

LECTURES ON THE LANGLANDS PROGRAM AND CONFORMAL FIELD THEORY

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INTRODUCTION

These lecture notes give an overview of recent results in geometric Langlands correspondence which may yield applications to quantum field theory. It has long been suspected that the Langlands duality should somehow be related to various dualities observed in quantum field theory and string theory. Indeed, both the Langlands correspondence and the dualities in physics have emerged as some sort of non-abelian Fourier transforms. Moreover, the so-called Langlands dual group introduced by R. Langlands in [1] that is essential in the formulation of the Langlands correspondence also plays a prominent role in the study of S-dualities in physics and was in fact also introduced by the physicists P. Goddard, J. Nuyts and D. Olive in the framework of four-dimensional gauge theory [2].

In recent lectures [3] E. Witten outlined a possible scenario of how the two dualities – the Langlands duality and the S-duality – could be related to each other. It is based on a dimensional reduction of a four-dimensional gauge theory to two dimensions and the analysis of what this reduction does to “D-branes”. In particular, Witten argued that the t’Hooft operators of the four-dimensional gauge theory recently introduced by A. Kapustin [4] become, after the dimensional reduction, the Hecke operators that are essential ingredients of the Langlands correspondence. Thus, a t’Hooft “eigenbrane” of the gauge theory becomes after the reduction a Hecke “eigensheaf”, an object of interest in the geometric Langlands correspondence. The work of Kapustin and Witten shows that the Langlands duality is indeed closely related to the S-duality of quantum field theory, and this opens up exciting possibilities for both subjects.

The goal of these notes is two-fold: first, it is to give a motivated introduction to the Langlands Program, including its geometric reformulation, addressed primarily to physicists. I have tried to make it as self-contained as possible, requiring very little mathematical background. The second goal is to describe the connections between the Langlands Program and two-dimensional conformal field theory that have been found in the last few years. These connections give us important insights into the physical implications of the Langlands duality.

The classical Langlands correspondence manifests a deep connection between number theory and representation theory. In particular, it relates subtle number theoretic data (such as the numbers of points of a mod p reduction of an elliptic curve defined by a cubic equation with integer coefficients) to more easily discernible data related to automorphic forms (such as the coefficients in the Fourier series expansion of a modular form on the upper half-plane). We will consider explicit examples of this relationship (having to do with the Taniyama-Shimura conjecture and Fermat’s last theorem) in Part I of this survey. So the origin of the Langlands Program is in *number theory*. Establishing the Langlands correspondence in this context has proved to be extremely hard. But number fields have close relatives called *function fields*, the fields of functions on algebraic curves defined over a finite field. The Langlands correspondence has a counterpart for function fields, which is much better understood, and this will be the main subject of our interest in this survey.

Function fields are defined geometrically (via algebraic curves), so one can use geometric intuition and geometric technique to elucidate the meaning of the Langlands correspondence. This is actually the primary reason why the correspondence is easier to understand in the function field context than in the number field context. Even more ambitiously, one can now try to switch from curves defined over finite fields to curves defined over the complex field – that is to Riemann surfaces. This requires a reformulation, called the *geometric Langlands correspondence*. This reformulation effectively puts the Langlands correspondence in the realm of complex algebraic geometry.

Roughly speaking, the geometric Langlands correspondence predicts that to each rank n holomorphic vector bundle E with a holomorphic connection on a complex algebraic curve X one can attach an object called *Hecke eigensheaf* on the moduli space Bun_n of rank n holomorphic vector bundles on X :

$$\boxed{\text{holomorphic rank } n \text{ bundles} \\ \text{with connection on } X} \longrightarrow \boxed{\text{Hecke eigensheaves on } \text{Bun}_n}$$

A Hecke eigensheaf is a \mathcal{D} -module on Bun_n satisfying a certain property that is determined by E . More generally, if G is a complex reductive Lie group, and ${}^L G$ is the Langlands dual group, then to a holomorphic ${}^L G$ -bundle with a holomorphic connection on X we should attach a Hecke eigensheaf on the moduli space Bun_G of holomorphic G -bundles on X :

$$\boxed{\text{holomorphic } {}^L G\text{-bundles} \\ \text{with connection on } X} \longrightarrow \boxed{\text{Hecke eigensheaves on } \text{Bun}_G}$$

I will give precise definitions of these objects in Part II of this survey.

The main point is that we can use methods of two-dimensional *conformal field theory* to construct Hecke eigensheaves. Actually, the analogy between conformal field theory and the theory of automorphic representations was already observed a long time ago by E. Witten [5]. However, at that time the geometric Langlands correspondence had not yet been developed. As we will see, the geometric reformulation of the classical theory of automorphic representations will allow us to make the connection to conformal field theory more precise.

To explain how this works, let us recall that chiral correlation functions in a (rational) conformal field theory [6] may be interpreted as sections of a holomorphic vector bundle on the moduli space of curves, equipped with a projectively flat connection [7]. The connection comes from the Ward identities expressing the variation of correlation functions under deformations of the complex structure on the underlying Riemann surface via the insertion in the correlation function of the stress tensor, which generates the Virasoro algebra symmetry of the theory. These bundles with projectively flat connection have been studied in the framework of Segal's axioms of conformal field theory [8].

Likewise, if we have a rational conformal field theory with affine Lie algebra symmetry [9], such as a Wess-Zumino-Witten (WZW) model [10], then conformal blocks give rise to sections of a holomorphic vector bundle with a projectively flat connection on the moduli space of G -bundles on X . The projectively flat connection comes from the Ward identities corresponding to the affine Lie algebra symmetry, which are expressed via the insertions of the currents generating an affine Lie algebra, as I recall in Part III of this survey.

Now observe that the sheaf of holomorphic sections of a holomorphic vector bundle \mathcal{E} over a manifold M with a holomorphic flat connection ∇ is the simplest example of a holonomic \mathcal{D} -module on M . Indeed, we can multiply a section ϕ of \mathcal{E} over an open subset $U \subset M$ by any holomorphic function on U , and we can differentiate ϕ with respect to a holomorphic vector field ξ defined on U by using the connection operators: $\phi \mapsto \nabla_\xi \phi$. Therefore we obtain an action of the sheaf of holomorphic differential operators on the sheaf of holomorphic sections of our bundle \mathcal{E} . If ∇ is only projectively flat, then we obtain instead of a \mathcal{D} -module what is called a *twisted* \mathcal{D} -module. However, apart from bundles with a projectively flat connection, there exist other holonomic twisted \mathcal{D} -modules. For example, a (holonomic) system of differential equations on M defines a (holonomic) \mathcal{D} -module on M . If these equations have singularities on some divisors in M , then the sections of these \mathcal{D} -module will also have singularities along those divisors (and non-trivial monodromies around those divisors), unlike the sections of just a plain bundle with connection.

Applying the conformal blocks construction to a general conformal field theory, one obtains (twisted) \mathcal{D} -modules on the moduli spaces of curves and bundles. In some conformal field theories, such as the WZW models, these \mathcal{D} -module are bundles with projectively flat connections. But in other theories we obtain \mathcal{D} -modules that are more sophisticated: for example, they may correspond to differential equations with singularities along divisors, as we will see below. It turns out that the Hecke eigensheaves that we are looking for can be obtained this way. The fact that they do not correspond to bundles with projectively flat connection is perhaps the main reason why these \mathcal{D} -modules have, until now, not caught the attention of physicists.

There are in fact at least two known scenarios in which the construction of conformal blocks gives rise to \mathcal{D} -modules on Bun_G that are (at least conjecturally) the Hecke eigensheaves whose existence is predicted by the geometric Langlands correspondence. Let us briefly describe these two scenarios.

In the first scenario we consider an affine Lie algebra at the critical level, $k = -h^\vee$, where h^\vee is the dual Coxeter number. At the critical level the Segal-Sugawara current becomes commutative, and so we have a “conformal field theory” without a stress tensor. This may sound absurd to a physicist, but from the mathematical perspective this liability actually turns into an asset. Indeed, even though we do not have the Virasoro symmetry, we still have the affine Lie algebra symmetry, and so we can apply the conformal blocks construction to obtain a \mathcal{D} -module on the moduli space of G -bundles on a Riemann surface X (though we cannot vary the complex structure on X). Moreover, because the Segal-Sugawara current is now commutative, we can force it to be equal to any numeric (or, as a physicist would say, “ c -number”) projective connection on our curve X . So our “conformal field theory”, and hence the corresponding \mathcal{D} -module, will depend on a continuous parameter: a projective connection on X .

In the case of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ the Segal-Sugawara field generates the center of the chiral algebra of level $k = -2$. For a general affine Lie algebra $\widehat{\mathfrak{g}}$, the center of the chiral algebra has $\ell = \text{rank } \mathfrak{g}$ generating fields, and turns out to be canonically isomorphic to a classical limit of the \mathcal{W} -algebra associated to the Langlands dual group ${}^L G$, as shown in [11, 12]. This isomorphism is obtained as a limit of a certain isomorphism of \mathcal{W} -algebras that naturally arises in the context of *T-duality* of free bosonic theories compactified on

tori. I will recall this construction below. So from this point of view the appearance of the Langlands dual group is directly linked to the T-duality of bosonic sigma models.

The classical \mathcal{W} -algebra of ${}^L G$ is the algebra of functions on the space of gauge equivalence classes of connections on the circle introduced originally by V. Drinfeld and V. Sokolov [13] in their study of the generalized KdV hierarchies. The Drinfeld-Sokolov construction has been recast in a more geometric way by A. Beilinson and V. Drinfeld, who called these gauge equivalence classes ${}^L G$ -opers [14]. For a general affine Lie algebra $\widehat{\mathfrak{g}}$ the procedure of equating the Segal-Sugawara current to a numeric projective connection becomes the procedure of equating the generating fields of the center to the components of a numeric ${}^L G$ -oper E on X . Thus, we obtain a family of “conformal field theories” depending on ${}^L G$ -opers on X , and we then take the corresponding \mathcal{D} -modules of conformal blocks on the moduli space Bun_G of G -bundles on X .

A marvelous result of A. Beilinson and V. Drinfeld [15] is that the \mathcal{D} -module corresponding to a ${}^L G$ -oper E is nothing but the sought-after Hecke eigensheaf with “eigenvalue” E ! Thus, “conformal field theory” of the critical level $k = -h^\vee$ solves the problem of constructing Hecke eigensheaves, at least for those ${}^L G$ -bundles with connection which admit the structure of a ${}^L G$ -oper (other flat ${}^L G$ -bundles can be dealt with similarly). This is explained in Part III of this survey.

In the second scenario one considers a conformal field theory with affine Lie algebra symmetry of integral level k that is less than $-h^\vee$, so it is in some sense opposite to the traditional WZW model, where the level is a positive integer. In fact, theories with such values of level have been considered by physicists in the framework of the WZW models corresponding to non-compact Lie groups, such as $SL_2(\mathbb{R})$ (they have many similarities to the Liouville theory, as was pointed out already in [16]). Beilinson and Drinfeld have defined explicitly an extended chiral algebra in such a theory, which they called the *chiral Hecke algebra*. In addition to the action of an affine Lie algebra $\widehat{\mathfrak{g}}$, it carries an action of the Langlands dual group ${}^L G$ by symmetries. If G is abelian, then the chiral Hecke algebra is nothing but the chiral algebra of a free boson compactified on a torus. Using the ${}^L G$ -symmetry, we can “twist” this extended chiral algebra by any ${}^L G$ -bundle with connection E on our Riemann surface X , and so for each E we now obtain a particular chiral conformal field theory on X . Beilinson and Drinfeld have conjectured that the corresponding sheaf of conformal blocks on Bun_G is a Hecke eigensheaf with the “eigenvalue” E . I will not discuss this construction in detail in this survey referring the reader instead to [17], Sect. 4.9, and [18] where the abelian case is considered and the reviews in [19] and [20], Sect. 20.5.

These two examples show that the methods of two-dimensional conformal field theory are powerful and flexible enough to give us important examples of the geometric Langlands correspondence. This is the main message of this survey.

These notes are split into three parts: the classical Langlands Program, its geometric reformulation, and the conformal field theory approach to the geometric Langlands correspondence. They may be read independently from each other, so a reader who is primarily interested in the geometric side of the story may jump ahead to Part II, and a reader who wants to know what conformal field theory has to do with this subject may very well start with Part III and later go back to Parts I and II to read about the origins of the Langlands Program.

Here is a more detailed description of the material presented in various parts.

Part I gives an introduction to the classical Langlands correspondence. We start with some basic notions of number theory and then discuss the Langlands correspondence for number fields such as the field of rational numbers. I consider in detail a specific example which relates modular forms on the upper half-plane and two-dimensional representations of the Galois group coming from elliptic curves. This correspondence, known as the Taniyama-Shimura conjecture, is particularly important as it gives, among other things, the proof of Fermat's last theorem. It is also a good illustration for the key ingredients of the Langlands correspondence. Next, we switch from number fields to function fields underscoring the similarities and differences between the two cases. I formulate more precisely the Langlands correspondence for function fields, which has been proved by V. Drinfeld and L. Lafforgue.

Part II introduces the geometric reformulation of the Langlands correspondence. I tried to motivate every step of this reformulation and at the same time avoid the most difficult technical issues involved. In particular, I describe in detail the progression from functions to sheaves to perverse sheaves to \mathcal{D} -modules, as well as the link between automorphic representations and moduli spaces of bundles. I then formulate the geometric Langlands conjecture for GL_n (following Drinfeld and Laumon) and discuss it in great detail in the abelian case $n = 1$. This brings into the game some familiar geometric objects, such as the Jacobian, as well as the Fourier-Mukai transform. Next, we discuss the ingredients necessary for formulating the Langlands correspondence for arbitrary reductive groups. In particular, we discuss in detail the affine Grassmannian, the Satake correspondence and its geometric version. At the end of Part II we speculate about a possible non-abelian extension of the Fourier-Mukai transform and its "quantum" deformation.

Part III is devoted to the construction of Hecke eigensheaves in the framework of conformal field theory, following the work of Beilinson and Drinfeld [15]. I start by recalling the notions of conformal blocks and bundles of conformal blocks in conformal field theories with affine Lie algebra symmetry, first as bundles (or sheaves) over the moduli spaces of pointed Riemann surfaces and then over the moduli spaces of G -bundles. I discuss in detail the familiar example of the WZW models. Then I consider the center of the chiral algebra of an affine Lie algebra $\widehat{\mathfrak{g}}$ of critical level and its isomorphism with the classical \mathcal{W} -algebra associated to the Langlands dual group ${}^L G$ following [11, 12]. I explain how this isomorphism arises in the context of T-duality. We then use this isomorphism to construct representations of $\widehat{\mathfrak{g}}$ attached to geometric objects called opers. The sheaves of coinvariants corresponding to these representations are the sought-after Hecke eigensheaves. I also discuss the connection with the Hitchin system and a generalization to more general flat ${}^L G$ -bundles, with and without ramification.

