We define (P, P) and (C_i, C_i) by means of the relation $(C_i, P + \sum m_j C_j) = 0$. Then $P + \sum m_j C_j$ is orthogonal to all the basis elements and we can apply the preceding lemma. Conditions $a), b), c)$ are satisfied. Hence $(,)$ is negative definite on the space spanned by $C_1, \ldots C_s$. This shows that the restriction of $(,)$ to the group ϑ_x of divisors on Y having support contained in $f^{-1}(x)$ is negative definite. This is a theorem due to Mumford. Suppose now that X is a surface over an algebraically closed field K and x a normal singular point. In this case, we can find an $f : Y \to X$ such that f is birational and proper Y regular by the Theorem of reduction of singularities of a surface ([1]). Using the negative definiteness of $(,)$ on the fibre, Mumford shows ([17]) that there is a unique inverse image operation $f^* : \vartheta(X) \otimes_{\mathbb{Z}} Q \to \vartheta(Y) \otimes_{\mathbb{Z}} Q$ and a unique intersection theory of (properly intersecting) divisorial cy-
97 cle on X, where the intersection multiplicities are in general fractions, such that the 'projection formula' is valid. This intersection theory is independent of the choice of Y, and enjoys all the 'good' properties of intersection theory on a non-singular variety. The denominator of the intersection multiplicities is det $((C_i, C_j))$.

Let X, Y and f be as in the beginning of the last paragraph, and suppose further that x is also regular. We define the 'projection' homomorphism $f_* : \vartheta(Y) \to \vartheta(X)$ by putting for an irreducible divisor C on Y,

$$f_*(C) = \begin{cases} 0 \quad \text{if } \dim f(C) = 0 \\ [k(z) : k(f(z))] f(C), \text{ if } \dim f(C) = 1, \text{ where} \\ z \quad \text{is a generic point of } C. \end{cases}$$

Let x be a closed point of X and C a divisor on Y with $|C| \subset f^{-1}(x)$. Then for any divisor D on X, we have

$$(C.f^{*}(D)) = 0, \text{ for } |C| \subset f^{-1}(x) \tag{6}$$

since D is defined in a neighbourhood of x by a rational function. Next, suppose that C is any divisor on Y such that C and $f^{*}(D)$ intersect properly at all points of their intersection on $f^{-1}(x)$. Then we have the projection formula

$$(f_{*}(C).D)_x = \sum_{z \in f^{-1}(x)} [k(z) : k(x)](C.f^{*}(D))_z \tag{7}$$

To prove this, we may assume X affine, $D = div(g)$ where g is regular on X, C irreducible and not a component of a fibre over X. By the theorem of Chevalley mentioned earlier, C and $C' = f(C)$ are affine one dimensional schemes and $A = \Gamma(C, \mathscr{O}_C)$ is a $\Gamma(C', \mathscr{O}'_C) = A'$-module of finite type. If g_1 denotes the restriction of g to $C', \mathfrak{M}'$ the maximal ideal of x in A' and $\mathfrak{M}_i$ the maximal ideals of A lying over $\mathfrak{M}'$, we have **98**

$$(f_{*}(C).D)_x = n(C'.D)_x = nl_{A'}\left(\frac{A'_{\mathfrak{M}'}}{g_1.A'_{\mathfrak{M}'}}\right), n = [R(C) : R(C')],$$

and $$\sum_{z \in f^{-1}(x)} [k(z) : k(x)](C.f^{*}(D))_z = \sum_i [k(z_i) : k(x)] l_{A_{\mathfrak{M}_i}}\left(\frac{A_{\mathfrak{M}_i}}{g.A_{\mathfrak{M}_i}}\right)$$
$$= l_{A'_{\mathfrak{M}'}}\left(\frac{A \otimes_{A'} A'_{\mathfrak{M}'}}{gA \otimes_{A'} A'_{\mathfrak{M}'}}\right).$$

If we put $B = A \otimes_{A'} A'_{\mathfrak{M}'}$, and if $\theta \in A$ generates $R(C)$ over $R(C')$, $B/A'_{\mathfrak{M}'}[\theta]$ is of finite length over $A'_{\mathfrak{M}'}$ and $A'_{\mathfrak{M}'}[\theta]$ if free of rank n over $A'_{\mathfrak{M}'}$. We have therefore

$$l_{A'_{\mathfrak{M}'}}(B/gB) = l_{A'_{\mathfrak{M}'}}(B/gA'_{\mathfrak{M}'}[\theta]) - l_{A'_{\mathfrak{M}'}}\left(\frac{gB}{gA'_{\mathfrak{M}'}[\theta]}\right)$$
$$= l_{A'_{\mathfrak{M}'}}\left(\frac{B}{A'_{\mathfrak{M}'}[\theta]}\right) + l_{A'_{\mathfrak{M}'}}\left(\frac{A'_{\mathfrak{M}'}[\theta]}{gA'_{\mathfrak{M}'[\theta]}}\right) - l_{A'_{\mathfrak{M}'}}\left(\frac{B}{A'_{\mathfrak{M}'}[\theta]}\right)$$

$$= l_{A'_{\mathfrak{M}'}}\left(\frac{A'_{\mathfrak{M}'}}{gA'_{\mathfrak{M}'}} \otimes_{A'_{\mathfrak{M}'}} A'_{\mathfrak{M}'}[\theta]\right) = nl_{A'_{\mathfrak{M}'}}\left(\frac{A'_{\mathfrak{M}'}}{gA'_{\mathfrak{M}'}}\right)$$

which prove (7). One deduces immediately from (6), and (7) that when 99
X (and hence also Y) is *proper over a field* K, we have

$$(f_*(C).D) = (C.f^*(D)), \tag{8}$$
$$(f^*(D).f^*(D')) = n(D.D')$$

for $C, C' \in \vartheta(Y)/\vartheta_l(Y)$ and $D, D' \in \vartheta(X)/\vartheta_l(X)$, where n is the degree of f, that is, $n = [k(y) : k(f(y))]$, y being a generic point of Y.

Let us apply (6) and (7) to the case when $Y = X'$ is the dilatation of X at a closed point x and $f = \sigma$ is the associated morphism. We put $L = \sigma^{-1}(x) \simeq \mathbb{P}'(k)$, where k denotes the residue field at x as usual. If (u, v) is a system of uniformising parameters at x, so that $Aw = A\left[\frac{v}{u}\right]$
100 is the ring of an affine open set Y_o on X' (Spec A being a suitable affine neighbourhood of x on X) and if $D = \text{div}\ (v)$ on X, we have $\sigma^*(D) = \sigma'(D) + L, \sigma'(D)$ being defined in Y'_o by the function $w = \dfrac{v}{u} \in A[w]$. But now,

$$(\sigma'(D).L) = l_A\left(\frac{A[w]}{(w, u)}\right) = l_A\left(\frac{A[w]}{(u, v, w)}\right) = l\left(\frac{A}{\mathfrak{M}}\right) = 1.$$

Thus we obtain (6),

$$0 = (\sigma^*(D).L) = 1 + (L^2), \tag{9}$$
$$(L^2) = 1$$

Further, for any divisor D on X of multiplicity l at x, we have proved in lecture 2 that $\sigma^*(D) = \sigma'(D) + l.L$. We thus obtain from (6),

$$0 = (\sigma^*(D).L) = (\sigma'(D).L) + l(L^2) = (\sigma'(D).L) - l,$$
$$(\sigma'(D).L) = l = \text{ Multiplicity of } D \text{ at } x \tag{10}$$

Suppose $D_i (i = 1, 2)$ are two divisors on X of respective multiplicities $l_i (i = 1, 2)$, and suppose D_i intersect properly at x. Since $\sigma_*(\sigma'(D_1)) = D_1$, we have by (7),

$$(D_1.D_2)_x = l_2(\sigma'(D_1).L) + \sum_{z \in L}[k(z) : k(x)](\sigma'(D_1).\sigma'(D_2))_z,$$

so that by (10),

$$\sum_{z\in L}[k(z):k(x)](\sigma'(D_1)).\sigma'(D_2))z = (D_1.D_2)_x - l_1 l_2 \tag{11}$$

Using this formula (11) we can very easily give a proof of the elimination of indeterminacies of a rational function f on X, without the hypothesis that X be a Japanese scheme. (This was promised in Lecture **101**
4). Suppose in fact that div $(f) = D_1 - D_2$ where D_i are effective divisors without common component. We have shown in Lecture 4 that a closed point x of X is an indeterminacy point of f if and only if $x \in |D_1| \cap |D_2|$. Let us put in this case $\nu(x, f) = (D_1.D_2)_x$. On the blown up scheme X', we have div $(f\circ\sigma) = \sigma^*(D_1) - \sigma^*(D_2) = \sigma'(D_1) - \sigma'(D_2) + (l_1 - l_2).\ L$, where l_i is the multiplicity of D_i at x. Suppose for instance that $l_1 \geq l_2$, so that div $(f\circ\sigma) = (\sigma'(D_1) + (l_1 - l_2)L) - \sigma'(D_2)$, and $\sigma'(D_1) + (l_1 - l_2)L$ and $\sigma'(D_2)$ are effective and have no common components. We have therefore

$$\begin{aligned}\sum_{z\in\bar{\sigma}^1(x)} \nu(z, f\circ\sigma)[k(z):k(x)] &= \sum_{z\in\bar{\sigma}^1(x)} [k(z):k(x)](\sigma'(D_1).\sigma'(D_2))_z \\ &\qquad + (l_1 - l_2)(L.\sigma'(D_2)) \\ &= (D_1.D_2)_x - l_1 l_2 + (l_1 - l_2).l_2 \\ &= \nu(x, f) - l_2^2,\end{aligned}$$

so that in particular $\nu(z, f\circ\sigma) < \nu(x, f)$ for any $z \in \sigma^{-1}(x)$. Since a decreasing sequence of non-negative integers must terminate, it follows that after a finite number of blowings up at points lying over x, we must have $\nu(*, f) = o$ at all points lying over x, or what is the same f has no indeterminates at point lying over x. This completes the proof of the **102**
theorem on elimination of indeterminates in the general case. This proof is due to Averbouh.

We have proved that when X is a locally noetherian regular, two dimensional prescheme, X' the prescheme obtained by blowing up a closed point x of X and L the fibre over x, then $L \simeq \mathbb{P}'(k(x))$ and $(L^2) = -[k(x):k(b)]$. The converse of this theorem also holds under suitable assumptions of projectivity, and this is a celebrated theorem due to Castelnuovo.

Theorem (Castelnuovo). *Let B be a locality noetherian prescheme and X' a projective regular B-scheme with structural morphism $\pi' : X' \to B$. Let b be a closed point of B and $\mathscr{I}$ an invertible sheaf of ideals of $\mathscr{O}_{X'}$, such that*

(i) *the closed subscheme L defined by $\mathscr{I}$ (with structure sheaf as the restriction of $\mathscr{O}_{X'}/\mathscr{I}$) is contained in the fibre $\pi'^{-1}(b)$, and is isomorphic to a projective line $\mathbb{P}'(K)$ over an extension K of $k(b)$. (K is necessarily a finite algebraic extension of $k(b)$, by Hilbert's Nullstellensatz);*

(ii) *the restriction $\mathscr{I} \otimes_{X'} \mathscr{O}_L = \mathscr{I}/_{\mathscr{I}^2}|L$ of $\mathscr{I}$ to L is the unique inversible sheaf on $\mathbb{P}'(K)$ of degree $+1$.*

Then there is a unique (upto isomorphism) projective B-scheme X and a projective. B-morphism $\sigma : X' \to X$ such that $\sigma(L)$ is a closed point x of X with the local ring $\mathscr{O}_{x,X}$ regular, two dimensional, and residue field $k(x) = K$, and σ induces an isomorphism of $X' - L$ onto $X - \{x\}$.

103 *Proof.* We may assume that $X' = \operatorname{Proj}(\mathscr{S}')$ where $\mathscr{S}' = \sum_{n \geq o} \mathscr{S}'_n$ is a quasi - coherent sheaf of graded $\mathscr{O}_B$-algebras of positive degrees such that $\mathscr{S}'_1$ is $\mathscr{O}_B$-coherent and generates $\mathscr{S}'$ over $\mathscr{O}_B$. Let $\mathscr{O}_{X'}(1)$ be the fundamental sheaf on X (EGA, II, (3.2.5.1)). Because of the theorem of Serre (EGA, III, 2.2.1), we may assume, be replacing $\mathscr{S}'$ by $\bigoplus_{n \geq o} \mathscr{S}nd$ for a suitably large d, if necessary, that $R'\pi'_*(\mathscr{O}_{X'}(1))$ is zero in a neighbourhood of b. Since the restriction $\mathscr{O}_L \otimes_{\mathscr{O}_{X'}} \mathscr{O}_{X'}(1)$ of $\mathscr{O}_{X'}(1)$ to L is very ample for the morphism $L \to B$ (EGA, II, (4.4.10)), $\mathscr{O}_{X'}(1) \otimes_{\mathscr{O}_{X'}} \mathscr{O}_L$ is an inversible sheaf of *positive* degree k on L. We denote by $\mathscr{L}$ inversible sheaf

$$\mathscr{L} = \mathscr{O}_{X'}(1) \otimes_{\mathscr{O}_{X'}} \mathscr{I}^{\otimes(-k)}.$$

The direct image $\pi'_*(\bigoplus_{n \geq o} \mathscr{L}^{\otimes n})$ is a quasi- coherent sheaf of graded $\mathscr{O}_B$-algebras and each $\pi'_*(\mathscr{L}^{\otimes n})$ is $\mathscr{O}_B$-coherent (EGA, III, 2.2.1). Let $\mathscr{S}$ be the subsheaf of graded algebras of $\pi'_*(\bigoplus_{n \geq o} \mathscr{L}^{\otimes n})$ generated over $\mathscr{O}_B$ by $\pi'_*(\mathscr{L})$, so that $\mathscr{S}$ is again quasi-coherent and is generated over $\mathscr{O}_B$

by $\mathscr{S}_{1'}$, which is coherent. We put $X = \text{Proj}\,(\mathscr{S})$ and denote by π the structural morphism $X \to B$, so that X is a projective B-scheme. □

The inclusion $\mathscr{S} \hookrightarrow \pi'_*(\underset{n\geq o}{\oplus}\mathscr{L}^{\otimes n})$ induces a homomorphism

$$\psi : \pi^{'*}(\mathscr{S}) \to \underset{n\geq o}{\oplus}\mathscr{L}^{\otimes n}$$

of graded $\mathscr{O}_{X'}$-algebras (EGA, 0, 4.4.3), and this in turn induces an X'-morphism **104**

$$\sigma_1 : G(\psi) \to Proj(\pi^{'*}(\mathscr{S}))$$

of an open subset $G(\psi)$ of Proj $(\underset{n\geq o}{\oplus}\mathscr{L}^{\otimes n})$, which is canonically determined by ψ. Now, $\mathscr{L}$ being an inversible sheaf on X', the structural morphism Proj $(\underset{n\geq o}{\oplus}\mathscr{L}^{\otimes n}) \to X'$ is an isomorphism, (EGA II, (3.1.7) and (3.1.8), (iii)), and we may identify $G(\psi)$ with an open subset of X'. Further, we have a canonical isomorphism of Proj $(\pi^{'*}(\mathscr{S}))$ with $X' X_B$ Proj $(\mathscr{S}) = X' \times_B X$ as X'- schemes, and composing σ_1 with the projection Proj $(\pi^{'*}(\mathscr{S})) \to X$, we get a B-morphism

$$\gamma_{\mathscr{L},\psi} = \sigma : G(\psi) \to X.$$

We shall show successively that (a) $G(\psi)$ contains $X' - L$ and σ restricted to $X' - L$ is an open immersion; (b) $G(\psi)$ contains L, so that $G(\psi) = X'$, and $\sigma(L)$ is a closed point of X not belonging to $\sigma(X' - L)$; $\sigma(X' - L) \cup \sigma(L) = X$; and ($c$) $\mathscr{O}_{x,X}$ is a regular (noetherian) local ring of dimension two with residue field $k(x) = K$.

Since the definitions of $\mathscr{S}$, $X, G(\psi)$ and σ are compatible with restriction to an open subset of B, we can restrict ourselves to the case when $B = \text{Spec}\,A$ is affine with A a noetherian ring. We shall make no further assumptions in the proof of (a), but for the proof of (b), and (c), we shall assume $H'(X', \mathscr{O}_{X'}(1)) = 0$. It is clearly sufficient to give the **105**
proofs in these cases.

