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LECTURES ON RELATIVITY THEORY

John Ray

Report R-311 August, 1966

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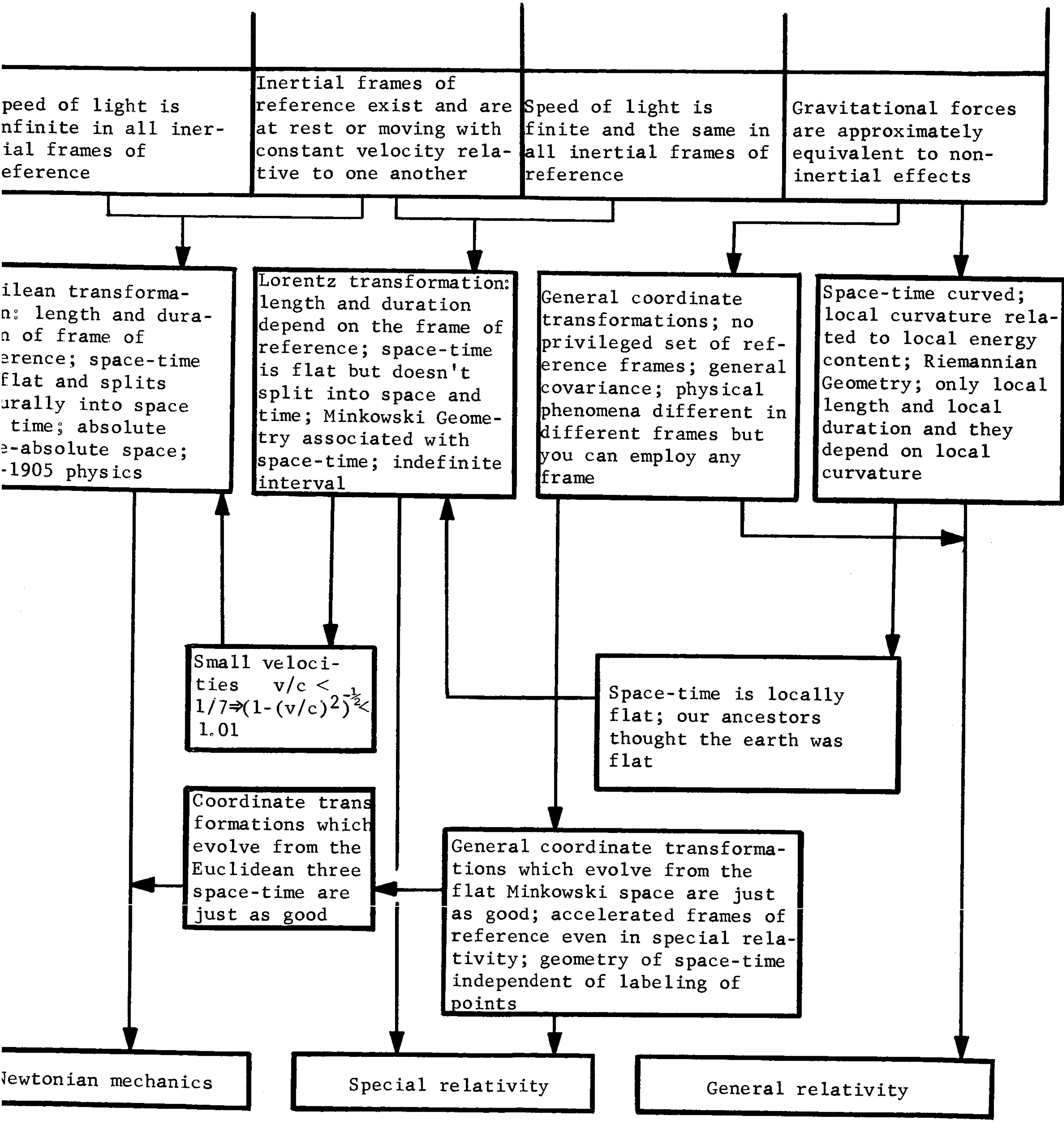
ABSTRACT

Lectures on Relativity Theory

These lectures were given to graduate and undergraduate engineering and science students during the spring semester, 1966. The lectures cover both the special and general theories. The coverage of the special theory is fairly complete while that of the general theory is somewhat abbreviated due to lack of time. The lectures on the general theory stopped with the derivation of the Schwartzschild solution and a discussion of the linear theory.

TABLE OF CONTENTS

FORMS OF CLASSICAL MECHANICS



Relativity Theory

Chapter 1

Newton's Theories

A. Newtonian Mechanics:

Postulates or assumptions

1) In inertial frames particles which are not acted on by forces remain at rest or continue in uniform motion. (This helps define an inertial frame.)

2) In inertial frames when particles are acted upon by applied forces, $F^\alpha = \{F_x, F_y, F_z\} = \{F^1, F^2, F^3\}$, the particle moves in accordance with

$$\frac{dP^\alpha}{dt} = F^\alpha. \tag{1.1}$$

Here $P^\alpha(t)$ is the linear momentum of the particle and t is the time. For speeds slow with respect to the speed of light in vacuum, it is an empirical fact that

$$\left.\begin{array}{l} P^\alpha \propto \dfrac{dx^\alpha}{dt} = \dot{x}^\alpha \\ F^\alpha \rightarrow F^\alpha(x,\dot{x},\ t\ ) \end{array}\right]\ v/c \ll 1\ ,$$

where $c = 2.99776 \times 10^{10}$ cm $\sec^{-1}$ is the speed of light in vacuum. $x(t) = \{x^1, x^2, x^3\}$ is the position of the particle. The proportionality constant is called the inertial mass of the particle m. Hence, we can write

$$P^\alpha = m\,\frac{dx^\alpha}{dt} \quad (v/c \ll 1). \tag{1.2}$$

Newton's second law is now in its popular form

$$m\,\frac{d^2x^\alpha}{dt^2} = F^\alpha(x,\dot{x},\ t) \quad (v/c \ll 1)\ . \tag{1.3}$$

These are a set of three second-order, ordinary, quasi-linear, differential equations for the position of the particle.

3) To every force there is an opposite and equal force. (action = reaction) Forces occur in pairs, that is, there is no such thing as a single force. To avoid confusion we point out that the action force and the reaction force always act on different particles i.e. (different bodies).

4) Forces obey the principle of linear summation. If two forces F_1^α, F_2^α act on a particle, then they may be replaced by a single force $F^\alpha = F_1^\alpha + F_2^\alpha$, that is, forces are vectors. (Page 88: Resnick Halliday for a popular critique of Newton's laws).

B. Galilean*, Transformations and Principle of Relativity.

Newtonian mechanics is formulated in an inertial frame, however, the precise definition of an inertial frame is not given. At present we assume that such reference frames exist and discuss their properties. The main features of inertial frames are:

1) Distances are determined by using Euclidean three dimensional geometry.

2) Time is measured by comparison with clocks reading a universal time in terms of which "free" particles remain at rest or continue to move with uniform velocity.

The transformations which leave Newton's laws invariant must be distance preserving transformations of Euclidean three dimensional space.

---

*The Galilean group is a 10 parameter Lie group; for details of group structure (infinitesimal generators and structure constants) see the book Group Theory by M Hammermesh.

These are the Galilean transformations, g, defined by,

$$
g: \quad \begin{aligned} x'^{\alpha} &= R^{\alpha}{}_{\beta}\, x^{\beta} - v^{\alpha} t + c^{\alpha} \\ c^{\alpha} &= \text{constant} \\ t' &= t + b \qquad \alpha, \beta, \gamma, \text{ etc.} = 1, 2, 3. \end{aligned} \tag{1.4}
$$

Distance preserving implies

$$
ds^2 = dx^{\alpha}\, dx_{\alpha} = dx'^{\alpha}\, dx'_{\alpha} = ds'^2 . \tag{1.5}
$$

This condition restricts $R^{\alpha}{}_{\beta}$

$$
dx'^{\alpha}\, dx'_{\alpha} = R^{\alpha}{}_{\beta} R_{\alpha}{}^{\gamma}\, dx^{\beta}\, dx_{\gamma}
$$

$$
R^{\beta}{}_{\alpha} R_{\gamma}{}^{\alpha} = \delta^{\beta}{}_{\gamma} \longleftrightarrow R^{\alpha}{}_{\beta} R_{\alpha}{}^{\gamma} = \delta_{\beta}{}^{\gamma} \qquad (R^{(\text{row})}_{(\text{col})} \text{ is an orthogonal matrix}).
$$

The Galilean principle of relativity is just the statement that Newton's laws 1), 2), 3), and 4) are valid in all inertial frames. One says that Newton's laws are <u>covariant</u> (form invariant) under Galilean transformations. In two inertial frames $S(x)$, $S'(x')$ we have

$$
\begin{aligned} S(x) &: \quad m\frac{d^2x^{\alpha}}{dt^2} = F^{\alpha} \\ S'(x') &: \quad m\frac{d^2x'^{\alpha}}{dt'^2} = F'^{\alpha} \end{aligned} \qquad \underline{\text{Covariance}} \tag{1.6}
$$

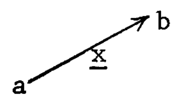
$$
F'^{\alpha} = R^{\alpha}{}_{\beta} F^{\beta} \qquad \frac{d^2x'^{\alpha}}{dt'^2} = R^{\alpha}{}_{\beta} \frac{d^2x^{\beta}}{dt^2} .
$$

C. Newtonian Gravitational Theory:

Postulates or assumptions

1) A particle with active gravitational mass Aa sets up in its vicinity a gravitational potential ϕ_a,

$$\phi_a(\underline{x}) = \frac{-Aa}{|\underline{x}|}\ , \quad |\underline{x}| = \sqrt{(x')^2+(x^2)^2+(x^3)^2} \tag{1.7}$$



2) Another particle brought into the vicinity experiences a force

$$F_{ba} \propto - \nabla\phi_a \ .$$

The constant of proportionality P_b is called the passive gravitational mass.

$$\underline{F}_{ba} = - P_b \nabla\phi_a = \frac{-P_b Aa\ \underline{x}}{|\underline{x}|^3} \quad . \tag{1.8}$$

The three masses introduced can be shown to be proportional if one assumes (<u>principle of equivalence</u>) 1) the motion of a particle in a gravitational field depends only on its initial position and velocity and is independent of the type of "material" the particle is constructed from, 2) Newton's laws of mechanics. We have

$$m_b \frac{d^2\underline{x}}{dt^2} = - P_b \nabla\phi_a \qquad \therefore$$

$\frac{m_b}{P_b}$ = constant for all particles => can choose units such that $m_b = P_b$. If we now interchange the particles, then by Newton's third law of motion,

$$\underline{F}_{ba} = - \underline{F}_{ab},$$

$$\underline{F}_{ba} = \frac{-m_b Aa\ \underline{x}}{|\underline{x}|^3}\ , \quad \underline{F}_{ab} = \frac{+m_a A_b\ \underline{x}}{|\underline{x}|^3} \qquad \therefore$$

$$\frac{m_b}{A_b} = \frac{m_a}{Aa} = \text{constant (for all particles).}$$

Hence, we can choose

$$A = km$$

and Newton's law of gravitation becomes

$$\underline{F}_{ba} = \frac{-k\, m_a\, m_b}{|\underline{x}|^3}\,\underline{x}\ . \tag{1.9}$$

$k = 6.670 \times 10^{-8}\ gm^{-1}\ cm^3\ sec^{-2}$. Newton's law of gravitation (1.9) is an empirical law. What is the range of validity of (1.9)? The potential at a distance $|\underline{x}|$ from a mass m is

$$\phi = \frac{-km}{|\underline{x}|}\ . \tag{1.10}$$

Of course the same result is true for a spherically symmetric uniform mass distribution of radius R as long as $|\underline{x}| > R$. For such a body a characteristic dimensionless number is

$$\frac{km}{Rc^2} \tag{1.11}$$

We shall find later that (1.10) is only an approximate expression and is "modified" in the following way in Einstein's gravitational theory,

$$\phi' = \frac{-km}{|\underline{x}|} + 0\left[\left(\frac{km}{c|\underline{x}|}\right)^2\right]\ . \tag{1.12}$$

The corrections will become important when

$$\frac{\text{(Correction)}}{\text{(Newtonian)}} = \frac{\left(\dfrac{km}{c|\underline{x}|}\right)^2}{\dfrac{km}{|\underline{x}|}} = \frac{km}{c^2|\underline{x}|} \approx 1\ .$$

Thus, we may take the number $\frac{km}{Rc^2}$ as a measure of the deviation from Newtonian gravitational theory. Another way to say this is that we may use (1.10) when

$$\delta = \frac{km}{c^2 R} \ll 1 \tag{1.13}$$

where R is a characteristic size. Examples:

Table I

Body	$\delta = \frac{km}{c^2R}$
earth	$\sim 7 \times 10^{-10}$
sun	$\sim 2 \times 10^{-6}$
moon	$\sim 3 \times 10^{-11}$
neutron	$\sim 10^{-39}$ ($R \sim 1\text{fm} = 10^{-13}$ cm)
electron	$= \frac{km}{c^2 e^2/mc^2} = \frac{km^2}{e^2} \sim 10^{-42}$ ($R \approx \frac{e^2}{m_e c^2} \sim 3 \times 10^{-13}$cm).

Thus, it is not in the atomic or subatomic world where we must look for deviations from Newtonian gravitational theory. For material of a given density ρ

$$m \sim \rho R^3 \text{ and}$$

$$\delta = \frac{km}{c^2 R} \sim \frac{k\rho R^2}{c^2} \quad .$$

Therefore for a given density the size of the body must be large. Further examples:

Body	δ	
Neutron star	~ 0.1	$\rho \sim 10^{13}$ gm cm^{-3}; $R \approx 10^7$ cm
Universe	$k\rho T^2 \sim 0.1$	$T \sim 10^{10}$ years; $\rho \sim 10^{-28}$ gm cm^{-3} .

Therefore, we expect the deviations from the Newtonian model to play a more important role in astrophysics and cosmology. Because the electrical force F_e is much larger than gravitational force

$$\text{(Proton)} \qquad \frac{F_g}{F_e} = \frac{km^2}{e^2} \sim 10^{-36}$$

when gravity does play a major role in phenomena, the bodies considered must be very nearly electrically neutral. If one could free the electrons in 10 tons* of water on the earth's surface and a like number of positions on the moon's surface, the electrical force of attraction would be the same as the gravitational attraction. If we are considering a continuous distribution of matter, then the potential at a point $\underline{x}$ produced by the body is from (1.10)

$$\phi(\underline{x}) = -k\int \frac{dm}{|\underline{x} - \underline{x}|} = -k\int \frac{\rho(\underline{x}')\, d^3x'}{|\underline{x} - \underline{x}'|} \; .$$

If we operate with the Laplacian ∇^2 on this equation and use

$$\nabla_{(\underline{x})}^{2} \left(\frac{1}{|\underline{x} - \underline{x}'|}\right) = -\, 4\pi\, \delta(\underline{x} - \underline{x}')$$

(where $\delta(\underline{x} - \underline{x}')$ is the Dirac delta)

we obtain

$$\nabla^2\, \phi(\underline{x}) = +\, 4\pi k\rho \tag{1.14}$$

which is called Poisson's equation.

---

*This is about the amount of water in a cube 7 ft. on a side.

D. Einstein's Generalizations of Newton's Theories.

In 1905 Einstein generalized equation (1.3) so that it would hold for arbitrary values of v/c. The theory he constructed in doing this is called the special theory of relativity. In 1915 Einstein formulated a theory of gravitation which superseded Newton's theory. In this latter theory, which is known as general relativity, Einstein generalized his ideas of 1905 and discovered an equation which represented a generalization of Poisson's equation in Newton's gravitational theory.

Chapter 2

A. Einstein's Special Relativity

One can find a hint of where to start this discussion by taking a look at the title of Einstein's paper of 1905, "On the Electrodynamics of Moving Bodies." As we have mentioned before, Newton's laws are covariant under the Galilean group. This led us to the Galilean or Newtonian principle of relativity. When one considers electrodynamics on the other hand, one finds that Maxwell's equations are not covariant under the Galilean group, i.e.

$$S \quad \nabla^2\phi - 1/c^2 \frac{\partial^2\phi}{\partial t^2} = 0 \quad \text{(Maxwell's equation)} \tag{2.1}$$

(ϕ = electric or magnetic field strengths)

under Galilean transformation

$$x'^{\alpha} = x^{\alpha} - v^{\alpha} t$$

$$t' = t$$

(2.1) becomes

$$S' \quad \nabla^2\phi - \frac{v^{\alpha}v^{\beta}}{c^2}\frac{\partial^2\phi}{\partial x'^{\alpha}\partial x'^{\beta}} - \frac{2v^{\alpha}}{c}\frac{\partial^2\phi}{\partial x'^{\alpha}\partial t'} - c^{-2}\frac{\partial^2\phi}{\partial t'^2} = 0 \,. \tag{2.2}$$

Therefore, by measuring the field ϕ alone in the new reference frame S', one can determine its velocity with respect to the frame S. From this point of view there is one frame which has a privilege position, namely $v^{\alpha} = 0 \Rightarrow$ Maxwell's equations have their simplest form, namely (2.1). In this frame ("ether" frame) the velocity of light was supposed to be c. (As the velocity of sound is independent of the velocity of the source, so also it was supposed that the velocity of light was independent of the

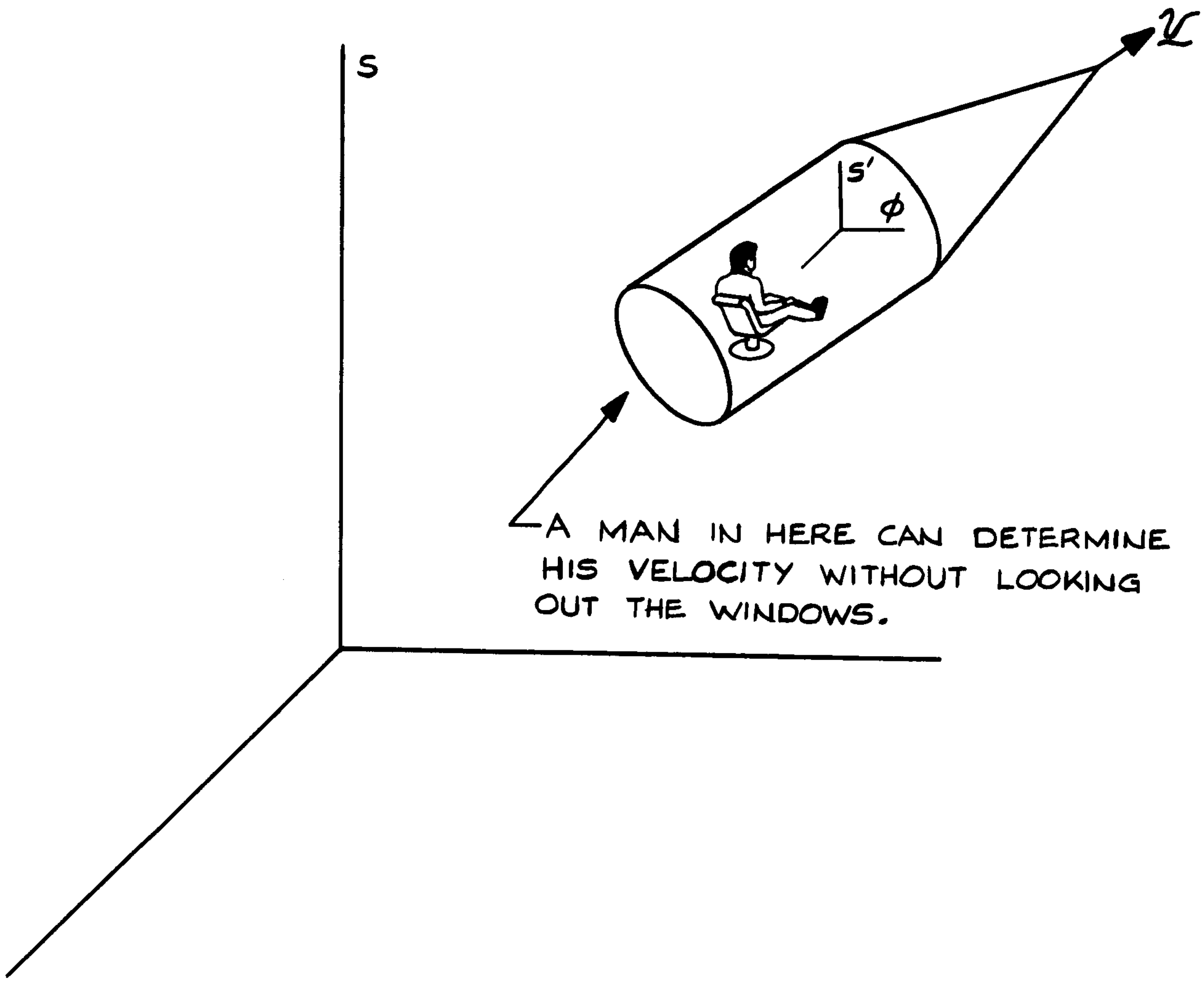

S
S'
ϕ
v
A MAN IN HERE CAN DETERMINE HIS VELOCITY WITHOUT LOOKING OUT THE WINDOWS.

Fig. 2.1

motion of the source in the "ether" frame.) (For this part we shall pick units such that $c = 1$.)

Some possibilities open in 1900:

1) There is a principle of relativity for mechanics (Galilean) but not for electrodynamics. A preferred frame exists for electrodynamics.

2) Maxwell's equations wrong and there exists a principle of relativity for both mechanics and electrodynamics.

3) Newton's laws (1.3) wrong and a principle of relativity exists for both mechanics and electrodynamics.

In physics such choices must be made by the experimentalists. Three types of experiments suggest themselves.

1) Attempt to detect the preferred frame of electrodynamics.

2) Attempt to find faulty predictions from Maxwell's equations.

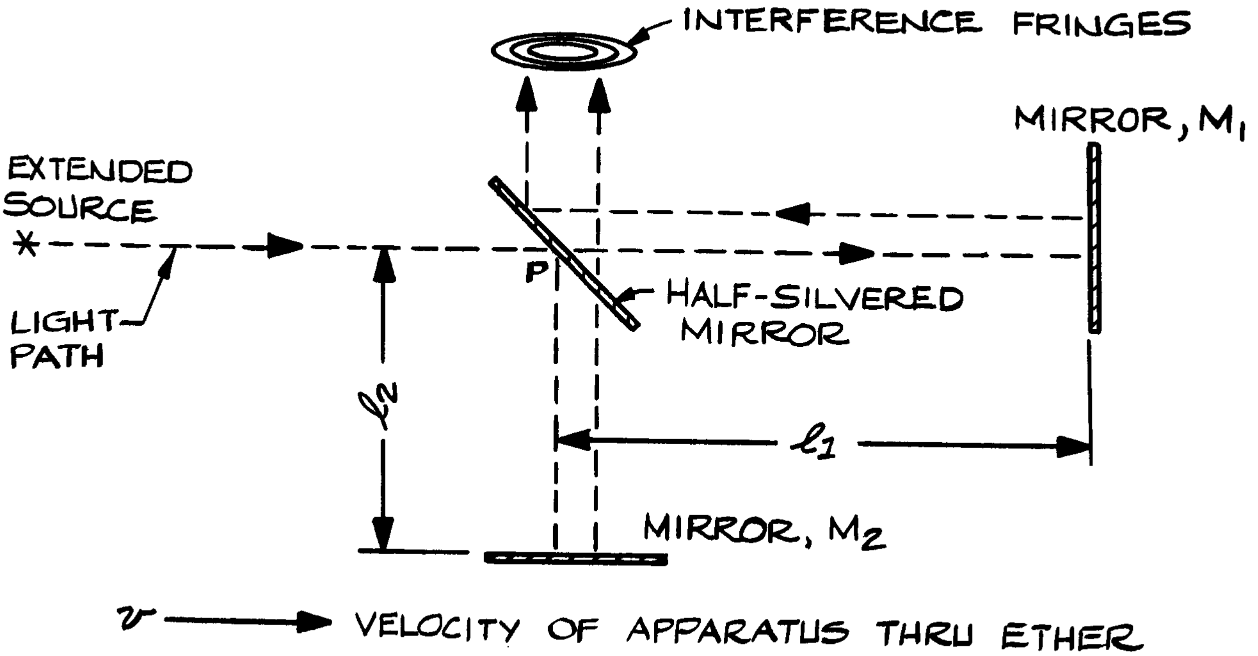
3) Attempt to find faulty predictions from Newton's equation (1.3).

The first and most famous experiment performed was of the type 1). This was the Michelson-Morley experiment. This made use of an interferometer (Michelson) on the earth which was used to compare the velocity of light along and perpendicular to the earth's velocity through the ether.

Let t_1 = time $PM_1P = \ell_1\left(\frac{1}{1-v} + \frac{1}{1+v}\right) = \frac{2\ell_1}{(1-v^2)}$ be the time for the light to travel from the half silvered mirror to M_1 and back. To calculate t_2 the time for the light to go from P to M_2 and back, we must consider the light traveling over the extended path PM_2P.

The time t_2 follows from $\ell_2{}^2 + \left(\frac{vt_2}{2}\right) = \left(\frac{t_2}{2}\right)$ or

$$t_2 = \frac{2\ell_2}{(1-v^2)^{1/2}} .$$


INTERFERENCE FRINGES
MIRROR, M_1
EXTENDED SOURCE
P
HALF-SILVERED MIRROR
LIGHT PATH
ℓ_2
ℓ_1
MIRROR, M_2
v VELOCITY OF APPARATUS THRU ETHER

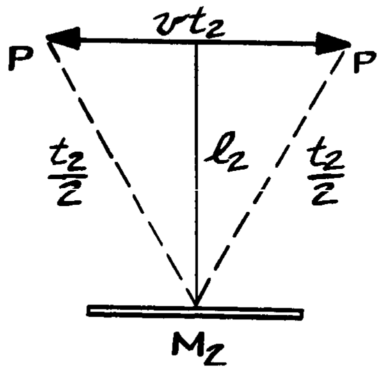

vt_2
P
P
$\frac{t_2}{2}$
ℓ_2
$\frac{t_2}{2}$
M_2

Fig. 2.2

The optical path difference of light reflected from the two mirrors is

$$\Delta = t_2 - t_1 = \frac{2}{(1-v^2)^{1/2}}\left[\frac{\ell_1}{(1-v^2)^{1/2}} - \ell_2\right] .$$

Rotate instrument by 90^o, then ℓ_1 and ℓ_2 change places and one has in the new configuration

$$t'_1 = \frac{2\ell_1}{(1-v^2)^{1/2}} \qquad t'_2 = \frac{2\ell_2}{(1-v^2)} .$$

The path difference is now

$$\Delta' = t'_1 - t'_2 = \frac{2}{(1-v^2)^{1/2}}\left[\ell_1 - \frac{\ell_2}{(1-v^2)^{1/2}}\right] .$$

Thus, when one rotates the apparatus a fringe shift should occur. The number of fringes is given by

$$n = \frac{\Delta-\Delta'}{\lambda} , \quad \text{or}$$

$$n = \frac{2(\ell_1+\ell_2)}{\lambda(1-v^2)^{1/2}}\left[\frac{1}{(1-v^2)^{1/2}} - 1\right] ,$$

expanding the square root $(1-v^2)^{-1/2} = 1 + 1/2\ v^2 \ldots$ gives

$$n = \frac{\ell_1+\ell_2}{\lambda} v^2 .$$

In most cases $\ell_1 = \ell_2$ and, therefore,

$$n = \frac{2\ell}{\lambda} v^2 .$$

Experimental results

Observer	Year	ℓ	$n = \frac{2\ell}{\lambda} v^2$	Measured
Michelson & Morley	1887	1100 cm	.40	$\leq$ .005
Joos	1930	2100 cm	.75	$\leq$ .001 .

Many modifications of the Michelson-Morley experiment were devised. The measurements were made at 12-hour intervals at 3- and 6-month intervals with the arms of the interferometer in all positions. The results are unequivocal - there is no fringe shift observed. As an explanation, Lorentz and Fitzgerald postulated ("Ether Theory") that all bodies that move with respect to the "ether" are contracted by the amount $\ell = (1-v^2)^{1/2}$ ℓ (rest), in the direction of motion. Thus, we must modify our calculations concerning the Michelson-Morley experiment. The necessary modifications are:

$$\Delta \rightarrow \frac{2\ell_1(1-v^2)^{1/2}}{(1-v^2)} - \frac{2\ell_2}{(1-v^2)^{1/2}}$$

$$\Delta' \rightarrow \frac{2\ell_1}{(1-v^2)^{1/2}} - \frac{2\ell_2(1-v^2)^{1/2}}{(1-v^2)}$$

$\therefore$ $\Delta' = \Delta$ and $n = 0$.

Thus, the null result of the Michelson-Morley experiment is explained away by the contraction hypothesis. By another experiment we can also show that time does not escape unscathed. Imagine in Fig. (2.2) another rod ℓ'_2 identical to ℓ_2 moving such that it is at rest with respect to the ether and parallel to ℓ_2 (Fig. 2.3).

We measure the time for the emission of light and its return to P and P' respectively. For PM_2P we have the same time as calculated as t^2 in the Michelson-Morley experiment,

$$\text{time } PM_2P = \frac{2\ell_2}{(1-v^2)^{1/2}} = t_2 .$$

Also, since ℓ'_2 is at rest in the ether

$$t_o = \text{time } PM_2P = 2\ell'_2 .$$

Since $\ell_2 = \ell'_2$ (because ℓ_2 perpendicular to motion) then

$$t_o = (1-v^2)^{1/2} t_2 \ . \tag{2.4}$$

If the time actually measured were different, this could be used to measure the velocity of ℓ_2 through ether. But we have seen from the results of the Michelson-Morley experiment that one cannot detect motion through the ether. An explanation of (2.4) is then that all clocks moving in the ether are slowed. This means that if a clock at rest in the ether has n ticks, then when it moves through the ether it will only have $(1-v^2)^{1/2}n$ ticks. Thus, although the times in (2.4) are actually different, they will be measured to be the same because the moving clocks slow down. Thus, in the ether theory the appearance of length contraction and time dilatation are not demonstrable because all bodies including the measuring scales and clocks undergo the same phenomena. Besides the length contraction hypothesis of Lorentz-Fitzgerald many modifications of Maxwell's equations were attempted. For example, one of these (emission theory) assumed that light travels at speed c with respect to its source instead of through the ether. All these attempts were shown to be in contradiction with experimental facts. Hence, in 1905 one had learned from the experimentalists

1) The existence of motion through the ether is not demonstrable.

2) Any alteration of Maxwell's equations is contradicted by experiment.

With these conclusions there was only one logical choice:

3) Newton's laws (1.3) were wrong and a principle of relativity exists for mechanics and electrodynamics.

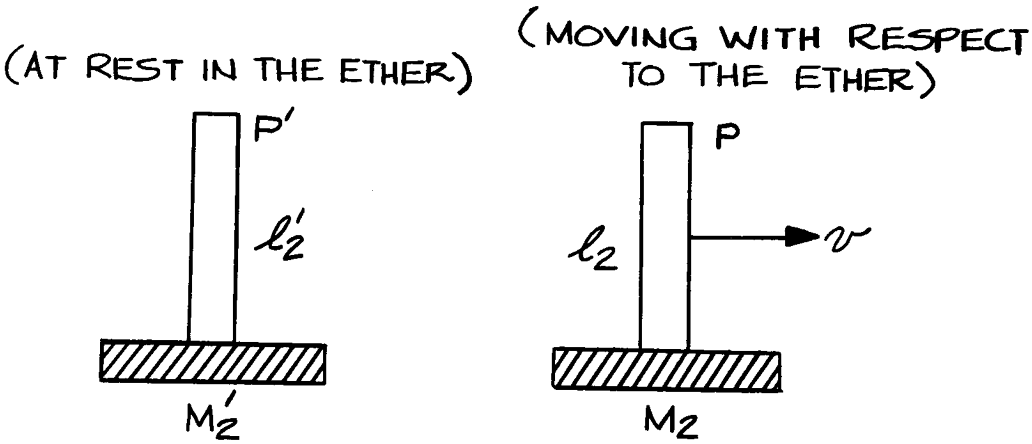

(AT REST IN THE ETHER)
(MOVING WITH RESPECT TO THE ETHER)
P'
ℓ_2'
M_2'
P
ℓ_2
v
M_2

Fig. 2.3

Einstein's reaction to the failure to detect the ether was radical. He proposed the following two postulates as a solution.

Einstein's 1905 postulates of special relativity:

1) Principle of Relativity (Einstein's) - all inertial frames are equivalent for the formulation of all physical laws. (Covariance of all physical laws under transformation of coordinates between inertial frames.)*

2) Light signals (in vacuum) are propagated rectilinearly with the same constant velocity c, at all times, in all directions, and in all inertial frames. That is, (2.1) holds in all inertial frames.

Remarks concerning the Einstein postulates:

1° 1) is just a generalization of the relativity principle of Newton to the whole of physics. (Electrodynamics must be covariant under the transformations from one inertial frame to another, while Newtonian mechanics (1.3) cannot be.)

2° 1) and 2) are not completely independent - it was known that the velocity of light was independent of the motion of the source. (de Sitter - binary stars) So there exists in one frame (at least) an effect that is propagated through vacuum at a velocity independent of the motion of the source. Thus, by the principle of relativity all inertial observers must find the same result. It is an empirical fact that this actually is the case. It is quite irrelevant here that this velocity happens to be equal to the velocity of light in vacuum.

---

*This postulate still suffers in that it picks out the inertial frames as privileged.

B. The Lorentz Transformations:

The transformation equations which replace the Galilean transformation equations can be derived immediately from Einstein's postulates in several ways. One could find the set of all transformations which leaves Maxwell's equations forms invariant. Poincaré was the first to do this. Here, however, we take a less sophisticated approach. We briefly review the properties of inertial frames:

1) spatial distance determined by Euclidean three dimensional geometry.

2) universal time defined throughout.

3) Newton's laws hold in the form stated in the first chapter.

4) spatially homogeneous and isotropic and temporally homogeneous with respect to all physical phenomena. (This means that space is the same at each point and in each direction and that time is the same at each instant or "a second is a second."

Remark: In practice there are no extended inertial frames. Whether or not a frame can be treated as inertial depends upon the particular situation. How do we define the universal time in our inertial frame? This is arbitrary to a certain extent, but we take the following definition: Two stationary clocks A and B are said to be synchronized if when a light signal is dispatched from A, it reaches clock B when B reads $t+\ell$ $(c=1)$ where ℓ is the distance between A and B, measured by rods at rest in the frame of the clocks. To set up an inertial frame we imagine that an origin is chosen and three dimensional cartesian coordinates are set up in this frame using a very short measuring rod. Next, at each coordinate point in three dimensional space, identical clocks are placed and synchronized with each other. Each point at each time is then characterized by an event $\mathcal{E}(x^1, x^2, x^3, t)$, that is, where and when. When we speak of an "observer" who measures the path of a given particle, we are speaking of giving the position in space together with the time as measured by the local clock which is located at that point.

One can then imagine space being filled with small "observers" each one wearing a watch (Fig. 2.4). If the particle comes very near a particular observer, then he measures t on his watch at that instant and records it with his position in the form of an event (x^1, x^2, x^3, t). The set of all these observations furnish the path of the particle. It is very important to keep in mind that events $\mathcal{E}(x^1, x^2, x^3, t)$ are measured by the local observer. Later when we speak of "observers" making measurements it must be borne in mind how the observations are made; if this is not kept in mind then much of what follows is meaningless. Spatial points in inertial frames will be denoted by capital Latin letters A, B, C, ..., while events will be characterized by a symbol $\mathcal{E}$ telling where and when $\mathcal{E}(x^1, x^2, x^3, x^4 = t)$. The universal time as defined above would also serve as an appropriate time for Newtonian mechanics. On the other hand because Newtonian mechanics allows arbitrarily large signal velocities, simultaneity can be defined so that events which are simultaneous in one inertial frame are simultaneous in all inertial frames. That is, simultaneity can be defined in an absolute way. That this is true follows directly from the Galilean transformation property of time in Newtonian theory [$t' = t + b$ equation (1.4)]. In Einstein's special relativity theory, on the other hand, simultaneity cannot be defined in an absolute way. Two events $\mathcal{E}_1$ and $\mathcal{E}_2$ are said to be simultaneous in a given inertial frame if a light signal emitted at the two events meets halfway between them.