Even in a long survey it is impossible to cover all essential aspects of the Langlands Program. To get a broader picture, I recommend the interested reader to consult the informative reviews [21]–[27]. My earlier review articles [28, 29] contain some of the material of the present notes in a more concise form as well as additional topics not covered here.

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Part I. The origins of the Langlands Program

In the first part of this article I review the origins of the Langlands Program. We start by recalling some basic notions of number theory (Galois group, Frobenius elements, abelian class field theory). We then consider examples of the Langlands correspondence for the group GL_2 over the rational adèles. These examples relate in a surprising and non-trivial way modular forms on the upper half-plane and elliptic curves defined over rational numbers. Next, we recall the analogy between number fields and function fields. In the context of function fields the Langlands correspondence has been established in the works of V. Drinfeld and L. Lafforgue. We give a precise formulation of their results.

1. THE LANGLANDS CORRESPONDENCE OVER NUMBER FIELDS

1.1. Galois group. Let us start by recalling some notions from number theory. A *number field* is by definition a finite extension of the field \mathbb{Q} of rational numbers, i.e., a field containing \mathbb{Q} which is a finite-dimensional vector space over \mathbb{Q} . Such a field F is necessarily an algebraic extension of \mathbb{Q} , obtained by adjoining to \mathbb{Q} roots of polynomials with coefficients in \mathbb{Q} . For example, the field

$$\mathbb{Q}(i) = \{a + bi \mid a \in \mathbb{Q}, b \in \mathbb{Q}\}$$

is obtained from \mathbb{Q} by adjoining the roots of the polynomial $x^2 + 1$, denoted by i and $-i$. The coefficients of this polynomial are rational numbers, so the polynomial is *defined over* \mathbb{Q} , but its roots are not. Therefore adjoining them to \mathbb{Q} we obtain a larger field, which has dimension 2 as a vector space over \mathbb{Q} .

More generally, adjoining to \mathbb{Q} a primitive N th root of unity ζ_N we obtain the N th *cyclotomic field* $\mathbb{Q}(\zeta_N)$. Its dimension over \mathbb{Q} is $\varphi(N)$, the Euler function of N : the number of integers between 1 and N such that $(m, N) = 1$ (this notation means that m is relatively prime to N). We can embed $\mathbb{Q}(\zeta_N)$ into \mathbb{C} in such a way that $\zeta_N \mapsto e^{2\pi i/N}$, but this is not the only possible embedding of $\mathbb{Q}(\zeta_N)$ into \mathbb{C} ; we could also send $\zeta_N \mapsto e^{2\pi im/N}$, where $(m, N) = 1$.

Suppose now that F is a number field, and let K be its finite extension, i.e., another field containing F , which has finite dimension as a vector space over F . This dimension is called the *degree* of this extension and is denoted by $\deg_F K$. The group of all field automorphisms σ of K , preserving the field structures and such that $\sigma(x) = x$ for all $x \in F$, is called the *Galois group* of K/F and denoted by $\text{Gal}(K/F)$. Note that if K' is a normal extension of K , then any field automorphism of K' will preserve K (although not pointwise), and so we have a natural homomorphism $\text{Gal}(K'/F) \rightarrow \text{Gal}(K/F)$. Its kernel is the normal subgroup of those elements that fix K pointwise, i.e., it is isomorphic to $\text{Gal}(K'/K)$.

For example, the Galois group $\text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ is naturally identified with the group

$$(\mathbb{Z}/N\mathbb{Z})^\times = \{[n] \in \mathbb{Z}/N\mathbb{Z} \mid (n, N) = 1\},$$

with respect to multiplication. The element $[n] \in (\mathbb{Z}/N\mathbb{Z})^\times$ gives rise to the automorphism of $\mathbb{Q}(\zeta_N)$ sending ζ_N to ζ_N^n , and hence ζ_N^m to ζ_N^{mn} for all m . If M divides N , then $\mathbb{Q}(\zeta_M)$ is contained in $\mathbb{Q}(\zeta_N)$, and the corresponding homomorphism of the Galois groups $\text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \rightarrow \text{Gal}(\mathbb{Q}(\zeta_M)/\mathbb{Q})$ coincides, under the above identification, with the natural surjective homomorphism

$$p_{N,M} : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow (\mathbb{Z}/M\mathbb{Z})^\times,$$

sending $[n]$ to $[n] \bmod M$.

The field obtained from F by adjoining the roots of all polynomials defined over F is called the *algebraic closure* of F and is denoted by \overline{F} . Its group of symmetries is the Galois group $\text{Gal}(\overline{F}/F)$. Describing the structure of these Galois groups is one of the main questions of number theory.

1.2. Abelian class field theory. While at the moment we do not have a good description of the entire group $\text{Gal}(\overline{F}/F)$, it has been known for some time what is the *maximal abelian quotient* of $\text{Gal}(\overline{F}/F)$ (i.e., the quotient by the commutator subgroup). This quotient is naturally identified with the Galois group of the maximal abelian extension F^{ab} of F . By definition, F^{ab} is the largest of all subfields of \overline{F} whose Galois group is abelian.

For $F = \mathbb{Q}$, the classical Kronecker-Weber theorem says that the maximal abelian extension \mathbb{Q}^{ab} is obtained by adjoining to \mathbb{Q} all roots of unity. In other words, \mathbb{Q}^{ab} is the union of all cyclotomic fields $\mathbb{Q}(\zeta_N)$ (where $\mathbb{Q}(\zeta_M)$ is identified with the corresponding subfield of $\mathbb{Q}(\zeta_N)$ for M dividing N). Therefore we obtain that the Galois group $\text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q})$ is isomorphic to the inverse limit of the groups $(\mathbb{Z}/N\mathbb{Z})^\times$ with respect to the system of surjections $p_{N,M} : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow (\mathbb{Z}/M\mathbb{Z})^\times$ for M dividing N :

$$(1.1) \quad \text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q}) \simeq \varprojlim (\mathbb{Z}/N\mathbb{Z})^\times.$$

By definition, an element of this inverse limit is a collection (x_N) , $N > 1$, of elements of $(\mathbb{Z}/N\mathbb{Z})^\times$ such that $p_{N,M}(x_N) = x_M$ for all pairs N, M such that M divides N .

This inverse limit may be described more concretely using the notion of *p-adic numbers*.

Recall (see, e.g., [30]) that if p is a prime, then a *p-adic number* is an infinite series of the form

$$(1.2) \quad a_k p^k + a_{k+1} p^{k+1} + a_{k+2} p^{k+2} + \dots,$$

where each a_k is an integer between 0 and $p - 1$, and we choose $k \in \mathbb{Z}$ in such a way that $a_k \neq 0$. One defines addition and multiplication of such expressions by “carrying” the result of powerwise addition and multiplication to the next power. One checks that with respect to these operations the *p-adic numbers* form a field denoted by \mathbb{Q}_p (for example, it is possible to find the inverse of each expression (1.2) by solving the obvious system of recurrence relations). It contains the subring \mathbb{Z}_p of *p-adic integers* which consists of the above expressions with $k \geq 0$. It is clear that \mathbb{Q}_p is the field of fractions of \mathbb{Z}_p .

Note that the subring of \mathbb{Z}_p consisting of all finite series of the form (1.2) with $k \geq 0$ is just the ring of integers \mathbb{Z} . The resulting embedding $\mathbb{Z} \hookrightarrow \mathbb{Z}_p$ gives rise to the embedding $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$.

It is important to observe that \mathbb{Q}_p is in fact a *completion* of \mathbb{Q} . To see that, define a norm $|\cdot|_p$ on \mathbb{Q} by the formula $|p^k a/b|_p = p^{-k}$, where a, b are integers relatively prime to

p . With respect to this norm p^k becomes smaller and smaller as $k \rightarrow +\infty$ (in contrast to the usual norm where p^k becomes smaller as $k \rightarrow -\infty$). That is why the completion of \mathbb{Q} with respect to this norm is the set of all infinite series of the form (1.2), going “in the wrong direction”. This is precisely the field \mathbb{Q}_p . This norm extends uniquely to \mathbb{Q}_p , with the norm of the p -adic number (1.2) (with $a_k \neq 0$ as was our assumption) being equal to p^{-k} .

In fact, according to Ostrowski’s theorem, any completion of \mathbb{Q} is isomorphic to either \mathbb{Q}_p or to the field \mathbb{R} of real numbers.

Now observe that if $N = \prod_p p^{m_p}$ is the prime factorization of N , then $\mathbb{Z}/N\mathbb{Z} \simeq \prod_p \mathbb{Z}/p^{m_p}\mathbb{Z}$. Let $\widehat{\mathbb{Z}}$ be the inverse limit of the rings $\mathbb{Z}/N\mathbb{Z}$ with respect to the natural surjections $\mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{Z}/M\mathbb{Z}$ for M dividing N :

$$(1.3) \quad \widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/N\mathbb{Z}.$$

It follows that

$$\widehat{\mathbb{Z}} \simeq \prod_p \left(\varprojlim \mathbb{Z}/p^r\mathbb{Z} \right),$$

where the inverse limit in the brackets is taken with respect to the natural surjective homomorphisms $\mathbb{Z}/p^r\mathbb{Z} \rightarrow \mathbb{Z}/p^s\mathbb{Z}$, $r > s$. But this inverse limit is nothing but \mathbb{Z}_p ! So we find that

$$(1.4) \quad \widehat{\mathbb{Z}} \simeq \prod_p \mathbb{Z}_p.$$

Note that $\widehat{\mathbb{Z}}$ defined above is actually a ring. The Kronecker-Weber theorem (1.1) implies that $\text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q})$ is isomorphic to the multiplicative group $\widehat{\mathbb{Z}}^\times$ of invertible elements of the ring $\widehat{\mathbb{Z}}$. But we find from (1.4) that $\widehat{\mathbb{Z}}^\times$ is nothing but the direct product of the multiplicative groups \mathbb{Z}_p^\times of the rings of p -adic integers where p runs over the set of all primes. We thus conclude that

$$\text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q}) \simeq \widehat{\mathbb{Z}}^\times \simeq \prod_p \mathbb{Z}_p^\times.$$

An analogue of the Kronecker-Weber theorem describing the maximal abelian extension F^{ab} of an arbitrary number field F is unknown in general. But the *abelian class field theory* (ACFT – no pun intended!) describes its Galois group $\text{Gal}(F^{\text{ab}}/F)$, which is the maximal abelian quotient of $\text{Gal}(\overline{F}/F)$. It states that $\text{Gal}(F^{\text{ab}}/F)$ is isomorphic to the group of connected components of the quotient $F^\times \backslash \mathbb{A}_F^\times$. Here \mathbb{A}_F^\times is the multiplicative group of invertible elements in the ring \mathbb{A}_F of *adèles* of F , which is a subring in the direct product of all completions of F .

We define the adèles first in the case when $F = \mathbb{Q}$. In this case, as we mentioned above, the completions of \mathbb{Q} are the fields \mathbb{Q}_p of p -adic numbers, where p runs over the set of all primes p , and the field \mathbb{R} of real numbers. Hence the ring $\mathbb{A}_\mathbb{Q}$ is a subring of the direct product of the fields \mathbb{Q}_p . More precisely, elements of $\mathbb{A}_\mathbb{Q}$ are the collections $((f_p)_{p \in P}, f_\infty)$, where $f_p \in \mathbb{Q}_p$ and $f_\infty \in \mathbb{R}$, satisfying the condition that $f_p \in \mathbb{Z}_p$ for all but finitely many p ’s. It follows from the definition that

$$\mathbb{A}_\mathbb{Q} \simeq (\widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}) \times \mathbb{R}.$$

We give the ring $\widehat{\mathbb{Z}}$ defined by (1.3) the topology of direct product, \mathbb{Q} the discrete topology and \mathbb{R} its usual topology. This defines $\mathbb{A}_{\mathbb{Q}}$ the structure of topological ring on $\mathbb{A}_{\mathbb{Q}}$. Note that we have a diagonal embedding $\mathbb{Q} \hookrightarrow \mathbb{A}_{\mathbb{Q}}$ and the quotient

$$\mathbb{Q} \backslash \mathbb{A}_{\mathbb{Q}} \simeq \widehat{\mathbb{Z}} \times (\mathbb{R}/\mathbb{Z})$$

is compact. This is in fact the reason for the above condition that almost all f_p 's belong to \mathbb{Z}_p . We also have the multiplicative group $\mathbb{A}_{\mathbb{Q}}^{\times}$ of invertible adèles (also called idèles) and a natural diagonal embedding of groups $\mathbb{Q}^{\times} \hookrightarrow \mathbb{A}_{\mathbb{Q}}^{\times}$.

In the case when $F = \mathbb{Q}$, the statement of ACFT is that $\text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q})$ is isomorphic to the group of connected components of the quotient $\mathbb{Q}^{\times} \backslash \mathbb{A}_{\mathbb{Q}}^{\times}$. It is not difficult to see that

$$\mathbb{Q}^{\times} \backslash \mathbb{A}_{\mathbb{Q}}^{\times} \simeq \prod_p \mathbb{Z}_p^{\times} \times \mathbb{R}_{>0}.$$

Hence the group of its connected components is isomorphic to $\prod_p \mathbb{Z}_p^{\times}$, in agreement with the Kronecker-Weber theorem.

For an arbitrary number field F one defines the ring \mathbb{A}_F of adèles in a similar way. Like \mathbb{Q} , any number field F has non-archimedean completions parameterized by prime ideals in its *ring of integers* \mathcal{O}_F . By definition, \mathcal{O}_F consists of all elements of F that are roots of monic polynomials with coefficients in F ; monic means that the coefficient in front of the highest power is equal to 1. The corresponding norms on F are defined similarly to the p -adic norms, and the completions look like the fields of p -adic numbers (in fact, each of them is isomorphic to a finite extension of \mathbb{Q}_p for some p). There are also archimedean completions, which are isomorphic to either \mathbb{R} or \mathbb{C} , parameterized by the real and complex embeddings of F . The corresponding norms are obtained by taking the composition of an embedding of F into \mathbb{R} or \mathbb{C} and the standard norm on the latter.