Hence, suppose that $B = \text{Spec}\,A$ where A is a noetherian ring so that $X' =$ Proj (S'), where S' is a graded A-algebra of positive degrees such that S'_1 is an A-module of finite type which generates S' over A. With $\mathscr{L}$ as above, let S be the A-subalgebra of $\underset{n\geq 0}{\oplus}\Gamma(X', \mathscr{L}^{\otimes^n})$ generated over

A by $\Gamma(X', \mathscr{L})$ so that $X = \operatorname{Proj} S$. Let $i : S \to \underset{n\geq 0}{\otimes} \Gamma(X', \mathscr{L}^{\otimes^n})$ be the inclusion homomorphism of A-algebras. For an element $f \in S'_d (d > 0)$, we denote as usual by $D_+(f)$ the affine open set of X' consisting of those prime ideals in Proj (S) which do not contain f. If S'_f is the ring of quotients of S' with respect to the multiplicatively closed set $1, f, f^2, \ldots, S'_f$ is a graded A-algebra, and $S'_{(f)}$ shall denote the A-algebra of elements of degree $\circ$ in S'_f. We then have a canonical isomorphism of affine schemes $D_+(f) \simeq \operatorname{Spec} S'_{(f)}$, by the very definition of Proj (S'). Let us denote by $\circ$ the restrictions ϱ the restriction homomorphism $\underset{n\geq 0}{\oplus} \Gamma(X', \mathscr{L}^{\otimes n}) \to \underset{n\geq 0}{\oplus} \Gamma(D_+(f), \mathscr{L}^{\otimes n})$. If f is such that the inversible sheaf $\mathscr{L}$ is trivial on $D_+(f)$, then

$$\underset{n\geq 0}{\oplus} \Gamma(D_+(f), \mathscr{L}^{\otimes n}) \xrightarrow{\sim} \Gamma(D_+(f), \mathscr{O}_{X'})[T] \simeq S'_{(f)}[T]$$

as graded A-algebras, so that Proj $(\underset{n\geq 0}{\oplus} \Gamma(D_+(f), \mathscr{L}^{\otimes n}))$ is $\simeq$ Proj $(S'_{(f)}[T]) \simeq \operatorname{Spec} S'_{(f)} \simeq D_+(f)$ (EGA, II, (3.1.7)). The homomorphism $\varrho \circ i$:
106 $S \to \underset{n\geq 0}{\oplus} \Gamma(D_+(f), \mathscr{L}^{\otimes n})$ of graded algebras induces a morphism of an open set of $D_+(f)$ into $X = \operatorname{Proj} S$, and by definition, this open set is precisely $G(\psi) \cap D_+(f)$ and the morphism is the restriction of σ to $G(\psi) \cap D_+(f)$.

Suppose x, y are two distinct points of $X' - L$. One can then find a $g \in S'_1$ such that $x, y \in D_+(g)$ and a $h \in S'_d$ such that $D_+(h) \cap L = \phi$ and $x, y \in D_+(h)$. Since $\mathscr{L}$ is isomorphic to $\mathscr{O}_{X'}(1)$ on $X' - L$ and $\mathscr{O}_{X'}(1)$ is trivial on $D_+(g)$, it follows that $\mathscr{L}$ is trivial on $D_+(gh)$ and $x, y \in D_+(gh)$. Thus, in order to prove (a), it is sufficient to show that for any $f \in S'_d$ such that $D_+(f) \cap L = \phi$ and $\mathscr{L}$ is trivial on $D_+(f), G(\psi) > D_+(f)$ and σ is an open immersion restricted to $D_+(f)$.

Let τ be the section of $\mathscr{I}^{-1} = \operatorname{Hom}_{\mathscr{O}_{X'}}(\mathscr{I}, \mathscr{O}_{X'})$ corresponding to the canonical inclusion of $\mathscr{I}$ in $\mathscr{O}_{X'}$, so that we have $\tau(x) \neq 0$ if $x \notin L$. Let $\theta : \underset{n\geq 0}{\oplus} \mathscr{O}_{X'}(1) \to \underset{n\geq 0}{\oplus} \mathscr{L}^{\otimes n}$ be the homomorphism of sheaves of graded algebras defined by $\theta_n(a_1 \otimes \cdots \otimes a_n) = (a_1 \otimes \tau^k) \otimes (a_2 \otimes \tau^k) \otimes \cdots \otimes (a_n \otimes \tau^k)$. Then the restriction of θ to $X' - L$ is an isomorphism.

Let $\alpha : S' \to \underset{n\neq o}{\oplus} \Gamma(X', \mathscr{O}_{X'}(n))$ be the canonical homomorphism

(EGA, II, (2.6.2.3)). Since S' is generated over A by S'_1 and S is the A-subalgebra of $\bigoplus_{n\geq 0}\Gamma(X',\mathscr{L}^{\otimes n})$ generated by $\Gamma(X',\mathscr{L})$, it follows that $\Gamma(\theta)o\alpha(S')\subset S$. We thus have a commutative diagram 107

$$\begin{array}{ccccc} S' & \xrightarrow{\alpha} & \sum_{n\geq 0}\Gamma(X',\mathscr{O}_{X'}(n)) & \xrightarrow{\varrho'} & \sum_{n\geq 0}\Gamma(D_+(f),\mathscr{O}_{X'}(n)) \\ \downarrow{\scriptstyle \gamma=\Gamma(\theta).\alpha} & & \downarrow{\scriptstyle \Gamma(\theta)} & & \downarrow{\scriptstyle \Gamma(D_+(f),\theta)} \\ S & \xrightarrow{i} & \sum_{n\geq o}\Gamma(X',\mathscr{L}^{\otimes^n}) & \xrightarrow{\varrho} & \sum_{n\geq o}\Gamma(D_+(f),\mathscr{L}^{\otimes n}) = C \end{array}$$

of A-algebras (ϱ' and ϱ are restriction maps). Suppose f satisfies the conditions mentioned at the end of the previous paragraph. Since θ is an isomorphism restricted to $D_+(f), \Gamma(D^+(f),\theta)$ is an isomorphism. If f is of degree $d, \Gamma(D_+(f),\mathscr{O}_{X'}(d))$ is generated over $\Gamma(D_+(f),\mathscr{O}_{X'}) = S'_{(f)}$ by the element $\varrho'\, o\alpha(f)$. Hence $\varrho\circ i(\gamma(f))$ generates $\Gamma(D_+(f),\mathscr{L}^{\otimes d})$ over $\Gamma(D_+(f),\mathscr{O}_{X'})$. But any homogeneous prime ideal of C which contains $C_d = \Gamma(D_+(f),\mathscr{L}^{\otimes d})$ necessarily contains $C^+ = \bigoplus_{n\geq 1} C_n$. It follows that the morphism $\sigma|D_+(f)\cap G(\psi)$ maps $D_+(f)\cap G(\psi)$ into the open subset $D_+(\gamma(f)) = \operatorname{Spec} S_{(\gamma(f))}$ of X. This induces a commutative diagram

$$\begin{array}{ccc} S'_{(f)} & \longrightarrow & \Gamma(D_+(f),\mathscr{O}_{X'}) \\ \downarrow & & \| \\ S_{(\gamma(f))} & \longrightarrow & \Gamma(D_+(f),\mathscr{O}_{X'}). \end{array}$$

The hupper homomorphism is an isomorphism, by definition of the structure sheaf $\mathscr{O}_{X'}$ on Proj (S') and the definition of α. The lower homomorphism is injective because of (EGA, I, (9.3.1)). Hence $S_{(\gamma(f))} \xrightarrow{\sim} \Gamma(D_+(f),\mathscr{O}_{X'})$ is an isomorphism, which means precisely that σ when restricted to $D_+(f)$ is an isomorphism onto $D_+(\gamma(f))\subset X$. Thus, σ restricted to $X'-L$ is an open immersion, and (a) is proved. 108

We now prove (b) under the assumption that $H'(X',\mathscr{O}_{X'}(1)) = 0$ (and of course $B = \operatorname{Spec} A$, affine). Because of (EGA, II, (3.7.4)), it is sufficient to show that there is an $s\in S_1 = \Gamma(X',\mathscr{L})$ such that $s(x)\neq 0$ for any $x\in L$. Now, the inversible sheaf $\mathscr{L}\otimes_{\mathscr{O}_{X'}}\mathscr{O}_L = \mathscr{O}_{X'}(1)\otimes_{\mathscr{O}_{X'}}$

$\mathscr{I}^{\otimes(-k)} \otimes_{\mathscr{O}_L} \mathscr{O}_L$ is of degree 0 on $L \simeq \mathbb{P}'(K)$, and hence is isomorphic to the 'trivial' inversible sheaf $\mathscr{O}_L$ on L. Thus, $H^o(L, \mathscr{L}_{\otimes_{\mathscr{O}_{X'}}} \mathscr{O}_L) \neq 0$ and any non-zero section of this sheaf on L does not vanish at any point of L. On the other hand, denoting by τ the 'canonical' section of $\mathscr{I}^{-1}$ as above we have the exact sequences

$$0 \to \mathscr{O}_{X'}(1) \otimes_{\mathscr{O}_{X'}} \mathscr{I}^{\otimes(-\mu)} \xrightarrow{\otimes\tau} \mathscr{O}_{X'}(1) \otimes_{\mathscr{O}_{X'}} \mathscr{I}^{\otimes} \xrightarrow{(-\mu-1)} \mathscr{D}_{k-\mu-1} \to 0$$

where $\mathscr{D}_r$ denotes the unique inversible on L an degree γ.

This leads to the cohomology exact sequence

$$\begin{aligned} H^o(X', \mathscr{O}_{X'}(1) \otimes \mathscr{I}^{\otimes(-\mu-1)}) &\to H^o(L, \mathscr{D}_{k-\mu-1}) \\ &\to H'(X', \mathscr{O}_{X'}(1) \otimes \mathscr{I}^{\otimes(-\mu)})) \\ &\to H^o(X', \mathscr{O}_{X'}(1) \otimes \mathscr{I}^{\otimes(-\mu-1)}) \\ &\to H'(L, \mathscr{D}_{k-\mu-1}) \end{aligned}$$

109 Now, for $0 \leq \mu \leq k-1$ we have $H'(L, \mathscr{D}_{k-\mu-1}) = 0$ (*EGA*, III, (2.1.12)) and since $H'(X', \mathscr{O}_{X'}(1)) = 0$ by assumption, it follows that $H'(X', \mathscr{O}_{X'}(1) \otimes \mathscr{I}^{\otimes(-r)}) = 0$ for $0 \leq r \leq k$ and consequently $H^o(X', \mathscr{O}_{X'}, (1) \otimes \mathscr{I}^{\otimes(-r)}) \to H^o(L, \mathscr{D}_{k-r})$ us surjective for $1 \leq r \leq k+1$. In particular, $H^o(X', \mathscr{L}) \to H^o(X', \mathscr{D}_o)$ is surjective, which proves that σ is defined on L in view of our earlier remark. To show that $\sigma(L)$ is a single point, we have to show that any section on X' of $\mathscr{L}^{\otimes n}(n \geq 1)$ vanishing at any point of L vanishes everywhere on L. But this follows from the fact that $\mathscr{L}$, and hence also $\mathscr{L}^{\otimes n}$ induces the 'trivial' invertible sheaf on L. Now by (EGA II (3.7.5)), σ is dominant, and since X' is proper over $B, \sigma(X')$ is closed so that $\sigma(X') = X$. Since $\pi \circ \sigma = \pi'$ is projective and π itself being projective and hence separated, σ is projective. Let x be any point of $X' - L$, and let $s \in \Gamma(X', \mathscr{O}_{X'}(1))$ such that $s(x) \neq 0$. Then the section $\theta(s) = s \otimes \tau^k$ of $\mathscr{L}$ vanishes on L but not at x, which shows that $\sigma(x) \neq \sigma(L)$. Hence the closed point $\sigma(L)$ does not belong to $\sigma(X' - L)$. This proves (*b*).

We shall now prove (*c*). The canonical homomorphism γ : $\sigma^*(\mathscr{O}_X(1)) \to \mathscr{L}$ (EGA II (3.7.9.1) is in our case an isomorphism since $\mathscr{L}$ is generated by its global sections as we have proved above. Further, if $\alpha : S_1 \to \Gamma(X, \mathscr{O}_X(1))$ is the natural map (EGA II (2.6.2.2.)) the composite $\Gamma(\gamma^b) o\alpha : \Gamma(X', \mathscr{L}) = S_1 \to (X', \mathscr{L})$ is the identity, so that $\Gamma(\gamma^b) : \Gamma(X, \mathscr{O}_x(1)) \to \Gamma(X', \mathscr{L})$ is surjective. **110**

By means of the given isomorphism $L \simeq \mathbb{P}'(K)$, we can identity $\Gamma(L, \mathscr{O}_L)$ with K, and we therefore have a monomorphism of fields $\frac{\mathscr{O}_{x,X}}{\mathfrak{M}_{x,X}} = k(x) \hookrightarrow \Gamma(L, \mathscr{O}_L) = K$. Let $i : \mathfrak{M}_x \mathscr{O}_X(1) \hookrightarrow \mathscr{O}_X(1)$ be the canonical injection. Since L is contained in the fibre $\sigma^{-1}(x)$, the composite map $\sigma^*(\mathfrak{M}_x \mathscr{O}_X(1)) \xrightarrow{\sigma^*(i)} \sigma^*(\mathscr{O}_X(1)) \xrightarrow{\Gamma(\gamma^b)} \mathscr{L} \to \mathscr{D}_o$ is zero, so that we obtain a homomorphism $\gamma_1 : \sigma^*(\mathfrak{M}_x \mathscr{O}_X(1)) \to \mathscr{I}\mathscr{L}$ We have the commutative diagram

$$\begin{array}{ccccccccc}
O & \longrightarrow & \Gamma(X, \mathfrak{M}_x \mathscr{O}_X(1)) & \longrightarrow & \Gamma(X, \mathscr{O}_X(1)) & \longrightarrow & \Gamma(X, k(x) \otimes \mathscr{O}_X(1)) & & \\
 & & \downarrow{\scriptstyle \Gamma(\gamma_1^b)} & & \downarrow{\scriptstyle \Gamma(\gamma^b)} & & \downarrow{\scriptstyle \varphi} & & \\
O & \longrightarrow & \Gamma(X', \mathscr{I}\mathscr{L}) & \longrightarrow & \Gamma(X', \mathscr{L}) & \longrightarrow & \Gamma(L, \mathscr{D}_\circ) & \longrightarrow & O
\end{array}$$

where the two rows are exact and $\Gamma(\gamma^b)$ is surjective. Hence φ is surjective. But φ is a homomorphism of a one-dimensional $k(x)$ vector space onto a one-dimensional K-vector space compatible with the inclusion $k(X) \hookrightarrow K$. It follows that $k(x) = K$ and φ is an isomorphism. Thus $\Gamma(\gamma_1^b)$ is also surjective.