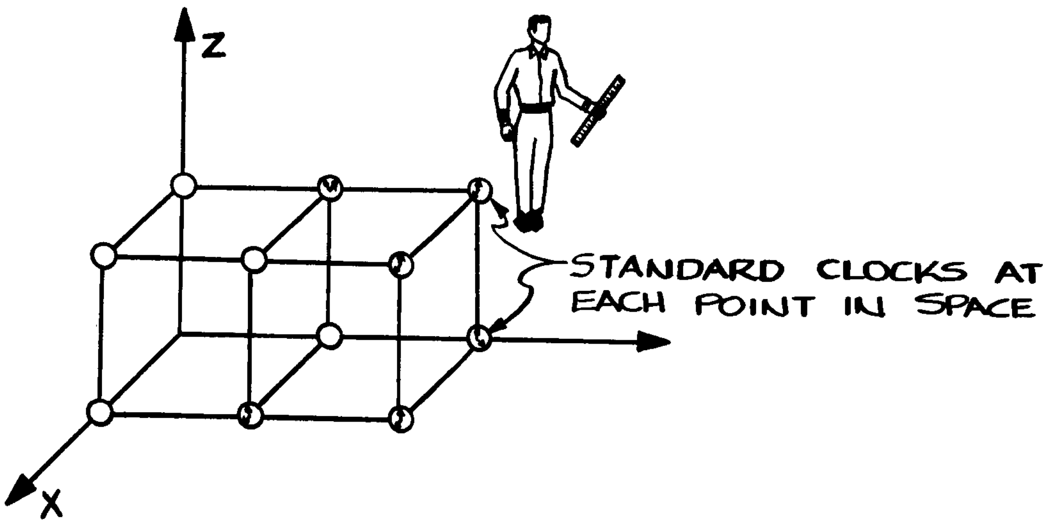


Fig. 2.4

Consider a flashbulb being set off in a reference frame S' at a point C. Suppose A and B are at rest and equidistant from C in S'. Then by our definition of simultaneity the illumination from the flash will reach A and B at the same instant in S'. Then by our definition of simultaneity the illumination from the flash will reach A and B at the same instant in S'. Now suppose S' is moving with respect to another frame S. To an observer in S the point B is moving away from the origin of the light flash and A is approaching the origin of the light flash. Therefore in S t precedes t_B (see Fig. 2.5). Therefore, as defined in special relativity simultaneity is relative.

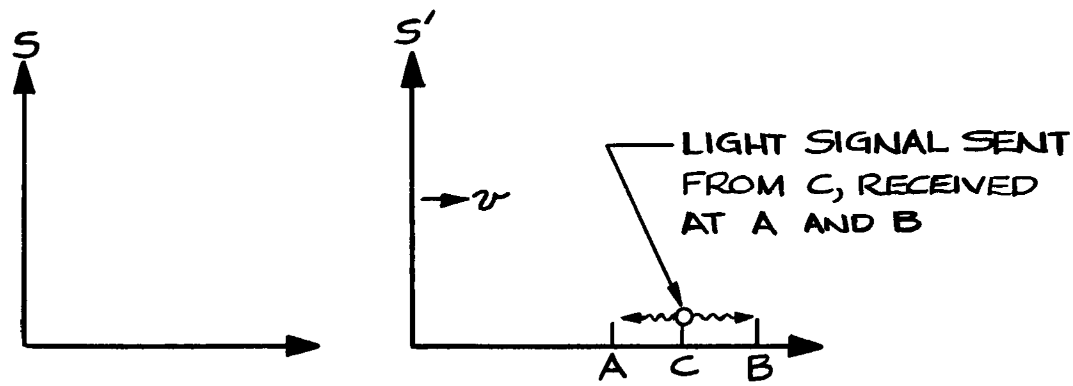


Fig. 2.5

See Fig. 4.5b on a further explanation of simultaneity.

Suppose a flashbulb is set off at (x^1, x^2, x^3) at time t = x^4 in S, then it illuminates a point a distance dx^1, dx^2, dx^3 from the flash at a time dx^4 from it.

S: Flash $\mathcal{E}_1(x^1,x^2,x^3,x^4)$; illumination $\mathcal{E}_2(x^1+dx^1,x^2+dx^2,x^3+dx^3,x^4+dx^4)$

S': Flash $\mathcal{E}_1(x'^1,x'^2,x'^3,x'^4)$; illumination $\mathcal{E}_2(x'^1+dx'^1,x'^2+dx'^2,x'^3+dx'^3,$ $x'^4+dx'^4)$

Since this is an expanding light signal in both frames, it follows from Einstein's postulates that:

$$S: \quad (dx^4)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 = 0$$

$$S': \quad (dx'^4)^2 - (dx'^1)^2 - (dx'^2)^2 - (dx'^3)^2 = 0 \; .$$

Let $M(\eta_{\ell k}) = \begin{array}{c c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\ \begin{bmatrix} -1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & +1 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$ then we can write

$$S: \quad \eta_{\ell k}\, dx^{\ell}\, dx^{k} = 0$$

$$S': \quad \eta_{\ell k}\, dx'^{\ell} dx'^{k} = 0$$

Notation: $i,j,k,\ell, \ldots = 1,2,3,4.$

$M(\;) =$ Matrix $(\;)$

The Lorentz transformation is given by

$$x'^{\ell} = f^{\ell}(x) \quad \text{or}$$

$$dx'^{\ell} = f^{\ell}{}_{,k} dx^{k}$$

$$f^{\ell}{}_{,k} = \frac{\partial f^{\ell}}{\partial x^{k}} \; .$$

(transformation of differentials always linear for coordinate transformations)

Therefore, we have two polynomials

$$1) \quad \eta_{\ell k} dx^{\ell} dx^{k} = 0$$

$$2) \quad \eta_{\ell k}\, f^{\ell}{}_{,r} f^{k}{}_{,s} dx^{r} dx^{s} = 0$$

which have the same zeroes, this means the polynomials are multiples of one another, or

$$\eta_{\ell k} dx'^{\ell} dx'^{k} = k' \eta_{\ell k} dx^{\ell} dx^{k} \quad .$$

By homogeneity and isotropy of space and time, k' cannot depend upon space or time. By the symmetry of the two inertial frames we can write

$$\eta_{\ell k} dx^\ell dx^k = k' \eta_{\ell k} dx'^\ell dx'^k$$

therefore, $k'^2 = 1$, but for $v = 0$ $k' = 1$; therefore $k' = 1$ always. Thus, the square of the measure ds^2 between two events is an invariant under coordinate transformations

$$ds^2 = \eta_{\ell k}\, dx^\ell dx^k = \text{invariant}. \qquad (2.5)$$

One can show that a transformation $x'^\ell = f^\ell(x)$ which transforms the form $dx^2 = \eta_{\ell k}\, dx^\ell dx^k$ with constant coefficients $\eta_{\ell k}$ into the form $ds'^2 = ds^2 = \eta'_{\ell k}\, dx^\ell dx^k$ with constant coefficients $\eta'_{\ell k}$ must be linear. When two inertial frames S, S' are oriented so that their axis are parallel, their origins coincide at $t = t' = 0$, and their relative motion is along their common x, x' axis we shall say they are in standard configuration. The transformation connecting frames in standard configuration will be called standard Lorentz transformations. Consider a light pulse: $\mathcal{E}$ pulse($x = x' = 0$, $y = y' = 0$, $z = z' = 0$, $t = t' = 0$) and assume the two frames are in standard configuration. Then since coordinate differences transform in the same way as coordinate differentials under linear transformations, we have

$$-x^2 - y^2 - z^2 + t^2 = -(x')^2 - (y')^2 - (z')^2 + (t')^2 \qquad (2.6)$$

as the expression of the invariance of measure (2.5). Note $y = 0 \Longrightarrow y' = 0$ $\therefore$ $y' = Ay$, but by the same arguments as used for k' before, $A = 1$. The same arguments holds for z. Therefore

$$-(x)^2 + t^2 = -(x')^2 + t'^2, \qquad (2.7)$$

$x' = 0 \Longrightarrow x = vt$, where v is the relative velocity of the two frames.

$x' = B(x - vt)$ B, C, D constants

$t' = Cx + Dt.$

Substituting these expressions into (2.7) and solving for the constants yields

$$B = D = \frac{\pm 1}{(1-v^2)^{1/2}} \ , \ C = \frac{\mp v}{(1-v^2)^{1/2}} \ .$$

For transformations which evolve continuously from the identity, we must choose the signs $+ -$. Therefore, the transformations are

$$x' = \frac{x - vt}{(1-v^2)^{1/2}} \qquad y' = y$$
$$t' = \frac{t - vx}{(1-v^2)^{1/2}} \qquad z' = z \ . \tag{2.8}$$

One can write this in matrix notation

$$x' = \begin{matrix} x' \\ y' \\ z' \\ t' \end{matrix} = \begin{matrix} \gamma(v) & 0 & 0 & -v\gamma(v) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v\gamma(v) & 0 & 0 & \gamma(v) \end{matrix} \quad \begin{matrix} x \\ y \\ z \\ t \end{matrix} = L(v) \quad x \tag{2.9}$$

where $\gamma(v) = (1 - v^2)^{-1/2}$. We note that the Newtonian formulas can often be recovered from the formal limiting process $c \to \infty$. To show this in (2.8) return to the conventional units:

$$x' = \frac{x - vt}{(1-v^2/c^2)^{1/2}} \ , \ t' = \frac{t - v/c^2 \ x}{(1-v^2/c^2)^{1/2}} \ . \tag{2.8}$$

In the limit $c \to \infty$ these go over into

$$x' = x - vt \qquad t' = t$$

which are just the Galilean formula. Thus, when there are Newtonian laws corresponding to laws in special relativity they can be obtained from the latter by this limiting procedure. It should be remembered, however, that there are theories in special relativity which have no Newtonian counterparts, i.e. Maxwell's theory. We can extend the Lorentz transformations

(2.8) to a more general Lorentz transformation. Writing (2.9)

$$\xi' = L(v)\xi, \quad \xi = \begin{bmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \\ \xi^4 \end{bmatrix} \tag{2.9}$$

and operating on this with a three dimensional rotation matrix R we have

$$R\xi' = R\ L(v)\ R^{-1}\ R\xi.$$

Let $x' = R\xi'$ and $x = R\xi$. Then

$$x' = R\ L(v)\ R^{-1}x \qquad L'(v) = R\ L(v)\ R^{-1}$$

is a Lorentz transformation from x to x' where the x, x' frames differ from standard configuration by the rotation R. Example:

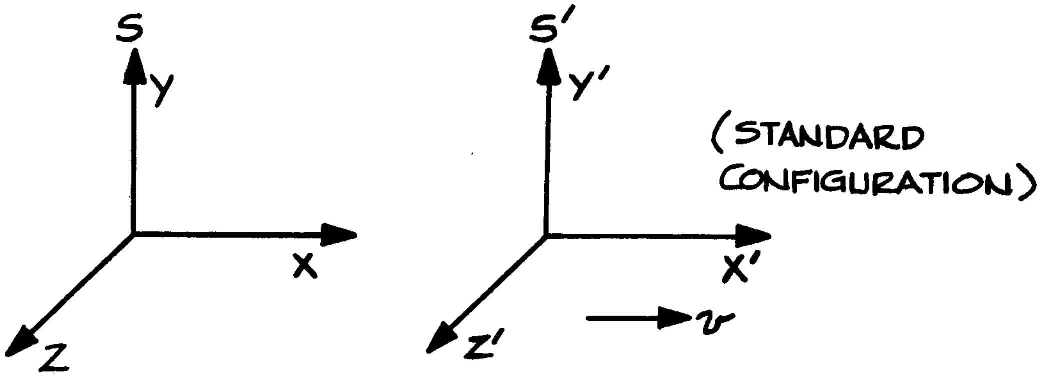


Fig. 2.6a

For $$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ the picture is:

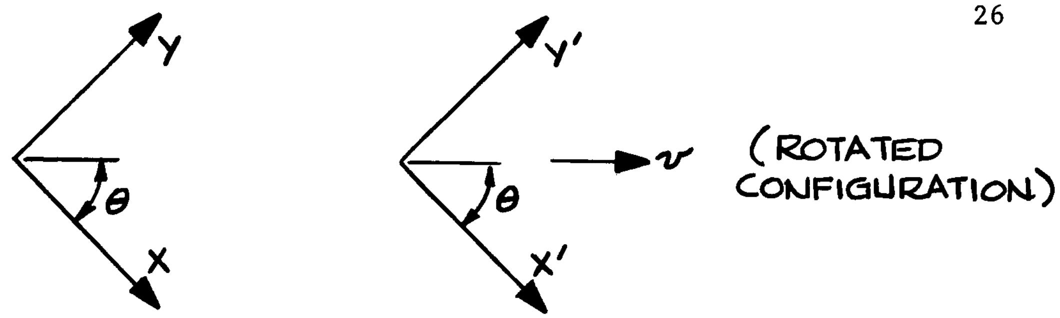


Fig. 2.6b

The Lorentz transformation for this case is:

$$L' = R\,L(v)\,R^{-1} = \begin{bmatrix} 1 - \frac{v_x^2}{v^2}(1-\gamma) & -\frac{v_x v_y}{v^2}(1-\gamma) & 0 & -v_x\gamma \\ -\frac{v_x v_y}{v^2}(1-\gamma) & 1 - \frac{v_y^2}{v^2}(1-\gamma) & 0 & -v_y\gamma \\ 0 & 0 & 1 & 0 \\ -v_x\gamma & -v_y\gamma & 0 & \gamma \end{bmatrix},$$

where $v_x = v\cos\theta$, $v_y = v\sin\theta$.

For the general case one allows R to be the most general three dimensional rotation:

$$R = \begin{bmatrix} \cos\Psi\cos\phi - \cos\theta\sin\phi\sin\Psi, & \cos\Psi\sin\phi + \cos\theta\cos\phi\sin\Psi, & \sin\Psi\sin\theta & 0 \\ -\sin\Psi\cos\phi - \cos\theta\sin\phi\cos\Psi, & -\sin\Psi\sin\phi + \cos\theta\cos\phi\cos\Psi, & \cos\Psi\sin\theta & 0 \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R^{-1} = R^{\text{Transpose}}$

Ψ, ϕ, and θ are the Euler angles* relating the frames ξ, x ; ξ', x' .

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*See the book Classical Mechanics, H. Goldstein, page 107.

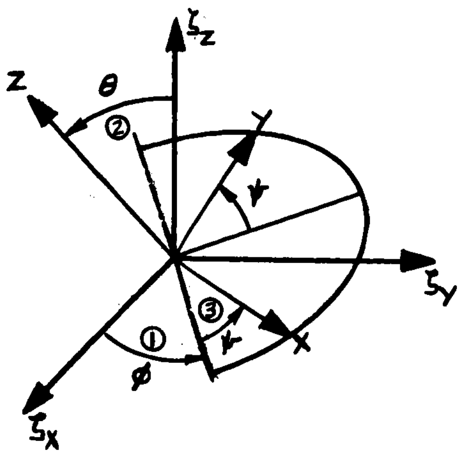


Fig. 2.7a

After a fairly lengthy calculation one finds

$$L' = RLR^{-1} = \begin{bmatrix} 1+\frac{v_x^2}{v^2}(\gamma-1) & \frac{v_x v_y}{v^2}(\gamma-1) & \frac{v_x v_z}{v^2}(\gamma-1) & -v_x\gamma \\ \frac{v_x v_y}{v^2}(\gamma-1) & 1+\frac{v_y^2}{v^2}(\gamma-1) & \frac{v_y v_z}{v^2}(\gamma-1) & -v_y\gamma \\ \frac{v_z v_x}{v^2}(\gamma-1) & \frac{v_z v_y}{v^2}(\gamma-1) & 1+\frac{v_z^2}{v^2}(\gamma-1) & -v_z\gamma \\ -v_x\gamma & -v_y\gamma & -v_z\gamma & \gamma \end{bmatrix} \tag{2.10}$$

where $\underline{\gamma = \gamma(v)}$

$$v_x = v(\cos\Psi\cos\phi - \cos\theta\sin\phi\sin\Psi)$$

$$v_y = -v(\sin\Psi\cos\phi - \cos\theta\sin\phi\cos\Psi)$$

$$v_z = v\sin\theta\sin\phi$$

(2.10) gives the most general Lorentz transformation without rotations.

We note that the components of $\underline{v}$ can be obtained from:

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = R \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix} .$$

One can write (2.10) in a three vector form

$$\underline{x}' = \underline{x} + 1/v^2 \ (\gamma-1) \ (\underline{x}\cdot\underline{v})\underline{v} - \gamma \ \underline{v}t \tag{2.10a}$$

$$t' = \gamma(t - \underline{x}\cdot\underline{v}) \quad .$$

The inverse transformation is obtained from (2.10a) by the substitutions $\underline{x} \longleftrightarrow \underline{x}'$, $\underline{v} \rightarrow \underline{v}'$

$$\underline{x} = \underline{x}' + 1/v^2 \ (\gamma-1)(\underline{x}'\cdot\underline{v}')\underline{v}' - \gamma \ \underline{v}'t$$

$$t = \gamma(t - \underline{x}'\cdot\underline{v}') \tag{2.10b}$$

$$\underline{v}' = - \underline{v} \quad .$$

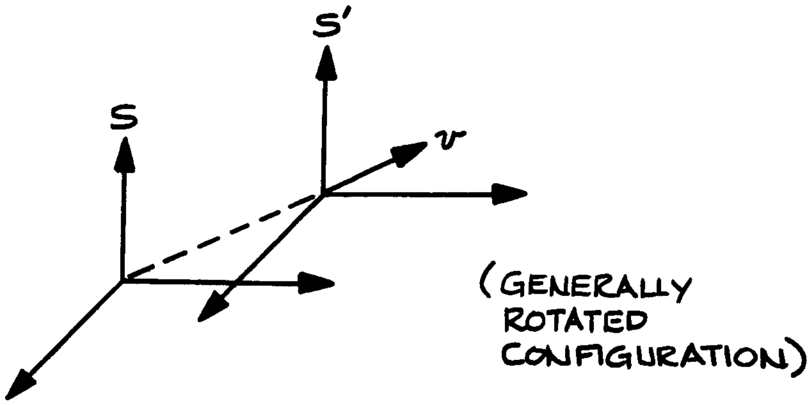


Fig. 2.7b

Appendix to Chapter 2

Standard Configuration

The Lorentz transformations (2.10) are the most general Lorentz transformations "without rotations". The general inhomogeneous Lorentz transformation can be written

$$x'^{\ell} = L^{\ell}{}_{r} x^{r} + a^{\ell} \, *$$

where $L^{\ell}{}_{r}$ are constant transformation coefficients. (2.9) and (2.10) are both examples of particular $L^{\ell}{}_{r}$s. The a^{ℓ}s are constants that correspond to a change in origin. We shall usually be concerned with the homogeneous Lorentz transformations $a^{\ell} = 0$. Proof that standard configuration is always possible between two arbitrary inertial frames S, S'. A plane fixed in S' has the equation

$$\alpha_{\ell}\, x'^{\ell} + P = 0 \;, \alpha_{4} = 0 \;;\; P = \text{const.} \tag{1}$$

Transforming this plane into S : $x'^{\ell} = L^{\ell}{}_{r} x^{r}$ gives

$$\alpha_{\ell} L^{\ell}{}_{k} x^{k} + P = 0 \tag{2}$$

or

$$\alpha_{\ell} L^{\ell}{}_{\beta} x^{\beta} + P + \alpha_{\ell} L^{\ell}{}_{4}\, t = 0 \;.$$

If the transformed plane is at rest in S then

$$\alpha_{\ell} L^{\ell}{}_{4} = 0$$

or

$$\alpha_{\beta} L^{\beta}{}_{4} = 0 \;. \tag{3}$$

---

*The set of all Lorentz transformations (L,a) is a ten parameter Lie group.

Now a point fixed in S has velocity in S' v'^{β}

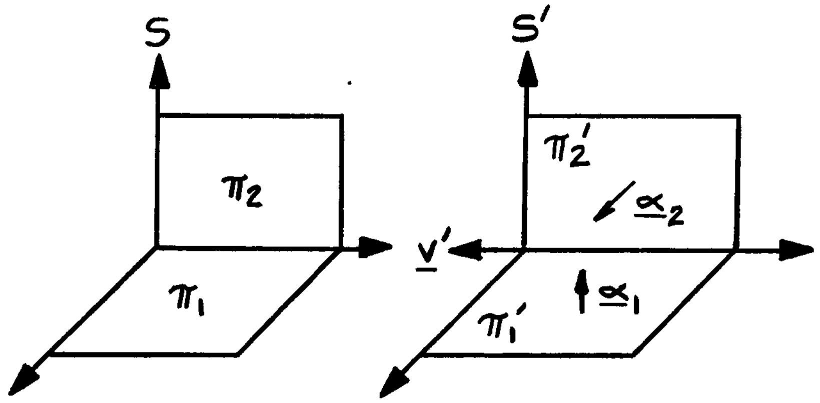
$$v'^{\beta} = \frac{dx'^{\beta}}{dx'^{4}} = \frac{L^{\beta}{}_{4}dx^{4}}{L^{4}{}_{4}dx^{4}} = \frac{L^{\beta}{}_{4}}{L^{4}{}_{4}} \quad \therefore \text{ the condition (3)}$$

becomes

$$\underline{\alpha}\cdot v' = 0 \; , \tag{3}$$

hence $\underline{\alpha}$ is $\perp$ to $\underline{v}'$.

$\underline{\alpha}$ is also perpendicular to the plane in S'. Choose two planes in S' at 90° to each other and parallel to $\underline{v}'$.



STATIONARY PLANES π_1', π_2', IN S' TRANSFORM INTO STATIONARY PLANES π_1, π_2 IN S.

Then these planes must also be at rest in S and they must by symmetry make 90° angles with one another. They must also be parallel to the original planes in S' by symmetry. (That is, there is no reason for them to move one way more than another.) We can now construct the coordinate axis of standard configuration in these fixed planes, taking the direction $-\underline{v}'$ as the x,x' axis.

Chapter 3

A. Lorentz Transformations

The transformations between two inertial frames which are in accord with Einstein's postulates have been shown to be:

$$x' = \gamma(v)(x - vt),\ y' = y,\ z' = z$$
$$t' = \gamma(v)(t - xv) \tag{2.8}$$

or written in the more general form

$$\underline{x}' = \underline{x} + \frac{1}{v^2}(\gamma - 1)(\underline{x} \cdot \underline{v})\ \underline{v} - \gamma\underline{v}t$$
$$t' = \gamma(t - \underline{x} \cdot \underline{v}). \tag{2.9}$$

These transformations are called homogeneous Lorentz transformations. They have some interesting consequences.

1) Length contraction

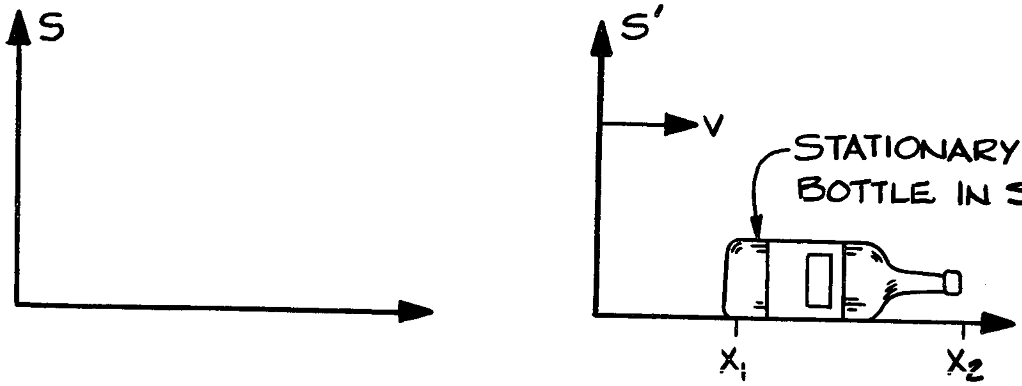


Fig. 3.1

The bottle is at rest in S' where its length is $L_o = x_2' - x_1'$. To measure the bottle's length in S requires two simultaneous markings of the end points in S: $\mathcal{E}_1(x_1,t)$, $\mathcal{E}_2(x_2,t)$. The particular way these simultaneous markings are made does not concern us here; they could be done by two observers by trial and error.

Using (2.8) we have

$$x' = \gamma(x - vt)$$

subtracting gives

$$x_2' - x_1' = L_o = \gamma L \quad \text{or}$$

$$L = (1 - v^2)^{\frac{1}{2}} L_o \quad \text{(Length contraction)} \tag{3.1}$$

(Moving bottles are shortened!)

As seen from (2.8) there is no contraction perpendicular to the motion. Thus, the result (3.1) is mathematically the same as the Lorentz-Fitzgerald contraction hypothesis and of course explains the null results of the various "experiments" to detect the ether. In general when a body having a rest volume V_o moves at velocity v, the volume is contracted to

$$V = (1 - v^2)^{\frac{1}{2}} V_o.$$

Let us carry out the same contraction calculations using the form (2.10) for the more general type of transformation "without rotation"

$$\underline{x}' = (\underline{x} + \tfrac{1}{v^2}(\gamma - 1)(\underline{x} \cdot \underline{v})\underline{v} - \gamma\underline{v}t).$$

Let $\underline{x}_1', \underline{x}_2'$ denote the position vectors of fixed points in S'; then

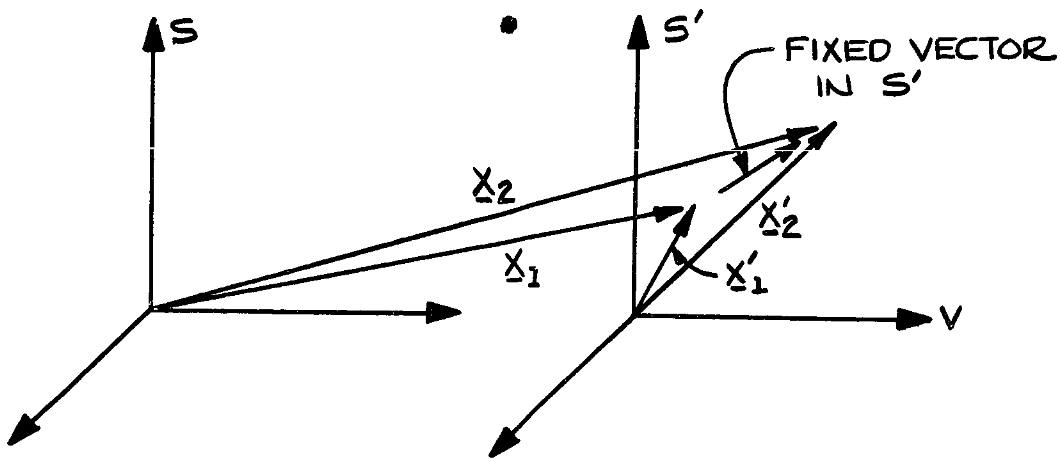


Fig. 3.2

these points have coordinates $\underline{x}_1$, $\underline{x}_2$ in S. If a measurement is carried out to measure the vector $\underline{x}_2 - \underline{x}_1$ then one has:

$$\underline{x}_2' - \underline{x}_1' = \Delta\underline{x}_o = \underline{x}_2 - \underline{x}_1 + \frac{1}{v^2}(\gamma - 1)(\underline{x}_2 - \underline{x}_1)\underline{v}$$

$$t_2 = t_1$$

$$\Delta\underline{x} = \underline{x}_2 - \underline{x}_1$$

$$\Delta\underline{x}_o = \Delta\underline{x} + \frac{1}{v^2}(\gamma - 1)(\Delta\underline{x} \cdot \underline{v})\underline{v}$$

$$\Delta\underline{x}_{o\parallel} = \gamma\Delta\underline{x}_{\parallel}, \quad \Delta\underline{x}_{\perp} = \Delta x_{\perp}$$

($\parallel \Rightarrow$ parallel to $\underline{v}$ $\perp \Rightarrow$ perpendicular to $\underline{v}$)

Inverting $\Delta\underline{x}_o$ one finds

$$\Delta\underline{x} = \Delta\underline{x}_o + \frac{1}{v^2}(\Delta\underline{x}_o \cdot \underline{v})[1/\gamma - 1]\underline{v} .$$

Thus, a vector along the x-axis in S' will not point along the x-axis in S. For two vectors (fixed in S') $\Delta\underline{x}_1'$, $\Delta\underline{x}_2'$ such that $\Delta\underline{x}_1' \cdot \Delta\underline{x}_2' = 0$, it does not follow that $\Delta\underline{x} \cdot \Delta\underline{x}_2 = 0$; therefore, the coordinate axis in S' as viewed from S will not be orthogonal. Although the two frames in Fig. 3.2 are both obtained from standard configuration by the same orthogonal transformation, their axis are not parallel. Parallelism is not transitive under Lorentz transformations.

2) Time dilation

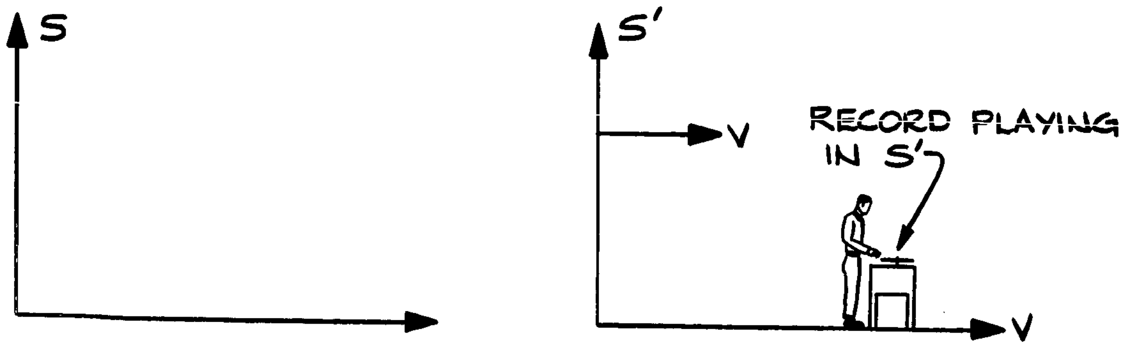


Fig. 3.3

Let $\mathcal{E}_1$ be the events of a person in S' starting a Beatle record on a phonograph. Let $\mathcal{E}_2$ signify the end of the record. These events have coordinates S: $\mathcal{E}_1(x_1,t_1)$; $\mathcal{E}_2(x_2,t_2)$; S' $\mathcal{E}_1(x_1',t_1')$, $\mathcal{E}_2(x_1',t_2')$. The Lorentz transformations of times for the two endpoints are:

$$t_1 = \gamma(t_1' + v\, x_1')$$

$$t_2 = \gamma(t_2' + v\, x_2')$$

and subtracting gives

$$\Delta t = \gamma \Delta t' \quad \text{(time dilation)}. \tag{3.2}$$

(Moving clocks run slow.) (Notice that these are the times corresponding to how long the record played. The details of what the record sounds like will have to await the discussion of wave phenomenon in relativity.) It is clear that (3.2) must be true for all repetitive processes, thus it is true for the life processes in particular. It is to be stressed that the relativity effects discussed here are not to be thought of as illusionary or merely results of our particular measuring methods. To an observer the effects are real in every possible sense of the word. One is often asked to explain why the "rod is shortened" or the "clock runs slow," it is important to realize that these effects cannot be explained by some underlying "magic mechanism" they are predicted in a straight forward manner in relativity theory. Relativity theory offers no explanation in terms of the structure of matter, etc., why these things occur, any theory which is Lorentz covariant must put up with such changes. Also to avoid the type of misunderstanding that has occasionally arisen, we point out that there is no absolute observer who can, so to say, "see things as they really are." As an example of the absurdity of this last statement consider the path of a bomb dropped from a bomber as seen by 1) the bombardier 2) a ground based observer. To the bombardier the path is a verticle straight line, while to the ground observer the path is a parabola. The details of what one sees depends on the frame of reference, a fact that is so obvious it is often overlooked. It is instructive to consider a particular type of clock and understand time dilation in a particular example. The clock we consider is the so called light clock. It consists of an electronic flash, a mirror and a photocell as shown in Fig. 3.4a.

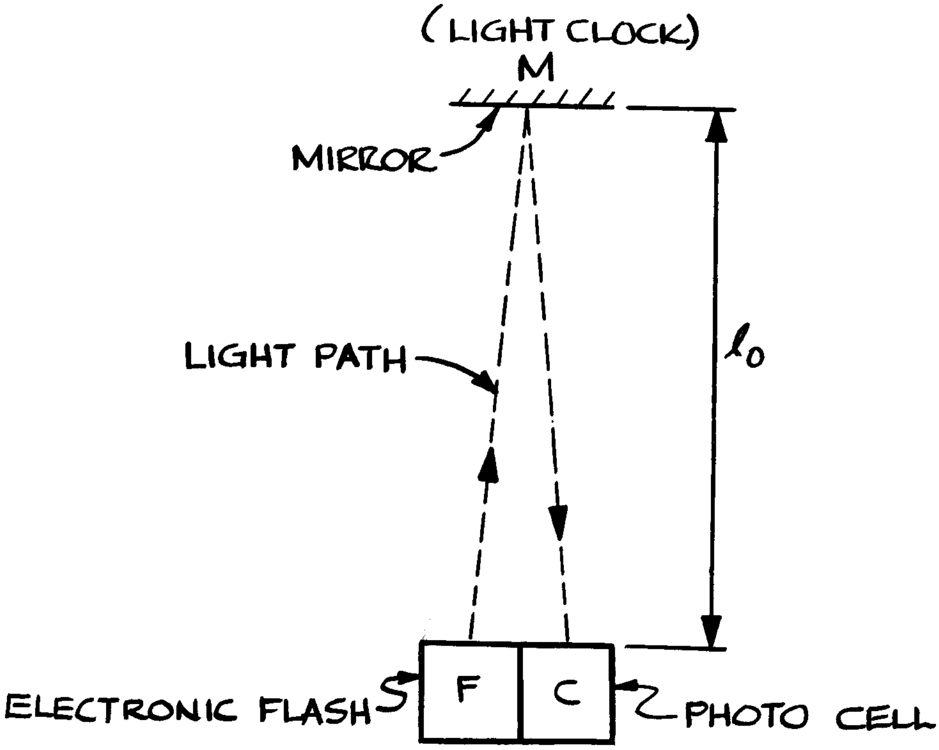


Fig. 3.4a

Light clock as seen by an observer at rest with respect to the clock.

Suppose the clock is at rest in S', then the time measured for a complete cycle is $T_o = 2\ell_o$. The observer in S sees the light travel in an extended path because of the clocks motion.

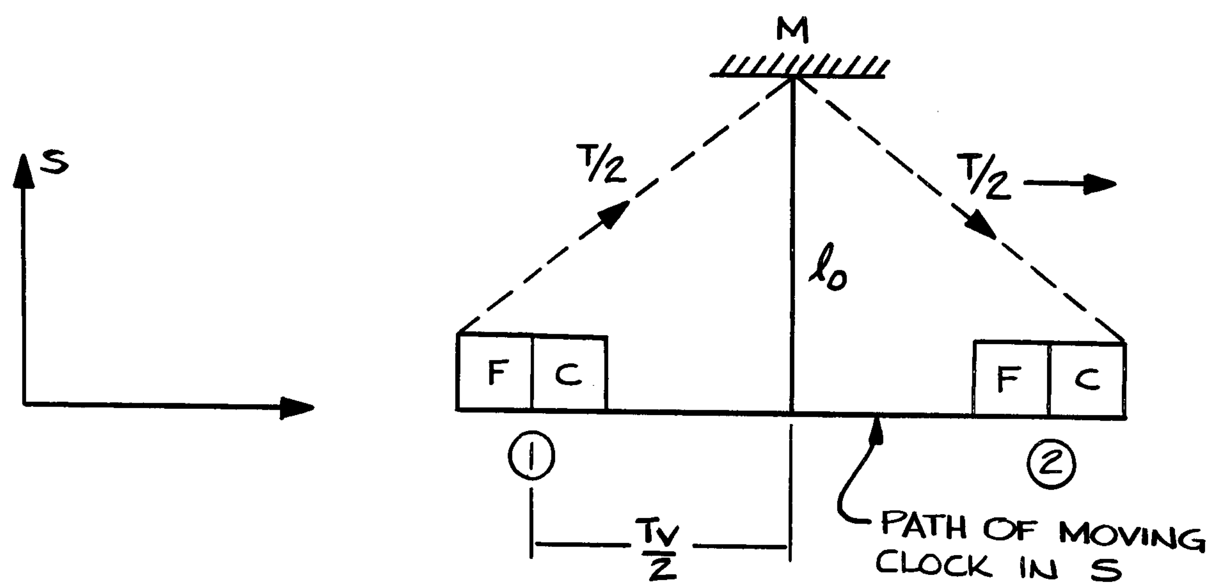


Fig. 3.4b

A clock at rest in S' as seen by S.