We denote these completions by F_v , where v runs over the set of equivalence classes of norms on F . Each of the non-archimedean completions contains its own “ring of integers”, denoted by \mathcal{O}_v , which is defined similarly to \mathbb{Z}_p . Now \mathbb{A}_F is defined as the restricted product of all (non-isomorphic) completions. Restricted means that it consists of those collections of elements of F_v which belong to the ring of integers $\mathcal{O}_v \subset F_v$ for all but finitely many v 's corresponding to the non-archimedean completions. The field F diagonally embeds into \mathbb{A}_F , and the multiplicative group F^{\times} of F into the multiplicative group \mathbb{A}_F^{\times} of invertible elements of \mathbb{A}_F . Hence the quotient $F^{\times} \backslash \mathbb{A}_F^{\times}$ is well-defined as an abelian group.

The statement of ACFT is now

$$(1.5) \quad \boxed{\text{Galois group } \text{Gal}(F^{\text{ab}}/F)} \simeq \boxed{\text{group of connected components of } F^{\times} \backslash \mathbb{A}_F^{\times}}$$

In addition, this isomorphism satisfies a very important property, which rigidifies it. In order to explain it, we need to introduce the Frobenius automorphisms, which we do in the next subsection.

1.3. Frobenius automorphisms. Let us look at the extensions of the finite field of p elements \mathbb{F}_p , where p is a prime. It is well-known that there is a unique, up to an isomorphism, extension of \mathbb{F}_p of degree $n = 1, 2, \dots$ (see, e.g., [30]). It then has $q = p^n$ elements and is denoted by \mathbb{F}_q . The Galois group $\text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ is isomorphic to the cyclic group $\mathbb{Z}/n\mathbb{Z}$. A generator of this group is the *Frobenius automorphism*, which sends $x \in \mathbb{F}_q$ to $x^p \in \mathbb{F}_q$. It is clear from the binomial formula that this is indeed a field automorphism of \mathbb{F}_q . Moreover, $x^p = x$ for all $x \in \mathbb{F}_p$, so it preserves all elements of \mathbb{F}_p . It is also not difficult to show that this automorphism has order exactly n and that all automorphisms of \mathbb{F}_q preserving \mathbb{F}_p are its powers. Under the isomorphism $\text{Gal}(\mathbb{F}_q/\mathbb{F}_p) \simeq \mathbb{Z}/n\mathbb{Z}$ the Frobenius automorphism goes to $1 \bmod n$.

Observe that the field \mathbb{F}_q may be included as a subfield of $\mathbb{F}_{q'}$ whenever $q' = q^n$. The algebraic closure $\overline{\mathbb{F}_p}$ of \mathbb{F}_p is therefore the union of all fields $\mathbb{F}_q, q = p^n, n > 0$, with respect to this system of inclusions. Hence the Galois group $\text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ is the inverse limit of the cyclic groups $\mathbb{Z}/n\mathbb{Z}$ and hence is isomorphic to $\widehat{\mathbb{Z}}$ introduced in formula (1.3).

Likewise, the Galois group $\text{Gal}(\mathbb{F}_{q'}/\mathbb{F}_q)$, where $q' = q^n$, is isomorphic to the cyclic group $\mathbb{Z}/n'\mathbb{Z}$ generated by the automorphism $x \mapsto x^q$, and hence $\text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ is isomorphic to $\widehat{\mathbb{Z}}$ for any q that is a power of a prime. The group $\widehat{\mathbb{Z}}$ has a preferred element which projects onto $1 \bmod n$ under the homomorphism $\widehat{\mathbb{Z}} \rightarrow \mathbb{Z}/n\mathbb{Z}$. Inside $\widehat{\mathbb{Z}}$ it generates the subgroup $\mathbb{Z} \subset \widehat{\mathbb{Z}}$, of which $\widehat{\mathbb{Z}}$ is a completion, and so it may be viewed as a topological generator of $\widehat{\mathbb{Z}}$. We will call it the Frobenius automorphism of $\overline{\mathbb{F}_q}$.

Now, the main object of our interest is the Galois group $\text{Gal}(\overline{F}/F)$ for a number field F . Can we relate this group to the Galois groups $\text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$? It turns out that the answer is yes. In fact, by making this connection, we will effectively transport the Frobenius automorphisms to $\text{Gal}(\overline{F}/F)$.

Let us first look at a finite extension K of a number field F . Let v be a prime ideal in the ring of integers \mathcal{O}_F . The ring of integers \mathcal{O}_K contains \mathcal{O}_F and hence v . The ideal (v) of \mathcal{O}_K generated by v splits as a product of prime ideals of \mathcal{O}_K . Let us pick one of them and denote it by w . Note that the *residue field* \mathcal{O}_F/v is a finite field, and hence isomorphic to \mathbb{F}_q , where q is a power of a prime. Likewise, \mathcal{O}_K/w is a finite field isomorphic to $\mathbb{F}_{q'}$, where $q' = q^n$. Moreover, \mathcal{O}_K/w is an extension of \mathcal{O}_F/v . The Galois group $\text{Gal}(\mathcal{O}_K/w, \mathcal{O}_L/v)$ is thus isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

Define the *decomposition group* D_w of w as the subgroup of the Galois group $\text{Gal}(K/F)$ of those elements σ that preserve the ideal w , i.e., such that for any $x \in w$ we have $\sigma(x) \in w$. Since any element of $\text{Gal}(K/F)$ preserves F , and hence the ideal v of F , we obtain a natural homomorphism $D_w \rightarrow \text{Gal}(\mathcal{O}_K/w, \mathcal{O}_L/v)$. One can show that this homomorphism is surjective.

The *inertia group* I_w of w is by definition the kernel of this homomorphism. The extension K/F is called *unramified* at v if $I_w = \{1\}$. If this is the case, then we have

$$D_w \simeq \text{Gal}(\mathcal{O}_K/w, \mathcal{O}_L/v) \simeq \mathbb{Z}/n\mathbb{Z}.$$

The Frobenius automorphism generating $\text{Gal}(\mathcal{O}_K/w, \mathcal{O}_L/v)$ can therefore be considered as an element of D_w , denoted by $\text{Fr}[w]$. If we replace w by another prime ideal of \mathcal{O}_K that occurs in the decomposition of (v) , then $D_{w'} = sD_w s^{-1}, I_{w'} = sI_w s^{-1}$ and $\text{Fr}[w'] = s \text{Fr}[w] s^{-1}$ for some $s \in \text{Gal}(K/F)$. Therefore the *conjugacy class* of $\text{Fr}[w]$ is a well-defined

conjugacy class in $\text{Gal}(K/F)$ which depends only on v , provided that $I_w = \{1\}$ (otherwise, for each choice of w we only get a coset in D_w/I_w). We will denote it by $\text{Fr}(v)$ and call it the Frobenius conjugacy class.

The Frobenius conjugacy classes $\text{Fr}(v)$ attached to the unramified prime ideals v in F contain important information about the extension K . For example, knowing the order of $\text{Fr}(v)$ we can figure out how many primes occur in the prime decomposition of (v) in K . Namely, if $(v) = w_1 \dots w_g$ is the decomposition of (v) into prime ideals of K^1 and the order of the Frobenius class is f ,² then $fg = \deg_F K$. The number g is an important number-theoretic characteristic, as one can see from the following example.

Let $F = \mathbb{Q}$ and $K = \mathbb{Q}(\zeta_N)$, the cyclotomic field, which is an extension of degree $\varphi(N) = |(\mathbb{Z}/N\mathbb{Z})^\times|$ (the Euler function of N). The Galois group $\text{Gal}(K/F)$ is isomorphic to $(\mathbb{Z}/N\mathbb{Z})^\times$ as we saw above. The ring of integers \mathcal{O}_F of F is \mathbb{Z} and $\mathcal{O}_K = \mathbb{Z}[\zeta_N]$. The prime ideals in \mathbb{Z} are just prime numbers, and it is easy to see that $\mathbb{Q}(\zeta_N)$ is unramified at the prime ideal $p\mathbb{Z} \subset \mathbb{Z}$ if and only if p does not divide N . In that case we have $(p) = \mathcal{P}_1 \dots \mathcal{P}_r$, where the \mathcal{P}_i 's are prime ideals in $\mathbb{Z}[\zeta_N]$. The residue field corresponding to p is now $\mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$, and so the Frobenius automorphism corresponds to raising to the p th power. Therefore the Frobenius conjugacy class $\text{Fr}(p)$ in $\text{Gal}(K/F)$ ³ acts on ζ_N by raising it to the p th power, $\zeta_N \mapsto \zeta_N^p$.

What this means is that under our identification of $\text{Gal}(K/F)$ with $(\mathbb{Z}/N\mathbb{Z})^\times$ the Frobenius element $\text{Fr}(p)$ corresponds to $p \bmod N$. Hence its order in $\text{Gal}(K/F)$ is equal to the multiplicative order of p modulo N . Denote this order by d . Then the residue field of each of the prime ideals \mathcal{P}_i 's in $\mathbb{Z}[\zeta_N]$ is an extension of \mathbb{F}_p of degree d , and so we find that p splits into exactly $r = \varphi(N)/d$ factors in $\mathbb{Z}[\zeta_N]$.

Consider for example the case when $N = 4$. Then $K = \mathbb{Q}(i)$ and $\mathcal{O}_K = \mathbb{Z}[i]$, the ring of Gauss integers. It is unramified at all odd primes. An odd prime p splits in $\mathbb{Z}[i]$ if and only if

$$p = (a + bi)(a - bi) = a^2 + b^2,$$

i.e., if p may be represented as the sum of squares of two integers.⁴ The above formula now tells us that this representation is possible if and only if $p \equiv 1 \pmod{4}$, which is the statement of one of Fermat's theorems (see [25] for more details). For example, 5 can be written as $1^2 + 2^2$, but 7 cannot be written as the sum of squares of two integers.

A statement like this is usually referred to as a *reciprocity law*, as it expresses a subtle arithmetic property of a prime p (in the case at hand, representability as the sum of two squares) in terms of a congruence condition on p .

1.4. Rigidifying ACFT. Now let us go back to the ACFT isomorphism (1.5). We wish to define a Frobenius conjugacy class $\text{Fr}(p)$ in the Galois group of the maximal abelian extension \mathbb{Q}^{ab} of \mathbb{Q} . However, in order to avoid the ambiguities explained above, we can really define it in the Galois group of the maximal abelian extension unramified at p , $\mathbb{Q}^{\text{ab},p}$.

¹each w_i will occur once if and only if K is unramified at v

²so that $\deg_{\mathcal{O}_F/v} \mathcal{O}_K/w = f$

³it is really an element of $\text{Gal}(K/F)$ in this case, and not just a conjugacy class, because this group is abelian

⁴this follows from the fact that all ideals in $\mathbb{Z}[i]$ are principal ideals, which is not difficult to see directly

This Galois group is the quotient of $\text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q})$ by the inertia subgroup I_p of p .⁵ While \mathbb{Q}^{ab} is obtained by adjoining to \mathbb{Q} all roots of unity, $\mathbb{Q}^{\text{ab},p}$ is obtained by adjoining all roots of unity of orders not divisible by p . So while $\text{Gal}(\mathbb{Q}^{\text{ab}}, \mathbb{Q})$ is isomorphic to $\prod_{p' \text{ prime}} \mathbb{Z}_{p'}^\times$, or the group of connected components of $\mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times$, the Galois group of $\mathbb{Q}^{\text{ab},p}$ is

$$(1.6) \quad \text{Gal}(\mathbb{Q}^{\text{ab},p}/\mathbb{Q}) \simeq \prod_{p' \neq p} \mathbb{Z}_{p'}^\times \simeq \left(\mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times / \mathbb{Z}_p^\times \right)_{\text{c.c.}}$$

(the subscript indicates taking the group of connected components). In other words, the inertia subgroup I_p is isomorphic to \mathbb{Z}_p^\times .

The reciprocity laws discussed above may be reformulated in a very nice way, by saying that under the isomorphism (1.6) the *inverse* of $\text{Fr}(p)$ goes to the double coset of the invertible adèle $(1, \dots, 1, p, 1, \dots) \in \mathbb{A}_{\mathbb{Q}}^\times$, where p is inserted in the factor \mathbb{Q}_p^\times , in the group $(\mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times / \mathbb{Z}_p^\times)_{\text{c.c.}}$.⁶ The inverse of $\text{Fr}(p)$ is the *geometric* Frobenius automorphism, which we will denote by Fr_p (in what follows we will often drop the adjective “geometric”). Thus, we have

$$(1.7) \quad \text{Fr}_p \mapsto (1, \dots, 1, p, 1, \dots).$$

More generally, if F is a number field, then, according to the ACFT isomorphism (1.5), the Galois group of the maximal abelian extension F^{ab} of F is isomorphic to $F^\times \backslash \mathbb{A}_F^\times$. Then the analogue of the above statement is that the inertia subgroup I_v of a prime ideal v of \mathcal{O}_F goes under this isomorphism to \mathcal{O}_v^\times , the multiplicative group of the completion of \mathcal{O}_F at v . Thus, the Galois group of the maximal abelian extension unramified outside of v is isomorphic to $(F^\times \backslash \mathbb{A}_F^\times / \mathcal{O}_v^\times)_{\text{c.c.}}$, and under this isomorphism the geometric Frobenius element $\text{Fr}_v = \text{Fr}(v)^{-1}$ goes to the coset of the invertible adèle $(1, \dots, 1, t_v, 1, \dots)$, where t_v is any generator of the maximal ideal in \mathcal{O}_v (this coset is independent of the choice of t_v).⁷ According to the Chebotarev theorem, the Frobenius conjugacy classes generate a dense subset in the Galois group. Therefore this condition rigidifies the ACFT isomorphism, in the sense that there is a unique isomorphism that satisfies this condition.

One can think of this rigidity condition as encompassing all reciprocity laws that one can write for the *abelian extensions* of number fields.

1.5. Non-abelian generalization? Having gotten an adèlic description of the abelian quotient of the Galois group of a number field, it is natural to ask what should be the next step. What about non-abelian extensions? The Galois group of the maximal abelian extension of F is the quotient of the absolute Galois group $\text{Gal}(\overline{F}/F)$ by its first commutator subgroup. So, for example, we could inquire what is the quotient of $\text{Gal}(\overline{F}/F)$ by the second commutator subgroup, and so on.

We will pursue a different direction. Instead of talking about the structure of the Galois group itself, we will look at its finite-dimensional representations. Note that for any group G , the one-dimensional representations of G are the same as those of its maximal abelian

⁵in general, the inertia subgroup is defined only up to conjugation, but in the abelian Galois group such as $\text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q})$ it is well-defined as a subgroup

⁶this normalization of the isomorphism (1.6) introduced by P. Deligne is convenient for the geometric reformulation that we will need

⁷in the case when $F = \mathbb{Q}$, formula (1.7), we have chosen $t_v = p$ for $v = (p)$

quotient. Moreover, one can obtain complete information about the maximal abelian quotient of a group by considering its one-dimensional representations.