Now, choose a section $s_o \in \Gamma(X, \mathscr{O}_X(1))$ such that $s_o(x) \neq 0$, so that $s_o(y) \neq 0$ for y belonging to an affine open neighbourhood U of x. Then $t_o = \Gamma(\gamma_1^b)(s_o)$ is a section of $\mathscr{L}$ over X' not vanishing at any point of $\sigma^{-1}(U)$. We then have isomorphisms $\mathscr{O}_X \Big| U \xrightarrow{\sim} \mathscr{O}_X(1) \Big| U$ and $\mathscr{O}'_X \Big| \sigma^{-1}(U) \xrightarrow{\sim} \mathscr{L} \Big| \sigma^{-1}(U)$ which carry the identity sections of $\mathscr{O}_X \Big| U$ and **111** $\mathscr{O}_{X'} \Big| \sigma^{-1}(U)$ into the sections s_o and t_o of $\mathscr{O}_X(1) \Big| U$ and $\mathscr{L} \Big| \sigma^{-1}(U)$ respectively. By means of these isomorphisms, γ^b transforms into the natural isomorphism $\sigma_U^*(\mathscr{O}_X \Big| U) \xrightarrow{\sim} \mathscr{O}_{X'} \Big| \sigma^{-1}(U)$, where $\sigma_U : \sigma^{-1}(U) \to U$ is the restriction of σ, and γ_1^b transforms into the natural homomorphism $\sigma_U^*(\mathfrak{M}_x \mathscr{O}_X \Big| U) \to \mathscr{I}\mathscr{O}_{X'} \Big| \sigma^{-1}(U)$. Now, since $\Gamma(\gamma_1^b) : \Gamma(X, \mathfrak{M}_x \mathscr{O}_X(1)) \to \Gamma(X', \mathscr{I}\mathscr{L})$ and $j : \Gamma(X', \mathscr{I}\mathscr{L}) \to \Gamma\left(L, \frac{\mathscr{I}\mathscr{L}}{\mathscr{I}^2\mathscr{L}}\right)$ are both surjective, so is the composite $jo\Gamma(\gamma_1^b) : \Gamma(X, \mathfrak{M}_x \mathscr{O}_X(1)) \to \Gamma\left(L, \frac{\mathscr{I}\mathscr{L}}{\mathscr{I}^2\mathscr{L}}\right)$ surjective. A fortiori,the corresponding map when we replace $\Gamma(X, \mathfrak{M}_x \mathscr{O}_X(1))$ by the 'bigger' $\Gamma(U, \mathfrak{M}_x \mathscr{O}_X(1))$ is also surjective. Since the image of $\Gamma\left(U, \mathfrak{M}_x^2 \mathscr{O}_X^{(1)}\right)$ in $\Gamma(L, \mathscr{I}/\mathscr{I}^2)$ is zero, we deduce that the 'characteristic map'

$$\psi_1 : \frac{\mathfrak{M}_x}{\mathfrak{M}_x^2} \to \Gamma(L, \mathscr{I}/\mathscr{I}^2)$$

is surjective. Now, $\dim \mathscr{O}_{x,X} = \dim_x X \geq 2$, since the projection $\sigma(C)$ of an irreducible closed subscheme C through a closed point p of L, such that $\dim_p C = 1$ and p is an isolated point of $C \cap L$, satisfies the conditions that $x \in \sigma(C)$, $\sigma(C)$ is closed irreducible but $\sigma(C)$ is not an irreducible component of X. Further, $\Gamma(L, \mathscr{I}/\mathscr{I}^2) \simeq \Gamma(L, \mathscr{D}_1)$ is a
112 two dimensional vector space over K. If we can show that ψ_1 is also injective, it would follow that $\mathscr{O}_{x,X}$ is a regular local ring of dimension 2.

Now, as a sheaf of $\mathscr{O}_L-$ algebras, $\underset{n\geq 0}{\oplus} \mathscr{I}^n/\mathscr{I}^{n+1} \simeq \underset{n\geq 0}{\oplus} \mathscr{D}_1^{\otimes n}$ since $\dfrac{\mathscr{I}}{\mathscr{I}^{n+1}} \simeq \mathscr{I}^n \otimes_{\mathscr{O}_X} \mathscr{O}_L \simeq (\mathscr{I}_{\otimes_{\mathscr{O}_X}} \mathscr{O}_L)^{\otimes n} \simeq \mathscr{D}_1^{\otimes n}$, Hence the K-algebra $\underset{n\geq 0}{\oplus} \Gamma(L, \mathscr{I}^n/_{\mathscr{I}^{n+1}}$ isomorphic to $\underset{n\geq 0}{\oplus} \Gamma(L, \mathscr{D}_1^{\otimes n})$, and hence to a polynomial ring in two variables over K. Now, for every $k \geq 0$, we have a K-homomorphism

$$\psi_k : \frac{\mathfrak{M}_x^k}{\mathfrak{M}_x^{k+1}} \to \Gamma(L, \mathscr{I}^k/_{\mathscr{I}^{k+1}}),$$

and hence a homomorphism K-algebras

$$\underset{n\geq 0}{\oplus} \frac{\mathfrak{M}_x^n}{\mathfrak{M}_x^{n+1}} \to \underset{n\geq 0}{\oplus} \Gamma\left(L, \frac{\mathscr{I}^n}{\mathscr{I}^{n+1}}\right).$$

Since the second algebra is generated by its first degree elements and ψ_1 is surjective, ψ is surjective.

Now, put $C = \Gamma(U, \mathscr{O}_X), C' = \Gamma(U, \sigma_{U^*}(\mathscr{O}_{X'})) = \Gamma(\sigma^{-1}(U), \mathscr{O}_{X'})$, and let $\mathfrak{M}$ be the maximal ideal in C defining x. Since $\sigma_{U*}(\mathscr{O}_{X'})$ is coherent by Serre's theorem, C' is a C-module of finite type and $\sigma_{U*}(\mathscr{O}_{X'})$ isomorphic to the sheaf $\tilde{C'}$ associated to C'. Further, since σ_U is an isomorphism $\sigma^{-1}(U) - \sigma^{-1}(x)$ onto $U - \{x\}$, the natural homomorphism $\mathfrak{M}_x^2 \mathscr{O}_X \to \sigma_*(\mathscr{O}_{X'})$ is an isomorphism when restricted to $X - \{x\}$. It

clearly follows that the C-module $C'/Im(\mathfrak{M}^2)$ is annihilated by $\mathfrak{M}^k$ for some suitably large k. On the other hand, if $\mathscr{O}_l$, is the ideal $\Gamma(\sigma^{-1}(U), \mathscr{I}^l)$ of C', we have by (EGA, III, (4.1.7)) that for l sufficiently large, **113**
$\mathscr{O}_l \subset \mathfrak{M}^k C' \subset \mathrm{Im}(\mathfrak{M}^2)$. Assuming for the moment that $C \to C'$ is injective, we shall show that $\psi_1 : \dfrac{\mathfrak{M}}{\mathfrak{M}^2} \to \Gamma\left(L, \dfrac{\mathscr{I}}{\mathscr{I}^2}\right) \simeq \dfrac{\mathscr{O}_1}{\mathscr{O}_2}$ is injective. We shall show by descending induction on k that the kernel of $\mathfrak{M} \to \dfrac{\mathscr{O}_1}{\mathscr{O}_k}$ is contained in $\mathfrak{M}^2$ for $k \geq 2$. Since $\mathscr{O}_l \subset \mathrm{Im}\,\mathfrak{M}^2$ for l sufficiently large, it is sufficient to show that if this assertion holds for $l > 2$, then it holds for $l - 1$. Now, we shown above that $\mathfrak{M}^{l-1} \to \dfrac{\mathscr{O}_{l-1}}{\mathscr{O}_l} = \Gamma(L, \mathscr{I}^{l-1}/_{\mathscr{I}^l})$ is surjective. If $f \in \mathfrak{M}$ is mapped into $\mathscr{O}_{l-1}$, we can thus find $g \in \mathfrak{M}^{l-1}$ such that $(f - g)$ goes into $\mathscr{O}_l$. But then, we have $f = (f - g) + g \in \mathfrak{M}^2 + \mathfrak{M}^{l-1} = \mathfrak{M}^2$. Thus shows that $\ker(\mathfrak{M} \to \mathscr{O}_l/_{\mathscr{O}_{l-1}}) \subset \mathfrak{M}^2$, completing the induction.

It only remains to show that $C \to C'$ is injective. Since $\mathscr{O}_X \to \sigma_*(\mathscr{O}_{X'})$ is an isomorphism on $U - \{x\}$, the kernel of $C \to C'$ is an ideal $\mathscr{O}$ annihilated by $\mathfrak{M}^k$ (k large). Let f by any element of $\mathscr{O}$. By definition of proj (S), f can be written as $\dfrac{\alpha}{\beta}$ with $\alpha, \beta \in S_d, \beta(x) \neq 0$ (β being considered as a section of $\Gamma(X, \mathscr{O}_X(d))$) is zero in a neighbourhood of x. This means that when α is considered as an element of $\Gamma(X', \mathscr{L})$ it vanishes on $V - L$, where V is a neighbourhood of L. Since the local rings $\mathscr{O}_{y,X'}$ are regular and hence reduced, α vanishes on the whole of V. Now, $\sigma(X' - V)$ is a closed subset of X not containing x, so that find a **114**
$\gamma \in S_{d'}$ such that $\gamma(x) \neq 0$ and $\gamma(y) = 0$ for $y \in \sigma(X - V)$. It then follows that $\alpha.\gamma^{d''} = 0$ in $S_{d+d'd''}$ for d'' sufficiently large (EGA, I, (9.3.1)). But this means that $f = \dfrac{\alpha}{\beta} = 0$ in $\mathscr{O}_{x,X}$, so that $\mathscr{O} = (0)$.

The proof of the theorem is complete.

Remark. 1. When X' is a projective variety over an algebraically closed field, the proof may be summarised as follows. Choose a projective imbedding such that $H'(X', \mathscr{O}_{X'}(1)) = 0$. If H is the linear system of hyperplane sections, the complete linear system $|H + kL|$, where $k = (H.L)$, has no base points, and thus defines a morphism of X' into a projective space. This morphism is an immersion restricted to $X' - L$ and contracts L to a simple point of the image variety.

It is only because we have not developed the machinery of projective imbedding by linear systems in sufficient generality in Lecture 3 that we have had to borrow freely the necessary apparatus from EGA, II.

2. The regularity of $\mathscr{O}_{X,x}$ can be proved as follows. By applying the fundamental theorem on proper morphisms (EGA, III, (4.1.5)), one shows first that $\sigma_*(\mathscr{O}_{X'})_x$ is a regular local ring of dimension 2. Since $\dfrac{\mathfrak{M}_x}{\mathfrak{M}_x^2} \xrightarrow{\psi} \Gamma(L, \mathscr{I}/_{\mathscr{I}^2})$ is surjective, we see that the image of $\mathfrak{M}_x$ in $\sigma_*(\mathscr{O}_{X'})_x$ generates the maximal ideal of this local ring. By Nakayama's lemma, $\mathscr{O}_x \to \sigma_*(\mathscr{O}_{X'})_x$ is surjective. But we
115 have also shown that it is injective, and this concludes the proof.

3. The full strength of our assumption (*ii*), that $\mathscr{I}/_{\mathscr{I}^2}$ is the line bundle of degree 1 on $\mathbb{P}'(K)$ was used only in the proof of the regularity of $\mathscr{O}_{X,x'}$. Suppose we only assume that $\mathscr{I}/_{\mathscr{I}^2}$ is of *positive* degree l. If we assume that $R'\pi'_*(\mathscr{O}_{X'}(l)) = 0$ in a neighbourhood of the point b and if we work with the inversible sheaf $\mathscr{L} = \mathscr{O}_{X'}(l) \otimes \mathscr{I}^{\otimes(-k)}$ where k is the degree of $\mathscr{O}_{X'}(1) \otimes_{\mathscr{O}_{X'}} \mathscr{O}_L$ on L, we get exactly as above a B-projective scheme X and a projective morphism $\sigma : X' \to X$ such that $\sigma(L)$ is a point x of X and the restriction of σ to $X' - L$ is an isomorphism onto $X - \{x\}$. We can only say that x is a *normal* point of X.

4. Grauert ([5]) has proves that if X' is a complex manifold of dimension 2 and $C_1, \cdots, C_n$ a system of compact, irreducible, one dimension analytic sets on X', such that the intersection matrix $(C_i.C_j)$ is negative definite, and $\bigcup_{i=1}^{n} C_i$ is connected, there is a normal complex space X and a proper holomorphic map $\sigma : X' \to X$ such that $\sigma(\bigcup_1^n C_i)$ is a point x and X and σ restricted to $X' - \bigcup_1^n C_i$ is an isomorphism onto $X - \{x\}$. Thus, for the contractibility of

a system of curves on a complex 2-manifold, the condition of Mumford on the negative definiteness of the intersection matrix is necessary and sufficient. The image point x of the set $\bigcup_{1}^{n} C_i$ will in general be a singular point.

However, ever if we assume that X' is a non-singular projective surface and C a non-singular curve on X' with $(C^2) < 0$, if the genus of C is ≥ 1, the analytic space X need not be algebraic. It is thus that Grauert constructs an example of a normal compact complex space of dimension 2 with an isolated singular point, which is not an algebraic surface but whose field of meromorphic function has transcendence degree 2 over $\mathbb{C}$. We shall give another due to Hironaka which is perhaps simpler. **116**

Let C be a non-singular cubic curve on $\mathbb{P}^2$, and $P_i (1 \leq i \leq 10)$ a set of 10 distinct points on C. Let X' be the non-singular surface obtained by blowing up the P_i, and let C' be the proper transform of C on X'. Since $(C^2) = 9$, we deduce by (12) that $(C'^2) = 9 - 10 = -1$ (blowing up a simple point of a curve diminishes self intersection by l as follows by (12). Hence, by Grauert's theorem, there is a proper holomorphic map σ of X' onto a normal complex space X such that $\sigma(C')$ is a point of X and σ is an isomorphism of $X' - C'$ onto $X - \{x\}$. If X were an algebraic variety, we can find a curve D on X not passing through x. The inverse image $\sigma^{-1}(D)$ would then be disjoint with C'. This means that the projection of $\sigma^{-1}(D)$ on $\mathbb{P}^2$ can intersect C at most at the points P_i. Let the intersection multiplicity of the projection with be C at P_i be n_i. Then not all n_i are zero, and if we regard C as an abelian group in the usual way with an inflexion point for 0, we must have the relation

$$\sum_{1}^{10} n_i P_i = 0.$$

If the P_i were chosen 'in general position', such a relation cannot hold, and thus X is not an algebraic surface. **117**

It may be mentioned in this connection that if X is a compact complex manifold such that there are two algebraically independent meromorphic functions on X over $\mathbb{C}$, then X is projective algebraic (Theorem of Chow-Kodaira).

We now deduce some corollaries from the theorem.

Corollary 1. *Let X' be a locally noetherian two dimensional regular prescheme, L a closed subscheme and U an open neighbourhood of L, such that*

(i) *U is quasi-projective over a noetherian ring A.*

(ii) *there is a closed point b of* $\operatorname{Spec} A$ *such that L is contained in the fibre over b, and is isomorphic as $k(b)$-scheme to $\mathbb{P}'(K)$, where K is an extension of $k(b)$.*

(iii) $(L^2) = -[K : k(b)]$.

Then there is a locally noetherian two dimensional regular prescheme X, a closed point x of X and a morphism $\sigma : X' \to X$ such that X' is X-isomorphic to the dilatation of X at x.

Proof. This results immediately from the theorem, in view of the theorem of decomposition of lecture 4, since a quasi-projective scheme over a quasi-compact base admits an open immersion in a projective scheme over same base (EGA, II, (5.3.3)), and since the 'contractibility' of L depends clearly only on an open neighbourhood of L. □

118 **Corollary 2** (Zariski)**.** *Let X be an algebraic scheme over a field K with isolated singular (i.e., non-singular) points. Then X is quasi-projective over K if and only if the set of singular points of X is contained in an affine open subset of X.*

Proof. Since a finite set of points of a quasi -projective K-scheme is contained in an affine open subset, the condition is clearly necessary. □

Suppose conversely that the condition is fulfilled. Since the set of singular points of X_{red} is contained in the singular set of X, and since X is quasi-projective over K if X_{red} is so (EGA, II, (4.5.14)), we may assume that X is reduced. Again, by (EGA, II, (5.3.6)), we may assume X connected. By assumption, there is a projective K-scheme Y, a dense

open subset U of Y, a non-void open neighbourhood V of the singular set S of X, and an isomorphism $\varphi : U \to V$. By normalising Y' outside $\varphi^{-1}(S)'$, we may assume that $Y - \varphi^{-1}(S)$ is normal (EGA, II, (6.1.11)). In particular the singular set of Y is again finite. By the theorem of resolution of singularities of a surface ([1]), we may actually assume that $Y - \varphi^{-1}(S)$ is regular. Since U is dense in Y, φ defines a rational map ψ of $Y - \varphi^{-1}(S)$ into X. This rational map has no indeterminacies in U, since φ is defined on U. By the theorem of elimination of indeterminacies of Lecture 4, we may further assume that ψ has no indeterminacies on $Y - \varphi^{-1}(S)$. (Blowing up is a projective morphism. One needs to blow up points outside of U, so that we still have an open subset of the blown up variety isomorphic to V). But this means that there is an open subset **119**
U' of Y and a morphism $\chi : U' \to X$ such that (i) χ is proper, (ii) there is an open subset U of U' and an isomorphism $\chi|U : U \to V$ onto an open neighbourhood V of S, and (iii) $U' - (\chi^{-1}(S) \cap U)$ is regular. Since X is connected, any irreducible component intersects S if $S \neq \phi$, and there is only one irreducible component if $S = \phi$, so that in any case V is dense in X. Hence χ is also surjective. Let $\xi : V \to U$ be the inverse of χ/U. The morphism $V \to U' \times_K V$ defined by $x \mapsto (\xi(x), x)$ is a closed immersion, being the graph of the composite morphism $V \xrightarrow{\xi} U \hookrightarrow U'$. Let Γ' be the closed subscheme of $U' \times_K V$ which is the image. By definition of χ, the graph of χ in $U' \underset{K_k}{X} X$ is the closure of Γ' in $U' X_k X$. Since Γ is already closed in $U' X_K V$, it follows that $\chi^{-1}(V) = U$, and hence $\chi^{-1}(S) \subset U$. Thus, $\chi|U' - \chi^{-1}(S)$ is a proper surjective morphism of $U' - \chi^{-1}(S)$ onto $X - S$. By the theorem of decomposition of Lecture 4, it follows that χ is a composite of dilatations at regular points of X and points lying over regular points of X. But now, by the theorem of Castelnuovo, if $\sigma : X' \to X$ is a dilatation at a regular point of X and X' is quasi-projective, X is also quasi -projective. This proves the corollary.