Since the clocks at (1) and (2) in S are synchronized, the time between clicks is T. The connection between the two is

$$T^2 = v^2T^2 + T_o^2.$$

$$T = \gamma T_o \tag{3.2}$$

or the light takes longer to travel the extended path than the up and down path. The formula for time dilation has been checked experimentally by the observation of muons at the surface of the earth. The life time of a muon is 2μ sec and traveling at the speed of light it could travel only $\sim$ .6km. However, even though the muons are produced at the top of the atmosphere $\sim$ 10km they are actually found at the surface of the earth. The answer is to be found in (3.2), they live 2μ sec in their rest frame but this corresponds to $\gamma \cdot 2\mu$ sec; since γ can be quite high the muons will penetrate to the earth's surface. When the experiments are carried our seriously the agreement with (3.2) is good.

Comments on time dilation: It will be noticed from the coordinates of the events in the Beatle record experiment, the clock at rest with respect to the record must really be compared with two clocks in S, one at x_1 and one at x_2. These clocks, of course have been synchronized by the methods discussed earlier. (The "clock" that lags is always the one which is being compared with different clocks in the other system.)

Chapter 4

A. Spacetime - Minkowski Space

Let us imagine the points of spacetime to be plotted on a four dimensional cartesian set of axis (x,y,z,t). Each point of the space is an event $\mathcal{E}$. We have seen that in cartesian coordinates the following measure

$$ds^2 = \eta_{\ell k}\, dx^\ell\, dx^k \tag{4.1}$$

is invariant in special relativity. The space made up of the points of a four dimensional Euclidean space together with the invariant measure (4.1) is called Minkowski space and is denoted by M_4. It is a flat (free particles move in straight lines) four space with signature --- +. The region about any event x_o of M_4 is divided into three sections. The null (light) cone of an event x_o consists of all points connected with x_o by light signals. It is the locus of all points

$$\Delta s^2 = \eta_{\ell k}\, \Delta x^\ell\, \Delta x^k = 0 \tag{4.2}$$

where

$$\Delta x^\ell = x^\ell - x_o^\ell\ . \tag{4.3}$$

The three sections are 1) absolute future of x_o; $\Delta s^2 \geq o$, $\Delta t > o$ 2) relative present of x_o; $\Delta s^2 < o$ and 3) absolute past $\Delta s^2 \geq o$ $\Delta t < o$. These three sections are depicted in Fig. 4.1.

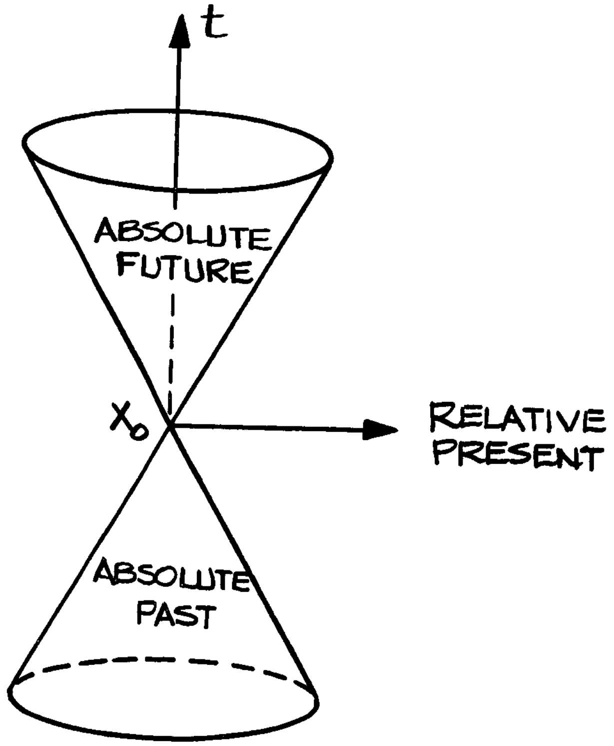


Fig. 4.1

If $x_o = 0$, then $\Delta s \to s$ and the measure from o is:

$$s^2 = - x^2 + t^2. \tag{4.4}$$

One can also introduce vectors into spacetime. Let V^ℓ be a four vector in spacetime. (This means that V^ℓ transforms in the same way that dx^ℓ does under Lorentz transformations.) The square of V^ℓ is defined as:

$$V^2 = \eta_{\ell k}\, V^\ell V^k = - (V^1)^2 - (V^2)^2 - (V^3)^2 + (V^4)^2. \tag{4.5a}$$

Since ds^2 is invariant one can show that V^2 is invariant. One can classify four vectors according to the magnitude of V^2 as shown in Fig. 4.2a:

1) $V^2 > o \Rightarrow V^\ell$ is a timelike vector

2) $V^2 = o \Rightarrow V^\ell$ is a lightlike vector

3) $V^2 < o \Rightarrow V^\ell$ is a spacelike vector

A curve in M_4 is classified by its tangent vector $\frac{dx^\ell}{d\lambda} = t^\ell$, where λ is a parameter along the curve. A general curve in M_4 will be a mixture of all three types. For example in Fig. 4.2b along portion A,

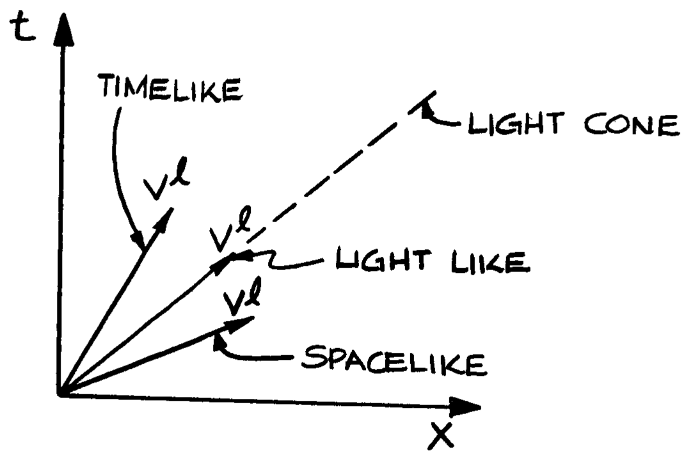


Fig. 4.2a

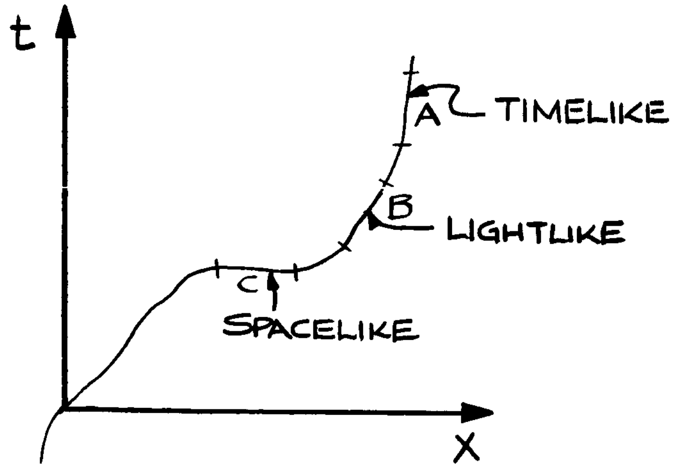


Fig. 4.2b

the curve is timelike, etc. The paths of a material particles when plotted in spacetime (that is M_4) are always timelike curves, and light signals are always lightlike curves. The dot product of two four A^ℓ, B^ℓ is defined as

$$\eta_{\ell k}\, A^\ell B^k = A \cdot B = AB. \tag{4.5b}$$

Two four vectors are called orthogonal if $A \cdot B = 0$. We shall next give a geometric interpretation of Lorentz transformations in spacetime. Two methods will be discussed. The first method introduces an imaginary forth coordinate into the metric

$$ds^2 = - dx^2 - dy^2 - dz^2 - (idt)^2 \qquad i = \sqrt{-1}$$

$$ds^2 = - dx^2 - dy^2 - dz^2 - d\xi^2 \qquad ,\xi = it. \tag{4.6}$$

(4.6) is formally the same as a Euclidean metric but, of course, its content is much different because of the imaginary forth coordinate. The Lorentz transformation (2.8) can now be written as a rotation by an imaginary angle θ in the spacetime described by (4.6).

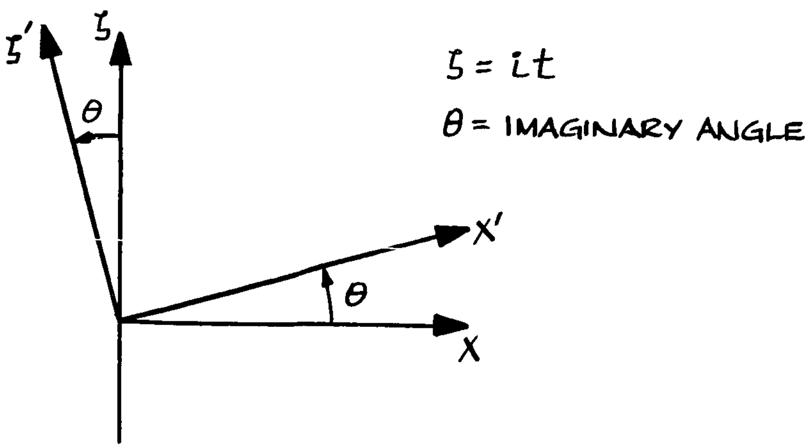


Fig. 4.3

$$x' = \gamma(x - vt) \quad \rightarrow \quad x' = \cos\theta \; x + \sin\theta \; \xi$$

$$t' = \gamma(t - vx) \quad \rightarrow \quad \xi' = -\sin\theta \; x + \cos\theta \; \xi \tag{4.7}$$

$$\cos\theta = \gamma \quad , \quad \sin\theta = iv\gamma .$$

Introducing a real angle $\theta = i\varphi$ and using the relations

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{e^{\varphi} + e^{-\varphi}}{2} = \cosh \varphi$$

and

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{-1}{2i} (e^{\varphi} - e^{-\varphi}) = i\sinh \varphi$$

the Lorentz transformations can be written in terms of the real quantities in the form

$$\begin{aligned} x' &= x \cosh \varphi - t \sinh \varphi \\ t' &= -x \sinh \varphi + t \cosh \varphi \end{aligned} \tag{4.8}$$

and $\sinh \varphi = v\gamma(v)$, $\cosh \varphi = \gamma(v)$, $\tanh \varphi = v$. φ is called the rapidity. In Minkowski space M_4 the transformation (4.8) has a different appearance than Fig. 4.3.

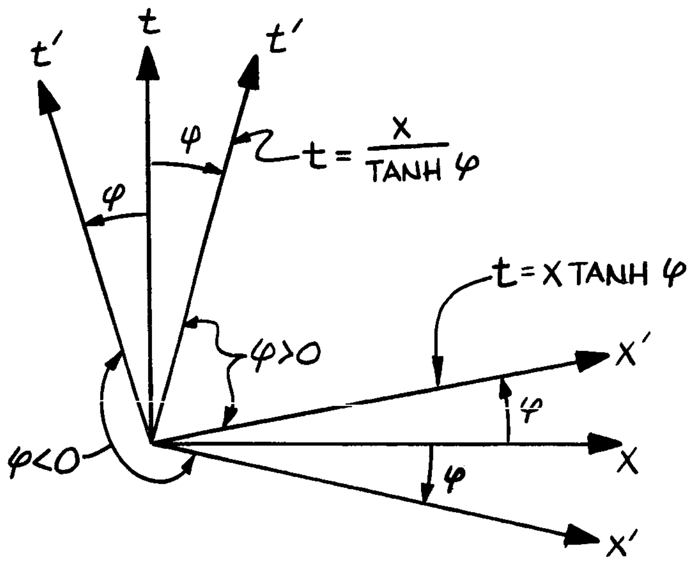


Fig. 4.4

Equations (4.8) can be written in a convenient form if we add and subtract them:

$$\begin{aligned} x' + t' &= e^{-\varphi}(x + t) \\ x' - t' &= e^{\varphi}(x - t). \end{aligned} \tag{4.9}$$

Using the definitions

$$e^{\pm\varphi} = \cosh\varphi \pm \sinh\varphi.$$

Under successive Lorentz transformations associated with coordinate systems in standard configuration rapidities are additive. Suppose we perform two standard Lorentz transformations.

$$\varphi_3 = \varphi_1 + \varphi_2$$

$$x \underset{\varphi_1}{\rightarrow} x' \underset{\varphi_2}{\rightarrow} x''$$

then from (4.9) the resultant transformation is

$$x'' + t'' = e^{-\varphi_1 - \varphi_2}(x + t)$$

which proves that $\varphi_3 = \varphi_1 + \varphi_2$. This example also shows that the "product" of two standard Lorentz transformations is another standard Lorentz transformation, that is, the set of all standard Lorentz transformations forms a group (abelian). Note the velocity of x" with respect to x is

$$v_3 = \tanh\varphi_3 = \tanh(\varphi_1 + \varphi_2) = \frac{\tanh\varphi_1 + \tanh\varphi_2}{1 + \tanh\varphi_1\,\tanh\varphi_2} = \frac{v_1 + v_2}{1 + v_1 v_2} \tag{4.10}$$

which is the Einstein law for composition of velocities.

B. Geometry of M_4

Can one picture length contraction and time dilation in Minkowski space? With certain reservations the answer is yes. Construct Fig. 4.5 using the invariant s^2 of equation (4.4).

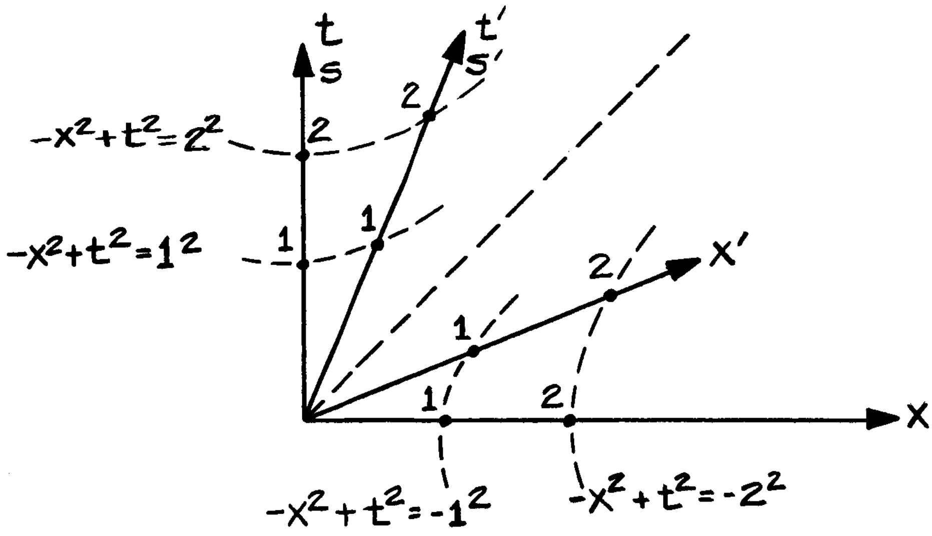


Fig. 4.5

$$x' = x \cosh \varphi - t \sinh \varphi \qquad \cosh \varphi = \gamma$$

$$t' = -x \sinh \varphi + t \cosh \varphi \qquad \sinh \varphi = \gamma v.$$

One sees that the scales on the two axis are not the same from the Euclidean point of view. This is because

$$-x^2 + t^2 = s^2 = -x'^2 + t'^2$$

which one must use to scale the two sets of axis.

1. Time dilation by diagrams.

Suppose we consider a clock at rest in a frame S' (at the origin) and compare it with clocks in S.

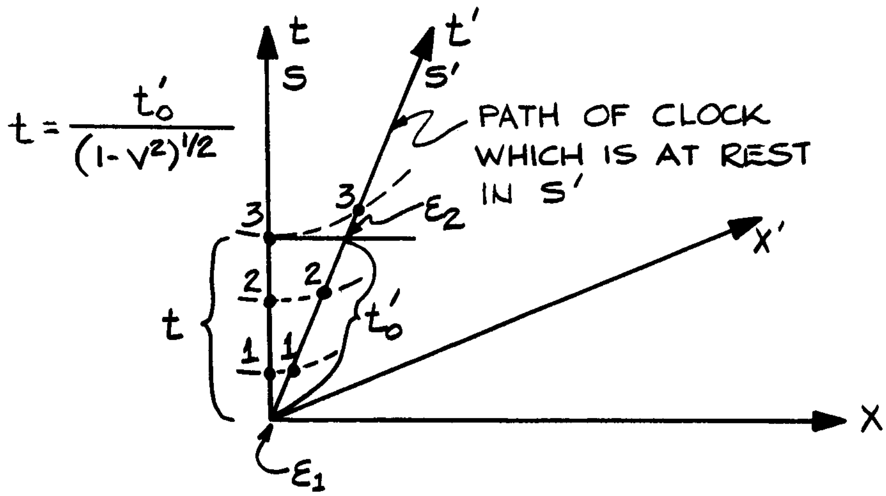


Fig. 4.6a

Time dilation

In S we observe the clock at the events $\mathcal{E}_1$ and $\mathcal{E}_2$, in S' the time elapsed on the clock is the time t_o', whereas in S the elapsed time is t. Note that it does not take S any time to read the clock at $\mathcal{E}_1$ and $\mathcal{E}_2$. This of course is not the case we have just idealized the comparison it would have to be done in reality by means of signals sent back and forth. Of course, the arguments can be reversed and a clock at rest in S compared with the clocks of S' will also run slow. The relativeness of simultaneity shown in Fig. 2.5 appears as in Fig. 4.6b in Minkowski space.

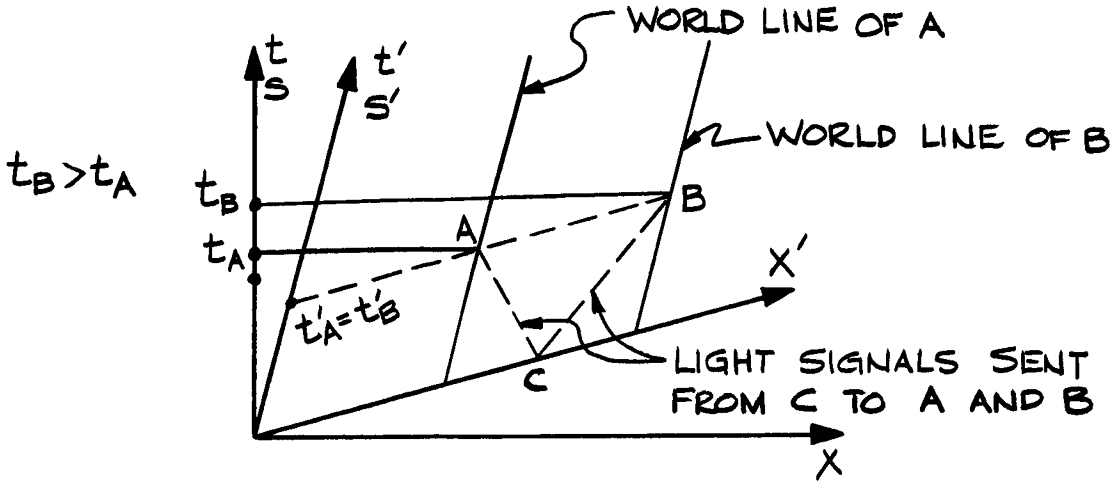


Fig. 4.6b

Simultaneity

2. Length contraction by diagrams

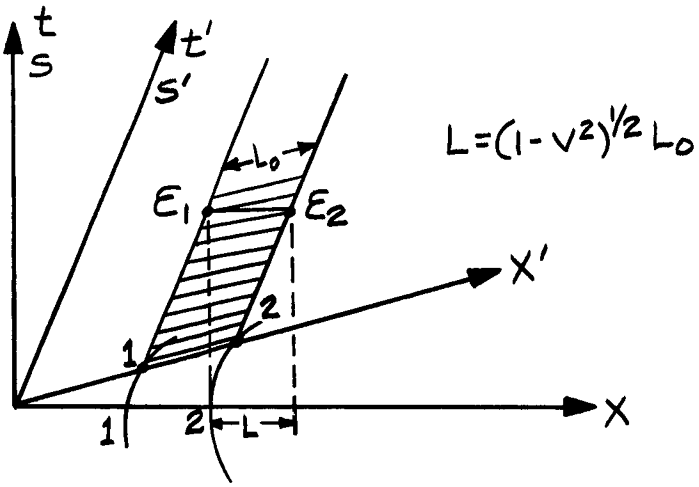


Fig. 4.7

Length contraction

Here the rod is assumed to be at rest in S'. It's end points are measured at the events $\mathcal{E}_1$ and $\mathcal{E}_2$ at the same time in S (definition of length measurement). The rod is seen to be on one meter length in S' but somewhat shorter than this in S.

Because Lorentz transformations are linear, straight lines will look like straight lines in M_4 and parallel lines will look like parallel lines. With these properties we can compare the measure Δs^2 of parallel lines in M_4. But where metrical properties (distances, angles) are concerned we can no longer trust our Euclidean intuition.

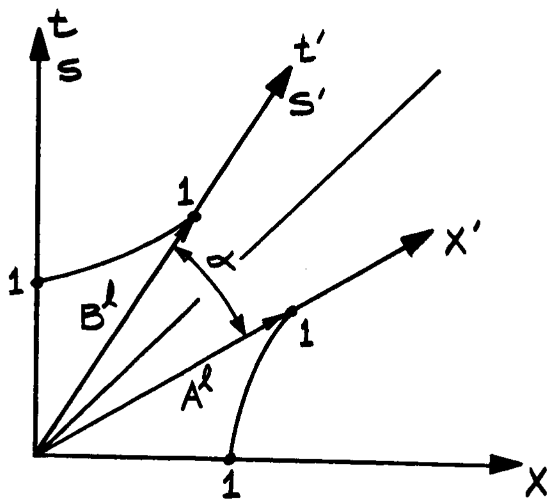


Fig. 4.8

For example, in Fig. 4.8, α is a "right angle" and the points drawn are all unit "distance" from the origin 0. In S': $A^\ell = (1,0,0,0)$, $B^\ell = (0,0,0,1)$ $\eta_{\ell k} A^\ell B^k = 0$.

C. Proper time

The path of a particle in spacetime is a single infinity of points; a curve characterized by some parameter λ defined along the curve. The equation of the curve Γ then consists in giving the coordinates of the line in some coordinate frame (which of course implies the curve is known in all coordinate systems)

$$x^{\ell} = x^{\ell}(\lambda) \quad : \quad \Gamma. \tag{4.11}$$

If the path is that of a material particle then the curve is at every point timelike. This means that the measure s can be used as the parameter along the curve. Of course, the time t could also be used for the path parameter. For the coordinate system in which the particle is at rest we have

$$ds^2 = dt^2 \quad ; \; dx^{\alpha} = 0$$
$$ds = d\tau = \text{"proper time"} \; .$$

Hence, ds may be interpreted as the time as measured by a clock "carried with particle." In an arbitrary frame

$$ds^2 = d\tau^2 = - dx^2 - dy^2 - d\tau^2 + dt^2$$
$$d\tau^2 = dt^2(1 - u^2)$$
$$dt = \frac{dt}{(1 - u^2)^{\frac{1}{2}}} = \gamma \; d\tau \tag{4.12}$$

which gives the relationship connecting the time as measured on the particle (proper time) to the time as measured in an arbitrary inertial frame. The arc length

$$\tau = \int_{A(\Gamma)}^{B(\Gamma)} d\tau \tag{4.13}$$

is the total time elapsed as measured by a clock moving with the particle between the points $A(\Gamma)$ and $B(\Gamma)$ along the curve Γ. In general the integral τ depends on the path Γ. This gives rise to the clock or "twin" effect of relativity theory. If two clocks have the same reading at point A, then if they are separated and have different motions and are reunited at B they will in general show different times.

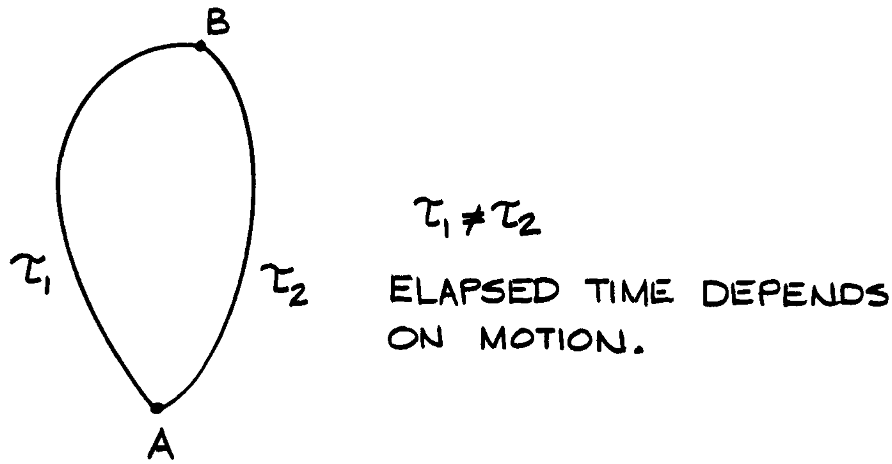
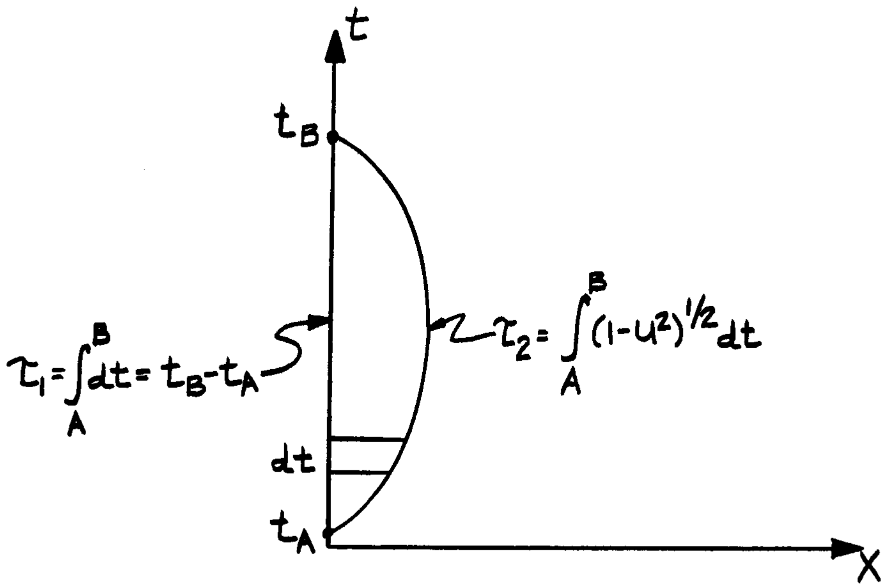


Fig. 4.9

The proper time $d\tau$ is therefore not an exact differential but it depends upon the path. The coordinate time dt is an exact differential, a fact which is obvious from the definition of time in an inertial frame.

If we have "twins" at rest in an inertial frame and one remains at rest while the second leaves and returns, then upon return the second twin will be younger. That this is true can be seen from spacetime picture.



CLOCK OR TWIN EFFECT

Fig. 4.10

Since $(1 - u^2)^{\frac{1}{2}} \leq 1$, it is clear that $\tau_1 > \tau_2$ and hence the returning twin is younger.

Chapter 5

A. Transformation of Kinematical Variables in Special Relativity

1) Velocity: The velocity of a material particle in two different frames is given by:

$$S: \quad \underline{u} = \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}$$

$$S': \quad \underline{u}' = \frac{dx'}{dt'}\hat{x}' + \frac{dy'}{dt'}\hat{y}' + \frac{dz'}{dt'}\hat{z}' ,$$

where the "hat" ^ signifies a unit vector.

We pick S and S' to be in standard configuration. Using the transformation formula (2.8) we find the transformation properties of the velocity to be

$$u'_x = \frac{dx'}{dt'} = \frac{\gamma(dx - vdt)}{\gamma(dt - vdx)} = \frac{u_x - v}{(1 - u_x v)}$$

$$u'_y = \frac{u_y}{\gamma(1 - u_x v)} \quad u'_z = \frac{u_z}{\gamma(1 - u_x v)} \quad . \tag{5.1}$$

It is easily seen that the equations (5.1) predict an ultimate speed for a material particle in an inertial frame. (Usually 1 + 1 = 2 but this is no longer true!) Suppose in a given inertial frame S one particle is traveling at a speed $(1 - \epsilon)$ and another particle is traveling in the opposite direction with a speed $-(1 - \epsilon)$. Then what is the velocity u'_x of the latter particle with respect to the former. (Imagine that particle creation occurs at a certain point.)

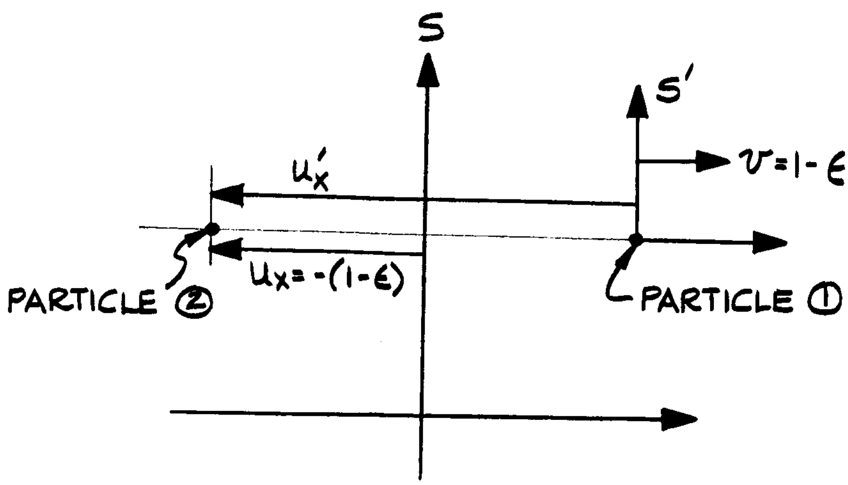


Fig. 5.1

$$u'_x = \frac{u_x - v}{1 - u_x v} \quad \therefore \quad u'_x = \frac{-(1 - \epsilon) - (1 - \epsilon)}{1 + (1 - \epsilon)^2}$$

$$u'_x = \frac{-2}{2 + \epsilon^2 + \epsilon^3 + \ldots}$$

Thus, $u'_x > -1$ and in the limit $\epsilon \to 0$

$$u'_x \xrightarrow[\epsilon \to 0]{} -1 .$$

When one adds velocities according to Einstein's rule 1 + 1 = 1. In order for the Lorentz transformations to make sense $v < 1$, that is the speed of an inertial frame with respect to another inertial frame is always less than the speed of light in free space. We see from the example just worked that when we add velocities according to Einstein's law of addition, a particle can never have a velocity in an inertial frame greater than that of light in vacuum. To avoid the confusion that sometimes results, we point out that this restriction applies by its derivation only to the velocity of a particle as measured in an inertial frame. One can also see directly from the transformation equations that there is an ultimate speed to any signal transmitted through an inertial frame.