Therefore describing the maximal abelian quotient of $\text{Gal}(\overline{F}/F)$ is equivalent to describing one-dimensional representations of $\text{Gal}(\overline{F}/F)$. Thus, the above statement of the abelian class field theory may be reformulated as saying that one-dimensional representations of $\text{Gal}(\overline{F}/F)$ are essentially in bijection with one-dimensional representations of the abelian group $F^\times \backslash \mathbb{A}_F^\times$.⁸ The latter may also be viewed as representations of the group $\mathbb{A}_F^\times = GL_1(\mathbb{A}_F)$ which occur in the space of functions on the quotient $F^\times \backslash \mathbb{A}_F^\times = GL_1(F) \backslash GL_1(\mathbb{A}_F)$. Thus, schematically ACFT may be represented as follows:

$$\boxed{\begin{array}{c} \text{1-dimensional representations} \\ \text{of } \text{Gal}(\overline{F}/F) \end{array}} \longrightarrow \boxed{\begin{array}{c} \text{representations of } GL_1(\mathbb{A}_F) \\ \text{in functions on } GL_1(F) \backslash GL_1(\mathbb{A}_F) \end{array}}$$

A marvelous insight of Robert Langlands was to conjecture, in a letter to A. Weil [31] and in [1], that there exists a similar description of *n-dimensional representations* of $\text{Gal}(\overline{F}/F)$. Namely, he proposed that those should be related to irreducible representations of the group $GL_n(\mathbb{A}_F)$ which occur in the space of functions on the quotient $GL_n(F) \backslash GL_n(\mathbb{A}_F)$. Such representations are called *automorphic*.⁹ Schematically,

$$\boxed{\begin{array}{c} \text{n-dimensional representations} \\ \text{of } \text{Gal}(\overline{F}/F) \end{array}} \longrightarrow \boxed{\begin{array}{c} \text{representations of } GL_n(\mathbb{A}_F) \\ \text{in functions on } GL_n(F) \backslash GL_n(\mathbb{A}_F) \end{array}}$$

This relation and its generalizations are examples of what we now call the *Langlands correspondence*.

There are many reasons to believe that Langlands correspondence is a good way to tackle non-abelian Galois groups. First of all, according to the ‘‘Tannakian philosophy’’, one can reconstruct a group from the category of its finite-dimensional representations, equipped with the structure of the tensor product. Therefore looking at the equivalence classes of *n*-dimensional representations of the Galois group may be viewed as a first step towards understanding its structure.

Perhaps, even more importantly, one finds many interesting representations of Galois groups in ‘‘nature’’. For example, the group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ will act on the geometric invariants (such as the étale cohomologies) of an algebraic variety defined over \mathbb{Q} . Thus, if we take an elliptic curve E over \mathbb{Q} , then we will obtain a two-dimensional Galois representation on its first étale cohomology. This representation contains a lot of important information

⁸The word ‘‘essentially’’ is added because in the ACFT isomorphism (1.5) we have to take not the group $F^\times \backslash \mathbb{A}_F^\times$ itself, but the group of its connected components; this may be taken into account by imposing some restrictions on the one-dimensional representations of this group that we should consider.

⁹A precise definition of automorphic representation is subtle because of the presence of continuous spectrum in the appropriate space of functions on $GL_n(F) \backslash GL_n(\mathbb{A}_F)$; however, in what follows we will only consider those representations which are part of the discrete spectrum, so these difficulties will not arise.

about the curve E , such as the number of points of E over $\mathbb{Z}/p\mathbb{Z}$ for various primes p , as we will see below.

Recall that in the abelian case ACFT isomorphism (1.5) satisfied an important “rigidity” condition expressing the Frobenius element in the abelian Galois group as a certain explicit adèle (see formula (1.7)). The power of the Langlands correspondence is not just in the fact that we establish a correspondence between objects of different nature, but that this correspondence again should satisfy a rigidity condition similar to the one in the abelian case. We will see below that this rigidity condition implies that the intricate data on the Galois side, such as the number of points of $E(\mathbb{Z}/p\mathbb{Z})$, are translated into something more tractable on the automorphic side, such as the coefficients in the q -expansion of the modular forms that encapsulate automorphic representations of $GL_2(\mathbb{A}_{\mathbb{Q}})$.

So, roughly speaking, one asks that under the Langlands correspondence certain natural invariants attached to the Galois representations and to the automorphic representations be matched. These invariants are the *Frobenius conjugacy classes* on the Galois side and the *Hecke eigenvalues* on the automorphic side.

Let us explain this more precisely. We have already defined the Frobenius conjugacy classes. We just need to generalize this notion from finite extensions of F to the infinite extension \bar{F} . This is done as follows. For each prime ideal v in \mathcal{O}_F we choose a compatible system \bar{v} of prime ideals that appear in the factorization of v in all finite extensions of F . Such a system may be viewed as a prime ideal associated to v in the ring of integers of \bar{F} . Then we attach to \bar{v} its stabilizer in $\text{Gal}(\bar{F}/F)$, called the decomposition subgroup and denoted by $D_{\bar{v}}$. We have a natural homomorphism (actually, an isomorphism) $D_{\bar{v}} \rightarrow \text{Gal}(\bar{F}_v, F_v)$. Recall that F_v is the non-archimedean completion of F corresponding to v , and \bar{F}_v is realized here as the completion of \bar{F} at \bar{v} . We denote by \mathcal{O}_v the ring of integers of F_v , by \mathfrak{m}_v the unique maximal ideal of \mathcal{O}_v , and by k_v the (finite) residue field $\mathcal{O}_v/\mathfrak{m}_v$. The kernel of the composition

$$D_{\bar{v}} \rightarrow \text{Gal}(\bar{F}_v, F_v) \rightarrow \text{Gal}(\bar{k}_v/k_v)$$

is called the inertia subgroup $I_{\bar{v}}$ of $D_{\bar{v}}$. An n -dimensional representation $\sigma : \text{Gal}(\bar{F}/F) \rightarrow GL_n$ is called unramified at v if $I_{\bar{v}} \subset \text{Ker } \sigma$.

Suppose that σ is unramified at v . Let Fr_v be the geometric Frobenius automorphism in $\text{Gal}(\bar{k}_v, k_v)$ (the inverse to the operator $x \mapsto x^{|k_v|}$ acting on \bar{k}_v). In this case $\sigma(\text{Fr}_v)$ is a well-defined element of GL_n . If we replace \bar{v} by another compatible system of ideals, then $\sigma(\text{Fr}_v)$ will get conjugated in GL_n . So its conjugacy class is a well-defined conjugacy class in GL_n , which we call the Frobenius conjugacy class corresponding to v and σ .

This takes care of the Frobenius conjugacy classes. To explain what the Hecke eigenvalues are we need to look more closely at representations of the adèlic group $GL_n(\mathbb{A}_F)$, and we will do that below. For now, let us just say that like the Frobenius conjugacy classes, the Hecke eigenvalues also correspond to conjugacy classes in GL_n and are attached to all but finitely many prime ideals v of \mathcal{O}_F . As we will explain in the next section, in the case when $n = 2$ they are related to the eigenvalues of the classical Hecke operators acting on modular forms.

The matching condition alluded to above is then formulated as follows: if under the Langlands correspondence we have

$$\sigma \longrightarrow \pi,$$

where σ is an n -dimensional representation of $\text{Gal}(\overline{F}/F)$ and π is an automorphic representation of $GL_n(\mathbb{A}_F)$, then the Frobenius conjugacy classes for σ should coincide with the Hecke eigenvalues for π for almost all prime ideals v (precisely those v at which both σ and π are unramified). In the abelian case, $n = 1$, this condition amounts precisely to the “rigidity” condition (1.7). In the next two sections we will see what this condition means in the non-abelian case $n = 2$ when σ comes from the first cohomology of an elliptic curve defined over \mathbb{Q} . It turns out that in this special case the Langlands correspondence becomes the statement of the Taniyama-Shimura conjecture which implies Fermat’s last theorem.

1.6. Automorphic representations of $GL_2(\mathbb{A}_{\mathbb{Q}})$ and modular forms. In this subsection we discuss briefly cuspidal automorphic representations of $GL_2(\mathbb{A}) = GL_2(\mathbb{A}_{\mathbb{Q}})$ and how to relate them to classical modular forms on the upper half-plane. We will then consider the two-dimensional representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ arising from elliptic curves defined over \mathbb{Q} and look at what the Langlands correspondence means for such representations. We refer the reader to [32, 33, 34, 35] for more details on this subject.

Roughly speaking, cuspidal automorphic representations of $GL_2(\mathbb{A})$ are those irreducible representations of this group which occur in the discrete spectrum of a certain space of functions on the quotient $GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A})$. Strictly speaking, this is not correct because the representations that we consider do not carry the action of the factor $GL_2(\mathbb{R})$ of $GL_2(\mathbb{A})$, but only that of its Lie algebra \mathfrak{gl}_2 . Let us give a more precise definition.

We start by introducing the maximal compact subgroup $K \subset GL_2(\mathbb{A})$ which is equal to $\prod_p GL_2(\mathbb{Z}_p) \times O_2$. Let \mathfrak{z} be the center of the universal enveloping algebra of the (complexified) Lie algebra \mathfrak{gl}_2 . Then \mathfrak{z} is the polynomial algebra in the central element $I \in \mathfrak{gl}_2$ and the Casimir operator

$$(1.8) \quad C = \frac{1}{4}X_0^2 + \frac{1}{2}X_+X_- + \frac{1}{2}X_-X_+,$$

where

$$X_0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad X_{\pm} = \frac{1}{2} \begin{pmatrix} 1 & \mp i \\ \mp i & -1 \end{pmatrix}$$

are basis elements of $\mathfrak{sl}_2 \subset \mathfrak{gl}_2$.

Consider the space of functions of $GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A})$ which are locally constant as functions on $GL_2(\mathbb{A}^f)$, where $\mathbb{A}^f = \prod'_p \mathbb{Q}_p$, and smooth as functions on $GL_2(\mathbb{R})$. Such functions are called *smooth*. The group $GL_2(\mathbb{A})$ acts on this space by right translations:

$$(g \cdot f)(h) = f(hg), \quad g \in GL_2(\mathbb{A}).$$

In particular, the subgroup $GL_2(\mathbb{R}) \subset GL_2(\mathbb{A})$, and hence its complexified Lie algebra \mathfrak{gl}_2 and the universal enveloping algebra of the latter also act.

The group $GL_2(\mathbb{A})$ has the center $Z(\mathbb{A}) \simeq \mathbb{A}^{\times}$ which consists of all diagonal matrices.

For a character $\chi : Z(\mathbb{A}) \rightarrow \mathbb{C}^{\times}$ and a complex number ρ let

$$\mathcal{C}_{\chi, \rho}(GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}))$$

be the space of smooth functions $f : GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) \rightarrow \mathbb{C}$ satisfying the following additional requirements:

- (K -finiteness) the (right) translates of f under the action of elements of the compact subgroup K span a finite-dimensional vector space;
- (central character) $f(gz) = \chi(z)f(g)$ for all $g \in GL_2(\mathbb{A}), z \in Z(\mathbb{A})$, and $C \cdot f = \rho f$, where C is the Casimir element (1.8);
- (growth) f is bounded on $GL_n(\mathbb{A})$;
- (cuspidality) $\int_{\mathbb{Q} \backslash N\mathbb{A}} f \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g \right) du = 0$.

The space $\mathcal{C}_{\chi, \rho}(GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}))$ is a representation of the group

$$GL_2(\mathbb{A}^f) = \prod_{p \text{ prime}} {}'GL_2(\mathbb{Q}_p)$$

and the Lie algebra \mathfrak{gl}_2 (corresponding to the infinite place), whose actions commute with each other. In addition, the subgroup O_2 of $GL_2(\mathbb{R})$ acts on it, and the action of O_2 is compatible with the action of \mathfrak{gl}_2 making it into a module over the so-called Harish-Chandra pair (\mathfrak{gl}_2, O_2) .

It is known that $\mathcal{C}_{\chi, \rho}(GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}))$ is a direct sum of irreducible representations of $GL_2(\mathbb{A}^f) \times \mathfrak{gl}_2$, each occurring with multiplicity one.¹⁰ The irreducible representations occurring in these spaces (for different χ, ρ) are called the *cuspidal automorphic representations of $GL_2(\mathbb{A})$* .

We now explain how to attach to such a representation a modular form on the upper half-plane \mathbb{H}_+ . First of all, an irreducible cuspidal automorphic representation π may be written as a *restricted* infinite tensor product

$$(1.9) \quad \pi = \bigotimes_{p \text{ prime}} {}'\pi_p \otimes \pi_\infty,$$

where π_p is an irreducible representation of $GL_2(\mathbb{Q}_p)$ and π_∞ is a \mathfrak{gl}_2 -module. For all but finitely many primes p , the representation π_p is *unramified*, which means that it contains a non-zero vector invariant under the maximal compact subgroup $GL_2(\mathbb{Z}_p)$ of $GL_2(\mathbb{Q}_p)$. This vector is then unique up to a scalar. Let us choose $GL_2(\mathbb{Z}_p)$ -invariant vectors v_p at all unramified primes p .

Then the vector space (1.9) is the restricted infinite tensor product in the sense that it consists of finite linear combinations of vectors of the form $\otimes_p w_p \otimes w_\infty$, where $w_p = v_p$ for all but finitely many prime numbers p (this is the meaning of the prime at the tensor product sign). It is clear from the definition of $\mathbb{A}^f = \prod_p {}'\mathbb{Q}_p$ that the group $GL_2(\mathbb{A}^f)$ acts on it.

Suppose now that p is one of the primes at which π_p is ramified, so π_p does not contain $GL_2(\mathbb{Z}_p)$ -invariant vectors. Then it contains vectors invariant under smaller compact subgroups of $GL_2(\mathbb{Z}_p)$.

Let us assume for simplicity that $\chi \equiv 1$. Then one shows that there is a unique, up to a scalar, non-zero vector in π_p invariant under the compact subgroup

$$K'_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \pmod{p^{n_p} \mathbb{Z}_p} \right\}$$

¹⁰the above cuspidality and central character conditions are essential in ensuring that irreducible representations occur in $\mathcal{C}_{\chi, \rho}(GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}))$ discretely.

for some positive integer n_p .¹¹ Let us choose such a vector v_p at all primes where π is ramified. In order to have uniform notation, we will set $n_p = 0$ at those primes at which π_p is unramified, so at such primes we have $K'_p = GL_2(\mathbb{Z}_p)$. Let $K' = \prod_p K'_p$.