The first example of a variety which is not quasi -projective is due to Nagata, who exhibits a complete, normal surface with exactly two singular points which is not projective. We give a construction. The first remark is the following. If C is an elliptic curve (over an alg. closed field K), x a point of C, X' the variety obtained from $X = \mathbb{P}' \times C$ by **120**
blowing up $(0, x)$ and C' the proper transform of $\{0\} \times C$ in X', C' can

be contracted to a point y of a normal projective variety Y, in the sense that there is a morphism $\tau : X' \to Y$, such that $\tau^{-1}(y) = C'$ and $\tau|X' - C'$ is an isomorphism onto $Y - \{y\}$. To prove this, observe first that for n large; (n, x) is very ample for C, so that $(0) \times C + n(\mathbb{P}' \times (x)) = D$ is very ample for X. (Segre imbedding). By what we have said in lecture 3, for m sufficiently large, the divisors of the complete linear system. $|mD|$ which contain the point $(0, x)$ give a projective imbedding of X', whose hyperplane sections are of the form $\sigma^*(D_1) - L$, where $\sigma : X' \to X$ is the canonical morphism, $L = \sigma^{-1}((0, x))$ and $D_1 \in |mD|, x \in SuppD_1$. Thus, if H is a hyperplane section of X', we have the *linear* (and not merely numerical) equivalence $H.C' \sim (mn-1)x'$ where x' is the point of C' lying over x. Replacing H by a sufficiently high multiple, we may assume that $H.C' = kx', k > 0$ and $H'(X', \mathscr{O}_{X'}(1)) = 0$. As in the proof of Castelnuovo theorem, we shall show that the complete linear system $|H + kC'|$ has no base points, and gives a morphism φ of X' into a projective space which is an immersion restricted to $X' - C'$ and which contracts C' to a point of the image. Now, we have the linear equivalence $C'^2 = -x'$ on C'. Hence, if $\mathscr{L}$ is the inversible sheaf $\mathscr{O}_{X'}(1) \otimes \mathscr{I}^{\otimes -(k)}$, where $\mathscr{I}$ is the defining sheaf of ideals of C', $\mathscr{L} \otimes_{\mathscr{O}_{X'}} \mathscr{O}_{C'}$ is trivial on C'. This means that any section of $\mathscr{L} \otimes_{\mathscr{O}_{X'}} \mathscr{O}_{C'}$ on
121 C' is either everywhere 0 on C' or does not vanish anywhere on C'. If we show that $H^o(X', \mathscr{L}) \to H^o(C', \mathscr{L} \otimes_{\mathscr{O}_{X'}} \mathscr{O}_{C'})$ is surjective, it would follow that φ is defined on C' and contracts C' to a point. Now, for any line bundle $\mathfrak{K}$ of *positive* degree on an elliptic curve C', we have $H'(C', \mathfrak{K}) = 0(\Omega' \simeq' \mathscr{O}_{C'}, H'(C', \mathfrak{K}) \simeq H^o(C', \mathfrak{K}^*) = 0$ by Serre duality). From the exact sequences $H'(X', \mathscr{O}_{\mathscr{X}'}(1) \otimes \mathscr{I}^{-\nu}) \to H^o(X', \mathscr{O}_{X'}(1) \otimes \mathscr{I}^{-\mu-1}) \to H'(C', \mathscr{O}_{X'}(1) \otimes \dfrac{\mathscr{I} - \mu - 1}{\mathscr{I} - \mu})$ we deduce by induction on μ that $H'(C', \mathscr{L} \otimes \mathscr{O}_{C'} = 0$, so that $H^o(X', \mathscr{L}) \to H^o(C', \mathscr{L} \otimes \mathscr{O}_{C'})$ is surjective. Now, φ is clearly defined and is an immersion on $X' - C'$ (See proof of theorem). By normalising the image variety if necessary, we obtain $\tau : X' \to Y$ having the required properties.

Now, let x, y be two points of C such that nx and ny are not linearly equivalent for any $n \in \mathbb{Z}$, $n \neq 0$. Let $X'' \xrightarrow{\sigma} X$ be the morphism obtained by blowing upto $(0, x)$ and (∞, y), and let C', C'' be the proper transforms of $\{0\} \times C$ and $\{\infty\} \times C$ on X''.

Since the contractibility of a curve to a normal point of an *abstract* variety depends only on an open neighbourhood of the curve, we see from above that there is an abstract normal variety Y'' and a proper morphism $\xi : X'' \to Y''$, such that $\xi(C') = y'$ and $\xi(C'') = y''$ are points of Y'', and ξ restricted to $X'' - C' - C''$ is an isomorphism onto $Y'' - \left\{y', y''\right\}$. We assert that Y'' is not projective.

For, if it were, there would be a curve D on Y'' which intersects 122
x the curves $\xi(\sigma^{-1}(0, x))$ and $\xi(\sigma^{-1}(\infty, y))$ but such that neither of the points y', y'' belongs to D. (One can take for D a hyperplane section not containing y' or y''). But then, $\xi^{-1}(D)$ is a curve on X'' intersecting both $\sigma^{-1}(0, x)$ and $\sigma^{-1}(\infty, y)$, but not intersecting C' or C''. Hence the curve $\sigma(\xi^{-1}(D)) = D'$ on X intersects $0 \times C$ only at the point $(0, x)$ and $\infty \times C$ only at the point (∞, y). It follows that if $D'.\ (0 \times C) = m(0, x)$ and $D'(\infty \times C) = n(\infty, y), m, n > 0$ the divisors mx and ny are linearly equivalent on C. This is impossible by assumption. Thus, Y'' is not projective.

Lecture 7

The existence of relatively minimal models

We need the following 123

Theorem . *Let Y be locally noetherian and $f : X \to Y$ a morphism of finite type; for any $x \in X$, put $\delta(x) = \dim_x f^{-1} f(x)$. Then for any integer $n \geq 0$, the set $\{x \in X/\delta(x) \geq n\} = E_n$ is closed in X. If further X is irreducible, x is a generic point of X and $e = \dim f^{-1}(f(x)) =$ tr.deg$_{k(f(x))}{}^{k(x)}$, then $E_e = X$ and $f(E_{e+1})^c$ contains a non-void open set of Y.*

For the proof, see (EGA, IV, 13.1.1).

We note however that under the assumptions of the theorem, it is *not* true that if X, Y are irreducible and f dominant, $\dim X = \dim Y + e$. This holds however when X, Y are irreducible algebraic schemes over a field K and f a K-morphism.

The following lemma is well known. but we add a proof since we could not find a ready reference. (For varieties over an algebraically closed filed, see [12]).

Lemma . *Let X, Y be irreducible noetherian schemes, and $f : X \to Y$ a morphism of finite type such that if y is a generic point of Y, the generic fibre $f^{-1}(y)$ is geometrically irreducible. Then there is a non-*

void open subset U of Y such that for any $z \in U$, $f^{-1}(z)$ is geometrically irreducible.

Proof. One may clearly assume X, Y integral, f dominant and $Y =$
124 $\operatorname{Spec} A$, where A is a noetherian domain. We first remark if U is a non-void open subset of X and g the restriction of f to U, the lemma holds for (X, Y, f) if and only if it holds for (U, Y, g). Indeed, if we put $F = X - U$ and $F_1, \ldots, F_r$ are the irreducible components of F, since each F_i intersects the generic fibre of f in a proper closed subset, it follows from the above theorem that there is a non-void open subset V of Y such that for any $y \in V$, we have

$$\begin{aligned} &\dim_x(f^{-1}(y)) = e, \text{ for every } x \in f^{-1}(y) \\ &\dim\ (f^{-1}(y) \cap F_i) < e \end{aligned}$$

It follows therefore that for any $y \in V$, $f^{-1}(y)$ is geometrically irreducible if and only if $g^{-1}(y) = f^{-1}(y) \cap U$ is geometrically irreducible. This proves our assertion. □

Let $T_1, \ldots, T_n$ be a transcendence basis of $R(X)$ over $R(Y)$, and T_{n+1} an element of $R(X)$ satisfying an irreducible separable monic equation over $R(Y)$ $(T_1, \ldots, T_n)$

$$T_{n+1}^r + f_1 T_{n+1}^{r-1} + \cdots + f_r = 0,\ f_i \in A[T_1, \ldots, T_n],$$

such that $R(X)$ is purely inseparable over $R(y)(T_1, \ldots, T_{n+1})$. It follows easily from (EGA, I (6.5.1)) that there is an open subset U of X, an open subset U' of $\operatorname{Spec} A[T_1, \ldots T_{n+1},]$ and a Y-morphism of U onto U' which is radical and finite. The fibres in U and U' of any point of Y are therefore both geometrically irreducible or both geometrically
125 reducible. In view of our earlier remarks, it is sufficient to prove the theorem in the case when $Y = \operatorname{Spec} A$, where A is a noetherian domain and $X = \operatorname{Spec} C$, where C is the A-algebra

$$C = \frac{A[T_1, \ldots, T_{n+1}]}{\varphi A[T_1, \ldots, T_{n+1}]},$$

where $\varphi = \varphi(T_1, \ldots, T_{n+1})$ is a polynomial in $T_1, \ldots, T_{n+1}$ which is irreducible over the quotient field $R(Y)$ of A, and considered as a polynomial in T_{n+1} over $R(Y)(T_1, \ldots, T_n)$, is separable. The assumption that the generic fibre of f is geometrically irreducible simply means that $\varphi(T_1, \ldots, T_{n+1})$ remains irreducible even over the algebraic closure of $R(Y)$. If for, any y $\in \operatorname{Spec} A$ we denote by $\varphi_y(T_1, \ldots, T_{n+1})$ the image of φ by the canonical homomorphism.

$A[T_1, \ldots T_{n+1},] \to k(y)[T_1, \ldots T_{n+1},]$ we have to show that the set of $y \in Y$ such that φ_y is irreducible over the algebraic closure $\overline{k(y)}$ of $k(y)$ contains an open subset of Y. Let d be the degree of φ, and for any integer $p > 0$, Let $N(p)$ denote the number of monomials of degree $\leq p$ in $(n+1)$ variables. For $p, q \geq 0$ with $p + q = d$, we have a morphism

$$\mathbb{A}^{N(p)}(A) \times_A \mathbb{A}_A^{N(q)} \xrightarrow{M_{p,q}} \mathbb{A}^{N(d)}\ (A)$$

which corresponds to multiplication of polynomials of degrees p and q. (Here , $\mathbb{A}^N(A)$) denotes the 'affine space' $\underline{\operatorname{Spec}}\ A[X_1, \ldots X_n,]$ over A. Now, for $y \in Y, \varphi_y$ over the algebraic closure $\overline{k(y)}$ of $k(y)$ into factors of degrees p and q if and only if the point of $\mathbb{A}^{N(d)}(k(y)) = \mathbb{A}^{N(d)}(A) \times_A k(y)$ **126**
corresponding to φ_y is in the image of the morphism

$$\mathbb{A}^{N(p)}(k(y)) \times_{k(y)} \mathbb{A}^{N(q)}(k(y)) \xrightarrow{M_{p,q} \times_A k(y)} \mathbb{A}^{N(d)}(k(y))$$

Now, φ itself defines a morphism $Y = \operatorname{Spec} A \xrightarrow{\Phi} \mathbb{A}^{N(d)}(A)$. Let $Z_{p,q}$ be the image of $M_{p,q}$, and put $Z = \bigcup_{p+q=d} Z_{p,q}$. Then Z is a constructable subset of $\mathbb{A}^{N(d)}(A)$, by a theorem of Chevalley (EGA, IV, (18.5)). Hence, $\Phi^{-1}(Z)$ is also constructable. Since the generic point of Y does not belong to $\Phi^{-1}(Z)$, there is an open subset of Y disjoint with $\Phi^{-1}(Z)$. This implies the Lemma.

We now come to the main theorem.

Theorem of existence of relatively minimal models

Let B be a noetherian prescheme, and $X_i (1 \leq i < \infty)$ B-schemes of finite type which are equi-two dimensional and regular. Suppose given B-

morphisms $\varphi_i : X_i \to X_{i+1} (1 \leq i < \infty)$ which are proper and birational. Then there is an integer n such that for $i \geq n, \varphi_i$ is an isomorphism.

Proof. Since each X_i has only a finite number of components (these
127 are disjoint and open in X_i), and each component of X_{i+1} is the image of one and only one component of X_i, we may assume without loss of generality that X_i are irreducible. Hence, we may also assume that B is irreducible and reduced. Since B is covered by a finite number of affine open set, we may further restrict ourselves to the case when $B = \operatorname{Spec} A$, where A is a noetherian domain. Since one may also assume that all the morphisms $X_i \to B$ are dominant, we have injections $A \to R(X_i)$ such that the diagrams □

$$\begin{array}{ccc} & A & \\ \swarrow & & \searrow \\ R(X_{i+1}) & \xrightarrow{\sim} & R(X_i) \end{array}$$

are commutative. Let A' be a finite type A-algebra contained in $R(X_1)$ such that the quotient field of A' is algebraically closed in $R(X_1)$. Since the X_i are regular and hence normal, $\Gamma(X_i, \mathscr{O}_{X_i})$ are integrally closed rings, so that the inverse image of A' by the composite isomorphism $R(X_i) \to R(X_{i-1}) \cdots \to R(X_1)$ is actually contained in $\Gamma(X_i, \mathscr{O}_{x_i})$. Hence, we may consider the X_i as schemes over $\operatorname{Spec} A'$ and φ_i are morphisms over $\operatorname{Spec} A'$. To sum up, we may assume that B-Spec A, such that the image of the quotient field of A in $R(X_i)$ is algebraically closed in $R(X_i)$ for every i. Finally, by the theorem of decomposition of lecture 4, we may assume that φ_i is either an isomorphism or (X_i, φ_i) is $X_{i+1}-$ isomorphic to a dilatation of X_{i+1} at a closed point of X_{i+1}.

128 Let r be the dimension of the generic fibre of X_1 over B. Since the generic fibre is homeomorphic to a subspace of X_1, we see that $r \leq 2$.

Case (i) $r \leq 1$.