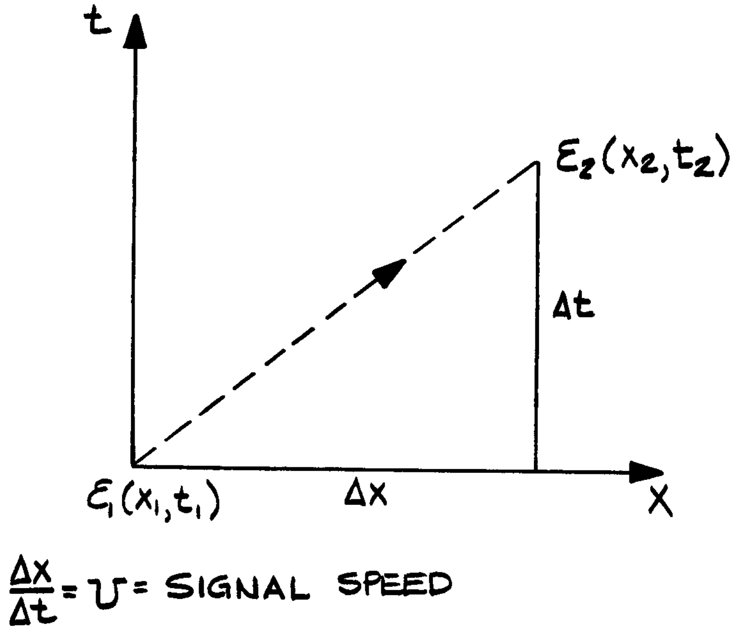


Fig. 5.2

Suppose $\mathcal{E}_1$ is a "cause" and $\mathcal{E}_2$ is an "effect". In S'

$$\Delta t' = \gamma(\Delta t - v\Delta x) = \Delta t\, \gamma(1 - vU).$$ Now if $U > 1$,

then there exists a frame of reference with $v < 1$ such that $1 - vU < 0$ $\therefore$ $\Delta t' = t'_2 - t'_1 < 0$, therefore we have found a frame in which cause and effect are interchanged. Hence, $U > 1$ is not possible. By using the transformations (2.10) for Lorentz transformations without rotations one can show the velocities are related by:

$$\underline{u}' = \left[\frac{\underline{u}}{\gamma} + \frac{\underline{v}(\underline{v}\cdot\underline{u})}{v^2}(1 - \gamma^{-1}) - \underline{v}\right] \Big/ (1 - \underline{u}\cdot\underline{v}) \quad . \tag{5.2}$$

Because the formulas for the transformation of dynamical variables become fairly complicated (i.e. (5.2) is not easy to remember) we introduce four-vectors. The four-velocity of a particle (also world velocity, proper velocity) is defined as the unit tangent to the world line of the particle

$$V^{\ell} = \frac{dx^{\ell}}{d\tau} = \frac{dt}{d\tau}\frac{dx^{\ell}}{dt} = \frac{1}{(1-u^2)^{1/2}} \frac{dx^{\ell}}{dt} \; . \tag{5.3}$$

Here $d\tau$ stands for the proper time elapsed during the particle's motion through dx^{ℓ}. $d\tau$ is a Lorentz invariant and since dx^{ℓ} is a four-vector V^{ℓ} is a four-vector. This means that under Lorentz transformation L:

$$x'^{\ell} = L^{\ell}_{\ k} x^k \Longleftrightarrow V'^{\ell} = L^{\ell}_{\ k} V^k \; . \tag{5.4}$$

In (5.3) u is the magnitude of the velocity of the particle in the frame where the four-velocity is V^{ℓ}. The first three components of V^{ℓ} are just $\gamma(u)$ times the ordinary velocity $\frac{dx^{\alpha}}{dt} = u^{\alpha}$ of the particle. The fourth component V^4 is given by:

$$V^4 = \frac{1}{(1-u^2)^{1/2}} = \gamma(u) \; . \tag{5.5}$$

We shall use u for the velocity of a particle in a reference frame and v for the velocity parameter between two Lorentz frames. (usually in standard configuration) Since the measure ds^2 is a Lorentz invariant the matrix $L^{\ell}_{\ k}$ must satisfy certain conditions;

$$ds'^2 = \eta_{\ell k}\, dx'^{\ell} dx'^{k} = \eta_{\ell k}\, L^{\ell}_{\ r}\, L^{k}_{\ s}\, dx^r\, dx^s$$

$$= \text{invariant} = ds^2 = \eta_{rs}\, dx^r\, dx^s \; .$$

Therefore,

$$L^{\ell}_{\ r}\, \eta_{\ell k}\, L^{k}_{\ s} = \eta_{rs} . \tag{5.6}$$

One can easily check to see that the standard Lorentz transformations (2.9) satisfy (5.6). Using (5.6) we can show that the square of the four-velocity is a Lorentz invariant, namely

$$V^2 = \eta_{\ell k} V^\ell V^k = V'^2 = \eta_{\ell k} V'^\ell V'^k .$$

Thus, if we know V^2 in one inertial frame, we know it in all inertial frames. In the rest frame of the particle: $V^\alpha = 0$, $V^4 = 1$, hence

$$V^2 = 1 . \tag{5.7}$$

(5.7) is called the relativity constraint on the four-velocity; it arises because there are really only three independent components of velocity. In order to check that the transformation law (5.4) contains the transformation properties of u, we consider the special Lorentz transformation (2.9). In matrix notation: $L^{(row)}_{(col)}$ we have

$$\begin{bmatrix} V'^1 \\ V'^2 \\ V'^3 \\ V'^4 \end{bmatrix} = \begin{bmatrix} \gamma(v) & 0 & 0 & -v\gamma(v) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v\gamma(v) & 0 & 0 & \gamma(v) \end{bmatrix} \begin{bmatrix} V^1 \\ V^2 \\ V^3 \\ V^4 \end{bmatrix} \quad \left\{ \begin{array}{l} V \text{ - four-velocity of particle} \\ u \text{ - ordinary velocity of particle} \\ v \text{ - relative velocity of inertial} \\ \quad \text{frames} \end{array} \right.$$

from which follows

$$V'^1 = \gamma(v)(V^1 - vV^4) , \quad V'^2 = V^2 , \quad V'^3 = V^3$$
$$V'^4 = \gamma(v)(V^4 - vV^1). \tag{5.8}$$

(5.8) holds for any four-vector A^ℓ under standard Lorentz transformation if the kernal symbol V is replaced by A. Since

$$V^\ell = \left(\gamma(u)\, \underline{u},\ \gamma(u)\right)$$

$$V'^\ell = \left(\gamma(u')\, \underline{u}',\ \gamma(u')\right)$$

we have from the second of equations (5.8):

$$\gamma(u') = \gamma(v)\left(\gamma(u) - v\gamma(u)\quad u^1\right)$$

$$\frac{\gamma(u')}{\gamma(u)} = \gamma(v)(1 - vu^1)\ , \tag{5.9}$$

and from the first of (5.8):

$$\gamma(u')\quad u'^1 = \gamma(v)\left(\gamma(u)\ u^1 - v\gamma(u)\right)$$

or

$$u'^1 = \gamma(v)\,\frac{\gamma(u)}{\gamma(u')}\,(u^1 - v)$$

$$u'^1 = \frac{u^1 - v}{(1 - vu^1)} \tag{5.1}$$

$$u'^{2,3} = \frac{u^{2,3}}{\gamma(v)(1 - vu^1)}$$

which are the same transformation laws as given before. Thus, the transformation properties of the ordinary velocity are contained in those for the four-velocity. With a given four-vector B^ℓ there is associated new quantities defined by

$$B_\ell = \eta_{\ell k}\ B^k\ . \tag{5.10}$$

The four-vector is characterized abstractly by the kernal symbol B; the components B^ℓ are called the contravariant components of B while the components B_ℓ are called the covariant components. The difference in B_ℓ and B^ℓ is that they transform differently under Lorentz transformations. B^ℓ transforms as:

$$B'^\ell = L^\ell{}_r\ B^r$$

or in matrix notation

$$\begin{bmatrix} B'^1 \\ B'^2 \\ B'^3 \\ B'^4 \end{bmatrix} = L \begin{bmatrix} B^1 \\ B^2 \\ B^3 \\ B^4 \end{bmatrix}$$

where L is the four by four matrix $L^{(\text{row})}_{(\text{col})}$. The transformation of B_ℓ is

$$B'_\ell = L_\ell{}^k \, B_k$$

where $L_\ell{}^k = \eta_{\ell p}\, \eta^{ks}\, L^k{}_s$ and η^{ks} is the inverse of $\eta_{\ell s}$ defined by

$$\eta^{\ell s}\, \eta_{ks} = \delta^\ell{}_k \; . \tag{5.11}$$

One finds $\eta^{\ell k}$ has the matrix representation

$$\text{matrix } \eta^{\ell k} = \begin{bmatrix} -1 & & & \\ & -1 & 0 & \\ 0 & & -1 & \\ & & & +1 \end{bmatrix} = \text{matrix } \eta_{\ell k} = \sqcap \; .$$

In matrix notation the constraint on the Lorentz matrix (5.6) can be written

$$L^\ell{}_r\, \eta_{\ell k}\, L^k{}_s = \eta_{rs} \iff L^T \sqcap L = \sqcap \, , \text{ by}$$

$$(T = \text{transpose})$$

matrix theory we can also write

$$L^T \sqcap L = \sqcap \iff L \sqcap L^T = \sqcap \iff L^r{}_s\, \eta^{sp} L^\ell{}_p = \eta^{r\ell} \; . \tag{5.12}$$

Using these results one can show that the transformation of B_ℓ may be written

$$\begin{bmatrix} B'_1 \\ B'_2 \\ B'_3 \\ B'_4 \end{bmatrix} = L^{T-1} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix}$$

That is, if B^ℓ is picked to transform under the usual representation of the Lorentz group L, then B_ℓ transforms under the contragredient representation L^{T-1}. These two representations are equivalent because $L^{T-1} = \prod L \prod$. In the terminology of modern algebra, if B^ℓ lies in the vector space M_4, then B_ℓ lies in the vector space M^*_4, dual to M_4. In terms of co- and contra-variant vectors we can write the scalor product as

$$(B\cdot C) = \eta_{\ell k} B^\ell C^k = C_k B^k = C^\ell B_\ell \ .$$

$$= -\underline{B}\cdot\underline{C} + B^4 C^4$$

where

$$B^\ell = (\underline{B},\ B^4) \text{ and}$$

$$B_\ell = (\underline{B},\ B^4) \ .$$

We shall now consider other four-vectors in M_4. The four-momentum P^ℓ of a particle is proportional to the four-velocity, the proportionality constant is called the rest mass, m

$$P^\ell = m\, V^\ell;\ P^\ell P_\ell = m^2 \ . \tag{5.13}$$

The rest mass of a particle is an invariant under Lorentz transformation but if you just heat a particle the rest mass changes as we shall later see. The acceleration four-vector Λ^ℓ is defined to be parallel to the first normal of the world line of a particle.

$$\Lambda^\ell = \frac{dV^\ell}{d\tau} = \frac{d^2x^\ell}{d\tau^2} \tag{5.14}$$

The four acceleration is normal to the four velocity (and four momentum)

$$(\Lambda\cdot V) = \Lambda^\ell V_\ell = \Lambda_\ell V^\ell = 0 \tag{5.15}$$

[this means that Λ must be space like as can be proven from (5.7)].

Since

$$V^{\ell} = \left(\gamma(u)\,\underline{u},\ \gamma(u)\right)$$

then

$$\Lambda^{\ell} = \left(\frac{d}{d\tau}(\gamma(u)\,\underline{u}),\ \frac{d}{d\tau}\,\gamma(u)\right)$$

which can be written

$$\Lambda^{\ell} = \left(\gamma(u)^2\,\underline{a} + \underline{u}\,\gamma(u)^4\,\underline{u}\cdot\underline{a},\ \gamma(u)^4\,\underline{u}\cdot\underline{a}\right), \tag{5.16}$$

where

$\underline{a} = \frac{d\underline{u}}{dt} = \frac{d^2\underline{x}}{dt^2}$ is the ordinary acceleration of the particle. The magnitude of Λ^{ℓ}

$$a_o = \sqrt{-\Lambda^{\ell}\Lambda_{\ell}} \tag{5.17}$$

is the curvature of the world line and physically is the acceleration which would determine the apparent weight of a pilot riding in a rocket along the world line.

It is clear that one can always make a Lorentz transformation such that the space components of V^{ℓ} are zero (instantaneous rest frame). That this be done for any time like four vector is easily proven. On the other hand, one can always make a transformation which eliminates the time component of a space like four-vector. In the case of the velocity and acceleration the same Lorentz transformation (instantaneous rest frame) brings them to the form:

$$\begin{aligned} V^{\ell}{}_{o} &= (0,\ 1) \\ \Lambda^{\ell}{}_{o} &= (\underline{a}_o,\ 0)\ . \end{aligned} \tag{5.18}$$

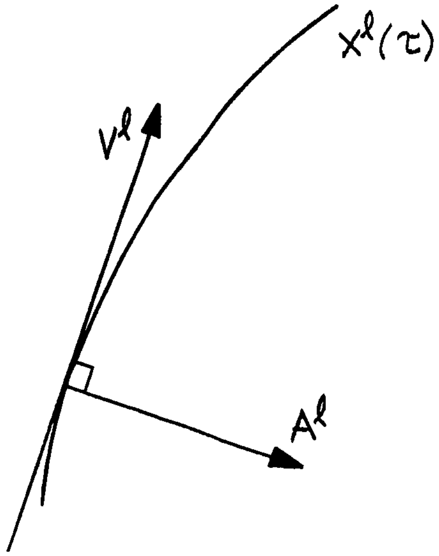


Fig. 5.3

For some reason the myth that one cannot treat acceleration in special relativity is widespread. This is just not the case. An accelerated frame of reference is a complicated thing and the transformation formula between an inertial frame and an accelerated frame will not be a Lorentz transformation as defined previously, but it does exist even in special relativity.*

B. Rectilinear Motion for Which the Acceleration a_o in the Instantaneous Rest Frame Remains Constant (Hyperbolic Motion)

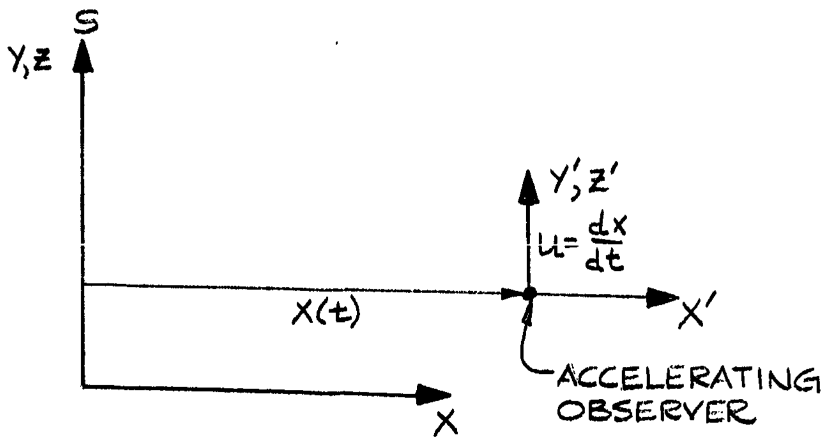


Fig. 5.4

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*L. Marsh, Am. Journ. Phys., page 934, 1965.

Since the square of the four acceleration is Lorentz invariant, we can write

$$\frac{dV^\ell}{d\tau}\frac{dV_\ell}{d\tau} = -a_o^{\,2} = \text{const} \quad . \tag{5.19}$$

In an inertial frame S

$$-\left(\frac{dV^1}{d\tau}\right)^2 + \left(\frac{dV^4}{d\tau}\right)^2 = -a_o^{\,2} \ . \tag{5.20}$$

Using

$$V^1 = \gamma(u)u \quad V^4 = \gamma(u)$$

$$d\gamma = \gamma^3\underline{u}\cdot d\underline{u} = \gamma^3 u du, \ d\gamma u = \gamma^3 du$$

$$\text{or } d\gamma = \gamma^3\underline{u}\cdot d\underline{u} = \underline{u}\cdot(d\gamma\underline{u}) \ ; \quad \gamma + u^2\gamma^3 = \gamma^3$$

(5.20) becomes

$$1/_\gamma \frac{d\gamma(u)u}{d\tau} = a_o \ .$$

In terms of coordinate time we write this as

$$\frac{d}{dt}\,(\gamma(u)u) = \frac{d}{dt}\frac{u}{(1-u^2)^{1/2}} = a_o$$

hence

$$\frac{u}{(1-u^2)^{1/2}} = a_o t + \text{const.}$$

If $u = 0$, $t = 0$ then const. = 0 and

$$\frac{u}{(1-u^2)^{1/2}} = a_o t = u\gamma(u)$$

solving for u one finds

$$u = \frac{dx}{dt} = \frac{a_o t}{(1 + a_o^{\,2}t^2)^{1/2}} \tag{5.22}$$

and integrating again gives

$$x(t) = {}^1/a_o \quad (1 + a_o^{\,2}t^2)^{1/2} + \text{const.}$$

If $x = 0$, $t = 0$, then const $= -{}^1/a_o$ and the solution is

$$x(t) = {}^1/a_o \left((1 + a_o^2 t^2)^{1/2} - 1\right) \tag{5.23}$$

(5.23) can be written in the form

$$x^2 + \frac{2}{a_o} x - t^2 = 0 \tag{5.24}$$

which is the equation for the path of the accelerated observer in the x,t plane. If we consider the observer at the origin of an accelerating coordinate system the coordinate transformation connecting the inertial frame $S(x,t)$ with the accelerating frame $S'(x',t')$ is

$$x = -{}^1/a_o + (x' + {}^1/a_o) \cosh a_o t' \tag{5.25a}$$

$$t = (x' + {}^1/a_o) \sinh a_o t' \tag{5.25b}$$

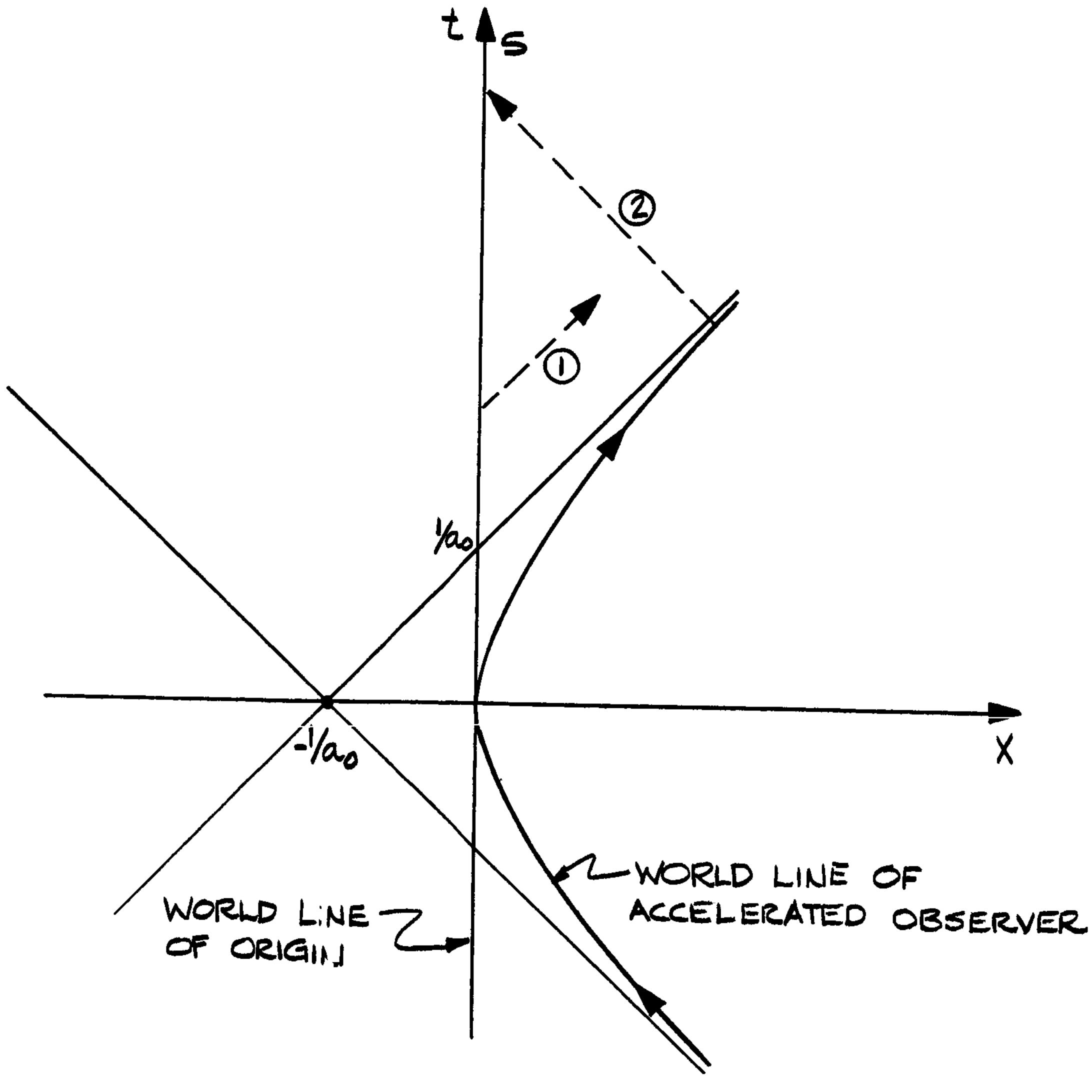


Fig. 5.5

Notice that light signals 1) emitted from the origin of S after $t \geq {}^1/a_o$ will not reach the accelerated observer, whereas light signals 2) emitted by the moving observer will always reach the origin of S. A clock carried by the accelerating observer will read a time $d\tau$ which is related to the coordinate time of S as in (4.12):

$$d\tau = (1 - u^2)^{1/2}\, dt \qquad \therefore \tag{4.12}$$

$$\tau = \int_o^t (1 - u^2)^{1/2}\, dt \; .$$

Using (5.22) this can be written

$$\tau = {}^1/a_o \int_o^t \frac{1}{(1 + a_o^2 t^2)^{1/2}}\, da_o t$$

and integrating

$$\tau = {}^1/a_o \sinh^{-1} a_o t$$

or

$$a_o t = \sinh a_o \tau \tag{5.26}$$

$$a_o t = \frac{e^{a_o t} - e^{-a_o t}}{2}$$

thus as $a_o t \to \infty$, $a_o \tau$ increases much more slowly than t:

$$2a_o t \approx e^{a_o \tau} \qquad \text{or}$$

$$a_o \tau \approx \ell n\, 2a_o t \; .$$

Thus, as $t \to \infty$, τ follows logarithmically.

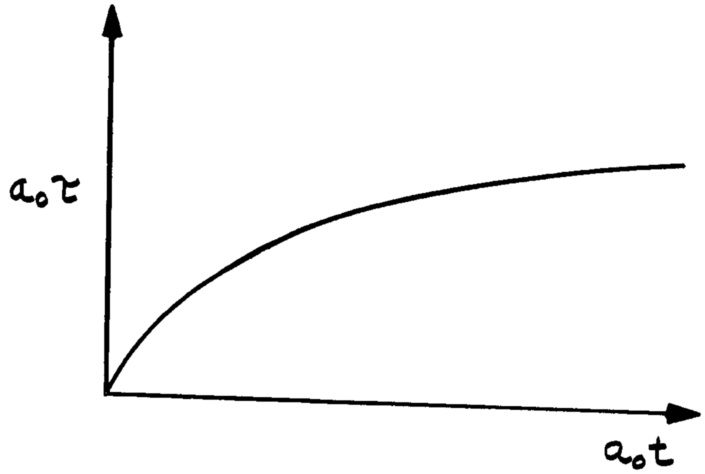


Fig. 5.6

From (5.26) and the identity

$$\cosh x^2 - \sinh x^2 = 1$$

follows:

$$u = \tanh a_o\tau \;, \quad \gamma(u) = \cosh a_o\tau$$
$$a_o t = \sinh a_o\tau \;, \quad x = {}^1/a_o(\cosh a_o\tau - 1) \tag{5.27}$$

If one considers deacceleration with initial conditions $t = 0$, $x = x_o$, $u = u_o$, $t = t_o$, $x = 0$, $u = 0$ then one has merely to replace t by $t_o - t$ in the above results. That is to say the moving clock is slowed both in acceleration and deacceleration. Some interesting numbers can be obtained from (5.27); if $a_o = g$ the acceleration of gravity at the earth's surface then for $x = 3.4 \times 10^9$ light years; $\tau = 22$ years. Therefore, if one could accelerate at g for 22 years he would be at the edge of the detectable universe. A real solution to this problem would require cosmological considerations. For a round trip to Andromeda, the most interesting nearby galaxy, $x = 2 \times 10^6$ light years; if the rocket accelerates at g for 14 years and then deaccelerates at g for 14 years it will arrive at Andromeda. If the rocket has the same type of return trip then $\tau = 56$ years for the round trip and the earth has aged about 2×10^6 years. (The rocket is traveling at very nearly the speed of light after $\tau = 1$ year as seen from the earth, the maximum speed of the rocket occurs at midpoint and is $u = .9999999999995C$.)

A further study of the rocket can be carried out using the same analysis as in the Newtonian case. Let I_o denote the relative velocity of the ejected mass as measured in the instantaneous rest frame of the rocket, r denote the burning rate and M the mass of the rocket in the instantaneous rest frame. Then by conservation of momentum in the instantaneous rest frame or Eq. (5.20)

$$I_o r = M a_o \tag{5.28}$$

where a_o is the acceleration. Substituting for r yields

$$\frac{dM}{M} = - \frac{a_o}{I_o} d\tau$$

or

$$M = M_o e - \frac{a_o}{I_o} \tau \tag{5.29}$$

which is the usual rocket equation. If $I_o = 1$, i.e., the ejection velocity is equal to the velocity of light in vacuum, then one finds for the trips previously discussed

$$\begin{aligned} \frac{M}{M_o} &\approx \frac{1}{2000} \qquad \text{(Alpha Centauri)} \\ \frac{M}{M_o} &\approx 10^{-25} \qquad \text{(Andromeda)} . \end{aligned} \tag{5.30}$$

In present day rockets, Saturn, etc., this ratio $\frac{M}{M_o}$ is between $\frac{1}{10}$ and $\frac{1}{50}$.

Chapter 6

A. Dynamics of Particles

From Einstein's first postulate for special relativity we know that all physical laws must have the same form in all inertial frames. Another way to state this is to say that all laws must be covariant (form invariant) under Lorentz transformations. Let us formulate this mathematically. Let

$$F(A,B,C, \ldots) = 0 \tag{6.1}$$

be a physical law in a reference frame S. Carry out a Lorentz transformation to a new reference frame S'. Let A',B',C', ... denote the transformed variables appearing in the physical law (6.1). Then by Einstein's postulate the physical law in S' must be

$$F(A',B',C', \ldots) = 0 \tag{6.2}$$

where F has exactly the same functional dependence on A',B',C' as it does on A,B,C, ... Now we can use Einstein's postulate along with the known transformation properties of four-vectors to write equations satisfying Einstein's postulates. The first such equation we shall look at will be Einstein's generalization of Newton's second law. When we wrote Newton's second law before

$$F^{\alpha} = \frac{dP^{\alpha}}{dt} \tag{1.1}$$

we left undefined the expression for P^{α}. We do know, however, that in the low velocity approximation

$$P^{\alpha} = m\, u^{\alpha} = m\frac{dx^{\alpha}}{dt}\,, \quad {}^{u}/c << 1 \tag{1.2}$$

where m is called the inertial mass. In analogy we assume the four-vector generalization of (1.1) is

$$f^{\ell} = \frac{dP^{\ell}}{d\tau} \tag{6.3}$$

where P^{ℓ} is the four momentum defined in (5.13) and f^{ℓ} will be called the four-force acting on the particle.

$$P^{\ell} = m\,V^{\ell} = \left(\frac{mu^{\alpha}}{(1-u^2)^{1/2}} , \frac{m}{(1-u^2)^{1/2}} \right) = \left(m\,\gamma(u)u^{\alpha}, m\,\gamma(u) \right) \tag{5.13}$$

(6.3) is certainly a covariant equation but we must now explore its physical meaning and connection with (1.1). Since $dt = \gamma(u) \; d\tau$ (6.3) can be written

$$f^{\ell} = \gamma(u) \frac{dP^{\ell}}{dt} .$$

The first three components of this equation can be written

$$1/\gamma(u) \; f^{\alpha} = \frac{dm\,\gamma(u)u^{\alpha}}{dt} . \tag{6.4}$$

Comparing (6.4) with (1.1) we see that

$$P^{\alpha} = m\,\gamma(u)u^{\alpha} \tag{6.5a}$$

$$\text{ordinary force} = F^{\alpha} = 1/\gamma(u) \; f^{\alpha} . \tag{6.5b}$$

In the low velocity limit the equation (6.3) will therefore yield the empirical Newton laws (1.3). It is an empirical fact that the associations (6.5) are indeed correct. Thus, the inertial mass of a particle m(u) depends on its speed u

$$m(u) = m\,\gamma(u). \tag{6.5}$$

The inertial mass in the rest frame of the particle m(o) = m is just what was defined in Newton's theory as the inertial mass. In relativity the inertial mass is m(u) and the mass m(o) or m is called the rest mass. The rest mass of a given particle is an invariant under Lorentz transformation but it of course can be changed by physical transformations such as heating the particle. Writing the law of motion in expanded form

$$f^{\ell} = m \frac{dV^{\ell}}{d\tau} + V^{\ell} \frac{dm}{d\tau} \tag{6.6}$$

we see that the four-force can be broken into two pieces; the non-mechanical $V^{\ell} \frac{dm}{d\tau}$ and the mechanical $m \frac{dV^{\ell}}{d\tau}$. If the rest mass is constant, then $\frac{dm}{d\tau} = 0$ and the force is purely mechanical.* In what follows we shall consider only the mechanical force unless explicitly stated otherwise.

$$f^{\ell} = m \frac{dV^{\ell}}{d\tau} = m \Lambda^{\ell}. \tag{6.7}$$

Note that since $\Lambda^{\ell} V_{\ell} = 0$, f^{ℓ} must satisfy the constraint condition

$$f^{\ell} V_{\ell} = 0 \tag{6.8}$$

when it is purely mechanical. We shall see later that the electromagnetic force acting on a charged particle satisfies (6.8). From (6.8) follows

$$f^{4} = f^{\alpha} u_{\alpha} \tag{6.9}$$

therefore for mechanical forces the four-force is

$$f^{\ell} = \left(f^{\alpha}, f^{4}\right) = \left(f^{\alpha}, f^{\alpha} u_{\alpha}\right) = \gamma(u) \left(F^{\alpha}, F^{\alpha} u_{\alpha}\right) . \tag{6.10}$$

We have accepted Newton's first, second, and fourth laws of mechanics, but what about the third? Newton's third law cannot be accepted in its Newtonian form. That this is true follows from the transformation properties of the ordinary force F^{α} in (6.10). By comparison with the transformation formula for the velocity (5.2) we can see that (for a

*The generalization of Newton's law to the case where the system expells mass (rocket) can be found in: K. Pomeranz, Am. J. of Phys. 32, 955 (1964).

Lorentz transformation "without rotation") the ordinary force transforms according to:

$$\underline{F}' = \frac{1}{(1-\underline{v}\cdot\underline{u})}\left[\frac{\underline{F}}{\gamma(u)} + \frac{\left(1 - {}^{1}/\gamma(v)\right)}{v^2}\,\underline{F}\cdot\underline{v} - (\underline{F}\cdot\underline{u})\ \underline{v}\right] . \tag{6.11}$$

Clearly, the force has a complicated transformation property and if Newton's third law holds in one frame it is not necessary that it hold in another frame. Short range forces (essentially contact forces) which are active only when the particles are in contact do imply Newton's third law true, since if the forces are equal and opposite in one frame it follows from (6.11) that they are equal and opposite in all frames. We also expect Newton's third law not to survive if we remark that it is the statement of the equality of two forces acting at different places but at the same time. However, as we have seen simultaneity is a relative concept.

B. Energy and Conservation Laws

The concept of energy arises in Newtonian physics as a first integral of Newton's equation (1.3). We now attempt the same analysis of the more general law (1.1):

$$F^{\alpha} = \frac{dP^{\alpha}}{dt} = m\,\frac{d\ \gamma(u)u^{\alpha}}{dt} \tag{1.1}$$

multiply by dx_{β} and summing $\alpha = \beta$ yields

$$F^{\alpha}\ dx_{\alpha} = m\left\{\frac{d\ \gamma(u)u^{\alpha}}{dt}\right\}\ u_{\alpha}\ dt \quad , \tag{6.12}$$

$$(dx_{\alpha} = \eta_{\alpha\beta}\ dx^{\beta}) \ ,$$

integrating from event $\mathcal{E}_1$ to $\mathcal{E}_2$ yields

$$\int_{\mathcal{E}_1}^{\mathcal{E}_2} F^{\alpha}\, dx_{\alpha} = - m \int_{\mathcal{E}_1}^{\mathcal{E}_2} \frac{d\,\gamma(u)}{dt}\, dt \ , \tag{6.13}$$

where use has been made of $d\gamma = \underline{u} \cdot d(\gamma \underline{u})$. Now the change in kinetic energy from $\mathcal{E}_1$ to $\mathcal{E}_2$ is defined to be the work done on the particle, hence,

$$T_2 - T_1 = m\,\gamma(u_2) - m\,\gamma(u_1)$$

where T_2, T_1 represent the kinetic energy of the particle at $\mathcal{E}_1$ and $\mathcal{E}_2$. We require $T|_{u=o} = 0$. Thus, the kinetic energy of a particle having rest mass m and speed u is

$$T(u) = m\,\gamma(u) - m \ , \tag{6.14}$$

independent of what type of forces act on the particle. For a free particle $f^{\ell} = 0$ (6.12) yields

$$\frac{dT}{dt} = 0 \quad T = m\,\gamma(u) - m = \text{const.} \tag{6.15}$$

as a first integral. In this case the total energy W is defined as $m\gamma(u)$ and we write

$$\text{const} = P^4 = W = T(u) + m = m\gamma(u) \ \left(\begin{matrix}\text{free}\\ \text{particle}\end{matrix}\right). \tag{6.16}$$

Consider a collection of particles. Let $F^{\alpha}_{AA'}$ be the force extended on the $A^{\underline{th}}$ by the $A'^{\underline{th}}$. Then the equation (6.13) becomes

$$\int_{t_1}^{t_2} \Sigma_{A,A'}\, F^{\alpha}_{AA'}\, dx^{A}_{\alpha} = - \int_{t_1}^{t_2} \Sigma_A\, m_A\, \frac{d\,\gamma(uA)}{dt}\, dt$$

$$\int_{t_1}^{t_2} \Sigma_{A,A'}\, F^{\alpha}_{AA'}\, dx^{A}_{\alpha} = - \Big[\Sigma_A\, W_A \Big]_{t_1}^{t_2} \tag{6.17}$$

where

$$W_A = m_A \gamma(u_A) = T_A + m_A \tag{6.18}$$

and m_A is the rest mass of the $A^{\underline{th}}$ particle. In the Newtonian theory

$$F^{\alpha}_{AA'}(t) = -F^{\alpha}_{A'A}(t) . \tag{6.19}$$

Relativistic mechanics is very complicated for the very reason that this equation cannot be true in a relativistic theory (effects cannot propagate at speeds greater than that of light). We shall discuss a scattering experiment where the complications arising from this are minimal.

1) Scattering: Suppose we are considering a number of particles, then (1.1) becomes

$$\sum_{A'} F^{\alpha}_{AA'} = \frac{dP^{\alpha}_A}{dt} \quad \therefore$$

summing over all particles and integrating yields,

$$\int_{-T}^{+T} \Sigma_{A,A'} F^{\alpha}_{AA'}\, dt = \int_{-T}^{+T} \Sigma_A \frac{dP^{\alpha}_A}{dt} \qquad (A \neq A') . \tag{6.20}$$

Even though we cannot use (6.19) it is safe to assume that if free particles in the initial state -T scatter into free particles in the final state +T then the integral

$$\lim_{T \to \infty} \int_{-T}^{+T} \Sigma_{AA'} F^{\alpha}_{AA'}\, dt = 0 \qquad (A \neq A') \tag{6.21}$$

(This is assumed to hold in all inertial frames!)

From (6.20) follows

$$\lim_{T\to\infty}\left[\left(\Sigma P_A^{\alpha}\right)_T = \left(\Sigma P_A^{\alpha}\right)_{-T}\right] \quad \therefore$$

$$\Sigma P_A^{\alpha} = P^{\alpha} = \text{const.} \tag{6.22}$$

Thus, the three momentum P^{α} is conserved in scattering processes.

Define

$$N^{\ell} = P_f^{\ell} - P_i^{\ell} \tag{6.23}$$

where P_f^{ℓ} is the final four-momentum and P_i^{ℓ} is the initial four-momentum. By (6.21) $N^{\alpha} = 0$ in all inertial frames. This implies that $N^4 = 0$ which implies

$$P_f^4 = P_i^4 \quad \text{or} \quad W_f = W_i \tag{6.24}$$

hence, under the assumption (6.21) the linear momentum and energy are conserved in a scattering processes. The conservation of four-momentum

$$\sum_f P_f^{\ell} = \sum_i P_i^{\ell} \tag{6.25}$$

is a fundamental law of physics which has been experimentally verified many times. 2) composite "particles".

The results obtained in (6.16) are also valid for a composite system of particles as long as the system is closed or isolated. This follows directly from (6.20) if we assume T is greater than the time taken for light to traverse the system under consideration then

$$\int_{-T}^{+T} \Sigma_{A,A'} F_{AA'}^{\alpha}\, dt = 0 \qquad T >> \text{(characteristic dimensions of the system)} \tag{6.26}$$

and

$$\frac{d}{dt} P^{\alpha} = 0 \quad \Longrightarrow \quad P^{\alpha} = \text{const.}$$

these conditions are the same as (6.22) and hence it follows that the energy momentum four-vector of the composite system is

$$P^{\ell} = MV^{\ell} = M\gamma(u)u^{\alpha},\ M\gamma(u) \tag{6.27}$$

or

$$P^{\ell} = (P^{\alpha},\ W).$$

Here u^{α} represents the velocity of the system as a whole and M its rest mass. Thus, the rest energy of a composite body is equal to its rest mass,

$$W_o = M. \tag{6.28}$$

In a composite body, say an atom, where does the rest energy reside? The energy resides in the rest masses of the constituent particles, the kinetic energy of the constituent particles, and in the energy of interaction between and among the constituent particles. Suppose we have a number of particles which form a composite body, then

$$M = W_o = \Sigma\, m_A + \Sigma\, T_A + \sum_{A,A'} V_{AA'} + \sum_{A,A',A''} V_{AA'A''} + \ldots.$$

$$(A \neq A' \neq A'' \ldots.)$$

Let $\quad \text{Vint} = \Sigma\, V_{AA'} + \Sigma\, V_{AA'A''} \quad + \ \ldots.$

Where the Vint is the interaction energies of the basic particles. Suppose we form a nucleus. To do this, we carry out the following steps:

2) Bring Z protons and N neutrons together with all interactions "turned off". The rest energy in this configuration is:

$$M' = W_o = \Sigma\, m_A + \Sigma\, T_A\ ,$$

2) Turn on the interactions. The rest energy now becomes

$$M = M' + Vint \tag{6.29}$$

therefore if the composite configuration is stable $\Longrightarrow$ Vint < 0

$$W_o = M = \Sigma\, m_A + \Sigma\, T_A - |Vint|. \tag{6.30}$$

For stable nuclei it is always observed that

$$M - \Sigma\, m_A = \Sigma\, T_A - |Vint| < 0 \tag{6.31}$$

indicating that the interaction energies are greater than the kinetic energies involved. Example:

$$\left.\begin{array}{rcl} m_{proton} & = & 938.211 \text{ Mev.} \\ m_{neutron} & = & 939.505 \text{ Mev.} \\ m_{electron} & = & .511 \text{ Mev.} \\ M = m\left({}_6c^{12}\right) & = & 11177.233 \text{ Mev.} \end{array}\right\} \text{experimental numbers.}$$

$$\Sigma\, m_A = [6(938.211) + 6(939.505) + 6(.511)]\text{Mev.}$$

$$\Sigma\, m_A = 11269.334 \text{ Mev.} \quad \therefore$$

$$\left.\begin{array}{c} |M - \Sigma\, m_A| = 92.101 \text{ Mev.} \approx 1.64 \times 10^{-25} \text{ gms.} \\ 1 \text{ cal.} = 2.613 \times 10^{13} \text{ Mev.} \end{array}\right\} \text{for}\left({}_6c^{12}\right)$$

Note: (The masses above are for the system $m_{\,{}_8O^{16}} = 16.000$ a.u. Now the official system is $M_{\,{}_6c^{12}} = 12.000$ a.u. and in this system $m_{\,{}_8O^{16}} = 15.995$)(no practical difference between these systems).