Thus, we obtain that the space of K' -invariants in π is the subspace

$$(1.10) \quad \tilde{\pi}_\infty = \otimes_p v_p \otimes \pi_\infty,$$

which carries an action of (\mathfrak{gl}_2, O_2) . This space of functions contains all the information about π because other elements of π may be obtained from it by right translates by elements of $GL_2(\mathbb{A})$. So far we have not used the fact that π is an automorphic representation, i.e., that it is realized in the space of smooth functions on $GL_2(\mathbb{A})$ left invariant under the subgroup $GL_2(\mathbb{Z})$. Taking this into account, we find that the space $\tilde{\pi}_\infty$ of K' -invariant vectors in π is realized in the space of functions on the double quotient $GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / K'$.

Next, we use the strong approximation theorem (see, e.g., [32]) to obtain the following useful statement. Let us set $N = \prod_p p^{n_p}$ and consider the subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \pmod{N\mathbb{Z}} \right\}$$

of $SL_2(\mathbb{Z})$. Then

$$GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / K' \simeq \Gamma_0(N) \backslash GL_2^+(\mathbb{R}),$$

where $GL_2^+(\mathbb{R})$ consists of matrices with positive determinant.

Thus, the smooth functions on $GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A})$ corresponding to vectors in the space $\tilde{\pi}_\infty$ given by (1.10) are completely determined by their restrictions to the subgroup $GL_2^+(\mathbb{R})$ of $GL_2(\mathbb{R}) \subset GL_2(\mathbb{A})$. The central character condition implies that these functions are further determined by their restrictions to $SL_2(\mathbb{R})$. Thus, all information about π is contained in the space $\tilde{\pi}_\infty$ realized in the space of smooth functions on $\Gamma_0(N) \backslash SL_2(\mathbb{R})$, where it forms a representation of the Lie algebra \mathfrak{sl}_2 on which the Casimir operator C of $U(\mathfrak{sl}_2)$ acts by multiplication by ρ .

At this point it is useful to recall that irreducible representations of $(\mathfrak{gl}_2(\mathbb{C}), O(2))$ fall into the following categories: principal series, discrete series, the limits of the discrete series and finite-dimensional representations (see [36]).

Consider the case when π_∞ is a representation of the discrete series of $(\mathfrak{gl}_2(\mathbb{C}), O(2))$. In this case $\rho = k(k-2)/4$, where k is an integer greater than 1. Then, as an \mathfrak{sl}_2 -module, π_∞ is the direct sum of the irreducible Verma module of highest weight $-k$ and the irreducible Verma module with lowest weight k . The former is generated by a unique, up to a scalar, highest weight vector v_∞ such that

$$X_0 \cdot v_\infty = -k v_\infty, \quad X_+ \cdot v_\infty = 0,$$

and the latter is generated by the lowest weight vector $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot v_\infty$.

Thus, the entire $\mathfrak{gl}_2(\mathbb{R})$ -module π_∞ is generated by the vector v_∞ , and so we focus on the function on $\Gamma_0(N) \backslash SL_2(\mathbb{R})$ corresponding to this vector. Let ϕ_π be the corresponding

¹¹if we do not assume that $\chi \equiv 1$, then there is a unique, up to a scalar, vector invariant under the subgroup of elements as above satisfying the additional condition that $d \equiv \text{mod } p^{n_p} \mathbb{Z}_p$

function on $SL_2(\mathbb{R})$. By construction, it satisfies

$$\begin{aligned}\phi_\pi(\gamma g) &= \phi_\pi(g), \quad \gamma \in \Gamma_0(N), \\ \phi_\pi\left(g \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}\right) &= e^{ik\theta} \phi_\pi(g) \quad 0 \leq \theta \leq 2\pi.\end{aligned}$$

We assign to ϕ_π a function f_π on the upper half-plane

$$\mathbb{H} = \{\tau \in \mathbb{C} \mid \text{Im } \tau > 0\}.$$

Recall that $\mathbb{H} \simeq SL_2(\mathbb{R})/SO_2$ under the correspondence

$$SL_2(\mathbb{R}) \ni g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{a + bi}{c + di} \in \mathbb{H}.$$

We define a function f_π on $SL_2(\mathbb{R})/SO_2$ by the formula

$$f_\pi(g) = \phi(g)(ci + d)^k.$$

When written as a function of τ , the function f satisfies the conditions¹²

$$f_\pi\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f_\pi(\tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

In addition, the “highest weight condition” $X_+ \cdot v_\infty = 0$ is equivalent to f_π being a *holomorphic* function of τ . Such functions are called *modular forms of weight k and level N* .

Thus, we have attached to an automorphic representation π of $GL_2(\mathbb{A})$ a holomorphic modular form f_π of weight k and level N on the upper half-plane. We expand it in the Fourier series

$$f_\pi(q) = \sum_{n=0}^{\infty} a_n q^n, \quad q = e^{2\pi i\tau}.$$

The cuspidality condition on π means that f_π vanishes at the cusps of the fundamental domain of the action of $\Gamma_0(N)$ on \mathbb{H} . Such modular forms are called *cuspidal forms*. In particular, it vanishes at $q = 0$ which corresponds to the cusp $\tau = i\infty$, and so we have $a_0 = 0$. Further, it can be shown that $a_1 \neq 0$, and we will normalize f_π by setting $a_1 = 1$.

The normalized modular cusp form $f_\pi(q)$ contains all the information about the automorphic representation π .¹³ In particular, it “knows” about the Hecke eigenvalues of π .

Let us give the definition of the Hecke operators. This is a local question that has to do with the local factor π_p in the decomposition (1.9) of π at a prime p , which is a representation of $GL_2(\mathbb{Q}_p)$. Suppose that π_p is unramified, i.e., it contains a unique, up to a scalar, vector v_p that is invariant under the subgroup $GL_2(\mathbb{Z}_p)$. Then it is an eigenvector of the *spherical Hecke algebra* \mathcal{H}_p which is the algebra of compactly supported $GL_2(\mathbb{Z}_p)$

¹²In the case when k is odd, taking $-I_2 \in \Gamma_0(N)$ we obtain $f_\pi(\tau) = -f_\pi(\tau)$, hence this condition can only be satisfied by the zero function. To cure that, we should modify it by inserting in the right hand side the factor $\chi_N(d)$, where χ_N is a character $(\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ such that $\chi_N(-1) = -1$. This character corresponds to the character χ in the definition of the space $\mathcal{C}_{\chi,\rho}(GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}))$. We have set $\chi \equiv 1$ because our main example is $k = 2$ when this issue does not arise.

¹³Note that f_π corresponds to a unique, up to a scalar, “highest weight vector” in the representation π invariant under the compact subgroup K' and the Borel subalgebra of \mathfrak{sl}_2 .

bi-invariant functions on $GL_2(\mathbb{Q}_p)$, with respect to the convolution product. This algebra is isomorphic to the polynomial algebra in two generators $H_{1,p}$ and $H_{2,p}$, whose action on v_p is given by the formulas

$$(1.11) \quad H_{1,p} \cdot v_p = \int_{M_2^1(\mathbb{Z}_p)} \rho_p(g) \cdot v_p dg,$$

$$(1.12) \quad H_{2,p} \cdot v_p = \int_{M_2^2(\mathbb{Z}_p)} \rho_p(g) \cdot \rho_p dg,$$

where $\rho_p : GL_2(\mathbb{Z}_p) \rightarrow \text{End } \pi_p$ is the representation homomorphism, $M_2^i(\mathbb{Z}_p)$, $i = 1, 2$, are the double cosets

$$M_2^1(\mathbb{Z}_p) = GL_2(\mathbb{Z}_p) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} GL_2(\mathbb{Z}_p), \quad M_2^2(\mathbb{Z}_p) = GL_2(\mathbb{Z}_p) \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} GL_2(\mathbb{Z}_p)$$

in $GL_2(\mathbb{Q}_p)$, and we use the Haar measure on $GL_2(\mathbb{Q}_p)$ normalized so that the volume of the compact subgroup $GL_2(\mathbb{Z}_p)$ is equal to 1.

These cosets generalize the \mathbb{Z}_p^\times coset of the element $p \in GL_1(\mathbb{Q}_p) = \mathbb{Q}_p^\times$, and that is why the matching condition between the Hecke eigenvalues and the Frobenius eigenvalues that we discuss below generalizes the ‘‘rigidity’’ condition (1.7) of the ACFT isomorphism.

Since the integrals are over $GL_2(\mathbb{Z}_p)$ -cosets, $H_{1,p} \cdot v_p$ and $H_{2,p} \cdot v_p$ are $GL_2(\mathbb{Z}_p)$ -invariant vectors, hence proportional to v_p . Under our assumption that the center $Z(\mathbb{A})$ acts trivially on π ($\chi \equiv 1$) we have $H_2 \cdot v_p = v_p$. But the eigenvalue $h_{1,p}$ of $H_{1,p}$ on v_p is an important invariant of π_p . This invariant is defined for all primes p at which π is unramified (these are the primes that do not divide the level N introduced above). These are precisely the *Hecke eigenvalues* that we discussed before.

Since the modular cusp form f_π encapsulates all the information about the automorphic representation π , we should be able to read them off the form f_π . It turns out that the operators $H_{1,p}$ have a simple interpretation in terms of functions on the upper half-plane. Namely, they become the classical Hecke operators (see, e.g., [32] for an explicit formula). Thus, we obtain that f_π is necessarily an eigenfunction of the classical Hecke operators. Moreover, explicit calculation shows that if we normalize f_π as above, setting $a_1 = 1$, then the eigenvalue $h_{1,p}$ will be equal to the p th coefficient a_p in the q -expansion of f_π .

Let us summarize: to an irreducible cuspidal automorphic representation π (in the special case when $\chi \equiv 1$ and $\rho = k(k-2)/4$, where $k \in \mathbb{Z}_{>1}$) we have associated a modular cusp form f_π of weight k and level N on the upper half-plane which is an eigenfunction of the classical Hecke operators (corresponding to all primes that do not divide N) with the eigenvalues equal to the coefficients a_p in the q -expansion of f_π .

1.7. Elliptic curves and Galois representations. In the previous subsection we discussed some concrete examples of automorphic representations of $GL_2(\mathbb{A})$ that can be realized by classical modular cusp forms. Now we look at examples of the objects arising on the other side of the Langlands correspondence, namely, two-dimensional representations of the Galois group of \mathbb{Q} . Then we will see what matching their invariants means.

As we mentioned above, one can construct representations of the Galois group of \mathbb{Q} by taking the étale cohomology of algebraic varieties defined over \mathbb{Q} . The simplest example

of a two-dimensional representation is thus provided by the first étale cohomology of an elliptic curve defined over \mathbb{Q} , which (just as its topological counterpart) is two-dimensional.

A smooth elliptic curve over \mathbb{Q} may concretely be defined by an equation

$$y^2 = x^3 + ax + b$$

where a, b are rational numbers such that $4a^3 + 27b^2 \neq 0$. More precisely, this equation defines an affine curve E' . The corresponding projective curve E is obtained by adding to E' a point at infinity; it is the curve in \mathbb{P}^2 defined by the corresponding homogeneous equation.

The first étale cohomology $H_{\text{ét}}^1(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ of $E_{\overline{\mathbb{Q}}}$ with coefficients in \mathbb{Q}_ℓ is isomorphic to \mathbb{Q}_ℓ^2 . The definition of étale cohomology necessitates the choice of a prime ℓ , but as we will see below, important invariants of these representations, such as the Frobenius eigenvalues, are independent of ℓ . This space may be concretely realized as the dual of the Tate module of E , the inverse limit of the groups of points of order ℓ^n on E (with respect to the abelian group structure on E), tensored with \mathbb{Q}_ℓ . Since E is defined over \mathbb{Q} , the Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts by symmetries on $H_{\text{ét}}^1(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$, and hence we obtain a two-dimensional representation $\sigma_{E,\ell} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(\mathbb{Q}_\ell)$. This representation is continuous with respect to the Krull topology¹⁴ on $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and the usual ℓ -adic topology on $GL_2(\mathbb{Q}_\ell)$.

What information can we infer from this representation? As explained in Sect. 1.5, important invariants of Galois representations are the eigenvalues of the Frobenius conjugacy classes corresponding to the primes where the representation is unramified. In the case at hand, the representation is unramified at the primes of “good reduction”, which do not divide an integer N_E , the conductor of E . These Frobenius eigenvalues have a nice interpretation. Namely, for $p \nmid N_E$ we consider the sum of their inverses, which is the trace of $\sigma_E(\text{Fr}_p)$. One can show that it is equal to

$$\text{Tr } \sigma_E(\text{Fr}_p) = p + 1 - \#E(\mathbb{F}_p)$$

where $\#E(\mathbb{F}_p)$ is the number of points of E modulo p (see, [33, 35]). In particular, it is independent of ℓ .

Under the Langlands correspondence, the representation σ_E of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ should correspond to a cuspidal automorphic representation $\pi(\sigma_E)$ of the group $GL_2(\mathbb{A})$. Moreover, as we discussed in Sect. 1.5, this correspondence should match the Frobenius eigenvalues of σ_E and the Hecke eigenvalues of $\pi(\sigma_E)$. Concretely, in the case at hand, the matching condition is that $\text{Tr } \sigma_E(\text{Fr}_p)$ should be equal to the eigenvalue $h_{1,p}$ of the Hecke operator $H_{1,p}$, at all primes p where σ_E and $\pi(\sigma_E)$ are unramified.

It is not difficult to see that for this to hold, $\pi(\sigma_E)$ must be a cuspidal automorphic representation of $GL_2(\mathbb{A})$ corresponding to a modular cusp form of weight $k = 2$. Therefore, if we believe in the Langlands correspondence, we arrive at the following startling conjecture: for each elliptic curve E over \mathbb{Q} there should exist a modular cusp form $f_E(q) = \sum_{n=1}^{\infty} a_n q^n$ with $a_1 = 1$ and

$$(1.13) \quad a_p = p + 1 - \#E(\mathbb{F}_p)$$

¹⁴in this topology the base of open neighborhoods of the identity is formed by normal subgroups of finite index (i.e., such that the quotient is a finite group)

for all but finitely many primes p ! This is in fact the statement of the celebrated Taniyama-Shimura conjecture that has recently been proved by A. Wiles and others [38]. It implies Fermat's last theorem, see [35] and references therein.

In fact, the modular cusp form $f_E(q)$ is what is called a *newform* (this means that it does not come from a modular form whose level is a divisor of N_E). Moreover, the Galois representation σ_E and the automorphic representation π are unramified at exactly the same primes (namely, those which do not divide N_E), and formula (1.13) holds at all of those primes [37]. This way one obtains a bijection between isogeny classes of elliptic curves defined over \mathbb{Q} with conductor N_E and newforms of weight 2 and level N_E with integer Fourier coefficients.

One obtains similar statements by analyzing from the point of view of the Langlands correspondence the Galois representations coming from other algebraic varieties, or more general motives.