Let $\psi_i : X_i \to B$ denote the structural morphism. By the theorem started at the beginning of this lecture and the next lemma, we can find a non-void open subset U of B such that for any $b \in U, \psi_1^{-1}(b)$ is geometrically irreducible and of dimension r (Note that the generic fibre of ψ_1

is geometrically irreducible since the quotient field of A is algebraically closed in $R(X_1)$ by assumption). Since $\psi_i^{-1}(b) \xrightarrow{\varphi_i X_B k(b)} \psi_{i+1}^{-1}(b)$ is surjective, (EGA, I, (3.5.2)) it follows from theorem above that for every $b \in U$, $\psi_i^{-1}(b)$ is irreducible and of dimension r. We assert that the restriction of φ_i to $\psi_i^{-1}(U)$ is an isomorphism of $\psi_i^{-1}(U)$ onto $\psi_{i+1}^{-1}(U)$. For otherwise, since (X_i, φ_i) is the dilatation of X_{i+1} at a point x, we must have

$$\varphi_i^{-1}(x) \simeq \mathbb{P}'(k(x)),$$

and $$\varphi_i^{-1}(x) \cap \psi_i^{-1}(U) \neq \Phi,$$

so that $$\varphi_i^{-1}(x) = \psi_i^{-1}(\psi_{i+1}(x))$$

since $$r \leq 1 \text{ and } \quad \psi_i^{-1}(\psi_{i+1}(x))$$

is irreducible of dimension r. It follows that $r = 1$ necessarily, and $\psi_{i+1}^{-1}(\psi_{i+1}(x)) = \varphi_i(\psi_i^{-1}(\psi_{i+1}(x))) = x$. This is impossible since we must have $\dim \psi_{i+1}^{-1}(\psi_{i+1}(x)) = 1$. Thus, φ_i induces an isomorphism of $\psi_i^{-1}(U)$ onto $\psi_{i+1}^{-1}(U)$. Now, $F_i = \psi_i^{-1}(B - U)$ is a proper closed subset of X_i, and hence has only a finite number s_i of components (of dimension $=$ 1). Since $\varphi_i(F_i) = F_{i+1}$, it follows that $s_i \geq s_{i+1}$. If further φ_i is not **129**
an isomorphism, φ_i must contract an irreducible component of F_i to a point of F_i to a point of F_{i+1}, so that $s_i > s_{i+1}$. Since $s_i \geq 0$, the sequence $s_1, s_2, \ldots$ must become stationary, which proves that φ_i, must be an isomorphism for i large.

Case (ii) $r = 2$.

In this case, we assert that $\psi_1(X_1)$ is a single (generic) point of B. If not, we can fine a $b \in B$ distinct from the generic point B such that $\psi_1^{-1}(b) \neq \Phi$ and $\neq X_1$ and $\dim \psi_1^{-1}(b) = 2$. Let $F_2 \supset F_1 \supset F_0$ be a sequence of distinct irreducible closed sets of $\psi_1^{-1}(b)$. Then $\overline{F_2} \supset \overline{F_1} \supset \overline{F_0}$, and $\overline{F}_{i+1} \neq \overline{F}_i$. Since $\overline{F}_{i+1} \cap \psi_1^{-1}(b) = F_{i+1} \neq F_i = \overline{F}_i \cap \psi_1^{-1}(b)$, and $\overline{F}_2 \neq X$ since $\overline{F}_2$ is contained in $\psi_1^{-1}(\overline{b})$, and $\overline{\{b\}} \neq B$ since b is not generic. Thus we have a strict chain $X \supset \overline{F}_2 \supset \overline{F}_1 \supset \overline{F}_0$ of irreducible closed sets, which contradicts the assumption that $\dim X = 2$.

Hence we must have $\psi_1(X_1) = b \in B$, and consequently $\psi_i(X_i) = b$. Since the X_i are reduced schemes, it follows that each ψ_i can be factored

as $X_i \to \operatorname{Spec} k(b) \to B$, so that the X_i can be considered as algebraic schemes over $k(b)$. Further, since $\operatorname{Spec} k(b) \to B$ is a monomorphism in the category of preschemes, the φ_i are actually $k(b)$- morphisms. Thus, we are reduced to the case when $B = \operatorname{Spec} K$, K being a field. By Corollary 2 to the theorem of Castelnuovo, we can find an irreducible reduced projective $k(b)$ -scheme which we denote by $\overline{X}_1$, such that X_1 is
130 isomorphic to an open subscheme of $\overline{X}_1$ (since the proof of this Corollary uses the reduction of singularities of a surface, we remark that it is sufficient to have any $\overline{X}$ which is $k(b)$ proper and contains an isomorphic copy of X_1; and the existence of such an $\overline{X}_1$ is ensured by the theorem of *Nagata* on the embedding of abstract varieties in a complete variety [19]). Now, there is a finite set F_i of closed points of X_i such that if we put $\chi_i = \varphi_{i-1} \circ \varphi_{i-2} \circ \cdots \circ \varphi_1 : X_1 \to X_i$, χ_i is an isomorphism of $X_1 - \chi_i^{-1}(F_i)$ onto $X_i - F_i$. Further, $\chi_i^{-1}(F_i)$ is closed in $\overline{X}_1$ since $\chi_i^{-1}(F_i)$ is k-proper. Let $\overline{X}_i$ be the variety obtained by 'patching up' the varieties X_i and $\overline{X}_1 - \chi_i^{-1}(F_i)$ by means of the isomorphism $X_i - F_i \xrightarrow{\chi_i^{-1}} X_1 - \chi_i^{-1}(F_i)$. Then each $\overline{X}_i$ is a $k(b)$-proper scheme, and φ_i extends to a morphism $\overline{\varphi}_i : \overline{X}_i \to \overline{X}_{i+1}$. Further, each $\overline{\varphi}_i$ is either an isomorphism, or is isomorphic to the dilatation of $\overline{X}_{i+1}$ at a regular point of $\overline{X}_{i+1}$.

Let $\Omega^1\frac{1}{\overline{X}_i}$ be the sheaf of 1-differentials on $\overline{X}_i$. Since $\Omega^1\frac{1}{\overline{X}_i}$ are coherent $\mathscr{O}_{X_i}$-modules, $\dim_K H^1(\overline{X}_i, \Omega\frac{1}{\overline{X}_i}) < \infty$. Further, we have seen in lecture 5 that if $\overline{\varphi}_i : \overline{\chi}_i \to \overline{X}_{i+1}$ is a dilatation at a closed point of $\overline{X}_{i+1}$,

$$\dim_K H^1(\overline{X}_{i+1}, \Omega^1\frac{1}{\overline{X}_{i+1}}) < \dim_K H^1(\overline{X}_i, \Omega^1\frac{1}{\overline{X}_i})$$

(The assumption that the varieties are everywhere regular plays no role
131 in the of this inequality). It follows that since the sequence of non-negative integers

$$\dim H^1(\overline{X}_1, \Omega^1\frac{1}{\overline{X}_1}) \geq \dim H^1(\overline{X}_2, \Omega^1\frac{1}{\overline{X}_2} \geq$$

must become stationary $\overline{\varphi}_i$ must be an isomorphism for i large. Consequently, φ_i is also an isomorphism for i large.

This completes the proof of the theorem.

We shall now state an equivalent formulation of the theorem. For this, we need the following

Definition . *A noetherian, two dimensional, regular scheme X over a noetherian base scheme B is said to be a relatively minimal model if for any B-scheme Y which is regular, any proper birational B-morphism $\varphi : X \to Y$ is an isomorphism.*

Corollary . *Let B be a noetherian scheme and X a B-scheme of finite type which is regular and of dimension 2. Then there is a relatively minimal model Y over B and a proper birational B-morphism of X onto Y.*

Indeed, if this were not true, we can clearly find a sequence $X \xrightarrow{\varphi_o} X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} ---$ of regular B-schemes X_i and proper B-morphisms φ_i such that no φ_i is an isomorphism. But this is impossible by the theorem.

In terms of the 'birationality class' $\mathscr{B}(X/B)$ of X introduced in lecture 4, we can restate the Corollary by saying that there is a Y in $\mathscr{B}(X/B)$ such that $X \geq Y$, but such that $Y \geq Z$ for any Z implies that $Y = Z$. This is indeed the justification for the terminology 'relatively minimal **132**
model'.

Remark . In the proof of the theorem when $B = \operatorname{Spec} K$ where K is an algebraically closed field, after imbedding the X_i as open subsets of projective normal surfaces $\overline{X}_i$, one could also use the fact that the Neron-Severi groups of $\overline{X}_i$ are of finite rank and that this rank is decreased by one when we blow down a line to a regular point, to show that the φ_i are isomorphisms for i large. When K is the complex field, the second Betti-numbers of the $\overline{X}_i$ would also serve the same purpose.

Minimal Models. Classification of complete, relatively minimal models over an algebraically closed field when minimal models do not exist.

Let the base scheme be arbitrary noetherian.

Let X be an irreducible, regular, proper B-scheme of dimension two, such that the structural morphism $X \to B$ is dominant and hence surjective. We say that X is a *relatively minimal model over B* if any

B-birational morphism $\varphi : X \to X'$ of X onto another regular, proper B-scheme X' is an isomorphism. It is said to be a *minimal model* if for any regular, B-proper scheme Y, any B-birational map $\psi : Y \to X$ is a B-morphism. In view of the theorem of existence of a minimal model dominated by any given model we see that a relatively minimal model X/B is a minimal model if and only if a birational map $\varphi : X \to X'$ of X
133 onto any other relatively minimal model X'/B is a B-isomorphism.

Our intention in this paragraph is to show that when $B = \operatorname{Spec} K$, where K is an algebraically closed field, except in comparatively few cases (rational and ruled surfaces), a minimal model exists, and to classify all the relatively minimal models in the exceptional cases.

Before we set out on this, we shall establish two results which ought to have found their places earlier. We now suppose that $B = \operatorname{Spec} K$, where K is an algebraically closed field.

The first concerns itself with the behaviour of the canonical class when the surface is blow up. Let X be a non-singular surface, and Ω^2 the line bundle on X whose sections are exterior 2 forms on X. The divisor class determined by this line bundle (i.e., divisor class of the divisor determined by a non-zero rational section of Ω^2) is called *the canonical class* K of X. Let x be any point of X, and $\sigma : X^1 \to X$ the dilatation of X at x. As usual, we denote the fibre $\sigma^{-1}(x)$ by L. We want the relationship between the canonical classes K_X and $K_{X'}$ of X and X' respectively. Let ω be a rational 2-form on X which is regular and non-vanishing at x, and $\sigma^*(\omega)$ its inverse image on X'. Since σ induces an isomorphism of $X' - L$ onto $X - \{x\}$, the zeros and poles of ω, and the multiplicities of the components of the zero and polar divisors are preserved. It only remains to compute the coefficient of L in the divisor
134 of $\sigma^*(\omega)$. Let u, v be uniformising parameters at x, regular in an affine neighbourhood of x with co-ordinate ring A. Then ω has an expression of the form $f\, du \wedge dv$ in this neighbourhood of x, where f is regular and non-vanishing at x. In the open subset of X' with co-ordinate ring $A[\frac{v}{u}]$, $\sigma^*(\omega)$ takes the form $(f \circ \sigma) d(u \circ \sigma) \wedge d(v \circ \sigma)$. Now, we know that at points of L which lie in this neighbourhood, $u \circ \sigma$ and $w = \dfrac{v \circ \sigma}{u \circ \sigma}$

are uniformising parameters. In terms of these, we have

$$\begin{aligned}\sigma^*(\omega) &= (f\circ\sigma)\, d(u\circ\sigma)\wedge\ d((u\circ\sigma)w)\\ &= (f\circ\sigma).\,(u\circ\sigma)\, d(u\circ\sigma)\wedge dw\end{aligned}$$

Since $f\circ\sigma$ is regular and does not vanish along L and since $u\circ\sigma$ vanishes to the first order on L, we obtain that

$$\text{div}\,(\sigma^*(\omega)) = \sigma^*(\,\text{div}\,(\omega)) + L,$$

and consequently we have between the canonical classes of X and X' the relation

$$K_{X'} = \sigma^*\,(K_X) + L \tag{1}$$

The second result that we need is concerned with the arithmetic (or virtual) genus of a (possibly reducible) curve D on a non-singular complete surface. The arithmetic genus is defined by the formula.

$$\pi(D) = 1-\chi(\mathscr{O}_D) = 1-\dim_K H^\circ(D,\mathscr{O}_D)+\dim_K H^1(D,\mathscr{O}_D)$$

where $\mathscr{O}_D = \mathscr{O}_{X/\mathscr{I}_D}$ and $\mathscr{I}_D$ is the invertible sheaf of ideals in $\mathscr{O}_X$ defin- **135**
ing the effective divisor D. Denoting as always by $\chi(F)$ the Euler characteristic $\sum_{i=0}^{2}(-1)^i\dim_K H^i(X,F)$ of a coherent sheaf F on X, the exact sequence

$$O\to\mathscr{I}_D\to\mathscr{O}_X\to\mathscr{O}_D\to O$$

gives the relation $\chi(\mathscr{O}_D)=\chi(\mathscr{O}_X)-\chi(\mathscr{I}_D)$. By Serre duality, $\chi(\mathscr{O}_X) = \chi(\Omega^2)$ and $\chi(\mathscr{I}_D) = \chi(\mathscr{I}_D^{-1}\otimes\Omega^2)$. Finally, the exact sequence $0\to\Omega^2\to\Omega^2\otimes\mathscr{I}_D^{-1}\to\Omega^2\otimes\mathscr{I}^{-1}_{D/\mathscr{O}_X}\to 0$ gives the relation

$$\begin{aligned}\chi(\mathscr{O}_D) &= \chi(\mathscr{O}_X)-\chi(\mathscr{I}_D)\\ &= \chi(\Omega^2)-\chi(\Omega^2\otimes\mathscr{I}_D^{-1})\\ &= -\chi(\Omega^2\otimes\frac{\mathscr{I}_D^{-1}}{\mathscr{O}_X})\end{aligned}$$

Now an elementary argument (used in the proof of the Riemann-Roch theorem for curves) shows that $\chi(\Omega^2\otimes\mathscr{I}^{-1}_{D/\mathscr{O}_X}) = ((K+D).D) + \chi(\mathscr{O}_D)$, and we obtain by substitution that

$$2\chi(\mathscr{O}_D) = 2-2\pi(D) = -((K+D).D)$$

$$\pi(D) = 1 + \frac{1}{2}((K + D).D) \tag{2}$$

As an application of formula (1) and (2) one can find how $\pi(D)$ changes under a single dilatation. Namely, applying formula (10) of lecture 6 one easily finds that

$$\pi(D') = \pi(D) - \frac{l(l-1)}{2},$$

136 $D' = \sigma'(D)$ and l is the multiplicity of D at the center of σ. In particular, if C is irreducible we can resolve the singularities of C by means of dilatations. We obtain a nonsingular curve $\overline{C}$ with genus g and $\pi(\overline{C}) = g$. The above formula gives

$$\pi(C) = g + \sum \frac{l_i(l_i - 1)}{2}$$

where l_i is the multiplicity of the image of C under the i^{th} consecutive dilatation.

We see that $\pi(C) \geq 0$ and even $\pi(C) \geq g$; if $\pi(C) = g$, then C is nonsingular.

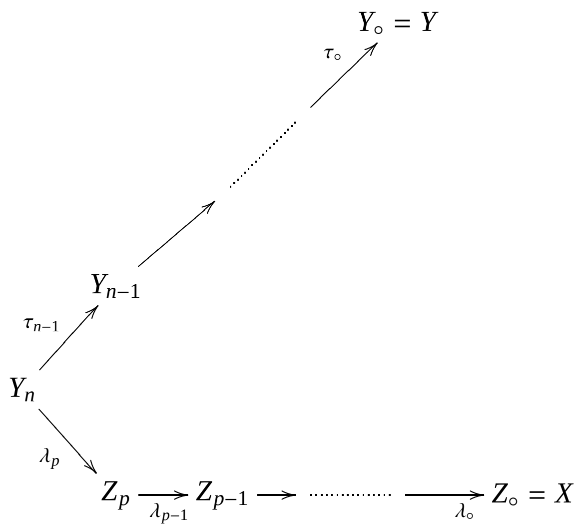
Let us return to the general case of $X \to B$. Suppose that X does not possess a minimal models. Then there are two relatively minimal models X and Y with function fields isomorphic to the given function field and a birational map $\varphi : X \to Y$ which is not regular either way.