From (5.13) it follows that for a free particle

$$P_\ell P^\ell = -\underline{P}^2 + W^2 = m^2 \quad \text{or}$$

$$W^2 = \underline{P}^2 + m^2 \tag{6.32}$$

which is the relativistic relationship between energy and momentum. Also the following relation between the energy and momentum holds

$$\underline{P} = W\,\underline{u}\ . \tag{6.33}$$

For $u = 1$ the energy and momentum of a particle become infinite, thus a particle with m different from zero cannot move with the speed of light. However, for zero rest mass particles we assume the limiting form of (6.33) to be

$$P = W\ ,\ m = 0\ . \tag{6.34}$$

Thus, the momentum four-vector of a zero mass particle is a lightlike vector. One can represent the four-momentum of a zero mass particle in the form

$$P^{\ell} = W(n^{\alpha}\ ,\ 1)\ . \tag{6.35}$$

Here n^{α} is a unit vector in the direction of the propagation of the particle. If one prefers to speak of the wave properties then the wave vector k^{ℓ} can be introduced by the definition

$$P^{\ell} = h\ k^{\ell}\ ,\ W = h\nu \tag{6.36}$$

ν is the frequency associated with energy W and his Plank's constant

$$h = 6.63 \times 10^{-34}\ \text{joule-sec.}$$

Relativistic mechanics of point particles is not nearly as successful as its Newtonian counterpart. Even though one knows the basic law of motion (6.3) one cannot write realistic forces f^{ℓ} easily because of the breakdown of Newton's third law. In general the form of f^{ℓ} makes (6.3) a nonlinear integro-differential equation and not much is known about such equations. In particular one has not been able to prove that unique solutions for these equations exist or not.

Chapter 7

A. Periodic Disturbances in Special Relativity

Let A represent a general periodic disturbance in a frame S. As an example A could be of the form

$$A = A_o \cos(\underline{k}.\underline{x} - \omega t) \tag{7.1}$$

A could have a general transformation property. $\underline{k}$ is called the propagation vector of the wave and ω is its frequency. These can be combined together to form the wave vector k^r:

$$k^r = (\underline{k}, \omega) \ . \tag{7.2}$$

Using the wave vector the distrubance A can be written

$$A = A_o \cos(k_r x^r) \ . \tag{7.3}$$

The quantity $\phi = k_r x^r = -\underline{k}.\underline{x} + \omega t$ is called the phase of the disturbance. It is clear that the phase ϕ must be an invariant under Lorentz transformation because A = 0 in S must imply A' = 0 in any other inertial frame S'. That is, a periodic wave in one inertial frame must be a periodic wave in every inertial frame. This is equivalent to asserting the wave vector k^r is a four-vector. Knowing this, we can write down the transformation properties of k^r immediately:

$$\left.\begin{aligned} \underline{k}' &= \underline{k} + {}^1/_{v^2}\ (\gamma-1)(\underline{k}.\underline{v})\underline{v} - \gamma\underline{v}\ \omega \\ \omega' &= \gamma(\omega - \underline{k}.\underline{v}) \end{aligned}\right\} \tag{7.4}$$

$\underline{v}$ = velocity of S' with respect to S ; $-\underline{v}$ = velocity of S wrt. S'. (7.4) gives the transformation law of k^r between two frames S, S' which differ from one another by a Lorentz transformation "without rotation". The speed of propagation of the wave disturbance (phase velocity) is

$W = \frac{\omega}{k}$. Suppose the source of the periodic disturbance is moving away from S at velocity $\underline{v}$ and sends the disturbance to S. In S' the kinematic characteristics of the disturbance are $\underline{k}'$, ω', and W' and in S they are $\underline{k}$, ω, and W. The connections between S and S' can be written

$$\left.\begin{aligned} k'_{11} &= k'\cos\alpha' = \gamma(k_{11} - \omega v) = \gamma k(\cos\alpha - Wv) \\ k'_{1} &= k'\sin\alpha' = k_{1} = k\sin\alpha \\ \omega' &= \gamma(\omega - k\cos\alpha\, v) = \gamma\omega(1 - {}^{v}/_{\omega}\cos\alpha) \,. \end{aligned}\right\} \qquad (7.5)$$

Here α is the angle between $\underline{k}$ and $\underline{v}$ in S and α' is the angle between $\underline{k}'$ and $\underline{v}$ in S'. It is convenient to consider the angles $\theta = \pi - \alpha$ and $\theta' = \pi - \alpha'$.

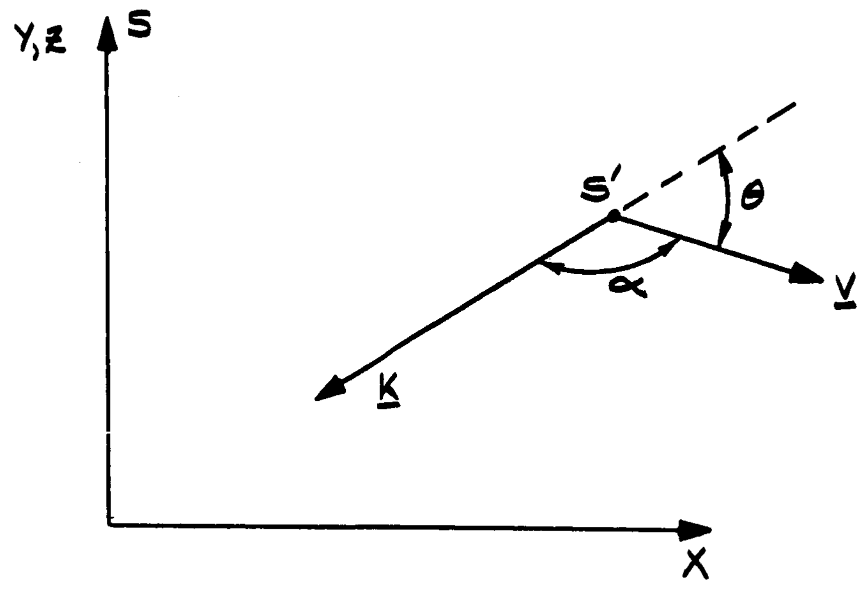


Fig. 7.1

The transformation formulas are:

$$\left.\begin{aligned} k' \cos\theta' &= \gamma k(\cos\theta + Wv) \\ k' \sin\theta' &= k \sin\theta \\ \omega' &= \gamma\omega(1 + {}^{v}/_{\omega} \cos\theta) \end{aligned}\right\} \tag{7.5}$$

From the top two equations follows the relativistic aberration formula:

$$\tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta + Wv)} \tag{7.6}$$

and the last equation

$$\omega' = \gamma\omega(1 + \frac{v}{W} \cos\theta) \tag{7.7}$$

is the relativistic Doppler equation.

From the invariance of $k_r k^r = -\underline{k}^2 + \omega^2$ under Lorentz transformation follows

$$\omega^2 \left(1 - {}^{1}/_{W^2}\right) = \omega'^2\left(1 - {}^{1}/_{W'^2}\right) . \tag{7.8}$$

Using (7.7) in (7.8) yields

$$1 - {}^{1}/_{W'^2} = \frac{\left(1 - {}^{1}/_{W^2}\right)\left(1 - v^2\right)}{\left(1 + {}^{v}/_{W} \cos\theta^{\;2}\right)} . \tag{7.9}$$

(7.9) gives the transformation properties of the disturbance velocity W. Since $v \cos\theta = v_R$ is the radial velocity of S' one can write the Doppler equation

$$\omega = \frac{\omega'}{\gamma\left(1 + \frac{v_R}{W}\right)} \tag{7.10}$$

In the classical limit this reduces to the classical Doppler formula. If a sound source S' is moving away from an observer S at rest with respect to the air, say, then the classical Doppler shift is

$$\omega = \frac{\omega'}{\left(1 + \frac{v_r}{W}\right)} \tag{7.11}$$

where W is the speed of sound in air. (7.11) differs from (7.10) only in the $\gamma(v)$ term which implies a differing of terms in v^2. Experimentally one cannot measure the difference between (7.10) and (7.11) for sound waves. In most applications one is interested in equations (7.6) and (7.7) for light W = 1,

$$\tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta + v)} \tag{7.12}$$

or

$$\tan\frac{\theta'}{2} = \left(\frac{1-v}{1+v}\right)^{\frac{1}{2}} \tan\frac{\theta}{2}$$

$$\omega' = \gamma\omega(1 + v\cos\theta). \tag{7.13}$$

Two cases for (7.13) suggest themselves 1) $\theta = 0^o$

$$\omega = \frac{\omega'}{\gamma(1+v)} = \frac{(1-v)^{\frac{1}{2}}}{(1+v)^{\frac{1}{2}}}\ \omega' ,$$

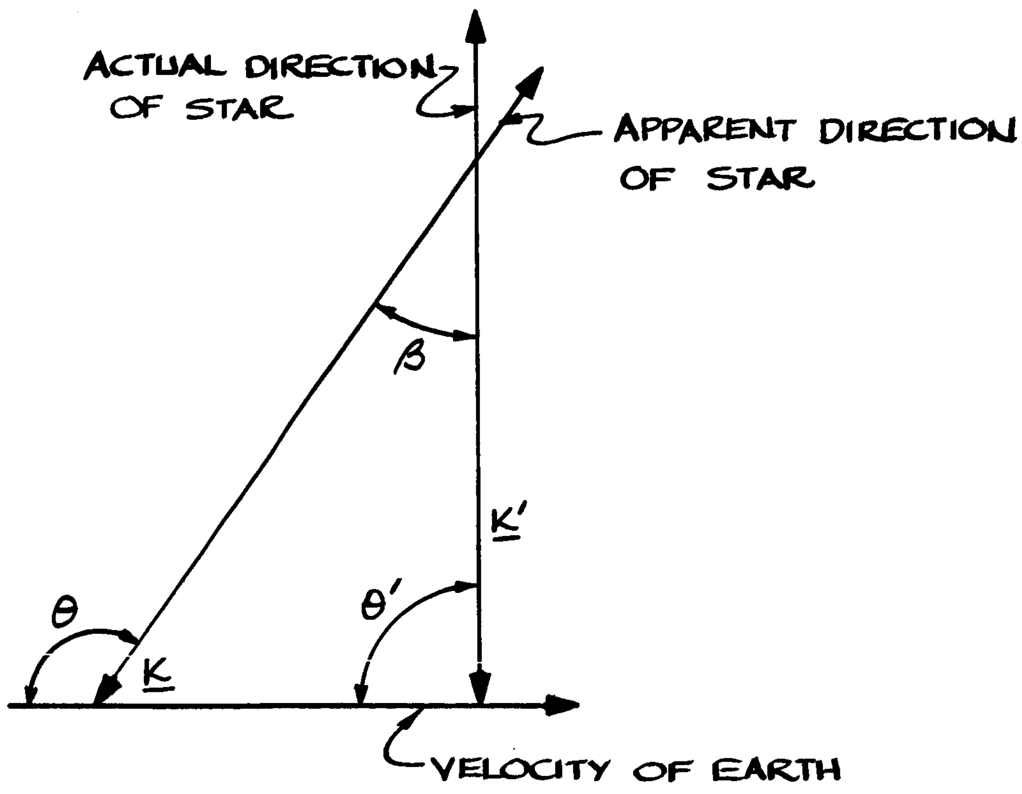
this is the longitudinal Doppler shift, $\omega < \omega' \Longrightarrow \lambda > \lambda'$ which is called the Doppler red shift: 2) $\theta = {}^{\pi}/_{2}$

$$\omega = \frac{\omega'}{\gamma} ,$$

this is the transverse Doppler effect which is just the inverse of time dilation, this has been checked using the Mössbauer effect. Inverting (7.12) we can write

$$\tan\theta = \frac{\sin\theta'}{\gamma(\cos\theta' - v)} . \tag{7.14}$$

The classical aberration formula can be obtained from (7.14) by taking $\gamma = 1$. If θ' is the angle between minus the earth's velocity and the actual direction to a star while θ is the angle between minus the earth's velocity and the apparent direction to the star, then (7.14) connects these two angles. The constant of aberration β is the apparent displacement of a star when the earth is revolving at average speed at right angles to the stars direction $\theta' = \pi/2$.



Aberration of Starlight

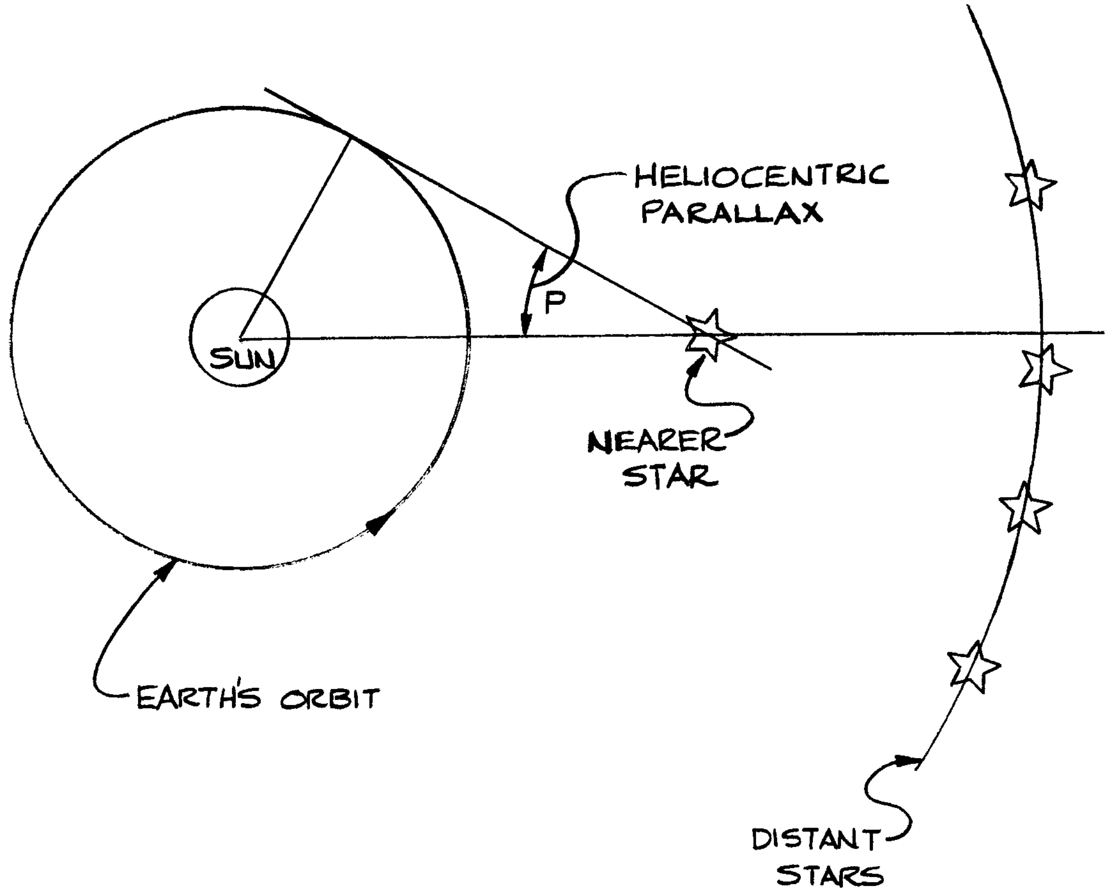
Fig. 7.2

$\tan\theta = -{}^{1}/_{v\gamma}$, $\theta' = {}^{\pi}/_{2}$ or

$$\tan\beta \approx \beta = v\gamma \approx v \tag{7.15}$$

$v \approx 10^{-4}$; $v = 3 \times 10^{4} \frac{m}{sec.}$ $\beta \approx 20.5$ sec. of arc.

The constant of aberration gives a measure of the size of apparent orbit of a star due to the aberration of starlight. The aberration orbit traced out by the star depends upon its location on the celestial sphere, it is a circle at the ecliptic poles, a straight line at the ecliptic equator and an ellipse in between. The constant of aberration is the same for all stars and measures the radius of the circle at the ecliptic poles, half the length of the straight line at the ecliptic equator, and half the major axis in between. The aberration of starlight was first noticed by Bradley in 1727 and was the first conclusive proof that the earth actually moves about the sun instead of the opposite. The aberration orbit of stars is independent of the size of the earth's orbit and depends only on the fact that the earth has a changing velocity. The parallax (heliocentric) of a star is due to the fact that since the earth is revolving around the sun, a nearer star seems to be describing a little orbit with respect to the more distant stars. This apparent orbit has almost the same shape as the aberration orbit. The parallax orbit is much smaller than the aberration orbit, being around .76" for the nearest star Alpha Centauri at 4.3 ly. and smaller than this for all other stars.



Heliocentric Parallax of a Star

Fig. 7.3

For completeness we might mention a third type of motion associated with stars. This is change in the stars position on the celestial sphere due to its actual motion. This is the so-called proper motion of the star. The known proper motions of only about 330 stars exceed 1" a year and the average for all naked eye stars is not greater than 0.1" per year. Barnard's star has the largest observer proper motion of 10.3" per year.

Chapter 8

A. Tensor Analysis I

Tensor analysis concerns the transformation of quantities from one frame of reference to another. The differential dx^ℓ is a tensor and its transformation law under coordinate transformation $x'^\ell = f^\ell(x)$* can be written

$$dx'^\ell = \left(\frac{\partial x'^\ell}{\partial x^r}\right) dx^r. \tag{8.1}$$

A scalar ϕ is also a tensor and its transformation law is the simplest

$$\phi'(x') = \phi(x). \tag{8.2}$$

The gradient of a scalar $\frac{\partial \phi}{\partial x^r} = \phi_{,r}$ is also a tensor and its transformation law is

$$\frac{\partial \phi'}{\partial x'^r} = \left(\frac{\partial x^\ell}{\partial x'^r}\right)\frac{\partial \phi}{\partial x^\ell}. \tag{8.3}$$

The two transformation laws (8.1) and (8.3) look very similar. If one considers linear transformations, then $x'^\ell = L_r{}^\ell x^r$ and (8.1) is the transformation law given previously for a contravariant vector, while (8.3) is the law given for a covariant vector. Thus, (8.1) is the transformation law for a contravariant tensor of rank 1 and (8.3) is the transformation law for a covariant tensor of rank 1. (The rank of a tensor is just the number of indicies it carries, i.e., $T^{\ell r}$ is a second rank tensor, a vector is a first rank tensor, etc.) This leads us to the

---

*For present we assume that all coordinate transformations are single valued, continuous, differentiable, and have non-vanishing Jacobian (which implies the inverse exists).

following definitions: 1) A set of quantities A^{ℓ} are said to form a contravariant vector if their transformation law is

$$A'^{\ell}(x') = \frac{\partial x'^{\ell}}{\partial x^{r}} A^{r}(x) \tag{8.4}$$

2) A set of quantities B_{ℓ} are said to form a covariant vector if their transformation law is

$$B'_{\ell}(x') = \frac{\partial x^{r}}{\partial x'^{\ell}} B_{r}(x). \tag{8.5}$$

For tensor transformations the "group property" holds. That is if we make two transformations $x \to x' \to x''$ then the transformation is the same as $x \to x''$. We shall show this in detail.

We have

$$x \to x' \qquad A'^{\ell} = \frac{\partial x'^{\ell}}{\partial x^{r}} A^{r}$$

$$x' \to x'' \qquad A''^{s} = \frac{\partial x''^{s}}{\partial x'^{p}} A'^{p},$$

therefore

$$A''^{s} = \frac{\partial x''^{s}}{\partial x'^{p}} \frac{\partial x'^{p}}{\partial x^{r}} A^{r},$$

and since

$$\frac{\partial x''^{s}}{\partial x'^{p}} \frac{\partial x'^{p}}{\partial x^{r}} = \frac{\partial x''^{s}}{\partial x^{r}} ,$$

then

$$A''^{s} = \frac{\partial x''^{s}}{\partial x^{r}} A^{r}, \tag{8.6}$$

which is the transformation $x \to x''$. Thus, the tensor transformation law implies the group property

$$f_{(T_2 T_1)} = f_{T_2} f_{T_1} .$$

The generalization of the transformation laws to higher rank tensors is to treat each covariant index as in (8.5) and each contravariant index as in (8.4). Thus, if $T^{rs\,\cdots\cdots}_{\cdot\cdot pq\cdots\cdot}$ is a r^{th} rank tensor it transforms under the law

$$T'^{rs\,\cdots\cdot}_{pq\,\cdots\cdot} = \underbrace{\frac{\partial x'^{s}}{\partial x^{\ell}} \frac{\partial x'^{s}}{\partial x^{m}} \cdots\cdots}_{r_1 \text{ factors}} \underbrace{\frac{\partial x^{n}}{\partial x'^{p}} \frac{\partial x^{i}}{\partial x'^{q}} \cdots\cdots}_{r_2 \text{ factors}} T^{\ell m\,\cdots\cdot}_{ni\,\cdots\cdot} \tag{8.7}$$

here r_1 is the number of contravariant indicies and r_2 is the number of covariant indicies, clearly $r_1 + r_2 = r$. The usefulness of tensors follows from the following fact: If a tensor equation holds in one coordinate system then it holds in all coordinate systems. This is because the transformation law for a tensor $T^{\cdots\cdot}_{\cdots\cdot}$ is linear and homogeneous in $T^{\cdots\cdot}_{\cdots\cdot}$. If a tensor equation is given in one frame

$$T^{r\,\cdots\cdot}_{s\,\cdots\cdot} = C^{\cdots r\cdot\cdot}_{\cdots s\cdot\cdot}$$

then it automatically follows that in another frame

$$T'^{r\,\cdots\cdot}_{s\,\cdots\cdot} = C'^{\cdots r\cdot\cdot}_{\cdots s\cdot\cdot} .$$

B. Algebra of Tensors

Given various tensors $T^{s\,\cdots\cdot}_{r\,\cdots\cdot}$, $S^{s\,\cdots\cdot}_{r\,\cdots\cdot}$, and $U^{s\,\cdots\cdot}_{r\,\cdots\cdot}$ one can define several algebraic combinations:

1) The sum of two tensors (having the same number of co- and contra-variant indicies) is a tensor

$$T^{s}_{r}\,{\cdots\atop\cdots} + S^{s}_{r}\,{\cdots\atop\cdots} = U^{s}_{r}\,{\cdots\atop\cdots}$$

2) The multiplication of two tensors is again a tensor

$$T^{s}_{r}\,{\cdots\atop\cdots}\; S^{p}_{q}\,{\cdots\atop\cdots} = U^{sp}_{rq}\,{\cdots\atop\cdots}$$

3) One may take the trace on any pair of one upper and one lower indicies thus reducing the rank of the tensor by 2, this process is called contraction.

$$T^{s}_{r}\,{\cdots\atop\cdots}\;\; r^{\underline{th}} \text{ rank tensor.} \xrightarrow[\substack{\text{contraction}\\ r\,=\,s}]{} T^{s}_{s}\,{\cdots\atop\cdots}\;\; (r-2)^{\underline{th}} \text{ rank tensor} \;.$$

Notice that there must be one upper and one lower index, a symbol such as T^{rr} or T_{rr} is not defined.

4) One may contract indicies across a tensor product. This process is called transvection.

$$T^{r}_{s}\,{\cdots\atop\cdots}\; S^{p}_{q}\,{\cdots\atop\cdots} \xrightarrow[\text{Transvection}]{} T^{r}_{s}\,{\cdots\atop\cdots}\; S^{p}_{r}\,{\cdots\atop\cdots}$$

5) Alternation (antisymmetrization) over indicies is defined as

$$T_{[i|j|k]} = \frac{1}{2!}\left[T_{ijk} - T_{kji}\right] ,$$

where the indicies set off by bars | are not acted on by the [] operator.

6) Mixing (symmetrization over indicies is defined as

$$T_{(i|j|k)} = \frac{1}{2!}\left[T_{ijk} + T_{kji}\right] .$$

Any second rank tensor T_{ij} can be written

$$T_{ij} = T_{(ij)} + T_{[ij]} .$$

A tensor is called symmetric if $T_{[ij]} = 0$ and antisymmetric if $T_{(ij)} = 0$. If one is given a set of functions T_{ij} then one can use the quotient rule to test for the tensorial character of the functions. The quotient rule is: If a set of functions T_{ij} when combined by a given type of multiplication with all tensors of a given type again yields a tensor, then the functions T_{ij} are tensors. The proof is elementary and can be found in any text on tensor analysis. (For example, Schaum's outline, Vector Analysis) In particular if $A_{k\;\cdots}^{\;\cdots\cdots} B^k = 0$ for all vectors B^k then $A_{k\;\cdots}^{\;\cdots\cdots} = 0$.

C. Differentiation of ensors

For general coordinate transformations $x'^{\ell} = f^{\ell}(x)$ the sum of tensors located at different points is not a tensor, however, for linear transformations the sum of a number of tensors located at different points is again a tensor. That is, let $A^{\ell}(x_1)$ and $A^{\ell}(x_2)$ stand for A^{ℓ} evaluated at two different points x_1 and x_2 then these transform as

$$A'^{\ell}(x'_1) = \left(\frac{\partial x'^{\ell}}{\partial x^r}\right)_{x_1} A^{\ell}(x_1)$$

$$A'^{\ell}(x'_2) = \left(\frac{\partial x'^{\ell}}{\partial x^r}\right)_{x_2} A^{\ell}(x_2) .$$

Adding these yields

$$A'^{\ell}(x'_1) + A'^{\ell}(x'_2) = \left(\frac{\partial x'^{\ell}}{\partial x^r}\right)_{x_1} A^{\ell}(x_1) + \left(\frac{\partial x'^{\ell}}{\partial x^r}\right)_{x_2} A^{\ell}(x_2), \tag{8.8}$$

$\therefore$ under linear transformations we have

$$\frac{\partial x'^{\ell}}{\partial x^r} = \text{const.} = a^{\ell}_r = A'^{\ell}(x'_1) + A^{\ell}(x'_2) = a^{\ell}_r\left(A^{\ell}(x_1) + A^{\ell}(x_2)\right)$$

or the sum transforms as a vector only under linear transformations. Because of (8.8) the derivative of a tensor of rank ≥ 1 is not a tensor under general coordinate transformations because it is constructed from tensor evaluated at two different points. Consider the derivative of a vector $A^{\ell}(x)$. How does dA^{ℓ} transform under coordinate transformations? Since

$$A'^{\ell}(x') = \partial_r x'^{\ell} A^r \qquad \partial_r = \frac{\partial}{\partial x^r}$$

we have

$$dA'^{\ell} = \partial_r x'^{\ell} dA^r + A^r(\partial_s \partial_r x'^{\ell}) dx^s , \tag{8.9}$$

thus only under linear transformations will dA^{ℓ} transform as a vector. The reason for this is clear, dA^r represents the difference of two vectors located at different points

$$dA^{\ell} = A^{\ell}(x + dx) - A^{\ell}(x). \tag{8.10}$$

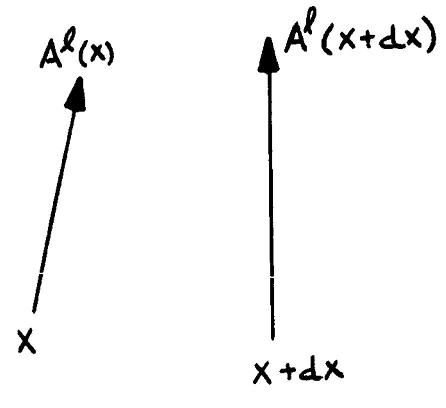


Fig. 8.1

What one needs is to replace (8.10) by the difference of vectors located at the same point. What is done is to carry $A^\ell(x)$ to $x + dx$ in a definite manner. The carrying of A^ℓ to $x + dx$ is by parallel propagation of A^ℓ from x to $x + dx$. (We shall define parallel propagation precisely later.) We write the parallel propagated vector in the form

$$A^\ell_{11}\ (x + dx).$$

The difference between the vectors

$$DA^\ell = A^\ell(x + dx) - A^\ell_{11}(x + dx) \tag{8.11}$$

is now a vector and is called the covariant differential of A^ℓ. Rewriting (8.11)

$$DA^\ell = A^\ell(x + dx) - A^\ell(x) - \left(A^\ell_{11}(x + dx) - A^\ell(x)\right)$$

or

$$DA^\ell = dA^\ell - d_{11}\ A^\ell\ . \tag{8.12}$$

The most general form for $d_{11}\ A^\ell$ (which is the change in A^ℓ under parallel propagation) is

$$d_{11}\ A^\ell = -\ L^\ell_{rs}\ A^r\ dx^s \tag{8.13}$$

where the $L^\ell_{rs}(x)$ is called the affine connection. (Sometimes L^ℓ_{rs} is called the linear connection or coefficient of affine displacement.) The covariant differential of A^ℓ can now be written:

$$DA^\ell = dA^\ell + L^\ell_{rs}\ A^r\ dx^s. \tag{8.14}$$

If A^ℓ is defined along a curve $x^s(\lambda)$ in the space, then

$$\frac{DA^\ell}{d\lambda} = \frac{dA^\ell}{d\lambda} + L^\ell_{rs}\ A^r\ \frac{dx^s}{d\lambda} \tag{8.15}$$

is called the absolute derivative of A^ℓ along the given curve. The fact that the absolute derivative (or covariant differential) of a vector (or tensor) is a tensor yields the transformation law for the affine connection using (8.14) and (8.9) one finds the connection transforms as:

$$L'^{\ell}_{rs}(x') = \frac{\partial x'^{\ell}}{\partial x^{p}} \frac{\partial x^{m}}{\partial x'^{r}} \frac{\partial x^{n}}{\partial x'^{s}} L^{p}_{mn}(x) - \left(\frac{\partial x^{m}}{\partial x'^{r}}\right)\left(\frac{\partial x^{n}}{\partial x'^{s}}\right) \frac{\partial^{2} x'^{\ell}}{\partial x^{m} \partial x^{n}} \quad . \tag{8.16}$$

Therefore, L^p_{rs} is not a tensor in general. In a like manner the covariant differential and absolute derivative of a covariant vector A_ℓ are given by

$$\begin{aligned} DA_\ell &= dA_\ell - L^p_{\ell r} A_p dx^r \\ \frac{DA_\ell}{d\lambda} &= \frac{dA_\ell}{d\lambda} - L^p_{\ell r} A_p \frac{dx^r}{d\lambda} \quad . \end{aligned} \tag{8.17}$$

Of course at this stage the affine connection for A^ℓ and A_ℓ does not have to be the same. It is easily verified that (8.17) are vectors. One generalizes these definitions to an arbitrary tensor $T^{k \dots\dots}_{\ell \dots\dots}$, where each index is treated as if it were a vector. One has

$$DT^{k \dots\dots}_{\ell \dots\dots} = dT^{k \dots\dots}_{\ell \dots\dots} + L^k_{rs} T^{r \dots\dots}_{\ell \dots\dots} dx^s - L^r_{\ell s} T^{k \dots\dots}_{r \dots\dots} dx^s + \dots\dots \tag{8.18}$$

Again one may prove that the covariant differential of a tensor is a tensor. For the case of a tensor field one may define the covariant derivative of a tensor with respect to x^r as $T^{k \dots\dots}_{\ell \dots\dots ;r}$ which is given by

$$DT^{k \dots\dots}_{\ell \dots\dots} = T^{k \dots\dots}_{\ell \dots\dots ;r} dx^r . \tag{8.19}$$

Let us summarize the various derivatives discussed:

1) Covariant differential

$$DT^{k\,\cdots}_{\ell\,\cdots} = dT^{k\,\cdots}_{\ell\,\cdots} + L^{k}_{rs}\, T^{r\,\cdots}_{\ell\,\cdots}\, dx^{s} - L^{r}_{\ell s}\, T^{k\,\cdots}_{r\,\cdots}\, dx^{s} \cdots$$

2) Absolute derivative along a curve $x^{\ell}(\lambda)$

$$\frac{DT^{k\,\cdots}_{\ell\,\cdots}}{d\lambda}$$

3) Covariant derivative

$$DT^{k\,\cdots}_{\ell\,\cdots} = T^{k}_{\ell;s}\, dx^{s} \,.$$

A tensor is said to be parallel propagated along a curve $x^{\ell}(\lambda)$ if

$$\frac{DT^{k\,\cdots}_{\ell\,\cdots}}{d\lambda} = 0 \tag{8.20}$$

is satisfied along the curve. For a vector A^{ℓ} this yields $DA^{\ell} = 0 \Longrightarrow dA^{\ell} = -L^{\ell}_{rs} A^{r} dx^{s} = d_{11}A^{\ell}$ which is the same as (8.13).

From the transformation properties for the linear connection one can prove the following facts: Let L^{ℓ}_{rs}, $L^{*\ell}_{rs}$ be two affine connections, then the sum

$$L^{\ell}_{rs} = \tfrac{1}{2}\left(L^{\ell}_{rs} + L^{*\ell}_{rs}\right)$$

in an affine connection and the difference

$$D^{\ell}_{rs} = L^{\ell}_{rs} - L^{*\ell}_{rs}$$

is a tensor. Let us specalize these results to a given affine connection. If L^{ℓ}_{rs} is an affine connection then L^{ℓ}_{sr} is also an affine connection and L^{ℓ}_{rs} can be decomposed as

$$L^{\ell}_{rs} = L^{\ell}_{(rs)} + L^{\ell}_{[rs]}.$$

Here $L^{\ell}_{(rs)} = \frac{1}{2}\left(L^{\ell}_{(rs)} + L^{\ell}_{sr}\right)$ is a symmetric affine connection and $L^{\ell}_{[rs]} = \frac{1}{2}\left(L^{\ell}_{rs} - L^{\ell}_{sr}\right) = \frac{1}{2}T_{rs}{}^{\ell}$ defines the torsion tnesor $T_{rs}{}^{\ell}$. In what follows we shall be interested primarily in spaces described by Riemannian geometry. Such a space is called a Riemannian space and is denoted by V_4. A Riemannian space is obtained from the general structure discussed so far by admitting a second rank symmetric tensor $g_{ij} = g_{ji}$ into the space. This tensor is called the metric tensor. In V_4 we may define a scalar product between pairs of vectors using the metric tensor

$$g_{ij}\, A^i\, B^j = \text{scalar}. \tag{8.21}$$

We now show that under several assumptions a unique linear connection is defined in V_4. First the concept of raising and lowering of indicies is explained. Instead of writing the scalar produce as (8.21) one can define a covariant vector associated with A^i or B^i by the relation

$$A_j = g_{ij}\, A^i \tag{8.22}$$

(the analogous result for Minkowski space in equation (5.10))
then (8.21) becomes

$$A_i\, B^i = \text{scalar}.$$

A_i and B^i are called associated vectors. Under the following assumptions there is a unique symmetric affine connection in a Riemannian space:

1) DA^{ℓ} and DA_{ℓ} are associated vectors
2) the torsion $T_{rs}{}^{\ell}$ is zero.

We first calculate the covariant derivative of the metric tensor and permute the symbols twice

$$\left.\begin{aligned} g_{ij;k} &= g_{ij,k} - L^{\ell}_{ik}\, g_{\ell j} - L^{\ell}_{jk}\, g_{i\ell} \\ g_{ki;j} &= g_{ki,j} - L^{\ell}_{kj}\, g_{\ell i} - L^{\ell}_{ij}\, g_{k\ell} \\ -g_{jk;i} &= -g_{jk,i} + L^{\ell}_{ji}\, g_{\ell k} + L^{\ell}_{ki}\, g_{j\ell} \end{aligned}\right\} \tag{8.23}$$

Adding the three equations yields

$$2[jk,i] = 2\, L^{\ell}_{(jk)} g_{\ell i} - T^{\ell}_{ki}\, g_{j\ell} - T_{ji}{}^{\ell}\, g_{\ell k} - g_{ij;k} - g_{ki;j} + g_{jk;i}\ . \tag{8.24}$$

Where the symbol

$$[jk,i] = \tfrac{1}{2}\left(g_{ij,k} + g_{ki,j} - g_{jk,i}\right) \tag{8.25}$$

is called the Christoffel symbol of the first kind and the other symbols have been defined. We show that assumption 1) implies $g_{\ell k;i} = 0$. The covariant differential of a scalar is the same as the ordinary differential because it is already a geometrical object (i.e., a covariant vector),

$$D(A_\ell B^\ell) = d(A_\ell B^\ell) = dA_\ell B^\ell + A_\ell dB^\ell$$

writing this product out one finds

$$D(A_\ell B^\ell) = DA_\ell B^\ell + A_\ell DB^\ell .$$

Applying the same analysis to the form $(g_{\ell k} A^\ell B^k)$ yields

$$\begin{aligned} D(A_\ell B^\ell) = {} & dg_{\ell k}\, A^\ell B^k + g_{\ell k} A^\ell DB^k - g_{\ell k} A^\ell L^k_{rs} B^r\, dx^s \\ & + g_{\ell k} B^\ell DA^k - g_{\ell k} B^k L^\ell_{rs}\, A^r\, dx^s . \end{aligned}$$

Under assumption 1) we can write $g_{\ell k} DA^k = DA_\ell$ therefore

$$D(A_\ell B^\ell) = D(A_\ell B^\ell) + \left(g_{\ell k,s} - L^p_{ks} g_{\ell p} - L^p_{\ell s} g_{pk} \quad A^\ell B^k \; dx^s\right)$$

or
$$\left(g_{\ell k,s} - L^p_{ks} g_{\ell p} - L^p_{\ell s} g_{pk}\right) A^\ell B^k \; dx^s = 0$$

by three applications of the quotient theorem we deduce that

$$g_{\ell k;s} = g_{\ell k,s} - L^p_{ks} g_{\ell p} - L^p_{\ell s} g_{pk} = 0 \ . \tag{8.26}$$

Using (8.26) and the fact that the torsion vanishes, we have

$$[jk,i] = g_{\ell i} \; L^\ell_{jk}$$

and

$$L^\ell_{jk} = g^{\ell i} \; [jk,i] = \Gamma^\ell_{jk} \tag{8.27}$$

where Γ^ℓ_{jk} is called the Christoffel symbol of the second kind.