2. FROM NUMBER FIELDS TO FUNCTION FIELDS

As we have seen in the previous section, even special cases of the Langlands correspondence lead to unexpected number theoretic consequences. However, proving these results is notoriously difficult. Some of the difficulties are related to the special role played by the archimedean completion \mathbb{R} in the ring of adèles of \mathbb{Q} (and similarly, by the archimedean completions of other number fields). Representation theory of the archimedean factor $GL_n(\mathbb{R})$ of the adèlic group $GL_n(\mathbb{A}_{\mathbb{Q}})$ is very different from that of the other, non-archimedean, factors $GL_2(\mathbb{Q}_p)$, and this leads to problems.

Fortunately, number fields have close cousins, called *function fields*, whose completions are all non-archimedean, so that the corresponding theory is more uniform. The function field version of the Langlands correspondence turned out to be easier to handle than the correspondence in the number field case. In fact, it is now a theorem! First, V. Drinfeld [39, 40] proved it in the 80's in the case of GL_2 , and more recently L. Lafforgue [41] proved it for GL_n with an arbitrary n .

In this section we explain the analogy between number fields and function fields and formulate the Langlands correspondence for function fields.

2.1. Function fields. What do we mean by a function field? Let X be a smooth projective connected curve over a finite field \mathbb{F}_q . The field $\mathbb{F}_q(X)$ of (\mathbb{F}_q -valued) rational functions on X is called the function field of X . For example, suppose that $X = \mathbb{P}^1$. Then $\mathbb{F}_q(X)$ is just the field of rational functions in one variable. Its elements are fractions $P(t)/Q(t)$, where $P(t)$ and $Q(t) \neq 0$ are polynomials over \mathbb{F}_q without common factors, with their usual operations of addition and multiplication. Explicitly, $P(t) = \sum_{n=0}^N p_n t^n$, $p_n \in \mathbb{F}_q$, and similarly for $Q(t)$.

A general projective curve X over \mathbb{F}_q is defined by a system of algebraic equations in the projective space \mathbb{P}^n over \mathbb{F}_q . For example, we can define an elliptic curve over \mathbb{F}_q by a cubic equation

$$(2.1) \quad y^2 z = x^3 + axz^2 + bz^3, \quad a, b, c \in \mathbb{F}_q,$$

written in homogeneous coordinates $(x : y : z)$ of \mathbb{P}^2 .¹⁵ What are the points of such a curve? Naively, these are the elements of the set $X(\mathbb{F}_q)$ of \mathbb{F}_q -solutions of the equations defining this curve. For example, in the case of the elliptic curve defined by the equation (2.1), this is the set of triples $(x, y, z) \in \mathbb{F}_q^3$ satisfying (2.1), with two such triples identified if they differ by an overall factor in \mathbb{F}_q^\times .

However, because the field \mathbb{F}_q is not algebraically closed, we should also consider points with values in the algebraic extensions \mathbb{F}_{q^n} of \mathbb{F}_q . The situation is similar to a more familiar situation of a curve defined over the field of real numbers \mathbb{R} . For example, consider the curve over \mathbb{R} defined by the equation $x^2 + y^2 = -1$. This equation has no solutions in \mathbb{R} , so naively we may think that this curve is empty. However, from the algebraic point of view, we should think in terms of the *ring of functions* on this curve, which in this case is $\mathcal{R} = \mathbb{R}[x, y]/(x^2 + y^2 + 1)$. Points of our curve are maximal ideals of the ring \mathcal{R} . The quotient \mathcal{R}/I by such an ideal I is a field F called the residue field of this ideal. Thus, we have a surjective homomorphism $\mathcal{R} \rightarrow F$ whose kernel is I . The field F is necessarily a finite extension of \mathbb{R} , so it could be either \mathbb{R} or \mathbb{C} . If it is \mathbb{R} , then we may think of the homomorphism $\mathcal{R} \rightarrow F$ as sending a function $f \in \mathcal{R}$ on our curve to its value $f(x)$ at some \mathbb{R} -point x of our curve. That's why maximal ideals of \mathcal{R} with the residue field \mathbb{R} are the same as \mathbb{R} -points of our curve. More generally, we will say that a maximal ideal I in \mathcal{R} with the residue field $F = \mathcal{R}/I$ corresponds to an F -point of our curve. In the case at hand it turns out that there are no \mathbb{R} -points, but there are plenty of \mathbb{C} -points, namely, all pairs of complex numbers (x_0, y_0) satisfying $x_0^2 + y_0^2 = -1$. The corresponding homomorphism $\mathcal{R} \rightarrow \mathbb{C}$ sends the generators x and y of \mathcal{R} to x_0 and $y_0 \in \mathbb{C}$.

If we have a curve defined over \mathbb{F}_q , then it has F -points, where F is a finite extension of \mathbb{F}_q , hence $F \simeq \mathbb{F}_{q^n}$, $n > 0$. An \mathbb{F}_{q^n} -point is defined as a maximal ideal of the ring of functions on an affine curve obtained by removing a point from our projective curve, with residue field \mathbb{F}_{q^n} . For example, in the case when the curve is \mathbb{P}^1 , we can choose the \mathbb{F}_q -point ∞ as this point. Then we are left with the affine line \mathbb{A}^1 , whose ring of functions is the ring $\mathbb{F}_q[t]$ of polynomials in the variable t . The F -points of the affine line are the maximal ideals of $\mathbb{F}_q[t]$ with residue field F . These are the same as the irreducible monic polynomials $A(t)$ with coefficients in \mathbb{F}_q . The corresponding residue field is the field obtained by adjoining to \mathbb{F}_q the roots of $A(t)$. For instance, \mathbb{F}_q -points correspond to the polynomials $A(t) = t - a$, $a \in \mathbb{F}_q$. The set of points of the projective line is therefore the set of all points of \mathbb{A}^1 together with the \mathbb{F}_q -point ∞ that has been removed.¹⁶

¹⁵Elliptic curves over finite fields \mathbb{F}_p have already made an appearance in the previous section. However, their role there was different: we had started with an elliptic curve E defined over \mathbb{Z} and used it to define a representation of the Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in the first étale cohomology of E . We then related the trace of the Frobenius element Fr_p for a prime p on this representation to the number of \mathbb{F}_p -points of the elliptic curve over \mathbb{F}_p obtained by reduction of $E \bmod p$. In contrast, in this section we use an elliptic curve, or a more general smooth projective curve X , over a field \mathbb{F}_q that is *fixed* once and for all. This curve defines a function field $\mathbb{F}_q(X)$ that, as we argue in this section, should be viewed as analogous to the field \mathbb{Q} of rational numbers, or a more general number field.

¹⁶In general, there is no preferred point in a given projective curve X , so it is useful instead to cover X by affine curves. Then the set of points of X is the union of the sets of points of those affine curves (each of them is defined as the set of maximal ideals of the corresponding ring of functions), with each point on the overlap counted only once.

It turns out that there are many similarities between function fields and number fields. To see that, let us look at the completions of a function field $\mathbb{F}_q(X)$. For example, suppose that $X = \mathbb{P}^1$. An example of a completion of the field $\mathbb{F}_q(\mathbb{P}^1)$ is the field $\mathbb{F}_q((t))$ of formal Laurent power series in the variable t . An element of this completion is a series of the form $\sum_{n \geq N} a_n t^n$, where $N \in \mathbb{Z}$ and each a_n is an element of \mathbb{F}_q . We have natural operations of addition and multiplication on such series making $\mathbb{F}_q((t))$ into a field. As we saw above, elements of $\mathbb{F}_q(\mathbb{P}^1)$ are rational functions $P(t)/Q(t)$, and such a rational function can be expanded in an obvious way in a formal power series in t . This defines an embedding of fields $\mathbb{F}_q(\mathbb{P}^1) \hookrightarrow \mathbb{F}_q((t))$, which makes $\mathbb{F}_q((t))$ into a completion of $\mathbb{F}_q(\mathbb{P}^1)$ with respect to the following norm: write

$$\frac{P(t)}{Q(t)} = t^n \frac{P_0(t)}{Q_0(t)}, \quad n \in \mathbb{Z},$$

where the polynomials $P_0(t)$ and $Q_0(t)$ have non-zero constant terms; then the norm of this fraction is equal to q^{-n} .

Now observe that the field $\mathbb{F}_p((t))$ looks very much like the field \mathbb{Q}_p of p -adic numbers. There are important differences, of course: the addition and multiplication in $\mathbb{F}_p((t))$ are defined termwise, i.e., “without carry”, whereas in \mathbb{Q}_p they are defined “with carry”. Thus, $\mathbb{F}_p((t))$ has characteristic p , whereas \mathbb{Q}_p has characteristic 0. But there are also similarities: each has a ring of integers, $\mathbb{F}_p[[t]] \subset \mathbb{F}_p((t))$, the ring of formal Taylor series, and $\mathbb{Z}_p \subset \mathbb{Q}_p$, the ring of p -adic integers. These rings of integers are local (contain a unique maximal ideal) and the residue field (the quotient by the maximal ideal) is the finite field \mathbb{F}_p . Likewise, the field $\mathbb{F}_q((t))$, where $q = p^n$, looks like a degree n extension of \mathbb{Q}_p .

The above completion corresponds to the maximal ideal generated by $A(t) = t$ in the ring $\mathbb{F}_q[t]$ (note that $\mathbb{F}_q[t] \subset \mathbb{F}_q(\mathbb{P}^1)$ may be thought of as the analogue of $\mathbb{Z} \subset \mathbb{Q}$). Other completions of $\mathbb{F}_q(\mathbb{P}^1)$ correspond to other maximal ideals in $\mathbb{F}_q[t]$, which, as we saw above, are generated by irreducible monic polynomials $A(t)$ (those are the analogues of the ideals (p) generated by prime numbers p in \mathbb{Z}).¹⁷ If the polynomial $A(t)$ has degree m , then the corresponding residue field is isomorphic to \mathbb{F}_{q^m} , and the corresponding completion is isomorphic to $\mathbb{F}_{q^m}(\tilde{t})$, where \tilde{t} is the “uniformizer”, $\tilde{t} = A(t)$. One can think of \tilde{t} as the local coordinate near the \mathbb{F}_{q^m} -point corresponding to $A(t)$, just like $t - a$ is the local coordinate near the \mathbb{F}_q -point a of \mathbb{A}^1 .

For a general curve X , completions of $\mathbb{F}_q(X)$ are labeled by its points, and the completion corresponding to an \mathbb{F}_{q^n} -point x is isomorphic to $\mathbb{F}_{q^n}((t_x))$, where t_x is the “local coordinate” near x on X .

Thus, completions of a function field are labeled by points of X . The essential difference with the number field case is that all of these completions are non-archimedean¹⁸; there are no analogues of the archimedean completions \mathbb{R} or \mathbb{C} that we have in the case of number fields.

We are now ready to define for function fields the analogues of the objects involved in the Langlands correspondence: Galois representations and automorphic representations.

Before we get to that, we want to comment on why is it that we only consider curves and not higher dimensional varieties. The point is that while function fields of curves are

¹⁷there is also a completion corresponding to the point ∞ , which is isomorphic to $\mathbb{F}_q((t^{-1}))$

¹⁸i.e., correspond to non-archimedean norms $|\cdot|$ such that $|x + y| \leq \max(|x|, |y|)$

very similar to number fields, the fields of functions on higher dimensional varieties have a very different structure. For example, if X is a smooth surface, then the completions of the field of rational functions on X are labeled by pairs: a point x of X and a germ of a curve passing through x . The corresponding complete field is isomorphic to the field of formal power series in two variables. At the moment no one knows how to formulate an analogue of the Langlands correspondence for the field of functions on an algebraic variety of dimension greater than one, and finding such a formulation is a very important open problem. There is an analogue of the abelian class field theory (see [42]), but not much is known beyond that.

In Part III of this paper we will argue that the Langlands correspondence for the function fields of curves – transported to the realm of complex curves – is closely related to the two-dimensional conformal field theory. The hope is, of course, that there is a similar connection between a higher dimensional Langlands correspondence and quantum field theories in dimensions greater than two (see, e.g., [43] for a discussion of this analogy).

2.2. Galois representations. Let X be a smooth connected projective curve over $k = \mathbb{F}_q$ and $F = k(X)$ the field of rational functions on X . Consider the Galois group $\text{Gal}(\overline{F}/F)$. It is instructive to think of the Galois group of a function field as a kind of fundamental group of X . Indeed, if $Y \rightarrow X$ is a covering of X , then the field $k(Y)$ of rational functions on Y is an extension of the field $F = k(X)$ of rational functions on X , and the Galois group $\text{Gal}(k(Y)/k(X))$ may be viewed as the group of “deck transformations” of the cover. If our cover is unramified, then this group may be identified with a quotient of the fundamental group of X . Otherwise, this group is isomorphic to a quotient of the fundamental group of X without the ramification points. The Galois group $\text{Gal}(\overline{F}/F)$ itself may be viewed as the group of “deck transformations” of the maximal (ramified) cover of X .

Let x be a point of X with a residue field $k_x \simeq \mathbb{F}_{q_x}$, $q_x = q^{\deg x}$ which is a finite extension of k . We want to define the Frobenius conjugacy class associated to x by analogy with the number field case. To this end, let us pick a point \overline{x} of this cover lying over a fixed point $x \in X$. The subgroup of $\text{Gal}(\overline{F}/F)$ preserving \overline{x} is the decomposition group of \overline{x} . If we make a different choice of \overline{x} , it gets conjugated in $\text{Gal}(\overline{F}/F)$. Therefore we obtain a subgroup of $\text{Gal}(\overline{F}/F)$ defined up to conjugation. We denote it by D_x . This group is in fact isomorphic to the Galois group $\text{Gal}(\overline{F}_x/F_x)$, and we have a natural homomorphism $D_x \rightarrow \text{Gal}(\overline{k}_x/k_x)$, whose kernel is called the inertia subgroup and is denoted by I_x .

As we saw in Sect. 1.3, the Galois group $\text{Gal}(\overline{k}_x/k_x)$ has a very simple description: it contains the *geometric Frobenius element* Fr_x , which is inverse to the automorphism $y \mapsto y^{q_x}$ of $\overline{k}_x = \overline{\mathbb{F}}_{q_x}$. The group $\text{Gal}(\overline{k}_x/k_x)$ is the profinite completion of the group \mathbb{Z} generated by this element.

A homomorphism σ from G_F to another group H is called *unramified at x* , if I_x lies in the kernel of σ (this condition is independent of the choice of \overline{x}). In this case Fr_x gives rise to a well-defined conjugacy class in H , denoted by $\sigma(\text{Fr}_x)$.