By the theorem of elimination of indeterminacies and the theorem
137 of decomposition, there exist B-proper regular schemes $X_i (0 \leq i \leq m), Y_j (0 \leq j \leq n)$ and morphisms

$$\sigma_i : X_{i+1} \to X_i (0 \leq i < m), \tau_j : Y_{j+1} \to Y_j (0 \leq j < n)$$

such that $X_0 = X, Y_0 = Y, X_{m+1} = Z = Y_{n+1}$, and each (X_{i+1}, σ_i) is X_i isomorphic to the dilatations of X_i at a point and each (Y_{j+1}, τ_j) is Y_j-isomorphic to the dilatation of Y_j at a point. We choose the $X_i, Y_j, \sigma_i, \tau_j$ in such a way that the integer $m + n$ is minimal among all such possible choices. We can suppose $m > 0$, $n > 0$, because otherwise either X or Y would not be relatively minimal. Let us denote by L_{m+1} the line on Z which is contracted to a point on Y_n by τ_n. We assert that the

image of L_{m+1} by $\sigma_\circ \circ \sigma_1 \circ \cdots \circ \sigma_m$ in $X = X_\circ$ is not a point of X but a curve. For if it were, we obtain a morphism $f : Y_n \to X$ such that $f \circ \tau_n = \sigma_\circ \circ \cdots \circ \sigma_m$. If we write $f \circ \tau_n$ as the composite of a sequence of dilatations, say $f \circ \tau_n = \lambda_o \circ \lambda_1 \circ \cdots \circ \lambda_p$, then p equals the number of irreducible components of the closed subset of Y_n where f fails to be an isomorphism. From similar interpretation of the integer
m, it follows that $p \leq m$. Thus, we have a situation depicted by the **138**
following diagram:



But now, we have $p + (n-1) \leq m + n - 1$, which contradicts the minimality of $(m + n)$ for all choices of $X_i, Y_j, \sigma_i, \tau_j$. Thus we have shown that L_{m+1} is not contradicted to a point in X. Let us denote by $L_i (0 \leq i \leq m+1)$ the curve in X_i which is the image of L_{m+1} by $\sigma_1 \circ \sigma_{i+1} \circ \cdots \circ \sigma_m$. It is then clear that for $i > 0$, L_i is the proper transform $\sigma'_{i-1}(L_{i-1})$ of the curve L_{i-1} on X_{i-1}.

Now L_{m+1} is isomorphic to $\mathbb{P}'(K)$ and $(L^2_{m+1}) = -[K : k(b)]$, where K is a finite algebraic extension of $k(b)$, since L_{m+1} can be blown down to a point in Y_n (Lecture 6 (9)), By formula (11) of Lecture 6, if we denote by s_i the multiplicity of L_i at the point x_i of X_i which is blown up in X_{i+1}, we have

$$(L_{i+1})^2 = (L_i)^2 - s_i^2[k(x_i) : k(b)] \ (i = 0, 1. \ldots, m),$$

139 so that

$$-[K : k(b)] = (L^2_{m+1}) = (L^2_0) - \sum_{i=0}^{m} s_i^2[k(x_i) : k(b)]$$

(We put $s_i = 0$ if X_i does not lie on L_i). Now, suppose all the s_i are 0, so that x_i does not lie on L_i for any i; there is a neighbourhood of L_{m+1} which is mapped isomorphically a neighbourhood of L_o by $\sigma_\circ \circ \cdots \circ \sigma_m$. Hence $L_\circ$ can be contracted to a point of a regular B-scheme, which contradicts over assumption that $X_o = X$ is a relatively minimal model. Hence, at least one $s_i > 0$. Let i be the largest integer for which $s_i > 0$. Then, by (10) of Lecture 6,

$$(L_{i+1}, \sigma_i^{-1}(x_i)) = s_i[k(x_i) : k(b)],$$

and since $L_{i+1} \simeq L_{m+1} \simeq \mathbb{P}'(K)$ the residue field at every point of L_{i+1} contains K, so that the left side is $\geq [K : k(b)]$

It follows from the above equation for (L^2_0) that

$$\left.\begin{aligned} &(L^2_0) \geq 0, \\ &(L^2_0) > 0, if L_0 \text{ possess singularities} \end{aligned}\right\} \tag{3}$$

Thus, we have proved the following fundamental lemma

Lemma. *If X is a proper, regular, irreducible, two dimensional scheme over B, which is a relatively minimal model but not a minimal model,*
140 *then there is a closed, irreducible, one-dimensional subscheme L of X such that L is contained in the fibre over a closed point b of B. $(L^2) \geq 0$ and $(L^2) > 0$ if L is not regular and L is birationally isomorphic to $\mathbb{P}^1(K)$, where K is a finite algebraic extension of $k(b)$.*

Now, suppose that $B = \operatorname{Spec} K$, K an algebraically closed field. Apply formula (2) to the curve L_i. Since $\pi(L_i) = \dim H^1(\mathbb{P}^1, \mathscr{O}) = 0$ we have $(K_i, L_i) = -1$, where K_i are the canonical classes of X_i. By (1) we have

$$(K_i, L_i) = (\sigma^*(K_{i-1}) + L_i, L_i) = (K_{i-1}, L_{i-1}) + l$$

by formulae (8) and (10) of lecture 6. So,

$$(K_{i-1}, L_{i-1}) = -1 - l < 0.$$

From (1) it again follows that the same is true for L_0. We have thus the following supplement to the principal lemma:

In the case where X is a surface over an algebraically closed field, we have $(K.L) < 0$ where K is the canonical class of X

Upto the end of this lecture, we will consider the case $B = \operatorname{Spec} K, K$ an algebraically closed field. The case of a one-dimensional B will be discussed in the next lecture.

Now, for any complete non-singular surface X, the integers $P_n = \dim H^0(X, (\Omega^2)^{\otimes n})(n1)$ are called the *plurigenera* of X. From the theorem of elimination of indeterminacies and the theorem of decomposition, one deduces by a standard argument (using the fact that a section **141**
of a vector bundle in the complement of a closed subset of codimension ≥ 2 extends to a section on the entire surface (see Lecture 5 for P_1) that the integers $P_n(X)$ are birational invariants of X. Further, since any non-zero section s of $(\Omega^2)^{\otimes n}$ leads to a non-zero section $s^{\otimes m}$ of $(\Omega^2)^{\otimes mn}$ for $m \geq 1$, we see that $P_{mn} = 0$ implies that $P_n = 0$.

Now let X be as in the fundamental lemma above. We shall then show that $P_n(X) = 0$ for all $n \geq 1$ (or in mere classical terminology, the pluri-canonical systems do not exist). For, suppose $P_n(X) \geq 1$ for some $n \geq 1$, so that there is an effective divisor D belonging to the class nK. If we write $D = D_1 + r.L$ where $r \in \mathbb{Z}$ and D_1 does not have L as a component, we have

$$n(K.L) = (D_1.L) + r(L^2) \geq 0$$

since $(D_1.L) \geq 0$ and $(L^2) \geq 0$ by the fundamental lemma. But the above inequality contradicts the second inequality of the same Lemma. Thus, $P_n = 0$ for all $n \geq 1$.

We now make the following

Definition. *A* ruled surface *is the total space of a locally trivial fibre bundle with base a complete non-singular curve, typical fibre the projective line $\mathbb{P}^1$ and structure group the projective group $PGL(1)$*

It is an easy matter to show that for a ruled surface, all the plurigenera vanish. Indeed, the bundle of which it is the total space being locally **142**
trivial, the surface is birationally equivalent to a product $C \times \mathbb{P}^1$. If π_1

and π_2 are the projections of $C \times \mathbb{P}^1$ onto the first and second factors, we have $\Omega^2(C \times \mathbb{P}^1) = \pi_1^*(\Omega_C^1) \otimes \pi_2^*(\Omega_{\mathbb{P}^1}^1)$, so that

$$(\Omega^2(C \times \mathbb{P}'))^{\otimes n} \simeq \pi_1^*((\Omega_C')^{\otimes n}) \otimes \pi_2^*((\Omega_{\mathbb{P}'}')^{\otimes n})$$

It follows that $H^0(C \times \mathbb{P}', (\Omega^2)^{\otimes n}) = H^0(C, (\Omega_C')^{\otimes n})^{\otimes k} \otimes_K H^0(\mathbb{P}', (\Omega_{\mathbb{P}'}')^{\otimes n}) = 0$ since $(\Omega_{\mathbb{P}}')^{\otimes n}$ is a locally free sheaf whose rational section have degree $-2n$. This shows that $P_n(C \times \mathbb{P}') = 0$ for $n \geq 1$, and consequently, because of the birational invariance of the $P_n, P_n = 0$ for $n \geq 1$ for any ruled surface.

It is a theorem of Enriques ([2] Sh. IV) that conversely if X is a complete non-singular surface with $P_4 = P_6 = 0$ (or P_{12} alone $= 0$), then X is birationally equivalent to $C \times \mathbb{P}'$. The proof of this theorem is long and we shall not give it. We note however that this yields the result that any X as in the fundamental Lemma is birationally a product $C \times \mathbb{P}'$. We shall prove this (and more, namely that X is actually a ruled surface) directly below.

From now on till the end of this lecture, X and L will have the same connotations as in the fundamental Lemma. We consider two cases.

143 **Case (i). The Albanese variety A of X is non-trivial.**

In this case, choose a point x_0 of L, and denote by $\psi : X \to A$ the canonical morphism of X into the Albanese such that $\psi(x_\circ) = 0$. The image $\psi(X)$ of X in A must be an irreducible variety of dimension 1 or 2, since this image must generate A. Further since L is a rational curve, by a well known result on abelian, since L is a rational curve, by a well known result on abelian varieties ([15] p. 25), $\psi(L) = 0$. Suppose $\psi(X)$ is of dimension 2. In this case, the restriction of the bilinear form (.) (intersection number) to the irreducible components of dimension one of any fibre is negative definite, as proved in lecture 6. But then, since $L \subset \psi^{-1}(0)$, we must have $(L^2) < 0$, which is a contradiction.

Thus, $\psi(X)$ must be a curve in A. Identifying the function field $R(\psi(X))$ of $\psi(X)$ with a subfield of $R(X)$ by means of ψ, let F be the algebraic closure of $R(\psi(X))$ in $R(X)$, and let C be the normalisation of the curve $\psi(X)$ in F, with canonical morphism $\pi : C \to \psi(X)$. Since X is nonsingular (and hence normal), the morphism $\psi : X \to \psi(X)$ factorises $\psi = \pi \circ \psi'$ where $\psi : X \to C$ is a morphism. Since the fibres

of π are finite, and since L is contained in a fibre of ψ, L is also contained in a fibre of ψ',i.e., a fibre of ψ, L is also contained in a fibre of ψ',i.e., $\psi'(L) = y \in C$. Further since $R(C)$ is algebraically closed in $R(X)$, the generic fibre of ψ' is geometrically irreducible (see Lemma at the beginning of lecture) and any fibre of ψ' is geometrically connected **144**
by the connectedness theorem of Zariski. Further, we have seen in lecture 6 that if the fibres are connected, the restriction of the intersection number to the free abelian group generated by components of any fibre is negative semi-definite, and the null-space of this bilinear form is precisely the maximal subgroup of rank one containing the whole fibre(considered as a divisor). Since $(L^2) \leq 0$ and $(L^2) \geq 0$ by assumption $(L^2) = 0$, so that L is non-singular and is the only irreducible component of its fibre. Hence the divisor, $\psi'^{-1}(y)$ is of the form nL, where n is a positive integer. Our aim is to show that $n = 1$. Since L is isomorphic to $\mathbb{P}', \pi(L) = 0$ and hence $(KL) = -2$. For any $z \in C$, we have by what we have seen in Lecture 6 that

$$(K.\psi'^{-1}(z)) = (K, \psi'^{-1}(y)) = (K.nL) = -2n,$$

so that for the arithmetic genus of $\psi'^{-1}(z)$ we have

$$\pi(\psi'^{-1}(z)) = 1 + \frac{1}{2}(\psi'^{-1}(z).(\psi'^{-1}(z) + K)) = 1 - n$$

since $(\psi'^{-1}(z)^2) = 0$.

Now, we have the

Lemma. *Let R be a function field in one variable over a perfect field K, and suppose R is contained and is algebraically closed in a field S is separable over R.*

Proof. Since R is separably generated over K, if $\{x\}$ is a separating transcendence basis of R/K, we have that $R^{1/p} = R(x^{1/p})$ We have to show that S and $R^{1/p}$ are linearly disjoint over R in order to establish that S/R **145**
is separable. But now, $x^{1/p} \notin S$ since R is algebraically closed in S, and since $X \in R \subset S$, we have $[S(x^{1/p}) : S] = p = [R(x^{1/p}) : R]$, which proves the assertion. □

Now, $R(C)$ is algebraically closed in $R(X)$, so that $R(X)$ is also separable over $R(C)$, by the above lemma. But in this case, it can be shown as in the first lemma of this lecture that there is an open subset U of C such that for $z \in U, \psi'^{-1}(z)$ is an integral scheme, or equivalently that as a divisor, $\psi'^{-1}(z)$ has a single component and this component occurs with multiplicity one. Hence for $z \in U$, we have $H^0(\psi'^{-1}(z), \mathscr{O}_{\psi'^{-1}(z)}) \simeq K$, so that

$$\begin{aligned}\pi(\psi'^{-1}(z)) &= 1 - \dim H^0(\psi'^{-1}(z), \mathscr{O}_{\psi'^{-1}(z)}) + \dim H^1(\psi'^{-1}(z), \mathscr{O}_{\psi'^{-1}(z)})\\ &= \dim H^1(\psi'^{(-1)}(z), \mathscr{O}_{\psi'^{-1}(z)}) \geq 0\end{aligned}$$

It follows from our earlier result that $1 - n \geq 0$, so that $n = 1$. Thus the *divisor* $\psi'^{-1}(y)$ is equal to L, and $(K.\psi'^{-1}(z)) = -2$ for all $z \in C$. We now assert that *every* fibre $\psi'^{-1}(z)$ in the schematic sense is isomorphic to $\mathbb{P}'$. To prove this, first suppose that $\psi'^{-1}(z) = n.D$ where n is an integer > 1 and D irreducible. Then we have $(D^2) = 0$ and $-0 = (nD.K) = n(D.K)$, so that $n = 2$ and $(D.K) = -1$. But this is impossible, since $(D^2) + (D.K)$ must be an even integer. Thus if $\psi'^{-1}(z) = nD$ with D irreducible, then $n = 1$ and $(D.K) = -2$, so
146 that $\pi(D) = 0$. Now suppose that $\psi'^{-1}(z) = \sum\limits_1^r n_i D_i$, with $n_i > 0$ and D_i running through the irreducible components of $\psi'^{-1}(z)$, with $r \geq 2$. In this case, we must have $(D_i^2) \leq -1$, and $(D_i^2) + (D_i.K) \geq -2$ for $i = 1, \ldots, r$. We assert that there exists an i such that $(D_i^2) = -1$ and $(D_i^2) + (D_i.K) = -2$. If not, for every i, either $(D_i^2) \leq -2$ or $(D_i^2) + (D_i.K) \geq -1$, with $(D_i^2) = -1$, so that $(D_i.K) \geq 0$ for all i. But this is impossible since $(\psi'^{-1}(z).K) = \sum n_i(D_i K) = -2$. Thus, there is an i for which $(D_i^2) = -1$ and $D_i^2 + (D_i.K) = -2$, that is, $\pi(D_i) = 0$. But as we have seen in connection with formula (2) $\pi \geq g$, and $\pi = g$ only if the curve is nonsingular. So D_i is nonsingular and of genus 0 and consequently isomorphic to $\mathbb{P}'$. Since $(D_i^2) = -1$, it follows by the Theorem of Castelnuovo that D_i can be blown down to a point of a nonsingular surface, thus contradictions the assumption that X is a relatively minimal model. Thus we finally deduce that for any $z \in C, \psi'^{-1}(z)$ (in the schematic sense) is isomorphic to $\mathbb{P}'$.

We shall now deduce that $\psi' : X \to C$ is a locally trivial fibration with $PGL(1)$ as structure group and $\mathbb{P}'$ as fibre.

Let x be a generic point of C and $\psi'^{-1}(x)$ the generic fibre. By the lemma proved earlier, the function field $R(\psi'^{-1}(x))(\simeq R(X))$ is a regular extension of $k(x)(\simeq R(C))$. Thus, the curve $\psi'^{-1}(x)$ defined over $k(x)$ has genus 0. (EGA III, 79). It is then well-known that $\psi'^{-1}(x)$ is birationally isomorphic *over* $k(x)$ to a conic $Q(X_o X_1 X_2) = 0$ in $\mathbb{P}^2$ defined over $k(x)$ **147**
(see lecture 5). But by a theorem of Tsen, any form in n variable of degree in a function field of one variable over an algebraically closed field has a non-trivial (i.e. different from (0)) zero in that field provided that $n > d$. In particular $\psi'^{-1}(x)$ carries a $k(x)$-rational point is a rational section $\sigma : C \to X$ such that $\psi' \circ \sigma = Id_C$. However, since C is a normal curve and X is complete, σ extends to a C-morphism of C into X. If we denote by D the curve $\sigma(C)$ in $X, \sigma(C)$ intersects every fibre $\psi'^{-1}(z)$ transversally at a unique point $\sigma(z)$ for all $z \in C$, so that $(D.\psi'^{-1}(z)) = 1$ for all $z \in C$. Let $\mathcal{L}(u)$ denote the invertible sheaf defined by the divisor u and put $F = \psi'^{-1}(z)$, for some fixed $z \neq y$. For any m, we have the standard exact sequence:

$$0 \to \mathscr{L}(D+(m-1)F)(m-1)F) \to \mathscr{L}(D+mF) \to \mathscr{L}_F(D+mF).F \to 0$$

where the last homomorphism is the restriction to F. Obviously $\mathscr{L}((D+mF).F) = \mathscr{L}_F(D.F)$.