In general relativity one usually considers the Christoffel symbols to be equal to the connection (8.27); however, this is not necessary, a fact which Einstein used to construct some of his "unified field theories." In this work we deal only with general relativity in the usual form, we do not consider any connection except (8.27).

B. Geodesics

In Euclidean geometry one makes frequent use of straight lines, while in spherical geometry one uses great circles. Both of these curves are geodesics of the geometry they are used in. In both of these examples one knows that these geodesics are the curves of shortest length connecting two given points in the space. Again in Euclidean geometry if one considers an arbitrary curve C, then its tangent vector has an arbitrary direction as one proceeds along the curve. However, again the straight lines have the property that their tangent vectors from point to point are parallel. This leads us to the abstract definition of a geodesic curve. A geodesic curve is a curve whose tangent vector $t^{\ell} = \frac{dx^{\ell}}{d\lambda}$ undergoes parallel propagation along the curve. That is,

$$\frac{Dt^{\ell}}{d\lambda} = \frac{dt^{\ell}}{d\lambda} + \Gamma^{\ell}_{rs} t^r t^s = \frac{d^2x^{\ell}}{d\lambda^2} + \Gamma^{\ell}_{rs} \frac{dx^r}{d\lambda} \frac{dx^s}{d\lambda} = 0 . \tag{8.28}$$

Equation (8.28) is a set of four ordinary differential equations, and thus has a unique solution corresponding to the initial conditions $x^{\ell}(\lambda_o)$, $t^{\ell}(\lambda_o)$. If one makes a change of curve parameter, say $\sigma = f(\lambda)$, then, equation (8.28) becomes

$$\frac{d^2x^{\ell}}{d\sigma^2} + \Gamma^{\ell}_{rs} \frac{dx^r}{d\sigma} \frac{dx^s}{d\sigma} = C(\sigma) \frac{dx^{\ell}}{d\sigma}$$

where $C(\sigma)$ is a definite function of σ whose form does not concern us here. If a parameter is chosen such that the equation for the geodesic takes the form (8.28), then the parameter λ is called an affine parameter. (If we have an affine parameter associated with a geodesic, we can still change the scale and the origin $\lambda \rightarrow C_1 \lambda + C_2$ where C_1 and C_2 are constants.) The equation (8.28) has the first integral

$$g_{\ell k} \frac{dx^{\ell}}{d\lambda} \frac{dx^k}{d\lambda} = \text{const.} \tag{8.29}$$

When the scalar product is indefinite, one finds three classifications, i.e., const. > 0; = 0; or < 0. In the former and latter case λ can be normalized so that the const. = +1, -1. For this choice the parameter λ is related to the measure s, $dx^2 = g_{\ell k}\, dx^\ell\, dx^k$, in a simple way:

$$
\begin{array}{llll}
g_{\ell k} \dfrac{dx^\ell}{ds} \dfrac{dx^k}{ds} = +1 & \text{Timelike geodesic} & \lambda = s & \\
g_{\ell k} \dfrac{dx^\ell}{d\lambda} \dfrac{dx^k}{d\lambda} = 0 & \text{Lightlike geodesic} & \lambda \neq s & (8.30) \\
g_{\ell k} \dfrac{dx^\ell}{ds} \dfrac{dx^k}{ds} = -1 & \text{Spacelike geodesic} & \lambda = is & \\
 & & i = \sqrt{-1}\ . &
\end{array}
$$

It will be noticed that a given geodesic is either timelike, spacelike, or lightlike; it cannot change from one type to another in a continuous manner. Through a given point of V_4, there are many geodesics (i.e., tangent vectors) however, connecting two neighboring (in the sense of topological closeness) points in a single geodesic as is the case for the geometry of surfaces imbedded in Euclidean space. A property that these curves on the surface have is that they represent the curve of shortest length connecting these points. The geodesics (8.28) also have the property that the equation describing them can be derived from an action principle. Consider the integral

$$I = \int_{\mathcal{E}_1}^{\mathcal{E}_2} L(x,\dot{x})d\lambda = \int_{\mathcal{E}_1}^{\mathcal{E}_2} g_{\ell k} \frac{dx^\ell}{d\lambda} \frac{dx^k}{d\lambda}\, d\lambda .$$

If the action I between $\mathcal{E}_1$ and $\mathcal{E}_2$ is an extremum then

$$\delta \int_{\mathcal{E}_1}^{\mathcal{E}_2} L d\lambda = 0$$

for variations of the curve $x^{\ell}(\lambda)$ which vanish at the end points we have the necessary conditions for I to be an extremum: (Euler-Lagrange equations)

$$\frac{\partial L}{\partial x^{\ell}} - \frac{d}{d\lambda}\frac{\partial L}{\partial \frac{dx^{\ell}}{d\lambda}} = 0 \longrightarrow \frac{Dt^{\ell}}{d\lambda} = 0.$$

Therefore, the curves which extremize I are geodesics.

We can now give a geometrical interpretation of the parallel propagation of a vector A^{ℓ}(or tensor) from a point x^{ℓ} to $x^{\ell} + dx^{\ell}$ used to define the covariant differential (Eq. 9.11). One constructs the geodesic from x to $x + dx$; then carries the vector along this geodesic, always keeping the same "angle" between the vector A and the tangent vector t . This angle is given by

$$\theta = g_{\ell k} A^{\ell} t^{k}.$$

Taking the absolute derivative of θ yields: $\frac{D\theta}{d\lambda} = 0$ by construction and since $\frac{Dg_{\ell k}}{d\lambda} = 0$ and $\frac{Dt^{\ell}}{d\lambda} = 0$ we have

$$DA^{\ell} = 0 \Longrightarrow dA^{\ell} = d_{11}A^{\ell}.$$

This construction is shown in (Fig. 8.2)

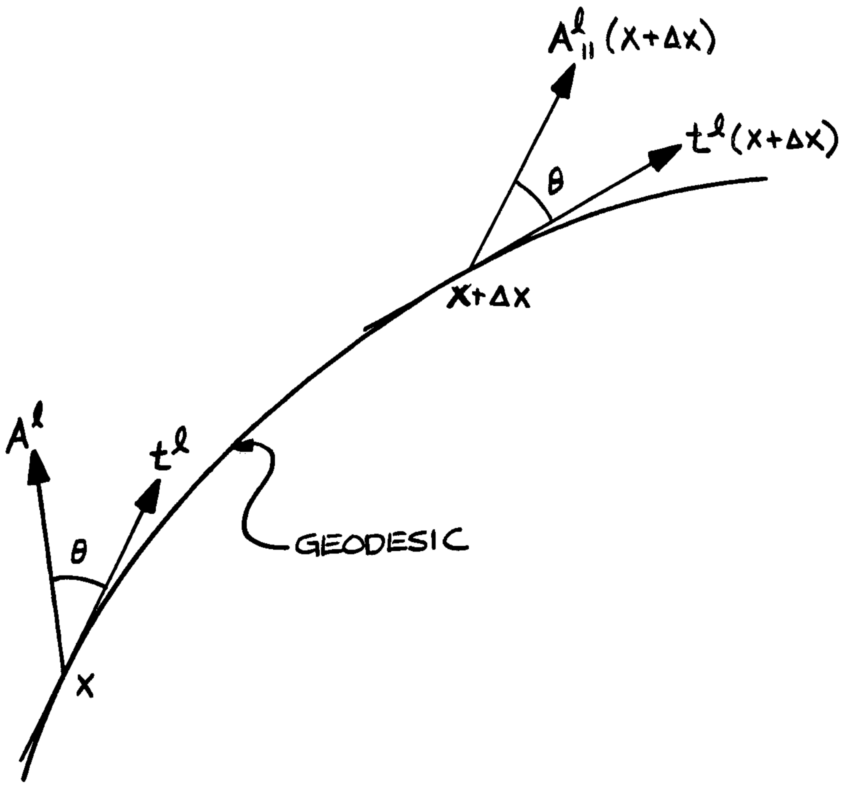


Fig. 8.2

We shall continue in the next chapter with more tensor analysis and an introduction to general relativity.

Chapter 9

A. Special Coordinate Systems

In Euclidean geometry one makes extensive use of Cartesian coordinates. Oblique coordinates could be used in Euclidean geometry but they would unduly complicate almost all the calculations carried out. Hence, in Euclidean geometry there exists a set of special coordinates which make the work easier. (One might say that Cartesian coordinates are "closer" to Euclidean geometry.) In a Riemannian space there also exists special coordinates systems in which some of the calculations are simplified.

The first set of special coordinates we consider are called (local) Cartesian coordinates. These arise in the following manner. At any point P the measure

$$ds^2 = g_{\ell k}(P)\ dx^{\ell}\ dx^{k} \tag{9.1}$$

is a quadratic form with constant coefficients $g_{\ell k}(P)$. At P the transformation to new variables dx'^{ℓ}

$$dx'^{\ell} = \frac{\partial x'^{\ell}}{\partial x^{r}}(P)\ dx^{r} \tag{9.2}$$

is a real linear transformation. From algebra one knows that a quadratic form can always be diagonalized by a linear transformation. After diagonalization one can rewrite (9.1) in the form:

$$ds^2 = \epsilon_1\left(dx^1\right)^2 + \epsilon_2\left(dx^2\right)^2 + \epsilon_3\left(dx^3\right)^2 + \epsilon_4\left(dx^4\right)^2 . \tag{9.3}$$

where the ϵi's are either +1 or -1. Such coordinates x^i are called Cartesian at the given point. The next special coordinates we consider are called Riemannian coordinates. Let x^r be a general coordinate system such that $P(x^r = 0)$ is the origin. Consider the family of geodesics emanating form P:

$$\frac{d^2x^\ell}{d\lambda^2} + \Gamma^\ell_{ks}\,\frac{dx^k}{d\lambda}\,\frac{dx^s}{d\lambda} = 0 \ . \qquad (9.4)$$

Let $t^\ell = \left(\frac{dx^\ell}{d\lambda}\right)_{x^\ell=0}$ be the tangent vectors of these geodesics. For points close to P we can write $x^r(\lambda)$ in a Taylor series about $\lambda=0$.

$$x^\ell(\lambda) = x^\ell(0) + \left(\frac{dx^\ell}{d\lambda}\right)_0 \lambda + \left(\frac{d^2x^\ell}{d\lambda^2}\right)_0 \frac{\lambda^2}{2} + \ldots. \qquad (9.5)$$

or using (9.4)

$$x^\ell = t^\ell\lambda - {}^1/_2\left(\Gamma^\ell_{ks}\right)_0 t^k\, t^s\, \lambda^2 + + \ldots. \qquad (9.6)$$

The Riemannian coordinates of points close to P are defined to be

$$x'^\ell = t^\ell\lambda \quad . \qquad (9.7)$$

Hence, (9.6) represents the coordinate transformation connecting the general coordinates x^ℓ with the Riemannian coordinates x'^ℓ:

$$\left.\begin{aligned} x^\ell &= x'^\ell - {}^1/_2\left(\Gamma^\ell_{ks}\right)_0 x'^k\, x'^s + \ldots. \\ x^\ell &= g^\ell(x') \ . \end{aligned}\right\} \qquad (9.8)$$

The inverse transformation $x'^\ell = f^\ell(x)$ connecting the Riemannian coordinates and the general coordinates x^ℓ, can be obtained by inverting the series (9.8). Carrying out the inversion one finds:

$$x'^\ell = x^\ell + {}^1/_2\left(\Gamma^\ell_{ks}\right)_0 x^k\, x^s + \ldots. \qquad (9.9)$$

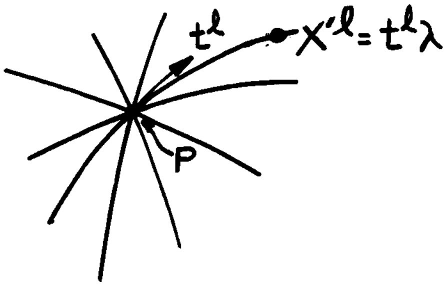


Fig. 9.1

Riemannian Coordinates

From (9.8) and (9.9) follows

$$\left(\frac{\partial x'^{\ell}}{\partial x^{r}}\right)_0 = \delta^{\ell}_{r} \quad , \quad \left(\frac{\partial x^{r}}{\partial x'^{\ell}}\right)_0 = \delta^{r}_{\ell} \tag{9.10}$$

These formulae (9.10) imply that the value of any tensor at the origin P is unchanged by the coordinate transformation (9.8). Also one finds from (9.8) and (9.9) that

$$\frac{\partial^2 x'^{\ell}}{\partial x^{r} \partial x^{s}} = \left(\Gamma^{\ell}_{rs}\right)_0$$

hence from (8.16) follows

$$\left(\Gamma'^{\ell}_{rs}\right)_0 = 0 \ . \tag{9.11}$$

Equation (9.11) says that at the origin of a Riemannian coordinate system the Christoffel symbols vanish. Since

$$g_{ik,\ell} = g_{mk}\,\Gamma^{m}_{i\ell} + g_{im}\,\Gamma^{m}_{k\ell}$$

it also follows from (9.11) that the first derivative of the metric tensor vanishes at the origin of a Riemannian coordinate system

$$\left(g_{ik,\ell}\right)_0 = 0 \quad . \tag{9.12}$$

Since the point P was a general point of our space it follows that at any point we can choose coordinates (Riemannian) such that

$$\left(g_{\ell k,r}\right)_0 = 0 \quad . \tag{9.13}$$

At the origin of Riemannian coordinates covariant differentiation is the same as ordinary differentiation. This can be used to prove (9.13) because the covariant derivative of $g_{\ell k}$ vanishes in all coordinate systems.

Combining the two types of coordinates discussed so far 1) Cartesian, 2) Riemannian, we can first transform to a coordinate system in which the metric tensor is diagonal at the point P (9.3) then we can transform from that set of coordinates to Riemannian coordinates at P. Since the value of any tensor is unaltered at the point P for this last transformation we have at the origin of our Cartesian-Riemannian coordinate system

$$\left(g_{\ell k,r}\right)_0 = 0 \quad , \quad \left(g_{\ell k}\right)_0 = \pm\, \delta_{\ell k} \quad . \tag{9.14}$$

Such coordinates are called geodesic at P. We shall make frequent use of geodesic coordinates in the future; if we have a tensor equation that we are trying to verify then if we can verify it in a geodesic coordinate system (9.14), then it holds in all coordinate systems.

B. The Curvature Tensor

The operators for ordinary differentiation $\frac{\partial}{\partial x^r}$, $\frac{\partial}{\partial x^s}$ commute with one another

$$\frac{\partial^2 A_\ell}{\partial x^s \partial x^r} - \frac{\partial^2 A_\ell}{\partial x^r \partial x^s} = 0 ,$$

however, covariant differentiations of a vector do not commute. One can prove the identity

$$A_{j[;k;\ell]} = {}^1/_2\, A_i\, R^i_{\cdot\ jk\ell}(x) \tag{9.15}$$

(We shall not write in the position dependence of $R^i{}_{jk\ell}$ usually) where

$$R^i_{\cdot\ jk\ell} = \Gamma^i_{j\ell ,k} - \Gamma^i_{jk,\ell} + \Gamma^r_{j\ell}\, \Gamma^i_{rk} - \Gamma^r_{jk}\, \Gamma^i_{r\ell} \tag{9.16}$$

by straight forward differentiation. By using the quotient rule in (9.15) one can show that $R^i_{\cdot\ jk\ell}$ is a forth rank tensor. $R^i{}_{jk\ell}$ is called the Riemann curvature tensor. By inspection we see that $R^i{}_{jk\ell}$ is antisymmetric in its last two indices.

$$R^i_{\cdot jk\ell} = R^i_{\cdot j[k\ell]} \ . \tag{9.17}$$

Equation (9.15) can be written

$$A_{j;k;\ell} - A_{j;\ell;k} = A_i\, R^i_{\cdot jk\ell} \ . \tag{9.18}$$

Permuting the indices of equation (9.18) twice and adding yields

$$\left(A_{j;k} - A_{k;j}\right)_{;\ell} + \left(A_{k;\ell} - A_{\ell;k}\right)_{;j} + \left(A_{\ell;j} - A_{j;\ell}\right)_{;k}$$

$$= A_i\left(R^i_{\cdot jk\ell} + R^i{}_{k\ell i} + R^i{}_{\ell jk}\right) \ . \tag{9.19}$$

Making use of the two identities

$$\left.\begin{aligned} V_{i;j} - V_{j;i} &= V_{i,j} - V_{j,i} \\ V_{ij;k} + V_{ki;j} + V_{jk;i} &= V_{ij,k} + V_{ki,j} + V_{jk,i} \end{aligned}\right\} \tag{9.20}$$

which hold for any vector $V_i (= A_i)$ and any second rank antisymmetric tensor V_{ij} $(= A_{i;j} - A_{j;i})$, equation (9.19) becomes

$$A_i \left(R^i_{\cdot\ jk\ell} + R^i_{\cdot k\ell j} + R^i_{\cdot\ \ell jk} \right) = 0 \ .$$

Since A_i is an arbitrary vector by the quotient theorem we have

$$R^i_{\cdot jk\ell} + R^i_{\cdot k\ell j} + R^i_{\ell jk} = 0$$

which can be written

$$R^i_{[jk\ell]} = 0 \ . \tag{9.21}$$

The identities of the Riemann tensor derived so far hold in a more general space if we replace Γ^ℓ_{kr} by $L^\ell_{(kr)}$, a general symmetric affine connection. When we introduce the metric we have one other identity

$$g_{ij;k;\ell} - g_{ij;\ell;k} = 0 = g_{nj} R^n_{\cdot\ ik\ell} + g_{ni}\ R^n_{\ jk\ell}$$

or

$$R_{ijk\ell} = R_{[ij]k\ell} \ . \tag{9.22}$$

Hence, in a Riemannian space the Riemann tensor is antisymmetric in its first two indices. Using (9.17), (9.21), and (9.22) one can prove the relation

$$R_{ijk\ell} = R_{k\ell ij} \ . \tag{9.23}$$

The equations

$$\left.\begin{array}{l} R_{ijk\ell} = R_{ij[k\ell]} \ ; \ R_{i[jk\ell]} = 0 \\ \\ R_{ijk\ell} = R_{[ij]k\ell} \ \ R_{ijk\ell} = R_{k\ell ij} \end{array}\right\} \tag{9.24}$$

which are the basic symmetry equations for the Riemann tensor, can be summarized in equations

$$R_{ijk\ell} = R_{[ij][k\ell]} \tag{9.25a}$$

$$R_{i[jk\ell]} = 0 \quad . \tag{9.25b}$$

A fourth rank tensor has in general n^4 independent components in a n dimensional space. However, because of (9.25a) the number of independent components of the Riemann tensor is reduced to $\left(1/2\ n(n-1)\right)^2$. The indices in (9.25b) can be selected in the following ways:

$$\begin{array}{cccc} i & j & k & \ell \\ n & (n-1) & n & (n-2) \end{array}$$

. Because permutation of the last three symbols does not change (9.25b), the total number of restrictions it places on the $\left(1/2\ n(n-1)\right)^2$ components is $1/6\ n(n-1)\ n(n-2)$. Therefore we have

$$\left(1/2\ n(n-1)\right)^2 - 1/6\ n^2(n-1)(n-2) = 1/12\ n^2(n^2-1)$$

independent components of the Riemann tensor at every point of our space. In four dimensions this is twenty. From the Riemann tensor $R^{i}_{\cdot\ jk\ell}$ one can form other quantities by contraction; the once contracted Riemann tensor R_{jk}

$$R_{jk} = R^{i}_{\cdot\ jki} = g^{in}\ R_{njki} \tag{9.26}$$

is called the Ricci tensor. It easily can be seen that R_{jk} is symmetric. The twice contracted Riemann tensor is called the scalar curvature

$$R^{ij}{}_{ji} = R^{i}{}_{i} = R \; . \tag{9.27}$$

We now derive a very important differential identity satisfied by the Riemann tensor. The Bianchi identities:

Rewriting (9.15)

$$A_{j;k;\ell} - A_{j;\ell;k} = A_i \, R^{i}_{\cdot\; jk\ell}$$

and taking the covariant derivative of this with respect to m yields

$$A_{j;k;\ell;m} - A_{j;\ell;k;m} = A_{i;m} \, R^{i}_{\cdot\; jk\ell} + A_i \, R^{i}_{\cdot\; jk\ell;m} \tag{9.28}$$

One can also show in the same manner as (9.15) was proven that

$$A_{j;m;k;\ell} - A_{j;m;\ell;k} = A_{i;m} \, R^{i}_{\cdot\; jk\ell} + A_{j;i} \, R^{i}_{\cdot\; mk\ell} \; . \tag{9.29}$$

Subtracting (9.28) and (9.29) yields

$$\begin{aligned} A_{j;k;\ell;m} - A_{j;m;k;\ell} + A_{j;m;\ell;k} - A_{j;\ell;k;m} - A_{j;i} \, R^{i}_{\cdot\; mk\ell} \\ = A_i \, R^{i}_{\cdot\; jk\ell;m} \; . \end{aligned} \tag{9.30}$$

If we permute $k\ell m$ twice in this expression and add all three equations we can derive the identity

$$R^{i}_{\cdot\; jk\ell;m} + R^{i}_{\cdot\; jmk;\ell} + R^{i}_{\cdot\; j\ell m;k} = 0 \tag{9.31}$$

which is called the Bianchi identity. Contracting on i, m and j,k we obtain the contracted Bianchi identity:

$$\left(R^{k}_{\ell} - 1/2 \; \delta^{k}_{\ell} \; R\right)_{;k} = 0$$

or

$$G^{k}_{\ell;k} = 0 \tag{9.32}$$

where

$$G^k_\ell = R^k_\ell - 1/2\ \delta^k_\ell\ R \tag{9.33}$$

is called the Einstein tensor.

A space is called flat if the full Riemannian tensor $R_{ijk\ell}$ vanishes, otherwise the space is curved. Minkowski space M_4 is a flat space as is obvious from the fact that the metric tensor g_{ij} is in this case constant (η_{ij}) and hence

$$R_{ijk\ell} = 0 \qquad (M_4 \text{ is flat}) \quad . \tag{9.34}$$

To be sure one may perform coordinate transformations which will induce non-constant metric coefficients, $g_{\ell k}$ from constant metric coefficients $\eta_{\ell k}$:

$$x'^\ell = f^\ell(x)$$

$$g_{rs}(x') = f^\ell_{,r}\ f^k_{,s}\ \eta_{\ell k}$$

but since (9.34) is a tensor equation and it holds in one coordinate system, then it holds in all coordinate systems. Hence, if a space is flat the full Riemann tensor vanishes. The converse statement is also true, namely, that if the full Riemann tensor vanishes then the space is flat and there exists a coordinate system in which the metric coefficients are constants. (See the book <u>Synge and Schild Tensor Calculus</u>, page 105, for a proof of this latter result.)

B. An Introduction to Einstein's Theory of Gravitation; The Principle of Equivalence.

It is an empirical fact that in a gravitational field all particles are affected in the same way or, in other words, in a gravitational field all particles started with the same initial conditions have the same path of motion. (That this is true in Newtonian mechanics is due to the fact that the gravitational mass of a particle is proportional to its inertial mass, as we have seen in Chapter I.) Experimentally it is known to one part in 10^{11} that all particles accelerate equally in a gravitational field. Thus, as far as the motion of a particle is concerned one cannot distinguish between a uniform gravitational field and uniform acceleration. The principle of equivalence goes beyond this by saying there is no difference between the two, i.e., a uniform gravitational field is equivalent to uniform acceleration. In nature one does not find uniform gravitational fields, therefore the principle of equivalence must be interpreted to hold only over a region of space-time which is small enough so that the gravitational field may be treated as uniform. Later we shall give arguments which show that one cannot introduce a theory in M_4 which is in agreement with the principle of equivalence, since any reasonable gravitational theory must incorporate the principle of equivalence; this means that one cannot formulate a theory of gravity in M_4. On the other hand, one knows that there are gravitational phenomena. The way out of this dilemma is to give up M_4 and in its place introduce V_4. That is, we assume that space-time is a Riemannian space V_4 with metric $g_{\ell k}$. The principle of equivalence can now be stated in more precise form as follows: In any region of a gravitational field (V_4) one can choose coordinates such that this region is equivalent, up to first

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*For a description of the Eötvös experiment, see the book Experimental Relativity, by R. H. Dicke (Gordan Breach, 1965).

order in the dimensions of the region, to a small region in M_4. (Riemannian coordinates in V_4.) This means that in any small space-time region of a gravitational field one can choose coordinates that remove the first order effects of the gravitational field; and up to the first order everything like it would be in an inertial frame (M_4). The invariant forms are

$$ds^2 = \eta_{\ell k}\, dx^\ell dx^k \quad : \quad M_4 \qquad\qquad ds^2 = g_{\ell k}\, dx^\ell dx^k \quad : \quad V_4 \; .$$

Let us examine the content of the principle of equivalence in the neighborhood of a point P in V_4. Introducing the Riemannian coordinates about P we can write the metric $g_{\ell k}$ in the form of a Taylor series about $P = x^\ell_{\;o}$.

$$g_{\ell k}(x) = (g_{\ell k})_o + \frac{1}{2}\,(g_{\ell k,r,s})_o\,(x^r - x^r_{\;o})(x^s - x^s_{\;o}) + \ldots . \tag{9.35}$$

where $(g_{\ell k,r})_o = 0$ in Riemannian coordinates. Using the principle of equivalence we see that $(g_{\ell k})_o = \eta_{\ell k}$ and that "small region" means

$$\text{size of region} \approx \left| (x^r - x^r_{\;o})(x^s - x^s_{\;o}) \right| \ll \left| \frac{1}{g_{\ell k,r,s,}(o)} \right| . \tag{9.36}$$

Therefore, locally (small region) in any gravitational field (V_4) one can use special relativity (M_4) and make only second order errors. Furthermore <u>at any point</u> in a gravitational field one can reduce the metric to the form

$$g_{\ell k}(p) = \eta_{\ell k} \; . \tag{9.37}$$

Thus, at the origin of geodesic coordinates in V_4 one has the same metric as in special relativity. Equation (9.37) places a restriction on the metric of a V_4 which describes a gravitational field, that is, at any point the matrix of the metric must have three negative eigenvalues and one positive eigenvalue. This set of signs --- + is called the signature of the space. According to Sylvester's law of inertia the difference between the number of positive coefficients and the number of negative coefficients is invariant for real transformations. In our case this means that a metric $g_{\ell k}$ which doesn't have signature --- + cannot correspond to a real gravitational field.

We can now use the principle of equivalence to derive the equation of motion of a free particle in V_4. We have the equation of a free particle in M_4

$$M_4: \qquad \frac{dV^\ell}{d\lambda} = 0 \tag{9.38}$$

where λ is some parameter along the world line of the particle. In V_4 this equation reads

$$\frac{DV^\ell}{d\lambda} = \frac{dV^\ell}{d\lambda} + \Gamma^\ell_{rs} \frac{dx^r}{d\lambda} \frac{dx^s}{d\lambda} = 0 \ . \tag{9.39}$$

Equation (9.39) is the equation of motion for a particle which is influenced only by gravity. In Einstein's theory of gravitation (which is what we are discussing here) the paths of particles which are acted on by gravity alone are geodesics of the V_4 describing the gravitational field. If another force is acting on the particles (say an electromagnetic force) (9.39) becomes

$$\frac{DV^\ell}{d\lambda} = f^\ell$$

where f^ℓ is the four-force action on the particle. The affect of gravity acting on a particle is to restrict the particle to move along a geodesic. An alternate statement of the principle of equivalence is that locally one can choose a coordinate system such that the first order ($g_{\ell k,i}$) effects of gravity are eliminated (Riemannian coordinates). If one interprets the term

$$-m\,\Gamma^{\ell}_{rs}\,\frac{dx^r}{d\lambda}\,\frac{dx^s}{d\lambda} \quad (m = \text{rest mass}) \tag{9.40}$$

as the "gravitational four-force" acting on a particle, then at any point we can make $\left(\Gamma^{\ell}_{rs}\right)_o = 0$ and eliminate the "gravitational force." Near the earth's surface this frame of reference represents Einstein's freely falling elevator. It is important to realize that removing the "gravitational force" on a particle in this manner is possible only because this "force" is homogeneous in $g_{ij,k}$ which vanishes in the "small region" considered, one does not remove the gravitational field in this small region because of the higher order terms in (9.35). In a similar manner one can usually write a theory which is known in M_4 so that it will be valid in V_4, for example: Maxwell's equations of electrodynamics, the equations of hydrodynamics, the theory of elasticity, etc. Some of these generalizations are well-known (Maxwell's equations) and others are not so well-known (elasticity). One particular example where a theory is known in M_4 but not in V_4 is quantum field theory. We shall be interested in studying these various theories at a later time; however, here we would like to carry through an assertion made earlier concerning the principle of equivalence and field theories in M_4.* Consider a small region of space near a static spherically symmetric gravitating body. The potential at a point r from the center of the body will be

$$\phi(r) = \frac{-kM}{r} \tag{9.41}$$

*A. Schild, Proc. of the International School of Physics, "E. Fermi" Course 20, pp. 69-115, Academic Press, 1963.

in the Newtonian approximation. Consider now the following events; $\mathcal{E}_1$: start with an electron-position pair at $\phi + \Delta\phi$, $\mathcal{E}_2$: the pair is allowed to fall to a potential ϕ, by the principle equivalence both particles accelerate downward and acquire a kinetic energy m $\Delta\phi$, $\mathcal{E}_3$: the pair is allowed to annihilate and produce two photons of energy $h\bar{\nu}$ which are reflected from heavy mirrors to the upper level $\phi + \Delta\phi$ where they have energy $h\nu$, $\mathcal{E}_4$: the photons at the upper level are brought back together so as to form another pair which must be at rest. Using the principle of equivalence we can write:

$$\begin{aligned} 2\,h\bar{\nu} &= 2m + 2m\Delta\phi && \text{at } \phi \\ 2m &= 2h\nu && \text{at } \phi + \Delta\phi\ . \end{aligned} \tag{9.42}$$

(h = Plank's constant)

Solving these equations we obtain

$$\frac{\bar{\nu} - \nu}{\nu} = \Delta\phi\ . \tag{9.43}$$

Therefore, light which moves from a lower potential ϕ to a higher potential $\phi + \Delta\phi$ is "redshifted." Suppose now that monochromatic light of frequency $\bar{\nu}$ is sent from ϕ to $\phi + \Delta\phi$ where it arrives with frequency ν. Since the number of wavelengths sent from ϕ equals the number received at $\phi + \Delta\phi$, we have

$$n = \bar{\tau}\,\bar{\nu} = \tau\nu \tag{9.44}$$

where $\bar{\tau}$ and τ are the times elapsed corresponding to sending and receiving the n wavelengths respectively. Combining (9.43) and (9.44) we see that

$$\frac{\tau - \bar{\tau}}{\tau} = \Delta\phi\text{(gravitational time dilation)} \tag{9.45}$$

which shows that the times are not the same. Equation (9.45) must be interpreted to mean that "identical clocks" located at different gravitational potentials run at different rates.

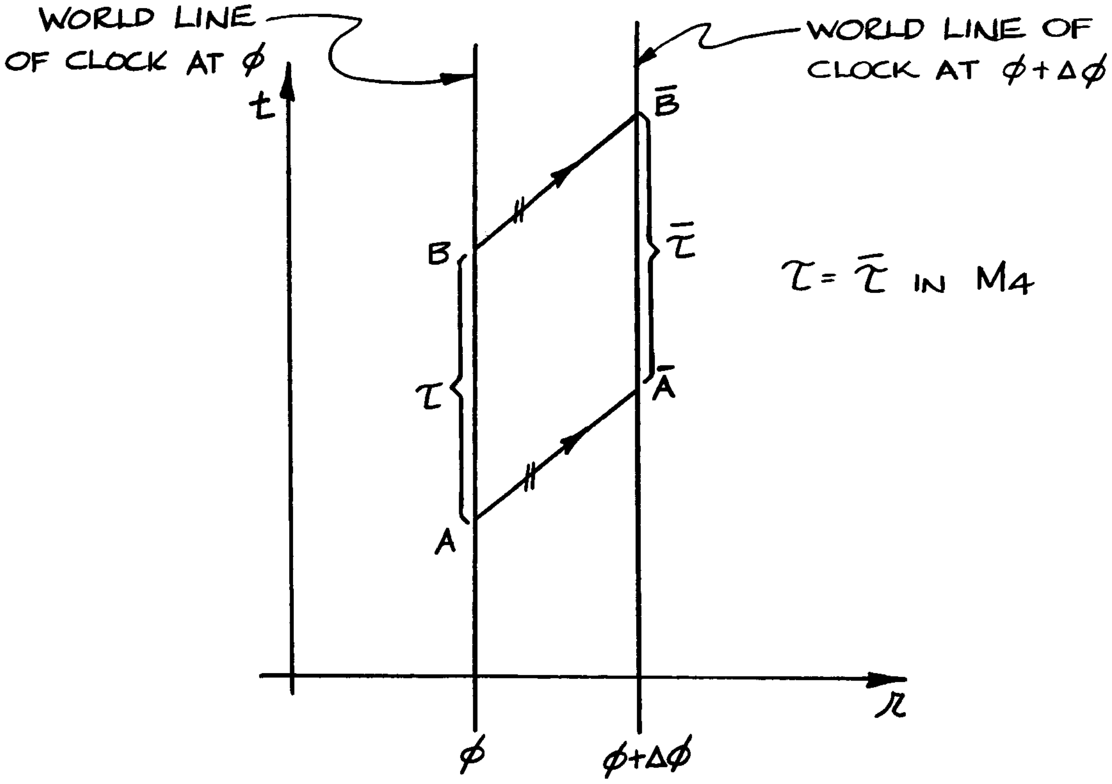

Fig. 9.2 Clocks at rest near a gravitating spherically symmetric mass.

In Fig. 9.2 is shown an experiment which is carried out in the region of space containing ϕ and $\phi + \Delta\phi$. At A an experiment is started at the lower level and a signal A $\bar{A}$ is sent to the higher level so as to compare initial clock readings. At B the experiment (say the Beatle record) is stopped and another signal B $\bar{B}$ is sent so as to compare the final clock readings. Since the gravitational field is static (does not change with time) the lines A $\bar{A}$ and B $\bar{B}$ are parallel and hence the prediction is that $\tau = \bar{\tau}$. However, by (9.45) $\tau \neq \bar{\tau}$ and therefore, the equivalence principle and/or the gravitational redshift cannot be formulated as a theory in special relativity, i.e. in M_4. But, both the principle of equivalence and the gravitational redshift are experimentally verified. This clearly separates theories such as electrodynamics and relativistic quantum theory which can be formulated in flat space (M_4) from gravitational theories which cannot be so formulated.