On the one side of the Langlands correspondence for the function field F we will have n -dimensional representations of the Galois group $\text{Gal}(\overline{F}/F)$. What kind of representations should we allow? The group $\text{Gal}(\overline{F}/F)$ is a profinite group, equipped with the Krull topology in which the base of open neighborhoods of the identity is formed by normal subgroups of finite index. It is natural to consider representations which are continuous with

respect to this topology. But a continuous finite-dimensional complex representation of a profinite group like $\text{Gal}(\overline{F}/F)$ necessarily factors through a finite quotient of $\text{Gal}(\overline{F}/F)$. To obtain a larger class of Galois representations we replace the field \mathbb{C} with the field \mathbb{Q}_ℓ of ℓ -adic numbers, where ℓ is a prime that does not divide q .

We have already seen in Sect. 1.7 that Galois representations arising from étale cohomology are realized in vector spaces over \mathbb{Q}_ℓ rather than \mathbb{C} , so this comes as no surprise to us. To see how replacing \mathbb{C} with \mathbb{Q}_ℓ helps we look at the following toy model.

Galois groups are profinite groups, and for the purposes of this toy model we will replace a Galois group by the simplest profinite group, the additive group \mathbb{Z}_ℓ of ℓ -adic integers,

$$\mathbb{Z}_\ell = \varprojlim \mathbb{Z}/\ell^n \mathbb{Z},$$

with the topology in which the open neighborhoods of the zero element are $\ell^n \mathbb{Z}$, $n \geq 0$. Suppose that we have a one-dimensional continuous representation of \mathbb{Z}_ℓ over \mathbb{C} . This is the same as a continuous homomorphism $\sigma : \mathbb{Z}_\ell \rightarrow \mathbb{C}^\times$. We have $\sigma(0) = 1$. Therefore continuity requires that for any $\epsilon > 0$, there exists $n \in \mathbb{Z}_+$ such that $|\sigma(a) - 1| < \epsilon$ for all $a \in \ell^n \mathbb{Z}_\ell$. In particular, taking $a = \ell^n$, we find that $\sigma(\ell^n) = \sigma(1)^{\ell^n}$. It is clear that the above continuity condition can be satisfied if and only if $\sigma(1)$ is a root of unity of order ℓ^N for some $N \in \mathbb{Z}_+$. But then σ factors through the finite group $\mathbb{Z}_\ell/\ell^N \mathbb{Z}_\ell = \mathbb{Z}/\ell^N \mathbb{Z}$.

Now let us look at a one-dimensional continuous representation σ of \mathbb{Z}_ℓ over \mathbb{Q}_ℓ . Given any ℓ -adic number μ such that $\mu - 1 \in \ell \mathbb{Z}_\ell$, we have $\mu^{\ell^n} - 1 \in \ell^{n+1} \mathbb{Z}_\ell$, and so $|\mu^{\ell^n} - 1|_\ell \leq \ell^{-n-1}$. This implies that for any such μ there exists a unique continuous homomorphism $\sigma : \mathbb{Z}_\ell \rightarrow \mathbb{Q}_\ell^\times$ such that $\sigma(1) = \mu$. Thus we obtain many representations that do not factor through a finite quotient of \mathbb{Z}_ℓ . The conclusion is that the ℓ -adic topology in \mathbb{Q}_ℓ^\times , and more generally, in $GL_n(\mathbb{Q}_\ell)$ is much better suited for the Krull topology on the Galois group $\text{Gal}(\overline{F}/F)$.

So let us pick a prime ℓ relatively prime to q . By an n -dimensional ℓ -adic representation of $\text{Gal}(\overline{F}/F)$ we will understand a continuous homomorphism $\sigma : \text{Gal}(\overline{F}/F) \rightarrow GL_n(\mathbb{Q}_\ell)$ which satisfies the following conditions:

- there exists a finite extension $E \subset \overline{\mathbb{Q}_\ell}$ of \mathbb{Q}_ℓ such that σ factors through a homomorphism $G_F \rightarrow GL_n(E)$, which is continuous with respect to the Krull topology on G_F and the ℓ -adic topology on $GL_n(E)$;
- it is unramified at all but finitely many points of X .

Let \mathcal{G}_n be the set of equivalence classes of irreducible n -dimensional ℓ -adic representations of G_F such that the image of $\det(\sigma)$ is a finite group.

Given such a representation, we consider the collection of the Frobenius conjugacy classes $\{\sigma(\text{Fr}_x)\}$ in $GL_n(\overline{\mathbb{Q}_\ell})$ and the collection of their eigenvalues (defined up to permutation), which we denote by $\{(z_1(\sigma_x), \dots, z_n(\sigma_x))\}$, for all $x \in X$ where σ is unramified. Chebotarev's density theorem implies the following remarkable result: if two ℓ -adic representations are such that their collections of the Frobenius conjugacy classes coincide for all but finitely many points $x \in X$, then these representations are equivalent.

2.3. Automorphic representations. On the other side of the Langlands correspondence we should consider automorphic representations of the adèlic group $GL_n(\mathbb{A})$.

Here $\mathbb{A} = \mathbb{A}_F$ is the ring of adèles of F , defined in the same way as in the number field case. For any closed point x of X , we denote by F_x the completion of F at x and by \mathcal{O}_x its

ring of integers. If we pick a rational function t_x on X which vanishes at x to order one, then we obtain isomorphisms $F_x \simeq k_x((t_x))$ and $\mathcal{O}_x \simeq k_x[[t_x]]$, where k_x is the residue field of x (the quotient of the local ring \mathcal{O}_x by its maximal ideal). As already mentioned above, this field is a finite extension of the base field k and hence is isomorphic to \mathbb{F}_{q_x} , where $q_x = q^{\deg x}$. The ring \mathbb{A} of adèles of F is by definition the *restricted* product of the fields F_x , where x runs over the set of all closed points of X . The word “restricted” means that we consider only the collections $(f_x)_{x \in X}$ of elements of F_x in which $f_x \in \mathcal{O}_x$ for all but finitely many x . The ring \mathbb{A} contains the field F , which is embedded into \mathbb{A} diagonally, by taking the expansions of rational functions on X at all points.

We want to define cuspidal automorphic representations of $GL_n(\mathbb{A})$ by analogy with the number field case (see Sect. 1.6). For that we need to introduce some notation.

Note that $GL_n(F)$ is naturally a subgroup of $GL_n(\mathbb{A})$. Let K be the maximal compact subgroup $K = \prod_{x \in X} GL_n(\mathcal{O}_x)$ of $GL_n(\mathbb{A})$. The group $GL_n(\mathbb{A})$ has the center $Z(\mathbb{A}) \simeq \mathbb{A}^\times$ which consists of the diagonal matrices.

Let $\chi : Z(\mathbb{A}) \rightarrow \mathbb{C}^\times$ be a character of $Z(\mathbb{A})$ which factors through a finite quotient of $Z(\mathbb{A})$. Denote by $\mathcal{C}_\chi(GL_n(F) \backslash GL_n(\mathbb{A}))$ the space of locally constant functions $f : GL_n(F) \backslash GL_n(\mathbb{A}) \rightarrow \mathbb{C}$ satisfying the following additional requirements (compare with the conditions in Sect. 1.6):

- (*K*-finiteness) the (right) translates of f under the action of elements of the compact subgroup K span a finite-dimensional vector space;
- (central character) $f(gz) = \chi(z)f(g)$ for all $g \in GL_n(\mathbb{A})$, $z \in Z(\mathbb{A})$;
- (cuspidality) let N_{n_1, n_2} be the unipotent radical of the standard parabolic subgroup P_{n_1, n_2} of GL_n corresponding to the partition $n = n_1 + n_2$ with $n_1, n_2 > 0$. Then

$$\int_{N_{n_1, n_2}(F) \backslash N_{n_1, n_2}(\mathbb{A})} \varphi(ug) du = 0, \quad \forall g \in GL_n(\mathbb{A}).$$

The group $GL_n(\mathbb{A})$ acts on $\mathcal{C}_\chi(GL_n(F) \backslash GL_n(\mathbb{A}))$ from the right: for

$$f \in \mathcal{C}_\chi(GL_n(F) \backslash GL_n(\mathbb{A})), \quad g \in GL_n(\mathbb{A})$$

we have

$$(2.2) \quad (g \cdot f)(h) = f(hg), \quad h \in GL_n(F) \backslash GL_n(\mathbb{A}).$$

Under this action $\mathcal{C}_\chi(GL_n(F) \backslash GL_n(\mathbb{A}))$ decomposes into a direct sum of irreducible representations. These representations are called *irreducible cuspidal automorphic representations* of $GL_n(\mathbb{A})$. A theorem due to I. Piatetski-Shapiro and J. Shalika says that each of them enters $\mathcal{C}_\chi(GL_n(F) \backslash GL_n(\mathbb{A}))$ with multiplicity one. We denote the set of equivalence classes of these representations by \mathcal{A}_n .

A couple of comments about the above conditions are in order. First, we comment on the cuspidality condition. Observe that if π_1 and π_2 are irreducible representations of $GL_{n_1}(\mathbb{A})$ and $GL_{n_2}(\mathbb{A})$, respectively, where $n_1 + n_2 = n$, then we may extend trivially the representation $\pi_1 \otimes \pi_2$ of $GL_{n_1} \times GL_{n_2}$ to the parabolic subgroup $P_{n_1, n_2}(\mathbb{A})$ and consider the induced representation of $GL_n(\mathbb{A})$. A theorem of R. Langlands says that an irreducible automorphic representation of $GL_n(\mathbb{A})$ is either cuspidal or is induced from cuspidal automorphic representations π_1 and π_2 of $GL_{n_1}(\mathbb{A})$ and $GL_{n_2}(\mathbb{A})$ (in that case it

usually shows up in the continuous spectrum). So cuspidal automorphic representations are those which do not come from subgroups of GL_n of smaller rank.

The condition that the central character has finite order is imposed so as to match the condition on the Galois side that $\det \sigma$ has finite order. These conditions are introduced solely to avoid some inessential technical issues.

Now let π be an irreducible cuspidal automorphic representation of $GL_n(\mathbb{A})$. One can show that it decomposes into a tensor product

$$\pi = \bigotimes'_{x \in X} \pi_x,$$

where each π_x is an irreducible representation of $GL_n(F_x)$. Furthermore, there is a finite subset S of X such that each π_x with $x \in X \setminus S$ is *unramified*, i.e., contains a non-zero vector v_x stable under the maximal compact subgroup $GL_n(\mathcal{O}_x)$ of $GL_n(F_x)$. This vector is unique up to a scalar and we will fix it once and for all. The space $\bigotimes'_{x \in X} \pi_x$ is by definition the span of all vectors of the form $\bigotimes_{x \in X} w_x$, where $w_x \in \pi_x$ and $w_x = v_x$ for all but finitely many $x \in X \setminus S$. Therefore the action of $GL_n(\mathbb{A})$ on π is well-defined.

As in the number field case, we will now use an additional symmetry of unramified factors π_x , namely, the spherical Hecke algebra.

Let x be a point of X with residue field \mathbb{F}_{q_x} . By definition, \mathcal{H}_x be the space of compactly supported functions on $GL_n(F_x)$ which are bi-invariant with respect to the subgroup $GL_n(\mathcal{O}_x)$. This is an algebra with respect to the convolution product

$$(2.3) \quad (f_1 \star f_2)(g) = \int_{GL_n(F_x)} f_1(gh^{-1})f_2(h) dh,$$

where dh is the Haar measure on $GL_n(F_x)$ normalized in such a way that the volume of the subgroup $GL_n(\mathcal{O}_x)$ is equal to 1. It is called the *spherical Hecke algebra* corresponding to the point x .

The algebra \mathcal{H}_x may be described quite explicitly. Let $H_{i,x}$ be the characteristic function of the $GL_n(\mathcal{O}_x)$ double coset

$$(2.4) \quad M_n^i(\mathcal{O}_x) = GL_n(\mathcal{O}_x) \cdot \text{diag}(t_x, \dots, t_x, 1, \dots, 1) \cdot GL_n(\mathcal{O}_x) \subset GL_n(F_x)$$

of the diagonal matrix whose first i entries are equal to t_x , and the remaining $n - i$ entries are equal to 1. Note that this double coset does not depend on the choice of the coordinate t_x . Then \mathcal{H}_x is the commutative algebra freely generated by $H_{1,x}, \dots, H_{n-1,x}, H_{n,x}^{\pm 1}$:

$$(2.5) \quad \mathcal{H}_x \simeq \mathbb{C}[H_{1,x}, \dots, H_{n-1,x}, H_{n,x}^{\pm 1}].$$

Define an action of $f_x \in \mathcal{H}_x$ on $v \in \pi_x$ by the formula

$$(2.6) \quad f_x \star v = \int f_x(g)(g \cdot v) dg.$$

Since f_x is left $GL_n(\mathcal{O}_x)$ -invariant, the result is again a $GL_n(\mathcal{O}_x)$ -invariant vector. If π_x is irreducible, then the space of $GL_n(\mathcal{O}_x)$ -invariant vectors in π_x is one-dimensional. Let $v_x \in \pi_x$ be a generator of this one-dimensional vector space. Then

$$f_x \star v_x = \phi(f_x)v_x$$

for some $\phi(f_x) \in \mathbb{C}$. Thus, we obtain a linear functional $\phi : \mathcal{H}_x \rightarrow \mathbb{C}$, and it is easy to see that it is actually a homomorphism.

In view of the isomorphism (2.5), a homomorphism $\mathcal{H}_x \rightarrow \mathbb{C}$ is completely determined by its values on $H_{1,x}, \dots, H_{n-1,x}$, which could be arbitrary complex numbers, and its value on $H_{n,x}$, which has to be a non-zero complex number. These values are the eigenvalues on v_x of the operators (2.6) of the action of $f_x = H_{i,x}$. These operators are called the *Hecke operators*. It is convenient to package these eigenvalues as an n -tuple of *unordered* non-zero complex numbers z_1, \dots, z_n , so that

$$(2.7) \quad H_{i,x} \star v_x = q_x^{i(n-i)/2} s_i(z_1, \dots, z_n) v_x, \quad i = 1, \dots, n,$$

where s_i is the i th elementary symmetric polynomial.¹⁹

In other words, the above formulas may be used to identify

$$(2.8) \quad \mathcal{H}_x \simeq \mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]^{S_n}.$$

Note that the algebra of symmetric polynomials on the right hand side may be thought of as the algebra of characters of finite-dimensional representations of $GL_n(\mathbb{C})$, so that $H_{i,x}$ corresponds to $q_x^{i(n-i)/2}$ times the character of the i th fundamental representation. From this point of view, (z_1, \dots, z_n) may be thought of as a semi-simple conjugacy class in $GL_n(\mathbb{C})$. This interpretation will become very useful later on (see Sect. 5.2).