We shall prove that, for m large enough the corresponding homomorphism

$$H^0(X, \mathscr{L}(D+mF)) \to H^0(F, \mathscr{L}_F(D.mF))$$

is surjective. Because of the cohomology exact sequence it is sufficient to prove that

$$H^1(X, \mathscr{L}(D+(m-1)F)) \to H^1(X, \mathscr{L}(D+(m-1)F))$$

is an isomorphic for large m. But as $H^1(F, \mathscr{L}(D.F)) = 0$ (note that **148**
$F \simeq \mathbb{P}'$ and $D.F$ is a point on F). This homomorphism is a certain surjective. So the numbers $\dim .H^1(X, \mathscr{L}(D+mF))$ form a decreasing sequences of positive integers and thus remain constant for large m. This is just what we need. As $D.F$ is a point on $\mathbb{P}^1, \dim H^0(F, \mathscr{L}(D.F)) = 2$.

Choose two section s_0 and s_1 of $\mathscr{L}(D+mF)$ over X whose images form a basis for $H^0(F, \mathscr{L}(D.F))$.

The linear system

$$\lambda_o s_o + \lambda_1 s_1 = 0(\lambda_\circ, \lambda_1 \in K, (\lambda_\circ, \lambda_1) \neq (0,0))$$

has no base points on $\psi'^{-1}(z)$, by choice of the s_i, so that because of the properness of ψ', we may, by choosing U smaller if necessary, assume that this linear system has no base points on $\psi'^{-1}(U)$. Thus, we get a morphism $\lambda : \psi'^{-1}(U) \to \mathbb{P}'$, and restricted to $\psi'^{-1}(z)$ is an isomorphism onto $\mathbb{P}'$. Let $\psi : \psi'^{-1}(U) \to U \times \mathbb{P}'$ denote the morphism $\chi = (\psi'|\psi'^{-1}(U), \lambda)$. For any $x \in \psi'^{-1}(z), \chi^{-1}(\chi(x) = \{x\}$, and the tangent mapping $d\chi$ is injective at x. Since χ is also proper, we deduce that χ is birational, and it follows by $z.M.T$ that χ is an isomorphism of a neighborhood of $\psi'^{-1}(z)$ onto a neighbourhood of $z \times \mathbb{P}'$ over U. Again because of the properness of ψ' and the projection $U \times \mathbb{P}' \to U$, it follows that there is a neighbourhood V of z such that there is a V-isomorphism of $\psi'^{-1}(V)$ onto $V \times \mathbb{P}'$. This establishes the local triviality of $X \xrightarrow{\psi} C$. It
149 can be deduced easily that this is indeed an associated fibre space on C to a principle fibre space with $PGL(1)$ as structure group. (We have only to show that if there are two 'trivializations' on an open subset, they are 'connected' by projective transformations 'varying algebraically' in the open subset).

Thus, our investigation in case (i) lead to the

Theorem . *Let X be a complete non-singular surface which is a relatively minimal model but which is not a minimal model. Assume that the Albanese variety of the surface is non-trivial (or in other words, the surface is irregular). Then X is a ruled surface over a curve of positive genus.*

Thus, the problem of classification of relatively minimal (but not minimal) models reduced to the classification of (locally trivial) principal $PGL(1)$ bundled on curves. Now, it is not difficult to show that any $PGL(1)$ bundle arises from a $GL(2)$-bundle by 'extension of structure group, and two $GL(2)$-bundles give rise to the same $PGL(1)$ bundle if and only if, of the corresponding vector line bundles of rank 2, each is

got from the other by tensoring with a line bundle. Let us fix once and for all a line bundle L of degree bundles of rank two one C whose determinantal bundle is the trivial line bundle (resp. is the line bundle L). Let π be the (finite) group of elements of order two on the Jacobian of the curve. Then it is clear that π acts on S_o and S_1 (by tensorisation). The quotient set of $S_o \cup S_1$ for the action of π can be canonically identified **150**
with the set of relatively minimal models or ruled surfaces on the give curve C.

We now turn to the case of regular surfaces which are relatively minimal but not minimal.

Case (ii). X **is (as in the fundamental lemma, and) such that the Albanese variety of** X **is trivial.**

In this case, by using a criterion due to *Castelnuovo* for the rationality of a projective surface, we shall show first that X is rational, and we shall then state without proof the theorem of classification of rational surfaces which are relatively minimal.

It is known that *when the first integer* P_1 *(usually called the geometric genus and denoted by* p_g*)* among the plurigenera P_i *of a non-singular surface is* 0*, the irregularity* q *(equal by definition to the dimension of the Albanese of the surface) is given by* ([8], 236, 2.10)

$$q = -p_a$$

It follows that for our relatively minimal model X, $p_a = o$. We now apply the following criterion of rationality ([2], [24], [25]).

Theorem (Castelnuovo). *If for a complete non-singular surface we have* $p_a = p_2 = 0$*, the surface is rational.*

Thus we deduce that X is rational. We want to describe here all relatively minimal models of a rational surface. First, we recall the definition of the line bundle $G(D)$ over a curve X associated with a divisor D.

Let $X = \bigcup_I U_i$ be a covering of X such that D is defined on each U_i **151**
by a function u_i. Then $G(D)$ is covered by open subset $U_i \times \mathbb{P}^1$ which are patched up together by means of the transition rule

$$x \times \xi \sim y \times \eta, \quad x \in U_i, y \in U_j,$$

$$\text{if} x = y \in U_i \cap U_j$$
$$\text{and } \eta = \xi \frac{u_i}{u_j}(x) \quad (3)$$

Here we understand that the point at ∞ on $\mathbb{P}^1$ remains unchanged on multiplication by $\dfrac{u_i}{u_j}(x)$. Thus, $G(D)$ has two obvious sections: $S_o = \{x \times o\}$ and $S_\infty = \{x \times \infty\}$. As is well- known $G(D) \simeq G(D')$ if $D \sim D'$. We shall apply this construction to $X = \mathbb{P}^1$. Since a divisor class on $\mathbb{P}^1$ is determined by its degree, we can take $D = nx$ and we shall denote the corresponding $G(D)$ by $\mathbb{F}_n$. One can check readily that

$$x \times \xi \longrightarrow \qquad x \times \xi^{-1}$$

defines an isomorphism between $G(D)$ and $G(-D)$. This explains why we can restrict ourselves to considering $\mathbb{F}_\mu, n \geq o$.

We shall first compute the group of divisor classes on $\mathbb{F}_n$. Note first that all the fibres of $\mathbb{F}_n$ are linearly equivalent. Indeed, let F_x and F_y be fibres over $x, y \in \mathbb{P}^1, \pi : \mathbb{F}_n \to \mathbb{P}^1$ be the projection and let f be a function on $\mathbb{P}^1$ such that $(f) = (x) - (y)$; then $F_x - F_y = (\pi^* f)$. We
152 shall prove that the group of divisor classes on $\mathbb{F}_n$ has two generators -any fibre F_o and the section S_o. Let D be any divisor on $\mathbb{F}_n$. Choose a proper open set U of the base $\mathbb{P}'$ such that if $\pi : \mathbb{F}_n \to \mathbb{P}'$ is the projection, $\pi^{-1}(U) \simeq U \times \mathbb{P}'$. Then $\pi^{-1}(U) - (S_o \cap U)$ is isomorphic to $U \times K$, hence to an open subset of K^2. Thus, the restriction of D to $\pi^{-1}(U)$ is a principal divisor (f) an $\pi^{-1}(U)$. The divisor $D - (f)$ has therefore components contained in $S_o \cup \pi^{-1}(\mathbb{P}' - U)$, that is, $D - (f)$ is a linear combination of S_o and a finite number of fibres. Since all fibre are linearly equivalent, our assertion follows.

We now derive an important equivalence between S_o, S_∞ and F_o. Let us look at $\mathbb{F}_n$ as being of the form $G(D), D$ a divisor of degree n on $\mathbb{P}^1$. Consider the function

$$f(x \times \xi) = \xi u_i(x)$$

on the open subset $U_i \times \mathbb{P}^1$. In view of the transition rule, it is obvious that we get a function f on the whole of $G(D)$. The divisor (f) can be easily computed, the answer is

$$(f) = \pi^{-1}(D) + S_o - S_\infty$$

In particular, we have on $\mathbb{F}_n$

$$nF_o + S_o - S_\infty \sim 0 \tag{4}$$

If we consider the intersections of both sides of (4) with S_o and S_∞ **153**
and use the equalities $(S_o.F_o) = (S_\infty.F_o) = 1$ we obtain

$$(S_o^2) = -n \text{ and } (S_\infty^2) = n$$

From this we can deduce that S_o is (if $n > o$) the only irreducible curve on $\mathbb{F}_n$ with negative self-intersection (while on $\mathbb{F}_o$ there are none).

Indeed, if C is another irreducible curve, $C \sim lF_\circ + mS_\circ$ and thus either $C \sim lF_\circ$ and $(C^2) = 0$

$$\begin{aligned} \text{or} \qquad m &= (C.F_o) > 0 \\ 1 - mn &= (C.S_o) \leq 0 \\ \text{and} \qquad (C^2) &= m(2l - n) = m(2(l - mn)) + (2m - 1)n) > 0 \end{aligned}$$

We see from this that the $\mathbb{F}_n$ are non-isomorphic not only as fibrations over $\mathbb{P}^1$ but also as surfaces. On the other hand all of them except $\mathbb{F}_1$ are relatively minimal models- they have no curves C with $(C^2) = -1$. Indeed, $\mathbb{F}_1$ has an exceptional curve of the first kind, namely S_o. From the Castelnuovo theorem proved in lecture 6 it follows that S_o on $\mathbb{F}_1$ can be contracted to a point. It is easy to verify directly that the resulting surfaces is the projective plane $\mathbb{P}^2$. Thus, we have obtained a series of surfaces.

$$\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1 (\simeq \mathbb{F}_o), \mathbb{F}_n (n > 1)$$

and have also' shown that all of them are relatively minimal models of
$\mathbb{P}^2$ and are non-isomorphic to one another. It can be proved that these are **154**
the only relatively minimal models of $\mathbb{P}^2$. The proof however is much complicated. The reader is referred to Nagata ([18]).

Lecture 8

Minimal Models in the case of a base scheme of dimension one

Throughout this section, we assume that the base scheme B is noethe- **155**
rian, regular and of dimension one. We shall further assume that for every closed b of B, the residue field $k(b)$ is perfect field.

We wish to investigate the nature of relatively minimal models which are not minimal in this case also. Let X be a relatively minimal model over B which is not minimal. We *assume* that $R(X)$ is a separably generated extension field of $R(B)$. Further, if $R(B)$ is not algebraically closed in $R(X)$, let $\tilde{B}$ be the normalisation of B in the algebraic closure of $R(B)$ in $R(X)$. Then the structural morphism of X onto B factories as $X \to \tilde{B} \to B$, so that X may be considered as a $\tilde{B}$-scheme; and as such, it continues to be relatively minimal but not minimal. Replacing B by $\tilde{B}$, we may therefore assume that $R(X)$ is a regular extension of $R(B)$. We shall show in this case that the generic fibre of $X \to B$ is a regular curve of genus 0. Thus, if X/B admits a rational section, $R(X)$ would be a simple transcendental extension of $R(B)$.

We shall now prove the above assertion. The proof is quite analogous to the proof of the corresponding result in the case of an irregular surface over an algebraically closed field and we shall only give those steps which are new to this context.

156 We have shown, in the fundamental lemma if lecture 7 that there is a closed irreducible one dimensional subscheme L of X such that it is contained in the fibre over a closed point b_o of B and satisfies $(i)(L^2) \geq \circ$, and $(L^2) > \circ$ if L is not regular and (ii) L is birationally isomorphic to $\mathbb{P}'(K)$, where K is a finite algebraic extension of $k(b_o)$. Now, since the restriction of the intersection product to the subgroup generated by the components of the fibre over b_o is negative semi-definite, and since its null space is generated by the entire fibre (see Lecture 6; the fibre is connected and even geometrically connected, since $R(B)$ is algebraically closed in $R(X)$) we deduce successively that $(L^2) = 0$ and that the fibre over b_o reduces to L. Since $(L^2) = 0, L$ must be regular, and hence L is isomorphic to $\mathbb{P}'(K)$, where K is a finite extension of $k(b)$. Since $k(b_o)$ is perfect by assumption, K is separably algebraic over $k(b_o)$. If we put $n = \left[K : k(b_o)\right]$, and if $\bar{k}$ denotes the algebraic closure of $k(b_o), \mathbb{P}'(K) \times_{k(b_o)} \bar{k}$ is isomorphic to the disjoint union of n copies of $\mathbb{P}'(\bar{k})$. Since L is geometrically connected over $k(b_o)$, it follows that we must have $n = 1$, so that $L \simeq \mathbb{P}'(k(h_o))$.

For any point $b \in B$, let us put F_b=Fibre b in X, and

$$\chi(F_b) = \sum_i (-1)^i \dim_{k(b)} H^i(F_b, \mathscr{O}_{F_b})$$

157 Because of (*EGA*, III 7.9), $\chi(F_b)$ is an integer independent of b. We shall evaluate $\mathcal{X}(F_{b_o})$. Let $\mathscr{I}$ be the sheaf of ideals defining the closed subscheme L. Since the sheaf $\mathscr{I}'$ of ideals defining the fibre F_{b_o} is principle (being in a neighbourhood of the fibre by a uniformising parameter for B at b_o) we must have $\mathscr{I}' = \mathscr{I}'^n$ for some integer $n > 0$. Further, since $(L^2) = 0$, the sheaves $\mathscr{I}^r / \mathscr{I}^{r+1} \simeq \mathscr{I}^r \otimes \mathscr{O}_L$ are all inversible and of degree 0 on $L \simeq \mathbb{P}'(k(b))$, and hence are isomorphic to $\mathscr{O}_L$. We deduce from, the exact sequence $o \to \mathscr{I}^r / \mathscr{I}^{r+1} \to \mathscr{O}_{X/\mathscr{I}^{r+1}} \to \mathscr{O}_{X/\mathscr{I}^r} \to 0$ the equation

$$\chi(\mathscr{O}_{X/\mathscr{I}^{r+1}}) = \chi(\mathscr{O}_{X/\mathscr{I}^r}) + \chi(\mathscr{O}_L) = 1 + \chi(\mathscr{O}_{X/\mathscr{I}^r})$$

so that

$$\chi(F_{b_o}) = \mathcal{X}(\mathscr{O}_{X/\mathscr{I}^n}) = n.$$

Hence for the generic point b^* of B, we have

$$n = \chi(F_{b^*}) = \dim_{k(b^*)} H^o(F_{b^*,\mathscr{O}_{F_b^*}}) - \dim_{k(b^*)} H^1(F_b^*, \mathscr{O}_{F_b*})$$

Since F_b^* is integral and $k(b^*)$-proper, and since $R(B)$ is algebraically closed in $R(X)$, we must have $H^o(F_b^*, \mathscr{O}_{f_b^*}) = k(b^*)$. It follows therefore that we must have $n = 1$ and $H^1(F_{b^*}, \mathscr{O}_{F_b^*}) = o$. Thus , F_{b^*} must be a regular curve of genus 0 over $k(b^*)$. This proves our assertion.

It would interesting to examine the different relatively minimal models in the case where the minimal model does not exist, as we have already done in the geometric case at the end of the previous lecture. Suppose for instance that $B = \operatorname{Spec} \mathscr{O}$, $\mathscr{O}$ a Dedekind domain. Let F_β be the generic fibre of X.

According to our principle theorem, there is a curve of genus 0 over **158**
$k(\beta) = R(B)$. One has to consider here two cases.