The non-relativistic limit of equation (9.39) must be the Newtonian equation

$$\frac{d^2x^\alpha}{dt^2} = -\phi,_\alpha \tag{9.46}$$

where ϕ is the gravitational potential at the particle. In order to find the non-relativistic limit of (9.39) we assume the metric can be written in the form

$$g_{\ell k} = \eta_{\ell k} + h_{\ell k} \ . \tag{9.47}$$

Where the $h_{\ell k}$ represent first order corrections to the Minkowski metric $\eta_{\ell k}$. Substituting this expression in the geodesic equation

$$\frac{dV^\alpha}{d\tau} + \Gamma^\alpha_{\ell r} V^r V^s = 0 \tag{9.39}$$

and making a low velocity $u \ll 1$ weak field approximation we have

$$\frac{du^\alpha}{dt} = -\Gamma^\alpha_{44} \ .$$

The first order Cristoffel symbol is

$$\Gamma^\alpha_{44} = 1/2\ \eta^{\alpha s}\left(2h_{4s,4} - h_{44,r}\right) .$$

Since $\frac{\partial}{\partial x^4} = 1/c \frac{\partial}{\partial t}$ and if the time variations in the field are not extremely large the first term is small with respect to the latter and we have

$$\frac{du^{\alpha}}{dt} = -1/2\, h_{44,\alpha} \cdot \text{(quasi-static)} \tag{9.48}$$

This equation is the same as the (9.46) if we associate

$$h_{44} = 2\phi + \text{const.} \tag{9.49}$$

Thus, Newton's equations follow from the geodesic equations. In Einstein's gravitational theory the metric $g_{\ell k}$ is a generalization of the Newtonian potential.

As previously mentioned in this chapter, if we know a given theory in M_4 we can usually generalize it to V_4 without a great deal of difficulty. There is one theory which we have seen cannot be formulated in M_4, that is a theory of gravitation; in the next chapter we formulate and begin our investigations of Einstein's theory of gravitation.

Chapter 10

A. Einstein's Gravitational Field Equations

In this chapter we shall include the constant c (speed of light in free space) in all formula. The reason for this will be explained later.

We have seen in the previous chapter that the principle of equivalence implies the formulation of the theory of gravitation in a Riemannian space V_4. In a Riemannian space the most important object is the metric tensor $g_{\ell k}$. We have shown heuristically that the $g_{\ell k}$'s are generalizations of the Newtonian potential. The missing item is the dynamical equations that will allow us to determine the gravitational field $g_{\ell k}$ from a given distribution of "sources." Thus, we need to know the generalization of Poisson's equation mentioned in Chapter 1. The generalization chosen by Einstein was

$$R^{\ell}_{k} - \tfrac{1}{2}\,\delta^{\ell}_{k}\,R = G^{\ell}_{k} = c_1\,T^{\ell}_{k} \quad , \; c_1 = \text{const.} \tag{10.1}$$

where G^{ℓ}_{k} is the Einstein tensor defined by (9.33) and T^{ℓ}_{k} is the energy-momentum tensor determined by the "sources." Einstein's field equation (10.1) thus generalizes Poisson's equation

$$\nabla^2\phi = 4\pi\, k\rho \; . \tag{1.14}$$

Equations (10.1) and (1.14) have some similarities. Both equations are second order partial differential equations for the gravitational field variables $g_{\ell k}$ and ϕ in terms of source distributions T^{ℓ}_{k} and ρ. Of course, (10.1) is a set of ten quasilinear partial differential equations while (1.14) is a simple linear partial differential equation. It follows immediately from (10.1) that the tensor $T^{\ell k} = g^{\ell r} T_r^k$ must be symmetric since $G^{\ell k}$ is symmetric. It also follows from the contracted Bianchi identities (9.32) that the tensor T_ℓ^k must be covariantly constant:

$$T_{\ell;k}^{k} = 0 . \tag{10.2}$$

By contracting (10.1) one finds that the field equations can be written in the equivalent form

$$R_{\ell k} = c_1 (T_{\ell k} - \tfrac{1}{2} g_{\ell k} T). \tag{10.3}$$

where $T = T_\ell^\ell$.

In regions where there are no sources ($T_k^\ell = 0 \Longleftrightarrow$ "vacuum") the field equations reduce to

$$R_k^\ell = 0. \quad \text{(vacuum field equations)} \tag{10.4}$$

Even if (10.4) is satisfied throughout V_4 it does not imply that V_4 is flat. A flat V_4 is characterized by the condition

$$R^\ell{}_{krs} = 0 \tag{10.5}$$

which is not necessarily implied by (10.4) (this is clear from the fact that (10.4) represents ten independent equations while (10.5) represents twenty). From (10.3) one can see the structure of the left-hand side of field equations

$$\begin{aligned} R_{\ell k} &= \tfrac{1}{2} g^{im} [g_{im,\ell,k} + g_{\ell k,i,m} - g_{ik,\ell,m} - g_{\ell m,i,k}] \\ &+ g^{im} g_{np} [\Gamma^n_{\ell k} \Gamma^p_{im} - \Gamma^n_{\ell k} \Gamma^p_{ik}] = c_1 (T_{\ell k} - \tfrac{1}{2} g_{\ell k} T) \end{aligned} \tag{10.6}$$

from which the above-mentioned quasilinear form of the field equations is obvious.

From the principle of equivalence one sees that the interpretation of the measure ds is the same as in special relativity, namely along the path of a material particle $ds = d\tau$ where $d\tau$ is the propertime increment. This proper time is the time that would be measured by a clock which is "carried by the particle." Light pulses are again characterized by null geodesics, $ds = 0$ along their trajectory. The proper time τ can again be used as a parameter to characterize the motion of a particle.

What is the form of the energy momentum tensor $T^{\ell k}$? It is a symmetric second rank tensor depending on the distribution and motion of the sources of the gravitational field. The simplest (or most complicated?) gravitational source one can imagine is a neutral point particle. Associated with such a particle is its four velocity V^ℓ and mass m,

$$V^\ell = \frac{dz^\ell}{d\tau} = \frac{dz^\alpha}{d\tau} \;,\; \frac{dz^4}{d\tau} \tag{10.7}$$

where $z^\ell(\tau)$ is the position of the particle. Since the mass m is assumed to be concentrated at a point we must have

$$T^{\ell k}(x) = 0 \qquad x^\ell \neq z^\ell \;. \tag{10.8}$$

A consistent form for $T^{\ell k}$ is arrived at by using the Dirac delta $\delta(x-z(\tau))$ (see Appendix A). The energy momentum tensor for a neutral (spiniess) point particle is defined as

$$T^{\ell k}(x) = \frac{mc}{\sqrt{-g}} \int_{-\infty}^{+\infty} V^\ell(\tau)\; V^k(\tau)\; \delta(x-z(\tau))\; d\tau \;. \tag{10.9}$$

Since $\sqrt{-g}$ and $\delta(x-z)$ are both relative tensors of weight plus one it is clear that $T^{\ell k}$ is an ordinary tensor (weight zero). The energy momentum tensor can be rearranged into the form

$$T^{\ell k}(x) = \frac{mc}{\sqrt{-g}} \int_{-\infty}^{+\infty} V^{\ell}(\tau) \frac{dz^k}{dz^4} \delta(x^4 - z^4)\, \delta(\underline{x}-\underline{z})\, dz^4$$

and after carrying out the integration over z^4

$$T^{\ell k}(x) = \frac{mc}{\sqrt{-g}} V^{\ell} \frac{dz^k}{dz^4} \delta(\underline{x}-\underline{z}(\tau)) \qquad (10.10)$$

$$; \qquad z^4(\tau)=x^4 \Rightarrow \tau=f(x^4).$$

The dirac delta therefore allows us to form a tensor field $T^{\ell k}(x)$ from quantities which are defined only along the world line of the particle. Later we shall use this energy momentum tensor to investigate the field equations.

In the last argument of Chapter 9 we showed that Newton's equation (9.46) follows from the geodesic equation (9.39) in a "weak field quasistatic" approximation. We now wish to perform the same analysis with the field equations (10.1) and show that Poisson's equation (1.14) follows as the first order approximation. The procedure to be followed here is to use a perturbation technique to solve the equation (10.1). Both sides of equation (10.1) are developed in terms of a parameter and the equations are solved successively. (Those familiar with perturbation theory in quantum mechanics will notice the similarity.) We assume the weak field metric can be written as in (9.47)

$$g_{\ell k} = \eta_{\ell k} + h_{\ell k} \qquad h_{\ell k} << 1 \qquad (9.47)$$

One may interpret (9.47) to mean that the geometry differs only slightly from M_4 in the coordinate system chosen. Far away from the matter, i.e. the particles producing the field, the space must be flat. That is

$$\lim_{r \to \infty} g_{\ell k}(x) = \eta_{\ell k} \qquad \lim_{r \to \infty} h_{\ell k} = 0 \qquad (10.11)$$

where $r^2 = \sum_{\alpha=1}^{3} x^\alpha x^\alpha$. The parameter expansion has the form

$$h_{\ell k} = \epsilon \underset{(1)}{h_{\ell k}} + \epsilon^2 \underset{(2)}{h_{\ell k}} + \ldots, \quad g_{\ell k} = \eta_{\ell k} + h_{\ell k} \tag{10.12}$$

$$h^{\ell k} = \epsilon \underset{(1)}{h^{\ell k}} + \epsilon^2 \underset{(2)}{h^{\ell k}} + \ldots, \quad g^{\ell k} = \eta^{\ell k} - h^{\ell k} . \tag{10.13}$$

We also assume the field to vary slowly with time or to be quasistatic. We shall choose the expansion parameter ϵ to be $1/c$ since we are interested in the Newtonian approximation to the field equations. In these approximations we are considering the coordinates $(x^1, x^2, x^3, x^4 = ct)$ have the interpretation of space and time in the usual Newtonian sense. As will be shown later this is a useful assumption, however, one should be warned that by doing this we are destroying the geometrical description of the gravitational field implied by general relativity. Because $\epsilon = 1/c$, differentiation with respect to x^4 increases the order of a quantity by one, that is,

$$\frac{\partial f}{\partial x^4} = 1/c \frac{\partial f}{\partial t} \tag{10.14}$$

is of one higher order in $1/c$ than f. Since the time variation of the field is directly related to the speed of the sources (10.14) implies

$$\left|\frac{v^\alpha}{c}\right| \ll 1 \quad \left\{ \begin{array}{l} \text{This is a so-called "slow" approximation in} \\ \text{contrast to other approximations which are} \\ \text{valid for arbitrary } \frac{v}{c} \text{ "fast" approximations.} \end{array} \right\} ,$$

where v^α is a characteristic speed associated with the sources. We have already determined the first term in h_{44} in equation (9.49). The additive constant in h_{44} must be zero since $g_{44} = +1$ as $r \to \infty$, i.e., condition (10.11). Thus putting in the correct factors of c we have

$$h_{44} = \frac{2\phi}{c^2} \ . \tag{9.49}$$

According to (10.12) this is $\underset{(2)}{h_{44}}$ which implies that $\underset{(1)}{h_{44}} = 0$.

$$\begin{aligned} \underset{(2)}{h_{44}} &= \frac{2\phi}{c^2} \\ h_{44} &= \frac{2\phi}{c^2} + 1/c^3 \underset{(3)}{h_{44}} + \dots. \end{aligned} \tag{10.15}$$

In order to determine the constant c_1 appearing in Einstein's field equation it is sufficient to know only equation (10.15) as we shall show later.

The expansions of $h_{\ell k}$ may be further narrowed if we note that in the limit $c \to \infty$ the general relativistic expressions must go over into the Newtonian expressions. There are basically two reasons why this is true 1) Newtonian mechanics does not contain the constant c in any fundamental way 2) Newtonian theory is a theory with infinite signal velocity. We now show how one may further restrict the expansion of $h_{\ell k}$. The Newtonian Lagrangian for a test particle* of mass m in a field ϕ can be written

$$L_{\text{Newtonian}} = \frac{1}{2}v^2 - \phi \tag{10.16}$$

and the Lagrangian for a test particle in general relativity is as we have seen on page 99.

$$L_{\text{G.R.}} = g_{\ell k} \frac{dx^\ell}{d\tau} \cdot \frac{dx^k}{d\tau} = g_{\ell k} \left(\frac{dt}{d\tau}\right)^2 \frac{dx^\ell}{dt} \frac{dx^k}{dt} \tag{10.17}$$

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*A test particle is a particle whose own gravitational field is negligible.

which can be written

$$L_{G.R.} = \left(\frac{dt}{d\tau}\right)^2 (\eta_{\ell k} + h_{\ell k}) \frac{dx^\ell}{dt} \frac{dx^k}{dt} \quad . \tag{10.18}$$

Expanding (10.18) one obtains in lowest order:

$$L_{G.R.} = (c^2 - v^2 + h_{44}\, c^2 + h_{4\alpha}\, cv^\alpha + h_{\alpha\beta}\, v^\alpha v^\beta + \ldots . \tag{10.19}$$

In the Newtonian limit $c \to \infty$ $L_{G.R.}$ must go over into (10.16), therefore as $c \to \infty$ we must at least have

$$h_{44} \to 1/c^2 \qquad h_{4\alpha} \to 1/c^2 \qquad h_{\alpha\beta} \to 1/c$$

$$\begin{aligned} h_{44} &= \underset{(2)}{h_{44}} + \ldots \\ h_{4\alpha} &= \underset{(2)}{h_{4\alpha}} + \ldots \\ h_{\alpha\beta} &= \underset{(1)}{h_{\alpha\beta}} + \ldots \end{aligned} \tag{10.20}$$

By using the field equations one can restrict the $h_{\ell k}$'s even more and prove the following expansions are generally true:

$$\begin{aligned} h_{44} &= \underset{(2)}{h_{44}} + \underset{(3)}{h_{44}} + \underset{(4)}{h_{44}} + \ldots \\ h_{4\alpha} &= \underset{(3)}{h_{4\alpha}} + \underset{(4)}{h_{4\alpha}} + \ldots \\ h_{\alpha\beta} &= \underset{(2)}{h_{\alpha\beta}} + \underset{(3)}{h_{\alpha\beta}} + \ldots \end{aligned} \tag{10.21}$$

(Actually one can even restrict the $h_{\ell k}$'s further by more detailed study of the field equations.) By making use of (10.21) one finds the following relations connecting the $h_{\ell k}$ with the $h^{\ell k}$. $\underset{(2)}{h^{44}} = \underset{(2)}{h_{44}}$, $\underset{(2)}{h^{4\alpha}} = - \underset{(2)}{h_{4\alpha}}$,

$\underset{(1)}{h^{\alpha\beta}} = \underset{(1)}{h_{\alpha\beta}}$, $\underset{(3)}{h^{44}} = \underset{(3)}{h_{44}}$, $\underset{(4)}{h^{44}} = (\underset{2}{h_{44}})^2 - \underset{4}{h_{44}}$. If we use the assumed metric (9.47), and expand the field equations in powers of $1/c$, the Riemann tensor through third order is given by

$$\underset{(\to 3)}{R_{i\ell km}} = \frac{1}{2}(h_{mi,\ell,k} - h_{m\ell,i,k} + h_{\ell k,i,m} - h_{ki,\ell,m}) \ . \tag{10.22}$$

Note that the right hand side has all the symmetries of the Riemann tensor $R_{ijk\ell}$. As one can prove from (10.21), indicies on the small quantities h_{ij} may be raised and lowered with the Minkowski metric as long as we are considering terms of third order and less. We shall often make use of this in the following. Contracting the Riemann tensor we obtain the Ricci tensor

$$\underset{(\to 3)}{R_{\ell k}} = \frac{1}{2}(h_{,k,\ell} - h^{i}_{\ell,i,k} + \eta^{im}\, h_{k\ell,i,m} - h^{i}_{k,\ell,i}) \tag{10.23}$$

where $\eta^{im}\, h_{im} = h^i_i = h$.

Define the new auxillary variables $\gamma_{\ell k}$

$$\gamma_{\ell k} = h_{\ell k} - 1/2\ \eta_{\ell k}\ h\ , \tag{10.24a}$$

$$h_{\ell k} = \gamma_{\ell k} - 1/2\ \eta_{\ell k}\ \gamma\ . \tag{10.24b}$$

If we expand $\gamma_{\ell k}$ in a power series in terms of $1/c$ we can make the following identifications:

$$\underset{(2)}{\gamma_{44}} = \underset{(2)}{h_{44}} - 1/2(\underset{(2)}{h_{44}} - \underset{(2)}{h_{\alpha\beta}}),\ \text{etc.} \tag{10.24c}$$

The Ricci tensor may be written in terms of γ_s

$$\underset{(\to 3)}{R_{\ell k}} = 1/2(\eta^{im}\, \gamma_{\ell k,i,m} - 1/2\ \eta^{im}\eta_{\ell k}\, \gamma_{,i,m} - \gamma^{i}_{k,\ell,i} - \gamma^{i}_{\ell,i,k}) . \tag{10.25}$$

Contracting the Ricci tensor and writing the Einstein tensor $G_{\ell k} = R_{\ell k} - 1/2 g_{\ell k} R$ in terms of γ_s yields

$$G_{\ell k} = 1/2\ (\eta^{im}\, \gamma_{\ell k,i,m} + \eta_{\ell k}\, \gamma^{ir}{}_{,i,r} - \gamma^{i}{}_{k,i,\ell} - \gamma^{i}_{\ell,i,k}). \qquad (10.26)$$

In the present coordinate system the $h_{\ell k}$ (or $\gamma_{\ell k}$) are small corrections to the $\eta_{\ell k}$. But the $h_{\ell k}$ will remain small if we perform an infinitesimal coordinate transformation

$$x'^{\ell} = x^{\ell} + \xi^{\ell}(x) \qquad \xi^{\ell} \text{ "small"} \qquad (10.27)$$

The transformation coefficients are given by

$$\frac{\partial x'^{\ell}}{\partial x^{r}} = \delta^{\ell}_{r} + \xi^{\ell}_{,r}\ , \quad \frac{\partial x^{r}}{\partial x'^{s}} = \delta^{r}_{s} - \xi^{r}_{,s} \qquad (10.28)$$

and the transformed metric is to first order in ξ^{ℓ},

$$g'_{rs} = g_{\ell k}(\delta^{\ell}_{r} - \xi^{\ell}_{,r})(\delta^{k}_{s} - \xi^{k}_{,s}) \qquad (10.29)$$

$$g'_{rs} = g_{rs} - g_{\ell s}\, \xi^{\ell}_{,r} - g_{rk}\, \xi^{k}_{,s}\ . \qquad (10.30)$$

If the two coordinate systems are equivalent we must have $g_{\ell k} = \eta_{\ell k} + h_{\ell k}$, $g'_{\ell k} = \eta_{\ell k} + h'_{\ell k}$, therefore, from (10.30) we find

$$h'_{\ell k} = h_{\ell k} - \eta_{p\ell}\, \xi^{p}_{,k} - \eta_{pk}\, \xi^{p}_{,\ell}\ . \qquad (10.31)$$

As can be seen by comparing (10.30) and (10.31) the $h_{\ell k}$'s transform as tensors under the infinitesimal transformations (10.27). We can deduce several properties of the transformation (10.27) from (10.31). Since $h'_{\alpha\beta}$ starts with second order terms (10.21) we must have $\xi^{\alpha} = \xi^{\alpha}_{(2)} + \ldots$ etc. Under this infinitesimal coordinate transformation the divergence of the γ's transforms in the following way:

$$\gamma'^{k}_{\ell,k} = \gamma^{k}_{\ell,k} - \eta^{kr}\, \xi_{\ell,k,r} \quad . \tag{10.32}$$

Since the only requirement on the $h_{\ell k}$'s is that they be small we can see from (10.31) that the $h_{\ell k}$'s are not determined uniquely. There are four arbitrary functions in (10.31), therefore we may impose four conditions on the $h_{\ell k}$. There conditions will be called "coordinate conditions" or "gauge" conditions. We choose the following coordinate conditions for our present analysis

$$\gamma^{k}_{\ell,k} = (h^{k}_{\ell} - 1/2\ \delta^{k}_{\ell}\ h)_{,k} = 0 \quad . \tag{10.33}$$

With this choice for the coordinate condition the Einstein tensor becomes

$$\underset{(\rightarrow 3)}{G_{\ell k}} = 1/2\ \eta^{im}\ \gamma_{\ell k,i,m}$$

$$\underset{(\rightarrow 3)}{G_{\ell k}} = -\ 1/2\ \Box\ \gamma_{\ell k}, \quad \Box = -\ \eta^{im}\,\partial_i\partial_m = \nabla^2 - 1/c^2\ \frac{\partial^2}{\partial t^2} \quad . \tag{10.34}$$

Even after specifying the coordinate conditions (10.33) $h_{\ell k}$ is still not unique since one can transform it by any infinitesimal vector ξ^{ℓ} such that $\Box\ \xi^{\ell} = 0$. Making use of (10.34) the field equations (10.1) become

$$1/2\ \Box\ \gamma_{\ell k} = -\ c_1\ \tau_{\ell k} \qquad \gamma_{\ell}{}^{k}{}_{,k} = 0 \quad . \tag{10.35}$$

Here $\tau_{\ell k}$ represents the effective energy momentum tensor obtained from the exact equations by our approximations. We shall return to (10.35) later. The equations (10.31) must be interpreted to mean that if we have an arbitrary solution $h_{\ell k}$ for the weak field metric $g_{\ell k} = \eta_{\ell k} + h_{\ell k}$ then all other $h_{\ell k}$'s of the form

$$h'_{\ell k} = h_{\ell k} - 2\ \xi_{(\ell,k)} \tag{10.31a}$$

are equally as valid. Thus, to the solution $h_{\ell k}$ is added a purely coordinate dependent part. However, if the Riemann tensor $R_{ijk\ell}$ is evaluated for the part of the metric $\xi_{p,\ell} + \xi_{\ell,p}$ one finds it vanishes identically (i.e. substitute $\xi_{(p,\ell)} \rightarrow h_{p\ell}$ in (10.22).) This means that the coordinate dependent part does not enter into the gravitation field equations (the equations are "gauge invariant"). In terms of the γ's the transformation corresponding to (10.31a) is

$$\gamma'_{\ell k} = \gamma_{\ell k} - 2\,\xi(\ell,k) + \eta_{\ell k}\,\xi^{p}{}_{,p} \;. \tag{10.31a}$$

Consider now a general solution, $\bar{\gamma}_{\ell k}$, to the field equations

$$\underset{(\rightarrow 3)}{G_{\ell k}} = c_1\,\tau_{\ell k}$$

with $\underset{(\rightarrow 3)}{G_{\ell k}}$ given by (10.26). We now write $\bar{\gamma}_{\ell k}$ as a sum of two parts

$$\bar{\gamma}_{\ell k} = \underline{\gamma}_{\ell k} + \gamma_{\ell k} \tag{10.31b}$$

where $\gamma^{\ell}_{k,\ell} = 0$ and $\underline{\gamma}_{\ell k} = +\,2\,\xi(\ell,k) - \eta_{\ell k}\,\xi^{p}{}_{,p}$. From (10.31c) follows

$$\bar{\gamma}^{\ell}_{k,\ell} = \underline{\gamma}^{\ell}_{k,\ell} = -\,\xi_{k}{}^{,\ell}{}_{,\ell} = \Box\,\xi_k \;. \tag{10.31c}$$

$\underline{\gamma}_{\ell k}$ is a part of the $\bar{\gamma}_{\ell k}$ which does not contribute to the curvature tensor while $\gamma_{\ell k}$ does contribute to the curvature tensor. We can next make a gauge transformation of the form (10.31b)

$$\gamma_{\ell k} = \bar{\gamma}_{\ell k} - \underline{\gamma}_{\ell k} \tag{10.31d}$$

which removes from $\bar{\gamma}_{\ell k}$ the part that does not contribute to the curvature tensor. One may refer to $\gamma_{\ell k}$ as the intrinsic part of $\bar{\gamma}_{\ell k}$ and $\underline{\gamma}_{\ell k}$ as the coordinate dependent part. The coordinate condition $\gamma^{\ell}_{k,\ell} = 0$ allows one to consider only the intrinsic part of $\bar{\gamma}_{\ell k}$.

In the general case the metric $g_{\ell k}$ contains two types of information: 1) describes the physical situation 2) describes the particular coordinate system employed. The coordinate conditions eliminate the second of these to a large extent.

The constant c_1 may be determined most easily by using (10.3). We write out the lowest order terms on both sides for $\ell, k = 4,4$. We have from (10.23) in the quasistatic case

$$\underset{(2)}{R_{44}} = -1/2\, \nabla^2 \underset{(2)}{h_{44}} \tag{10.36}$$

to the lowest order term in R_{44}. From (10.10) we see that the lowest order term in $T_{\ell k}$ is

$$T_{44} \rightarrow mc^2\, \delta(\underline{x}-\underline{z}) \; . \tag{10.37}$$

The trace $T = T^{\ell}_{\ell}$ is also equal to $mc^2\, \delta(\underline{x}-\underline{z})$ to this order. Hence, (10.3) becomes

$$-1/2\, \nabla^2 \underset{(2)}{h_{44}} = c_1\, 1/2\, mc^2\, \delta(\underline{x}-\underline{z}),$$

or rearranging and making use of (10.15) we have

$$-\nabla^2 \phi = c_1\, 1/2\, mc^4\, \delta(\underline{x}-\underline{z}) \; . \tag{10.38}$$

To determine c_1 we need only compare (10.38) with Poisson's equation for a point particle

$$\nabla^2 \phi = 4\pi\, km\, \delta(\underline{x}-\underline{z}) \; . \tag{10.39}$$

Comparing the constants in (10.38-39) we find

$$c_1 = \frac{-8\pi k}{c^4} \tag{10.40}$$

for the value of the constant c_1. Thus, the field equations can be written

$$G_{\ell k} = \frac{-8\pi k}{c^4} T_{\ell k} \quad . \tag{10.41}$$

We shall sometimes choose units such that c = 1 and k = 1. (This may be interpreted as measuring time in seconds and choosing units for length and mass such that c = 1 and k = 1.)

We can use equation (10.35) to determine the remaining second order corrections to the metric, that is, $\underset{(2)}{h_{\alpha\beta}}$ (remember that we have chosen a coordinate system such that $\underset{(2)}{h_{\alpha 4}} = 0$). To determine $\underset{(2)}{h_{\alpha\beta}}$ we need only calculate $\underset{(2)}{\gamma_{44}}$ from (10.35) and use equation (10.24). Equation (10.35) in second order becomes

$$1/2\ \nabla^2 \underset{(2)}{\gamma_{44}} = \underbrace{\frac{8\pi k}{c^4}}_{-c_1} \cdot \underbrace{mc^2\ \delta(\underline{x}-\underline{z})}_{T_{44}} \tag{10.42a}$$

$$1/2\ \nabla^2 \underset{(2)}{\gamma_{\alpha\beta}} = 0 \tag{10.42b}$$

$$1/2\ \nabla^2 \underset{(2)}{\gamma_{\alpha 4}} = 0 \quad . \tag{10.42c}$$

The solutions of these equations (which tend to zero at infinity) are

$$\underset{(2)}{\gamma_{44}} = \frac{4\phi}{c^4}\,, \quad \underset{(2)}{\gamma_{\alpha\beta}} = 0 \quad \underset{(2)}{\gamma_{\alpha 4}} = 0 \tag{10.43}$$

Using (10.43) and (10.24b) we find

$$\begin{aligned} \underset{(2)}{h_{\alpha\beta}} &= \frac{2\phi}{c^2}\,\delta_{\alpha\beta} \\ \underset{(2)}{h_{4\alpha}} &= 0 \ . \end{aligned} \tag{10.44}$$

Thus, the metric to second order (in our particular coordinate system) can be written:

$$ds^2 = -\left(1 - \frac{2\phi}{c^2}\right)(dx^2 + dy^2 + dz^2) + \left(1 + \frac{2\phi}{c^2}\right) c^2\, dt^2 \ . \qquad (10.45)$$

If one assumes this form for the metric outside a non-rotating spherically symmetric gravitating body (the sun on the earth) and studies the consequences, the following predictions can be made: 1) gravitational red shift, 2) deflection of light passing the body 3) geodetic precession of a gyroscope. Another effect, the advance of the perihelion of an orbiting body depends upon higher order terms in the metric (namely $h_{44}^{(4)}$ in the present coordinate system). Later we shall give a more detailed examination of the experimental verification of general relativity.

It is interesting that $h_{\alpha\beta}^{(2)}$ correction to the metric is the same order of magnitude as the $h_{44}^{(2)}$ correction but does not show up in the Newtonian approximation equation (9.48). The reason for this is the presence of the c^2 in the timelike part of the metric (10.45). In the Lagrangian $L = \frac{ds^2}{c^2 d\lambda^2}$ the $g_{\alpha\beta}$ terms are multiplied by v^2/c^2 which is small in the Newtonian approximation.

The solution of the linearized gravitational equations (10.35)

$$\Box\, \gamma_k^{\ell} = + \frac{16\pi k}{c^4}\, \tau_k^{\ell} \ , \quad \gamma_{k,\ell}^{\ell} = 0 \qquad (10.35)$$

is solved in exactly the same way as the analogous equation in electromagnetic theory. The general solution to the equation

$$\Box\, \phi = -\,\alpha\rho \qquad (10.46)$$

is

$$\phi(x) = +\alpha \int D^{ret}(x-x')\ \rho(x')\ d^4x' \tag{10.47}$$

where $D^{ret}(x-x')$ is the retarded Green's function and $d^4x' = dV'_{(4)} = dx'^1\ dx'^2\ dx'^3\ dx'^4$. The retarded Green's function may be expressed in the form

$$D^{ret}(x-x') = \frac{1}{2\pi}\ \theta(t-t')\ \delta((x-x')^2) \tag{10.48}$$

where

$$\theta(t-t') = \begin{cases} 0 & t-t' < 0 \\ 1 & t-t' > 0 \end{cases}$$

$$\delta((x-x')^2) = \delta(\eta_{\ell k}(x^\ell - x'^\ell)(x^k - x'^k)) \ .$$

Substituting (10.48) into (10.47) and integrating yields the usual result

$$\phi(\underline{x},t) = + \frac{\alpha}{4\pi} \int \frac{\rho(\underline{x}', t' = t - |\underline{x}-\underline{x}'|/c}{|\underline{x}-\underline{x}'|}\ d^3x' \tag{10.49}$$

where $d^3x' = ds'_{(s)} = dx'^1\ dx'^2\ dx'^3$. Apply (10.49) to (10.35) we have

$$\gamma^\ell_k(x) = -\frac{4k}{c^4} \int \frac{T^\ell_k(\underline{x}',\ t-|\underline{x}-\underline{x}'|/c)}{|\underline{x}-\underline{x}'|}\ d^3x' \ . \tag{10.50}$$

One can show directly that (10.50) satisfies the coordinate conditions (10.33). We shall not continue the calculations of the linear theory further since from here on the calculations almost parallel those of electrodynamics. If one calculates the energy radiated* by an oscillating system then the result comes out

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*Landau, Lifshitz, The Classical Theory of Fields, (Second Edition p. 366).

$$\frac{dE}{dt} \approx \frac{k\ m^2\ r^3}{c^5}\ \omega^6 \tag{10.51}$$

when one uses (10.50). Here m is the mass of the system, r is the size of the system, and ω is the angular frequency of the system. For laboratory size objects the radiation calculated from (10.51) is completely negligible, i.e. m = 10gms, r = 10cm and $\omega = 10^6$/sec we find

$$\frac{dE}{dt} \approx 10^{-30}\ \frac{\text{gms}}{\text{year}}\ . \tag{10.52}$$

Thus, the measurement of gravitational radiation will be very difficult. For binary neutron stars the magnitude of the energy loss is more favorable although none of these objects have yet been observed.

As a final point, I want to mention what role the linear theory plays in Einstein's over-all theory of gravitation. Although the linear theory is non-geometrical and therefore very repugnant to many relativistics it does allow one to make numerical estimates as to the order of magnitude of various effects (i.e. such as gravitational radiation treated above). The full theory does not allow, at present, such numerical predictions. Thus, as far as the experimentalist is concerned the linear theory is probably the most important part of general relativity; however, to the person wishing to understand the theory in full, the linear theory is a very insignificant part indeed.

Chapter 11

A. General Relativity I

In this chapter we shall be interested in studying the exact equations of Einstein's gravitational theory (which we shall call general relativity). Let us review briefly the contents of the theory. We have a hyperbolic Riemannian space V_4 with signature (--- +) and a metric tensor $g_{\ell k} = g_{k\ell}$. The $g_{\ell k}$'s are related to the "sources" by the partial differential equations (of the hyperbolic type)

$$G_{\ell k}\ (g_{rs}) = \frac{-8\pi k}{c^4}\, T_{\ell k}\ (\Psi \text{ matter}) \tag{10.41}$$

where $T_{\ell k}$ (Ψ matter) represents the stress energy momentum tensor or the source of the g field. We have given a particular example of $T_{\ell k}$ for simple "point" particles in the previous chapter. This form is appropriate when one discusses the motion of particles, however, other forms for the stress tensor must be chosen if we want to treat a continuous system. The interpretation of the stress tensor (10.9) used in the previous chapter is relatively simple. The path of the particle described by (10.9) is some curve in V_4. We choose a geodesic coordinate system attached to the particle. (Such a coordinate system is called a Fermi coordinate system.) In this coordinate system we have (at the origin)

$$\begin{gathered} g_{44} = -g_{11} = -g_{22} = -g_{33} = 1 \\ v^{\alpha} = 0 \quad , \quad v^4 = c \quad . \end{gathered} \tag{11.1}$$

The non-zero component of $T_{\ell k}$ is:

$$T_{44} = mc^2\ \delta(\underline{x}-\underline{z}) \tag{11.2}$$

$$T_{\ell k} = 0 \qquad \ell, k \neq 4,4\ .$$

Hence, T_{44} is the energy density associated with the particle. Now for an arbitrary system $T_{\ell k}$, we shall retain the same interpretation of its stress-energy tensor as we have for point particles.

$$T_{44} = \text{energy density} > 0 \tag{11.3a}$$

we notice from (10.41) that this implies

$$G_{44} < 0\ . \tag{11.3b}$$

The conditions (11.3b) are conditions which must be satisfied in order for the energy density to be positive definite. The units of T_{44} are $(\frac{\text{energy}}{\text{time}})$ which is the same as $(\frac{\text{momentum}}{\text{time}\cdot\text{area}})$ (which we shall call momentum flux) from (10.10) we find that for "point" particles

$$T^{\alpha 4} = \frac{mc}{\sqrt{-g}}\, V^{\alpha}\ \delta(\underline{x}-\underline{z}) \tag{11.4}$$

and hence $T^{\alpha 4}$ represents the momentum flux in the α^{th} direction. For those acquainted with the electromagnetic field it might be of interest to quote some of the components of $T^{\ell k}$:

$$T_{44} = \frac{1}{8\pi}(\underline{E}^2 + \underline{B}^2)\ \ ,\ T^{4\alpha} = \frac{1}{4\pi}(\underline{E} \times \underline{B})\ \alpha\ \ , \tag{11.5}$$

where $\underline{E}$ is the electric field and $\underline{B}$ the magnetic field. The units of (force) are the same as those of $(\frac{\text{momentum}}{\text{time}})$, therefore, the units of $T^{\ell k}$ are also $(\frac{\text{force}}{\text{area}})$. In the case where we are considering a continuous system we may interpret the space components of $T^{\alpha\beta}$ as the ordinary stress tensor:

$$T^{\alpha\beta} = \begin{array}{l}\text{force in } \alpha \text{ direction per unit of area} \\ \text{having its normal in the } \beta \text{ direction.}\end{array} \tag{11.6}$$

The diagonal components T^{11}, T^{22}, T^{33} are stress components while the off diagonal terms T^{12}, T^{23}, T^{13} are shear type stress components. If we consider a unit cube in three space the following diagram illustrates some of the forces exerted on the cube:

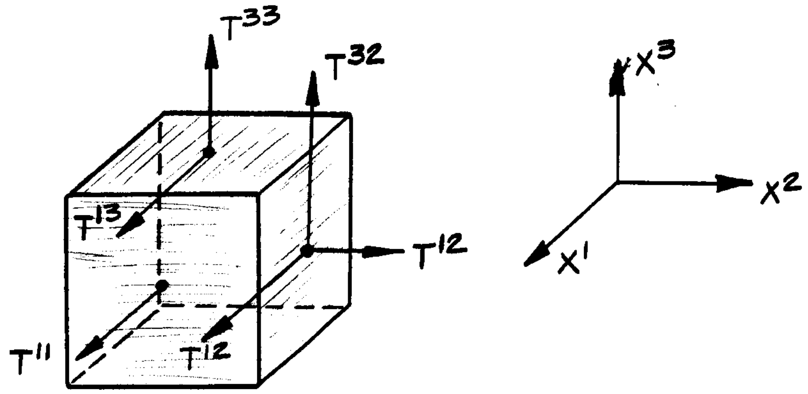

Fig. 11.1 A unit volume of continuous material

We now want to consider an exact solution to the field equations (10.41). First we define what we mean by a stationary space time. Besides coordinate transformations in V_4 one can also consider point transformations in a given coordinate system. Suppose we have a coordinate system S and we change to a coordinate system S':

$$CT: \quad S \to S' \qquad x'^{\ell} = f^{\ell}(x) \tag{11.7}$$

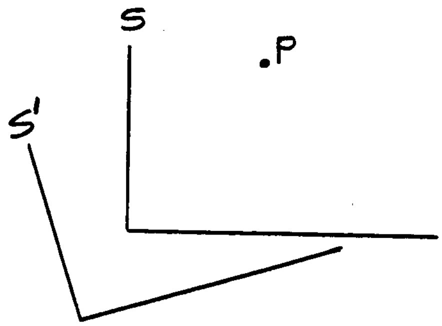

Fig. 11.2 Coordinate transformation CT.