Case 1. F_β has a rational point over $R(B)$. Then, as we have seen in lecture 5, $F_\beta \simeq \mathbb{P}^1(R(B))$. In this case, A. Bialynicki-Birula has proved that the relatively minimal models of X are in a $1-1$ correspondence with the elements of Cl/Cl^2 where Cl is the group of divisor classes of $\mathscr{O}$ (unpublished).

Case 2. F_β has no rational point over $R(B)$. In this case F_β is isomorphic to a quadric over $R(B)$. The structure of relatively minimal models in this case is unknown.

We now given an example of application of the main theorem.

An example

Let $F = O$ and $G = O$ be two non-singular cubic curves in $\mathbb{P}^2$ (over an algebraically closed fields) which intersect at nine distinct points. Then necessarily the two curves cannot touch at any one of these points, since their intersection number would exceed 9. We further assume that among these nine points, no six lie on a conic and no three on a line. One sees easily that this latter assumption is equivalent to the assumption that the pencil

$$\lambda F + \mu G = 0 \qquad (*)$$

of cubics contains no degenerate members. This incidentally proves the existence of such a pair of cubics, since the dimension of the projective space of cubics is 9, whereas the closed subset of degenerate cubics
159 is of dimension $2 + 5 = 7$, so that a general pencil of cubics, which is represented by *a* line in the space of cubics, contains no degenerate cubics.

Since F and G do not touch at any of their points of intersection, the members of the linear system ((∗)) have *a* variable tangent line at their base points $P_o, P_1, -, P_8$. It follows from what was seen in lecture 3 that if X is the surface obtained from $\mathbb{P}^2$ by blowing up these 9 points, the linear system ((∗)) defines *a morphism* f of X onto $\mathbb{P}'$ (given by the rational function G/F). We consider X as *a* scheme over $\mathbb{P}'$ by this morphism f. We denote by L_i the fibre over P_i in X. Then it is trivially seen that the restriction of f to each L_i is an isomorphism of L_i onto $\mathbb{P}'$. Now, the fibres of f are proper transforms in X of cubic curves of the pencil ((∗)). By our assumption on the pencil, no fibre contains *a* non-singular rational curve, so that X is *a* relatively minimal model over $\mathbb{P}'$. Since the generic fibre of f is *a* non- singular cubic and hence elliptic curve, X must actually be *a* minimal model over $\mathbb{P}'$. Note however that X is not relatively minimal over the base field, by our very construction of X.

Now, since f restricted to each L_i is an isomorphism of L_i onto $\mathbb{P}'$, we have section σ_i: $\mathbb{P}' \to X (0 \leq i \leq 8)$ such that $\sigma_i(\mathbb{P}') = L_i$. Let x be *a* generic point of $\mathbb{P}'$, and put $\sigma_i(x) = y_i \in f^{-1}(x)$. Since $f^{-1}(x)$ is an elliptic curve over $k(x)$ with *a* rational point $y_\circ$, it is well known that
160 we can define a group law on $f^{-1}(x)$ which makes of it an abelian group variety over $k(x)$ whose zero point is $y_\circ$ in *a* unique way. Let H be the discrete abelian group generated by the elements $y_i (1 \leq i \leq 8)$ for this group law. Each element of this group is a $k(x)$-rational point of $f^{-1}(x)$, and hence defines *a* translation morphism $f^{-1}(x)$ onto itself defined over $k(x)$. Further, an element of H different from the O-element acts on $f^{-1}(x)$ without fixed points (since it is a translation map of a group by an element of a group). Now, any automorphism of $f^{-1}(x)$ over $k(x)$ can also be considered as a birational automorphism of X *over* $\mathbb{P}'$. Since X is *a* minimal model, these birational automorphisms are actually biregular.

Thus, we have proved that H acts as a group of biregular automorphisms of X over $\mathbb{P}'$, such that for $g \in H$, $g \neq e$, g leaves no element of the generic fibre fixed.

Now, if F and G are 'general' cubic forms, it is easily shown that the group H generated by $y_1, \cdots, y_8$ is actually a free abelian group $\sum\limits_{1}^{8} \mathbb{Z} y_i$. Indeed, first choose F as *a* non singular cubic, and choose 8 general points $y_1 \cdots, y_8$ of $F = 0$ such that with respect to an inflexional point of $F = 0$ as origin the y_i are linearly independent. Then, choose G as a member of the pencil of cubic through $y_1, \cdots y_8$. If $y_\circ$ is the ninth point of intersection of F and G, we must have (with the inflexion point of $F = 0$ as origin) on $F = 0$, $\sum\limits_{0}^{8} y_i = 0$. If we now take $y_\circ$ as the origin instead of the inflexion point, we see that $y_1, \ldots, y_8$ must still be linearly **161**
independent with respect to this origin.

It follows that 'in general', H is a free abelian group on 8 generators. Since H acts as a group of biregular automorphisms of X, and since L_0 is an except ional curve on $X (i.e., L_o \simeq \mathbb{P}', (L_0)^2 = -1)$, the curves $h.L_0, h \in H$ are all distinct and are all exceptional curves.

Thus, we have an example of a non-singular surface carrying an infinity of exceptional curves.

There are many other interesting features in this example.

For instance, consider *a* fibre of f which has *a* singularity.

Since the fibre is a non-degenerate cubic, it must either have a node or a cusp, and no other singularity. In either case, it must be a rational curve. If the singularity is a node, the complement of this singular point is isomorphic to K^*, and if it is a cusp, the complement is isomorphic to K. In either case therefore, the fibre is isomorphic to a group variety. It can further be shown by simple considerations that the group law of the generic fibre specialises to the group law on the special fibres also. We shall try to explain this phenomenon in the next section, where we shall study general fibrations by elliptic curves of a surface.

Surfaces fibred by elliptic curves.

We shall merely state some of the results of this very interesting topic without proof.

162 We assume throughout that the base scheme B is irreducible, noetherian, regular and of dimension one. We also assume (though this is not essential in much of what follows) that for every closed point $b \in B$, the residue field $K(b)$ at b is perfect.

We say that an irreducible regular B-proper scheme X with surjective structural morphism $f : X \to B$ is a *fibration by elliptic curves over* B, if $(i) f^*(R(B))$ is algebraically closed in $R(X)$, and $R(X)$ is a separably generated extension of $R(B)$, (ii) the generic fibre $f^{-1}(b_o)$ (where $b_\circ$ is the generic point of B) is *a* non-singular curve of genus 1 on $k(b_\circ)$ which admits a rational point $x_\circ$ over $k(b_\circ)$ (iii) X is a relatively minimal (and hence minimal) model.

The rational point $x_\circ$ over $k(b_o)$ defines a rational section $\sigma_o : B \to X$, with $\sigma_o(b_o) = x_o$ which is actually regular on X, since X is B-proper and B is normal of dimension one. Since $f^{-1}(b_o)$ is a non-singular curve of genus one on $k(b_\circ)$ with a rational point $x_\circ$, it is well-known that there is a unique structure of a commutative group variety on $f^{-1}(b_o)$ such that x_o is the zero point. (It follows in particular, by 'extension of base' to the algebraic closure of $k(b_\circ)$, that $f^{-1}(b_\circ)$ is absolutely simple). Let $m : f^{-1}(b_\circ) \times_{k(b_\circ)} f^{-1}(b_\circ) \to f^{-1}(b_\circ)$ be the multiplication law in $f^{-1}(b_\circ)$, so that m is defined over $k(b_o)$. Then m defines a unique rational map $M : X \times_B X \to X$ whose 'restriction' to $f^{-1}(b_\circ) \times_{k(b_\circ)} f^{-1}(b_\circ)$ coincides with m.

163 Similarly, if $i : f^{-1}(b_\circ) \to f^{-1}(b_\circ)$ is the $k(b_\circ)$-morphism of $f^{-1}(b_\circ)$ corresponding to the inverse operation of the group, i defines a B-birational map $I : X \to X$, and I is actually a morphism since X is a minimal model. It is clear that M and I satisfy all the formal laws of multiplication and inverse in a group scheme, with σ_o as the identity section. (Note however that M is only *a* rational map).

Let $\mathscr{U}$ be the set of points in X where the morphism f is simple. By [S.G.A II, 2.1], f is simple at x if and only if $f^{-1}(f(x))$ is absolutely simple at x, since $\mathscr{O}_{x,X}$ is in any case $\mathscr{O}_{f(x),B}$ flat. In view of our earlier remark, $f^{-1}(b_\circ) \subset \mathscr{U}$, and since for a closed point b of $B; k(b)$ is a perfect field by assumption, if $f(x) \neq b_\circ$ then $x \in \mathscr{U}$ if and only if $f^{-1}(f(x))$ is regular at $x. \mathscr{U}$ is an open subset of X (SGA, II, 1.1).

Further, since for any point $x \in \sigma_\circ(B), \sigma_\circ(b)$ and $f^{-1}(f(x))$ intersect with multiplicity one at x, x is a simple point of $f^{-1}(f(x))$, so that $\sigma_\circ(B) \subset \mathscr{U}$.

Now, it can be shown that M is regular on the open subset $\mathscr{U} \times_B \mathscr{U}$ of $X \times_B X$, and that $M(\mathscr{U} \times_B \mathscr{U}) \subset \mathscr{U}$. Since it is clear that also $I(\mathscr{U}) \subset \mathscr{U}$ (I being *a* B-automorphism of X/B), we see that when restricted to $\mathscr{U}$, M, I and $\sigma_\circ$ define *a* structure of a *scheme of groups over* B on $\mathscr{U}$.

Suppose in particular that $B = \operatorname{Spec} A$, where A is *a* complete discrete valuation ring with maximal ideal $\mathfrak{M}$ (and perfect residue field k). **164**
For any integer $n > 0$, we get *a* group scheme $\mathscr{U}_n = \mathscr{U} \times_B \operatorname{Spec} A/\mathfrak{M}^n$ over $\operatorname{Spec} A/\mathfrak{M}^n$. Now, Greenberg ([16]) has shown how to associate canonically with any group scheme over an Artinian local ring an algebraic group over the residue field. Thus, for any $n > 0$, we get *a* commutative algebraic group G_n over K. The 'reduction' morphism

$$\mathscr{U}_{n+1} = \mathscr{U} \times_B \operatorname{Spec} A/\mathfrak{M}^{n+1} \leftarrow \mathscr{U} \times_B \operatorname{Spec} A/\mathfrak{M}^n = \mathscr{U}_n$$

give rise to rational homomorphisms $\varphi_n : G_{n+1} \to G_n$ of algebraic groups defined over k. Let X_n denote the group of K-rational points of G, and $\Psi_n : X_{n+1} \to X_n$ the homomorphism induced by φ_n. We thus obtain *a* projective system of groups $\{X_n, \psi_n\}_{n \geq 1}$, whose projective limit we denote by X. It can then be shown that

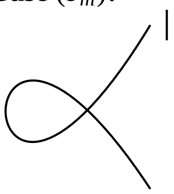
$$\begin{aligned} X &= \text{Group of } K\text{-rational points of generic fibre } f^{-1}(b_\circ) \\ &= \text{Group of regular section of } \mathscr{U} \text{ (or } X\text{) over } B. \end{aligned}$$

So we can introduce the structure of a proalgebraic group in the group I of $K(b_o)$-rational points on $f^{-1}(b_o)$. This makes it possible to introduce in I notions that are defined for proalgebraic groups, e.g., connected component, fundamental group and so on. They appear to be quite useful in the study of elliptic curves over complete fields.

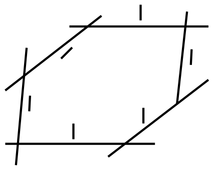
We next turn to another question. Given a fibration $f : X \to B$ by elliptic curves, what do the degenerate fibres (i.e. the fibres which are not regular curves) look like? Note that by what we have said above, the set of non-singular points of any fibre (not the fibre as a reduced scheme, **165**
but in the schematic sense) form a commutative algebraic group. A complete classification of all degeneracies has been by Kodaira (in the

'classical case', that is when B is a curve and X a surface over $\mathbb{C}$) and Néron (in the general case). We summarise this classification diagrammatically below. A line —- indicates a projective line, the figures $O, \prec, \propto$ indicate respectively a conic, a cuspidal cubic and a nodal cubic. The multiplicity with which a component occurs in the fibre is indicated by an integer written above the corresponding figure. Two components intersect or touch if and only if the corresponding figures do so. We have also indicated the nature of the group G of non- singular points in each case.

Case (b_m)**.**



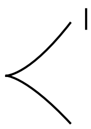
(m=1),



(m components, m arbitrary $\rangle$ 1)

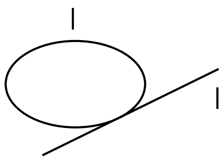
G is an extension of $\mathbb{Z}_{/m\mathbb{Z}}$ *by the multiplicative group* K^*.

Case $(C\ 1)$**.**



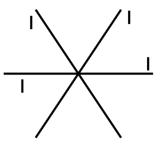
G is the additive group K.

Case $(C\ 2)$**.**



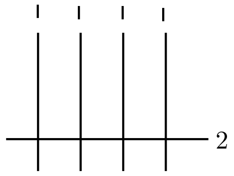
G is an extension of $\mathbb{Z}_{/2\mathbb{Z}}$ *by K*

Case $(C\ 3)$**.**



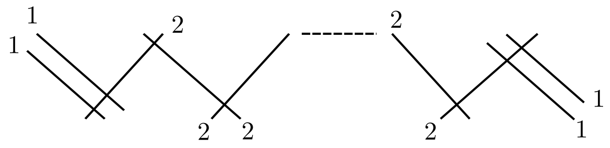
G is an extension of $\mathbb{Z}_{/3\mathbb{Z}}$ *by K*

166 **Case** (C 4)**.**



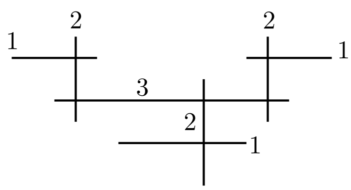
G is an extension of the Klein group $\mathbb{Z}_{/2\mathbb{Z}} x \mathbb{Z}_{/2\mathbb{Z}}$ by K

Case (C 5_m)**.**



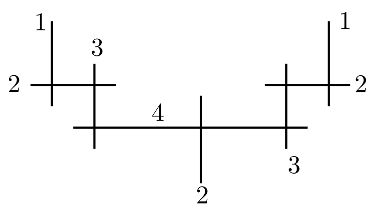
($m+1$ components occurring with multiplicity 2). G is an extension of $\mathbb{Z}/4\mathbb{Z}$ when m is odd (resp. the Klein group when m is even) by K.

Case (C 6)**.**



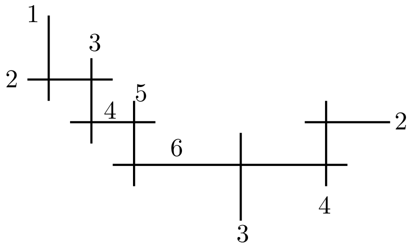
G is an extension of $\mathbb{Z}_{/3\mathbb{Z}}$ by K.

Case (C 7)**.**



G is an extension of $\mathbb{Z}_{/2\mathbb{Z}}$ by K.

Case (C 8)**.**



G is isomorphic to K.

This completes the list.

167 This description of all degenerate fibres was given by Kodaira in the geometric case and by Neron in the case $B = \text{Spec } \mathscr{O}$, $\mathscr{O}$ a valuation ring with an algebraically closed residue field. Their proofs are quite different. Neron starts with a model X' of X which is a subscheme of $\mathbb{P}^2(\mathscr{O})$ and which corresponds to the Weierstrass normal for of the generic fibre of X over $R(B)$. This model is in general not regular.

Resolving the singularities by means of dilatations, Néron arrives at the minimal model. This procedure is based on a detailed consideration of many different particular cases.

The proof of Kodaira is much easier. It is based on the consideration of the intersection multiplicities of the components of degenerate fibres and on formula (2) of lecture 7. It seems interesting to try to extend this proof to the general case considered by Neron -especially because such a proof has to be based on the consideration of some generalisation of the canonical class K to arbitrary schemes. This is not the only question for which it seems essential to have some notions corresponding to those of a canonical class.tangent bundle etc. in the case of arbitrary two dimensional schemes. For instance, one has the notion of a tangent plane at a point x (if X is regular at x) but can one introduce some structure in the totality of all such planes? The absence of such notions seems to be the principal hurdle if one tries to carry over Grauert's proof (Publications Mathematiques $No.25, 131-149$)of Manin's theorem to the case of a number field.

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