Associated with every coordinate transformation CT: $S \to S'$ there is a point transformation (PT) PT: $S \to S$ such that the transformed point $P \to P'$ in S has the same numerical value as the new coordinates of P in S'.

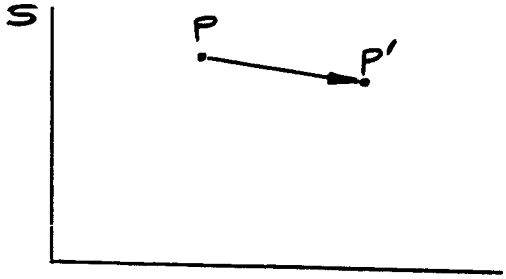

Fig. 11.3 Point transformation PT.

For the coordinate transformation (11.7) the point transformation is:

$$PT: \quad {}'x^{\ell} = f^{\ell}(x) = x'^{\ell}. \tag{11.8}$$

In point transformations the points of space are moved relative to the coordinate system (this is sometimes called an active coordinate transformation although this is not very accurate terminology) and under a coordinate transformation (passive) the labels attached to the points are changed. Let us consider an infinitesimal vector x^ℓ, $x^\ell + dx^\ell$ in terms of a picture such as Fig. 11.4.

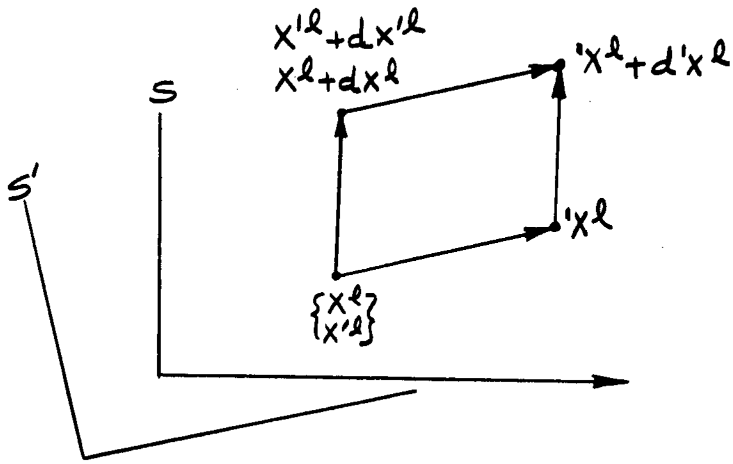

Fig. 11.4 CT and PT of an infinitesimal vector.

The relations (11.7) and (11.8) connect the coordinates in Fig. 11.4. A point transformation in V_4 is called a <u>motion</u> or <u>isometry</u> in V_4 if the measure $d'S^2$ of the displaced points $'x^\ell$, $'x^\ell + d'x^\ell$ is equal to the measure of the two original points x^ℓ $x^\ell + dx^\ell$.

$$d'S^2 = g_{\ell k}('x)\, d'x^\ell\, d'x^k \qquad (11.9a)$$

$$dS^2 = g_{\ell k}(x)\, dx^\ell\, dx^k \qquad (11.9b)$$

Using (11.8) we can write (11.9a) in the form

$$d'S^2 = g_{\ell k}(x')\ dx'^{\ell}\ dx'^{k} \ , \tag{11.10a}$$

and making use of (11.7) we can write (11.9b) in the form

$$dS^2 = g'_{\ell k}(x')\ dx'^{\ell}\ dx'^{k} \tag{11.10b}$$

where $g'_{\ell k}$ is the transformed metric

$$g'_{\ell k}(x') = f^{r}_{,\ell}\ f^{s}_{,k}\ g_{rs}(x) \ . \tag{11.11}$$

Therefore, if $dS^2 = d'S^2$ we have from (11.10a) and (11.10b)

$$g_{\ell k}(P) = g'_{\ell k}(P) \ , \tag{11.12}$$

that is, the two metrics must be the same when evaluated at the same numerical coordinates. Useful statements from (11.12) cannot be obtained for arbitrary point transformations, however, a kind of point transformation that yields useful statements from (11.12) is the infinitesimal type. When we consider infinitesimal point transformations we have

$$'x^{\ell} = \mathbf{x}'^{\ell} = x^{\ell} + \xi^{\ell}\delta t \tag{11.13}$$

where δt is treated as a first order infinitesimal (i.e., all powers of δt higher than first are neglected) and ξ^{ℓ} is an arbitrary vector field. From (11.11) follows (see equation (10.30))

$$g'_{\ell k}(x') = g_{\ell k}(x) - g_{s\ell}(x)\ \xi^{s}_{,k}\ \delta t - g_{sk}(x)\ \xi^{s}_{,\ell}\ \delta t \ . \tag{11.14}$$

Expanding $g'_{\ell k}(x') = g'_{\ell k}(x+\xi\delta t)$ about $\delta t = 0$ and retaining only first order terms (11.14) becomes

$$g'_{\ell k}(x) = g_{\ell k}(x) - g_{\ell k,s}\ \xi^{s}\ \delta t - g_{s\ell}\ \xi^{s}_{,k}\ \delta t - g_{sk}\ \xi^{s}_{,\ell}\ \delta t \ . \tag{11.15}$$

The difference

$$\mathcal{L}_\xi\ g_{\ell k} = g_{\ell k}(x) - g'_{\ell k}(x) = (g_{\ell k,s}\ \xi^s + g_{s\ell}\ \xi^s_{,k} + g_{sk}\ \xi^s_{,\ell})\ \delta t \qquad (11.16)$$

which can be written

$$\mathcal{L}_\xi\ g_{\ell k} = (\xi_{\ell;k} + \xi_{k;\ell})\ \delta t \qquad (11.17)$$

is called the Lie differential of $g_{\ell k}$. A necessary condition for ξ^ℓ to be a motion is for

$$\mathcal{L}_\xi\ g_{\ell k} = 0 \qquad (11.18a)$$

or

$$\xi_{\ell;k} + \xi_{k;\ell} = 0 \quad . \qquad (11.18b)$$

The equation (11.18b) is known as Killing's equation and a vector satisfying this equation is known as a Killing vector. Each solution of (11.18b) generates a one parameter group (this is a Lie group) of transformations which leave the metric invariant. The maximum number of solutions of (11.18b) in an n dimensional Riemannian space is n(n+1)/2 which in the case n = 4 is ten. In a general space-time V_4, however, the number of solutions will be less than ten, say r. Hence, in general, a space time will admit a r parameter group G_r which leaves the metric invariant in form, $0 \leq r \leq 10$. Physically the group G_r is the symmetry group of the space. A four dimensional space, admits four independent translations along the four coordinate axis and six independent rotations in the coordinate planes. In the case of maximum symmetry, r = 10 and one can show that the V_4 is of constant curvature; for vanishing curvature this is just M_4 and the ten parameter group is the Lorentz group. In the case r = 0 we have a V_4 which has no particular symmetry properties. Petrov has given an exhaustive classification of many V_4's for various values of r.* From (11.13) we see that the trajectories in V_4 of the one parameter group associated with t are given by integral curves of the ordinary differential equations:

$$\frac{dx^\ell}{dt} = \xi^\ell(x). \tag{11.19}$$

We shall give an example for r = 1. A V_4 is called stationary if it admits a one parameter group of motions with a time-like Killing vector. We can always choose a coordinate system in which a time-like vector ξ^ℓ has the form (this is because V_4 is of the hyperbolic type):

$$\xi^\ell = (0,0,0,1) .$$

In this coordinate system Killing's equation reduces to

$$g_{\ell k,4} = 0 , \qquad (11.\ \ )$$

that is, all the metric coefficients are independent of the time like coordinate x^4 in the chosen coordinate system. Furthermore the trajectories of the group are parallel to the Killing vector $\xi^\ell = \delta^\ell_4$.

$$\left.\begin{array}{c} \dfrac{dx^\ell}{dt} = \delta^\ell_4 \quad \text{or} \\ x^1,x^2,x^3 = \text{const.} \quad x = t \end{array}\right\} \qquad (11.2\ \ )$$

*See A. Petrov, <u>Einstein - Räume</u> Akademie-Verlog, Berlin, 1964 (pages 53-62 for an introduction).

A <u>static</u> space-time is a special case of a stationary space-time where the time-like Killing vectors are orthogonal to the family of surfaces x^4 = const. This means

$$g_{\ell k}\, dx^\ell\, \delta^k_4 = 0\ , \quad dx^\ell = (dx^\alpha, 0)\ ,$$

which yields

$$g_{\alpha 4} = 0\ . \tag{11.22}$$

A static V_4 is called "spherically symmetric" if in the coordinate system

$$g_{\ell k,4} = 0 \qquad g_{\alpha 4} = 0 \tag{11.23}$$

the surfaces x^4 = const. have "spherical symmetry." By "spherical symmetry" we mean that all directions (from the origin of the coordinate system) in the surface x^4 = const. are equivalent. If $d\sigma^2$ denotes the metric of this surface we have

$$d\sigma^2 = -\, g_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{11.24}$$

where x^α are the coordinates on the surface. We choose x^α as "orthogonal cartesian" coordinate system on the surface. Under a rotation of the three orthogonal space axis in the surface x^4 = const. we have

$$x'^\alpha = a^\alpha_\beta\, x^\beta \qquad \text{matrix } (a^\alpha_\beta) = a,\ a^{-1} = a^T, \tag{11.25}$$

where a^α_β is a three dimension rotation matrix. If the metric (11.24) is invariant under rotations it must have the form

$$d\sigma^2 = -\, f(r_1)\left((dx^1)^2 + (dx^2)^2 + (dx^3)^2\right)$$
$$r_1^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$$

where f is an arbitrary function. Introducing the "usual" spherical coordinates (r,θ,ϕ) we have

$$\begin{aligned} x^1 &= r_1 \sin\theta \cos\phi \\ x^2 &= r_1 \sin\theta \sin\phi \\ x^3 &= r_1 \cos\theta \ . \end{aligned} \tag{11.26}$$

Substitution of (11.26) into (11.25) yields

$$d\sigma^2 = - f(r_1)\, dr_1^2 - f(r_1)\, r_1^2\, (d\theta^2 + \sin\theta\, d\phi^2). \tag{11.27}$$

Using (11.27) the general form for the static, spherically symmetric line element can be written

$$ds^2 = g(r_1)(dx^4)^2 - f(r_1)\left(dr_1^2 + r_1^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right) \tag{11.28}$$

where g and f are arbitrary functions. The coordinates (r,θ,ϕ) are called isotropic spherical coordinates. The line element (11.28) can still be subjected to the transformations

$$\begin{aligned} r &= h(r_1) \\ x'^4 &= \ell(x4) \end{aligned} \tag{11.29}$$

and still retain the properties of being 1) static 2) spherically symmetric. Under the coordinate transformation

$$r = \sqrt{f(r_1)}\ t_1$$

the metric (11.28) transforms into

$$ds^2 = e^{\beta(r)}(dx^4)^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - e^{\alpha(r)}\, dr^2\ , \tag{11.30}$$

where

$$
\left.\begin{aligned}
e^{\beta(r)} &= g\big|_{r_1=r_1(r)} \\
e^{\alpha(r)} &= \left.\frac{f}{\left[\sqrt{f} + \frac{r_1}{2\sqrt{f}}\frac{df}{dr_1}\right]^2}\right|_{r_1=r_1(r)}
\end{aligned}\right\} \qquad (11.31)
$$

The form (11.30) is called the standard form of the line element. We shall now solve Einstein's equations for a static, spherically symmetric source centered at the origin of the coordinate system. The source is supposed to represent a model for a "star."

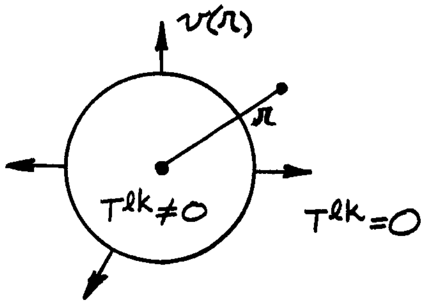

Fig. 11.1 A model "star".

(The final form tor the metric (11.31) can also be derived using group theory* or by geometrical construction.**) We now write the Einstein

*A. Petrov, Einstein-Räume, pages 331-335.

**J. Synge, Relativity: General Theory, pages 266-267.

equations (10.41) for this metric. The metric coefficients are

$$g_{11} = -e^{\alpha}, \quad g_{22} = -r^2, \quad g_{33} = -r^2\sin^2\theta, \quad g_{44} = e^{\beta}$$
$$g^{11} = -e^{-\alpha}, \quad g^{22} = -r^{-2}, \quad g^{33} = -r^{-2}\sin^{-2}\theta, \quad g^{44} = e^{-\beta} \tag{11.32}$$

In order to find the non-zero Christoffel symbols we make use of the variational principle equations for geodesics given on page 99. The Lagrangian is

$$L = -e^{\alpha}\left(\frac{dr}{ds}\right)^2 - r^2\left(\left(\frac{d\theta}{ds}\right)^2 + \sin^2\theta\left(\frac{d\phi}{ds}\right)^2\right) + e^{\beta}\left(\frac{dx^4}{ds}\right)^2. \tag{11.33}$$

The Lagrange equation for the $\dot{r}$ coordinate is

$$\frac{d}{ds}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0 \qquad \dot{r} = \frac{dr}{ds} \tag{11.34}$$

which becomes

$$\frac{d^2r}{ds^2} + \frac{\alpha'}{2}\left(\frac{dr}{ds}\right)^2 - re^{-\alpha}\left(\frac{d\theta}{ds}\right)^2 - re^{-\alpha}\sin^2\theta\left(\frac{d\phi}{ds}\right)^2 + 1/2\, e^{\beta-\alpha}\beta'\left(\frac{dx^4}{ds}\right)^2 = 0 \tag{11.35}$$

where $\alpha' = \frac{d\alpha}{dr}$, $\beta' = \frac{d\beta}{dr}$. From this we can pick off the non-zero Christoffel symbols of the form $\Gamma^1_{\ell k}$, namely,

$$\Gamma^1_{11} = \frac{\alpha'}{2}, \quad \Gamma^1_{22} = -re^{-\alpha}, \quad \Gamma^1_{33} = -re^{-\alpha}\sin^2\theta, \quad \Gamma^1_{44} = 1/2\, e^{\beta-\alpha}\beta'. \tag{11.36a}$$

The remaining Christoffel symbols can be determined in a similar manner:

$$\left.\begin{array}{c} \Gamma^2_{12} = 1/r, \quad \Gamma^2_{33} = -\sin\theta\cos\theta \\ \Gamma^3_{13} = 1/r, \quad \Gamma^3_{33} = \cot\theta \\ \Gamma^4_{14} = \beta'/2. \end{array}\right\} \tag{11.36b}$$

A further calculation yields the non-vanishing components of (10.41)

$$
\left.
\begin{aligned}
G^1_1 &= e^{-\alpha}\left(\frac{\beta'}{r} + 1/r^2\right) - 1/r^2 = -\, C_1 T^1_1 \\
G^3_3 = G^2_2 &= 1/2\ e^{-\alpha}\left(\beta'' + \beta'^2/2 + \frac{\beta'-\alpha'}{r} - \frac{\alpha'\beta'}{2}\right) = -\, C_1 T^2_2 \\
G^4_4 &= e^{-\alpha}(1/r^2 - \alpha'/r) - 1/r^2 = -\, C_1 T^4_4 \quad .
\end{aligned}
\right\}
\qquad (11.37)
$$

In the region external to the "star" we have

$$T^{\ell}_k = 0 \text{ (exterior solution)}$$

and the equations become:

$$e^{-\alpha}\left(\frac{\beta'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 0 \qquad (11.38a)$$

$$1/2\ e^{-\alpha}\left(\beta'' + \frac{\beta'^2}{2} + \frac{\beta'-\alpha'}{r} - \frac{\alpha'\beta'}{2}\right) = 0 \qquad (11.38b)$$

$$e^{-\alpha}\left(\frac{1}{r^2} - \frac{a'}{r}\right) - \frac{1}{r^2} = 0 \quad . \qquad (11.38c)$$

Adding the first and second equations yields

$$\alpha' + \beta' = 0$$

therefore

$$\alpha + \beta = \text{const.} \qquad (11.39)$$

By imposing the physical boundary condition

$$\lim_{r\to\infty} V_4 = M_4$$

The constant in equation (11.39) is easily seen to be zero. Multiply (11.38c) by r^2 we can write

$$r\frac{d}{dr}e^{-\alpha} + e^{-\alpha} - 1 = 0$$

which can be integrated to yield

$$e^{-\alpha} = 1 + \frac{\text{const.}}{r} \quad . \tag{11.40}$$

The constant is easily evaluated by requiring that far away from the body the field must reduce to the weak field form (10.45):

$$g_{44} \underset{r\to\infty}{\longrightarrow} 1 + \frac{2\phi}{c^2}$$

where ϕ is the Newtonian potential $\frac{-km}{r}$. Hence, the exterior solution for the spherical symmetric "star" is:

$$e^{-\alpha} = e^{\beta} = 1 - \frac{2km}{c^2 r} \tag{11.41}$$

and the measure is:

$$ds^2 = (1 - \frac{2km}{c^2 r})(dx^4)^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - \frac{dr^2}{(1 - \frac{2km}{c^2 r})} \quad . \tag{11.42}$$

Equation (11.42) is called the Schwartzschild solution and is one of the most useful <u>exact solutions</u> to the vacuum field equations. Although we have assumed the metric to be 1) static and 2) spherically symmetric it can be shown that any spherically symmetric vacuum solution of the field equations is static. The proof of this result is Birkhoff's theorem. This means that even if the "star" pulsates in the radial direction the exterior solution (11.42) remains valid.

APPENDIX A

In this appendix we shall consider more results from the tensor calculus.

1) Relative tensors: A relative tensor of weight N is a quantity which has the following transformation law under coordinate transformations:

$$T'^{\ell\ldots}_{k\ldots} = \left|\frac{\partial x}{\partial x'}\right|^N \frac{\partial x'^{\ell}}{\partial x^r} \ldots \frac{\partial x^s}{\partial x'^k} \ldots T^{r\ldots}_{s\ldots} \qquad (A.1)$$

Where $\left|\frac{\partial x}{\partial x'}\right|$ is the determinent of the matrix $\left(\frac{\partial x^\ell}{\partial x'^r}\right)$ or the Jacobian J of the transformation:

$$J = \det\left(\frac{\partial x^\ell}{\partial x'^r}\right) . \qquad (A.2)$$

If N = 0 then T is called an absolute tensor or just a tensor. If N = 1 then T is called a tensor density. Since

$$g'_{\ell k} = \frac{\partial x^r}{\partial x'^\ell}\frac{\partial x^s}{\partial x'^k} g_{rs}$$

we have

$$g' = J^2 g \qquad (A.3)$$

or

$$\sqrt{-g'} = J\sqrt{-g} \quad , \qquad (A.4)$$

that is $\sqrt{-g}$ is a relative scalor of weight +1. If $T^{\ell\ldots}_{k\ldots}$ is a tensor one can form a tensor density of plus one by writing

$$\underline{T}^{\ell\ldots}_{k\ldots} = \sqrt{-g}\; T^{\ell\ldots}_{k\ldots} . \qquad (A.5)$$

(We shall underline tensor densities of weight +1 sometimes.)

Under Coordinate transformations

$$\underline{T}'^{\ell\ldots}_{k\ldots} = J \frac{\partial x'^{\ell}}{\partial x^{r}} \ldots \frac{\partial x^{s}}{\partial x'^{k}} \ldots \underline{T}^{r\ldots}_{s\ldots} \quad . \tag{A.6}$$

Sometimes one finds quantities which transform with an odd power of J called pseudo (or axial) quantities. In these cases one is usually dealing with transformations for which $J = \pm 1$ and one must keep track of sign (J) under reflections. However, here such terminology would be redundant since we are treating tensor densities separately from the outset.

A less obvious tensor density is the permutation symbol

$$\varepsilon^{ijk\ell} \overset{*}{=} \varepsilon_{ijk\ell} = \begin{cases} +1, & i,j,k,\ell = \text{even permutation of } 1,2,3,4 \\ -1 & i,j,k,\ell = \text{odd} \quad \text{"} \quad \text{"} \quad \text{"} \\ 0 & \text{elsewhere} \end{cases} \tag{A.7}$$

(* => same value)

Any object with the symmetry of the permutation tensor is completely determined by only one of its non-zero components. That is if we know ε_{1234} we can generate all other components by merely switching indices on ε_{1234}. Form the quantity

$$f'_{rstu} = \frac{\partial x^{\ell}}{\partial x'^{r}} \frac{\partial x^{m}}{\partial x'^{s}} \frac{\partial x^{n}}{\partial x'^{t}} \frac{\partial x^{p}}{\partial x'^{u}} \varepsilon_{\ell mnp} \quad . \tag{A.8}$$

By switching indices on f'_{rstu} one can show that it has the same symmetry as ε_{rstu}, and therefore is determined from only one component, say, f'_{1234},

$$f'_{1234} = \frac{\partial x^{\ell}}{\partial x'^{1}} \frac{\partial x^{m}}{\partial x'^{2}} \frac{\partial x^{n}}{\partial x'^{3}} \frac{\partial x^{p}}{\partial x'^{4}} \varepsilon_{\ell mnp} \quad . \tag{A.9}$$

Now if one has a matrix a^{i}_{j} then the permutation tensor is related to the determinent of a^{i}_{j} in the following way

$$\det\left(a^i_j\right) = \varepsilon_{ijk\ell}\, a^i_1\, a^j_2\, a^k_3\, a^\ell_4 = \varepsilon^{ijk\ell}\, a^1_i\, a^2_j\, a^3_k\, a^4_\ell$$
$$= 1/4!\, \varepsilon^{ijk\ell}\, \varepsilon_{rstu}\, a^r_i\, a^s_j\, a^t_k\, a^u_\ell\ . \tag{A.10}$$

Returning to (A.9) we see that

$$f'_{1234} = J\ . \tag{A.11}$$

Define

$$f'_{rstu} = J\, \varepsilon'_{rstu}$$

where ε'_{rstu} is the permutation tensor in the primed frame. We then have

$$\varepsilon'_{rstu} = J^{-1} \frac{\partial x^\ell}{\partial x'^r} \frac{\partial x^m}{\partial x'^s} \frac{\partial x^n}{\partial x'^t} \frac{\partial x^p}{\partial x'^u}\, \varepsilon_{\ell mnp} \tag{A.12}$$

and hence the permutation tensor is a relative tensor of weight minus one. In a like manner one can show that ε'^{rstu} is a relative tensor of weight plus one.

2) Covariant differentiation of relative tensors.

Let A be a scalar density of weight N

$$A' = J^N A.$$

Taking the derivative of A' yields

$$\frac{dA'}{dx'^i} = J^N \cdot \frac{\partial x^\ell}{\partial x'^i} \left(\frac{dA}{dx^\ell}\right) + A \cdot NJ^{N-1} \frac{\partial J}{\partial x'^i}\ . \tag{A.13}$$

We have the following expression for J

$$J = \det\left(\frac{\partial x^i}{\partial x'^\ell}\right) = \Sigma_i \frac{\partial x^i}{\partial x'^\ell}\, C(i,\ell) \tag{A.14}$$

where $C(i,\ell)$ is the cofactor of $\dfrac{\partial x^i}{\partial x'^\ell}$. Thus

$$\frac{\partial J}{\partial x'^i} = \frac{\partial J}{\partial\left(\dfrac{\partial x^m}{\partial x'^n}\right)} \frac{\partial^2 x^m}{\partial x'^i \partial x'^n} , \tag{A.15}$$

or

$$\frac{\partial J}{\partial x'^i} = \frac{C(m,n)}{J} J \frac{\partial^2 x^m}{\partial x'^i \partial x'^n} . \tag{A.16}$$

The inverse of $\dfrac{\partial x^r}{\partial x'^s}$ is $\dfrac{\partial x'^\ell}{\partial x^k}$ and from matrix theory one know that $\dfrac{\partial x'^\ell}{\partial x^k}$ may be written

$$\frac{\partial x'^m}{\partial x^n} = \frac{C(n,m)}{J} . \tag{A.17}$$

Finally,

$$\frac{\partial J}{\partial x'^i} = \frac{\partial x'^n}{\partial x^m} \cdot J \frac{\partial^2 x^m}{\partial x'^i \partial x'^n} \tag{A.18}$$

and

$$\frac{dA'}{dx'^i} = J^N \frac{\partial x^\ell}{\partial x'^i} \frac{dA}{dx^\ell} + A \cdot NJ^N \frac{\partial x'^n}{\partial x^m} \frac{\partial^2 x^m}{\partial x'^i \partial x'^n} . \tag{A.19}$$

Multiplying

$$\Gamma'^n_{in} = \frac{\partial x^\ell}{\partial x'^i} \Gamma^n_{\ell n} + \frac{\partial x'^n}{\partial x^m} \circ \frac{\partial^2 x^m}{\partial x'^i \partial x'^n} \tag{A.20}$$

which can be derived from (8.16), by NA' and subtracting the resultant from (A.19) one obtains

$$\left(\frac{dA'}{dx^i} - NA'\Gamma'^n_{in}\right) = J^N \frac{\partial x^\ell}{\partial x'^i} \left(\frac{dA}{dx^\ell} - NA\,\Gamma^n_{\ell n}\right) . \tag{A.21}$$

Therefore the covariant derivative of a relative scalar of weight N is

$$A_{;i} = \frac{dA}{dx^i} - NA\,\Gamma^n_{in} . \tag{A.22}$$

Similarly for a relative tensor of weight N, $T^{\ell\ldots.}_{k\ldots.}$ the covariant derivative is

$$T^{\ell\cdots\cdot}_{k\cdots\cdot\,;i} = \frac{dT^{\ell\cdots\cdot}_{k\cdots\cdot}}{dx^i} + \Gamma^{\ell}_{pi}\, T^{p\cdots\cdot}_{k\cdots\cdot} + \cdots\cdot$$

$$-\Gamma^{p}_{ki}\, T^{\ell\cdots\cdot}_{p\cdots\cdot} \qquad \cdots\cdot - NT^{\ell\cdots\cdot}_{k}\, \Gamma^{s}_{si} \quad . \tag{A.23}$$

Using this result we can show that

$$\sqrt{-g}_{;r} = \frac{\partial\sqrt{-g}}{\partial x^r} - \sqrt{-g}\,\Gamma^{n}_{rn} = 0 \tag{A.24}$$

since

$$\frac{\partial\sqrt{-g}}{\partial x^r} = \sqrt{-g}\,\Gamma^{\ell}_{\ell r}$$

and hence $\sqrt{-g}$ is covariantly constant, a result which seems reasonable since $\sqrt{-g}$ only depends on g_{ij} which is covariantly constant. Because of (A.24) when covariantly differentiating a tensor density we can commute the $\sqrt{-g}$ with the differentiation

$$\left(\sqrt{-g}\; T^{k\cdots\cdot}_{\ell\cdots\cdot}\right)_{;r} = \sqrt{-g}\; T^{k\cdots\cdot}_{\ell\cdots\cdot\,;r} \quad . \tag{A.25}$$

3) Green's Theorem

The extension of a 4-cell spanned by the independent infinitesimal vectors $d_{(1)}x^{\ell}$, $d_{(2)}x^{\ell}$, $d_{(3)}x^{\ell}$, $d_{(4)}x^{\ell}$ is given by

$$dV_{(4)} = \begin{vmatrix} d_{(1)}x^1 & \cdots & d_{(4)}x^1 \\ \vdots & & \\ d_{(1)}x^4 & \cdots & d_{(4)}x^4 \end{vmatrix} \tag{A.26}$$

which can be written in terms of the permutation symbol

$$dV_{(4)} = \varepsilon_{\ell krs}\, d_{(1)}x^{\ell}\, d_{(2)}x^{k}\, d_{(3)}x^{r}\, d_{(4)}x^{s} \quad . \tag{A.27}$$

The orientation of the 4-cell is chosen so that $d_{(4)}V > 0$. The extension of a 3-cell span by the three independent infinitesimal vectors

$d_{(1)}x^\ell$, $d_{(2)}x^\ell$, and $d_{(3)}x^\ell$ is

$$dS_{(3)s} = \mathcal{E}_{s\ell kr}\, d_{(1)}x^\ell\, d_{(2)}x^k\, d_{(3)}x^r \quad . \tag{A.28}$$

Green's theorem states

$$\int_V T_{,r}\, dV_{(4)} = \oint_{Sv} T\, dS_{(3)r} \tag{A.29}$$

where V is a region in four space, Sv its surface and the orientation of $dS_{(3)r}$ is chosen so that

$$dS_{(3)r}\, \lambda^r > 0$$

for any vector λ^r pointing out of V as shown in Fig. A.1.

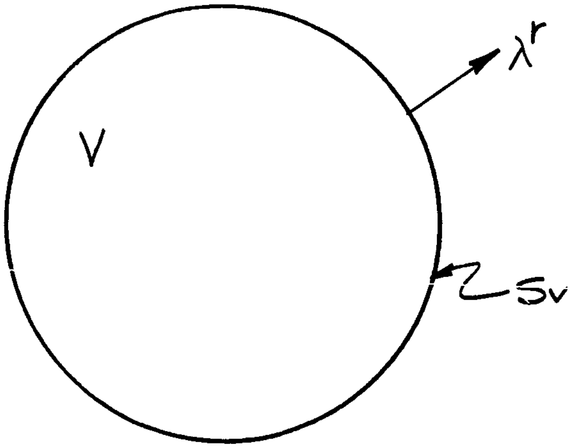

Fig. A.1

It will be noticed that Green's theorem (A.29) is independent of any metric considerations; it is merely the statement that one can "integrate by parts." From (A.12) one sees that the permutation symbol is a relative tensor of weight minus one, and therefore $dV_{(4)}$ $\left(dS_{(3)r}\right)$ is a relative scalar (vector) of weight minus one. If one has a relative tensor of weight plus on $\underline{T}^{k\ldots\ldots}_{\ell\ldots\ldots}$, then the products

$$\underline{T}^{k\ldots\ldots}_{\ell\ldots\ldots}\, dS_{(3)r}, \qquad \underline{T}^{k\ldots\ldots}_{\ell\ldots\ldots}\, dV_{(4)} \tag{A.30}$$

are absolute tensors. In particular since

$$\int \delta(x-y)\, dV_{(4)} = 1 \tag{A.31}$$

it follows that $\delta(x-y)$ is relative tensor of plus one or a scalar density. Here $\delta(x-y)$ is the four dimensional Dirac delta.

$$\delta(x-y) = \delta(x^1 - y^1)\, \delta(x^2 - y^2)\, \delta(x^3 - y^3)\, \delta(x^4 - y^4)\ .$$

If we pick the infinitesimal vectors which span the 4-cell to lie along the parametric coordinate lines then $dV_{(4)} = dx^1 dx^2 dx^3 dx^4$ and we can define the three dimensional Dirac delta as

$$\delta(\underline{x}-\underline{y}) = \int_{-\infty}^{+\infty} \delta(x-y)\, dx^4 \tag{A.32}$$

from (A.31) it then follows that

$$\int \delta(\underline{x}-\underline{y})\, dx^1 dx^2 dx^3 = 1$$

and therefore $\delta(\underline{x}-\underline{y})\, dx^1 dx^2 dx^3$ may be treated as an invariant. As an example we give the tensor formulas for curvilinear coordinates in Euclidean three dimensional space. In rectangular coordinates

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$g_{\ell k} = \delta_{\ell k} = (1,1,1)$$

therefore, a vector A can be written

$$A_\ell = (A_x, A_y, A_z) = A^\ell = (A^x, A^y, A^z) \; .$$

If we introduce orthogonal curvilinear coordinates $x'^\ell = f^\ell(x)$, then

$$g_{\ell k} = (g_{11}, g_{22}, g_{33}) \; ,$$

$$g^{\ell k} = (\frac{1}{g_{11}}, \frac{1}{g_{22}}, \frac{1}{g_{33}}) \; ,$$

$$A^\ell = (A^1, A^2, A^3)$$

$$A_\ell = (A_1, A_2, A_3)$$

and

$$A_\ell = g_{\ell k} A^k \; .$$

Using this last relation we find

$$A_1 = g_{11} A^1, \text{ etc. and } \frac{A_1}{\sqrt{g_{11}}} = \sqrt{g_{11}} A^1 \; .$$

The square of A is an invariant

$$A^2 = A_\ell A^\ell = A_1 A^1 + A_2 A^2 + A_3 A^3$$

$$A^2 = \frac{A_1}{\sqrt{g_{11}}} \sqrt{g_{11}} A^1 + \frac{A_2}{\sqrt{g_{22}}} \sqrt{g_{22}} A^2 + \frac{A_3}{\sqrt{g_{33}}} \sqrt{g_{33}} A^3 \; .$$

The "physical components" of the vector are defined as

$$A(1) = \frac{A_1}{\sqrt{g_{11}}} = \sqrt{g_{11}} A^1 \; , \text{ etc.}$$

If A is represented in the form

$$\underline{A} = A(\ell)\ \underline{u}^{\ell} ,$$

where $\underline{u}^{\ell}$ are unit vectors along the coordinate lines then

$$A^2 = \underline{A} \cdot \underline{A} = A(\ell)\ A(\ell) \qquad \text{(sum over } \ell\text{)}$$

the usual expression. In terms of the "physical components" the divergence can be written

$$\text{div } A = A^{\ell}{}_{;\ell} = A^{\ell}{}_{,\ell} + \Gamma^{\ell}_{r\ell}\ A^{r}$$

or

$$\text{div } A = \frac{1}{\sqrt{g}}\ \frac{\partial(\sqrt{g}\ A^{\ell})}{\partial x^{\ell}}$$

after using

$$\frac{\partial\sqrt{g}}{\partial x^{\ell}} = \sqrt{g}\ \Gamma^{k}_{\ell k}$$

which has been derived before. In terms of physical components

$$\text{div } A = \frac{1}{\sqrt{g}}\ \frac{\partial}{\partial x^{\ell}}\left(\frac{\sqrt{g}}{g_{\ell\ell}}\ A(\ell)\right) , \quad \text{(sum } \ell\text{)}$$

which can be checked by writing it out in spherical coordinates. If A^{ℓ} is the gradient of a scalar ϕ then

$$A^{\ell} = g^{\ell k}\ \frac{\partial\phi}{\partial x^{k}} \qquad \text{and}$$

$$\text{Laplacian } \phi = \text{Lap } \phi = \frac{1}{\sqrt{g}}\ \frac{\partial}{\partial x^{\ell}}\ (\sqrt{g}\ A^{\ell})$$

or

$$\text{Lap } \phi = \nabla^2\phi = \frac{1}{\sqrt{g}}\ \frac{\partial}{\partial x^{\ell}}\ (\sqrt{g}\ g^{\ell\ell}\ \frac{\partial\phi}{\partial x^{\ell}}) \qquad \text{(sum } \ell\text{)}$$

which can be easily checked.

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These lectures were given to graduate and undergraduate engineering and science students during the spring semester, 1966. The lectures cover both the special and general theories. The coverage of the special theory is fairly complete while that of the general theory is somewhat abbreviated due to lack of time. The lectures on the general theory stopped with the derivation of the Schwartzschild solution and a discussion of the linear theory.

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