So, using the spherical Hecke algebra, we attach to those factors π_x of π which are unramified a collection of n unordered non-zero complex numbers, which we will denote by $(z_1(\pi_x), \dots, z_n(\pi_x))$. Thus, to each irreducible cuspidal automorphic representation π one associates a collection of unordered n -tuples of numbers

$$\{(z_1(\pi_x), \dots, z_n(\pi_x))\}_{x \in X \setminus S}.$$

We call these numbers the *Hecke eigenvalues of π* . The strong multiplicity one theorem due to I. Piatetski-Shapiro says that this collection determines π up to an isomorphism.

2.4. The Langlands correspondence. Now we are ready to state the Langlands conjecture for GL_n in the function field case. It has been proved by Drinfeld [39, 40] for $n = 2$ and by Lafforgue [41] for $n > 2$.

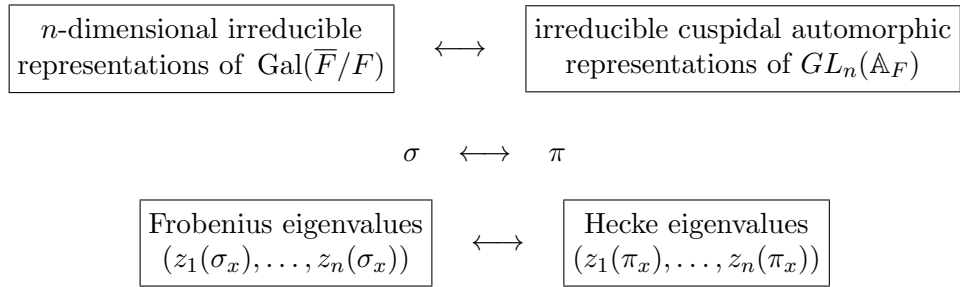
Theorem 1. *There is a bijection between the sets \mathcal{G}_n and \mathcal{A}_n defined above which satisfies the following matching condition. If $\sigma \in \mathcal{G}_n$ corresponds to $\pi \in \mathcal{A}_n$, then the sets of points where they are unramified are the same, and for each x from this set we have*

$$(z_1(\sigma_x), \dots, z_n(\sigma_x)) = (z_1(\pi_x), \dots, z_n(\pi_x))$$

up to permutation.

In other words, if π and σ correspond to each other, then the Hecke eigenvalues of π coincide with the Frobenius eigenvalues of σ at all points where they are unramified. Schematically,

¹⁹the factor $q_x^{i(n-i)/2}$ is introduced so as to make nicer the formulation of Theorem 1



The reader may have noticed a small problem in this formulation: while the numbers $z_i(\sigma_x)$ belong to $\overline{\mathbb{Q}}_\ell$, the numbers $z_i(\pi_x)$ are complex numbers. To make sense of the above equality, we must choose, once and for all, an isomorphism between $\overline{\mathbb{Q}}_\ell$ and \mathbb{C} , as abstract fields (note that such an isomorphism necessarily takes the subfield $\overline{\mathbb{Q}}$ of $\overline{\mathbb{Q}}_\ell$ to the corresponding subfield of \mathbb{C}). This is possible, as the fields $\overline{\mathbb{Q}}_\ell$ and \mathbb{C} have the same cardinality. Of course, choosing such an isomorphism seems like a very unnatural thing to do, and having to do this leads to some initial discomfort. The saving grace is another theorem proved by Drinfeld and Lafforgue which says that the Hecke eigenvalues $z_i(\pi_x)$ of π are actually algebraic numbers, i.e., they belong to $\overline{\mathbb{Q}}$, which is also naturally a subfield of $\overline{\mathbb{Q}}_\ell$.²⁰ Thus, we do not need to choose an isomorphism $\overline{\mathbb{Q}} \simeq \mathbb{C}$ after all.

What is remarkable about Theorem 1 is that it is such a “clean” statement: there is a *bijection* between the isomorphism classes of appropriately defined Galois representations and automorphic representations. Such a bijection is impossible in the number field case: we do not expect that all automorphic representations correspond to Galois representations. For example, in the case of $GL_2(\mathbb{A})$ there are automorphic representations whose factor at the archimedean place is a representation of the principal series of representations of (\mathfrak{gl}_2, O_2) .²¹ But there aren’t any two-dimensional Galois representations corresponding to them.

The situation in the function field case is so much nicer partly because the function field is defined geometrically (via algebraic curves), and this allows the usage of techniques and methods that are not yet available for number fields (surely, it also helps that F does not have any archimedean completions). It is natural to ask whether the Langlands correspondence could be formulated purely geometrically, for algebraic curves over an arbitrary field, not necessarily a finite field. We will discuss this in the next part of this survey.

²⁰moreover, they prove that these numbers all have (complex) absolute value equal to 1, which gives the so-called Ramanujan-Petersson conjecture and Deligne purity conjecture

²¹these representations correspond to the so-called Maass forms on the upper half-plane

Part II. The geometric Langlands Program

The geometric reformulation of the Langlands conjecture allows one to state it for curves defined over an arbitrary field, not just over finite fields. For instance, it may be stated for complex curves, and in this setting one can apply methods of complex algebraic geometry which are unavailable over finite fields. Hopefully, this formulation will eventually help us understand better the general underlying patterns of the Langlands correspondence. In this section we will formulate the geometric Langlands conjecture for GL_n . In particular, we will explain how moduli spaces of rank n vector bundles on algebraic curves naturally come into play. We will then show how to use the geometry of the simplest of these moduli spaces, the Picard variety, to prove the geometric Langlands correspondence for GL_1 , following P. Deligne. Next, we will generalize the geometric Langlands correspondence to the case of an arbitrary reductive group. We will also discuss the connection between this correspondence over the field of complex numbers and the Fourier-Mukai transform.

3. THE GEOMETRIC LANGLANDS CONJECTURE

What needs to be done to reformulate the Langlands conjecture geometrically? We have to express the two key notions used in the classical set-up: the Galois representations and the automorphic representations, geometrically, so that they make sense for a curve defined over, say, the field of complex numbers.

3.1. Galois representations as local systems. Let X be again a curve over a finite field k , and $F = k(X)$ the field of rational functions on X . As we indicated in Sect. 2.2, the Galois group $\text{Gal}(\bar{F}/F)$ should be viewed as a kind of fundamental group, and so its representations unramified away from a finite set of points S should be viewed as local systems on $X \setminus S$.

The notion of a local system makes sense if X is defined over other fields. The main case of interest to us is when X is a smooth projective curve over \mathbb{C} , or equivalently, a compact Riemann surface. Then by a local system on X we understand a locally constant sheaf \mathcal{F} of vector spaces on X , in the *analytic topology* of X in which the base of open neighborhoods of a point $x \in X$ is formed by small discs centered at x (defined with respect to a particular metric in the conformal class of X). This should be contrasted with the *Zariski topology* of X in which open neighborhoods of $x \in X$ are complements of finitely many points of X .

More concretely, for each open analytic subset U of X we have a \mathbb{C} -vector space $\mathcal{F}(U)$ of sections of \mathcal{F} over U satisfying the usual compatibilities²² and for each point $x \in X$ there is an open neighborhood U_x such that the restriction of \mathcal{F} to U_x is isomorphic to the

²²namely, we are given restriction maps $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$ for all inclusions of open sets $V \hookrightarrow U$ such that if $U_\alpha, \alpha \in I$, are open subsets and we are given sections $s_\alpha \in \mathcal{F}(U_\alpha)$ such that the restrictions of s_α and s_β to $U_\alpha \cap U_\beta$ coincide, then there exists a unique section of \mathcal{F} over $\cup_\alpha U_\alpha$ whose restriction to each U_α is s_α

constant sheaf.²³ These data may be expressed differently, by choosing a covering $\{U_\alpha\}$ of X by open subsets such that $\mathcal{F}|_{U_\alpha}$ is the constant sheaf \mathbb{C}^n . Then on overlaps $U_\alpha \cap U_\beta$ we have an identification of these sheaves, which is a *constant* element $g_{\alpha\beta}$ of $GL_n(\mathbb{C})$.²⁴

A notion of a locally constant sheaf on X is equivalent to the notion of a homomorphism from the fundamental group $\pi_1(X, x_0)$ to $GL_n(\mathbb{C})$. Indeed, the structure of locally constant sheaf allows us to identify the fibers of such a sheaf at any two nearby points. Therefore, for any path in X starting at x_0 and ending at x_1 and a vector in the fiber \mathcal{F}_{x_0} of our sheaf at x_0 we obtain a vector in the fiber \mathcal{F}_{x_1} over x_1 . This gives us a linear map $\mathcal{F}_{x_0} \rightarrow \mathcal{F}_{x_1}$. This map depends only on the homotopy class of the path. Now, given a locally constant sheaf \mathcal{F} , we choose a reference point $x_0 \in X$ and identify the fiber \mathcal{F}_{x_0} with the vector space \mathbb{C}^n . Then we obtain a homomorphism $\pi_1(X, x_0) \rightarrow GL_n(\mathbb{C})$.

Conversely, given a homomorphism $\sigma : \pi_1(X, x_0) \rightarrow GL_n(\mathbb{C})$, consider the trivial local system $\tilde{X} \times \mathbb{C}^n$ over the pointed universal cover (\tilde{X}, \tilde{x}_0) of (X, x_0) . The group $\pi_1(X, x_0)$ acts on \tilde{X} . Define a local system on X as the quotient

$$\tilde{X} \times_{\pi_1(X, x_0)} \mathbb{C}^n = \{(\tilde{x}, v)\} / \{(\tilde{x}, v) \sim (g\tilde{x}, \sigma(g)v)\}_{g \in \pi_1(X, x_0)}.$$

There is yet another way to realize local systems which will be especially convenient for us: by defining a complex vector bundle on X equipped with a flat connection. A complex vector bundle \mathcal{E} by itself does not give us a local system, because while \mathcal{E} can be trivialized on sufficiently small open analytic subsets $U_\alpha \subset X$, the transition functions on the overlaps $U_\alpha \cap U_\beta$ will in general be non-constant functions $U_\alpha \cap U_\beta \rightarrow GL_n(\mathbb{C})$. To make them constant, we need an additional rigidity on \mathcal{E} which would give us a *preferred* system of trivializations on each open subset such that on the overlaps they would differ only by *constant* transition functions. Such a system is provided by the data of a flat connection.

Recall that a *flat connection* on \mathcal{E} is a system of operations ∇ , defined for each open subset $U \subset X$ and compatible on overlaps,

$$\nabla : \text{Vect}(U) \rightarrow \text{End}(\Gamma(U, \mathcal{E})),$$

which assign to a vector field ξ on U a linear operator ∇_ξ on the space $\Gamma(U, \mathcal{E})$ of smooth sections of \mathcal{E} on U . It must satisfy the Leibniz rule

$$(3.1) \quad \nabla_\xi(fs) = f\nabla_\xi(s) + (\xi \cdot f)s, \quad f \in C^\infty(U), s \in \Gamma(U, \mathcal{E}),$$

and also the conditions

$$(3.2) \quad \nabla_{f\xi} = f\nabla_\xi, \quad [\nabla_\xi, \nabla_\eta] = \nabla_{[\xi, \eta]}$$

(the last condition is the flatness). Given a flat connection, the local horizontal sections (i.e., those annihilated by all ∇_ξ) provide us with the preferred systems of local trivializations (or equivalently, identifications of nearby fibers) that we were looking for.

Note that if X is a complex manifold, like it is in our case, then the connection has two parts: holomorphic and anti-holomorphic, which are defined with respect to the complex structure on X . The anti-holomorphic (or $(0, 1)$) part of the connection consists of the operators ∇_ξ , where ξ runs over the anti-holomorphic vector fields on $U \subset X$. It gives us

²³for which the space $\mathcal{F}(U)$ is a fixed vector space \mathbb{C}^n and all restriction maps are isomorphisms

²⁴these elements must satisfy the cocycle condition $g_{\alpha\gamma} = g_{\alpha\beta}g_{\beta\gamma}$ on each triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$

a holomorphic structure on \mathcal{E} : namely, we declare the holomorphic sections to be those which are annihilated by the anti-holomorphic part of the connection. Thus, a complex bundle \mathcal{E} equipped with a flat connection ∇ automatically becomes a holomorphic bundle on X . Conversely, if \mathcal{E} is already a holomorphic vector bundle on a complex manifold X , then to define a connection on \mathcal{E} that is compatible with the holomorphic structure on \mathcal{E} all we need to do is to define a *holomorphic flat connection*. By definition, this is just a collection of operators ∇_ξ , where ξ runs over all holomorphic vector fields on $U \subset X$, satisfying conditions (3.1) and (3.2), where f is now a holomorphic function on U and s is a holomorphic section of \mathcal{E} over U .

In particular, if X is a complex curve, then locally, with respect to a local holomorphic coordinate z on X and a local trivialization of \mathcal{E} , all we need to define is an operator $\nabla_{\partial/\partial z} = \frac{\partial}{\partial z} + A(z)$, where $A(z)$ is a matrix valued holomorphic function. These operators must satisfy the usual compatibility conditions on the overlaps. Because there is only one such operator on each open set, the resulting connection is automatically flat.

Given a vector bundle \mathcal{E} with a flat connection ∇ on X (or equivalently, a holomorphic vector bundle on X with a holomorphic connection), we obtain a locally constant sheaf (i.e., a local system) on X as the sheaf of horizontal sections of \mathcal{E} with respect to ∇ . This construction in fact sets up an equivalence of the two categories if X is compact (for example, a smooth projective curve). This is called the *Riemann-Hilbert correspondence*.

More generally, in the Langlands correspondence we consider local systems defined on the non-compact curves $X \setminus S$, where X is a projective curve and S is a finite set. Such local systems are called *ramified* at the points of S . In this case the above equivalence of categories is valid only if we restrict ourselves to holomorphic bundles with holomorphic connections with regular singularities at the points of the set S (that means that the order of pole of the connection at a point in S is at most 1). However, in this paper (with the exception of Sect. 9.8) we will restrict ourselves to unramified local systems. In general, we expect that vector bundles on curves with connections that have singularities, regular or irregular, also play an important role in the geometric Langlands correspondence, see [44]; we discuss this in Sect. 9.8 below.

To summarize, we believe that we have found the right substitute for the (unramified) n -dimensional Galois representations in the case of a compact complex curve X : these are the rank n local systems on X , or equivalently, rank n holomorphic vector bundles on X with a holomorphic connection.

3.2. Adèles and vector bundles. Next, we wish to interpret geometrically the objects appearing on the other side of the Langlands correspondence, namely, the automorphic representations. This will turn out to be more tricky. The essential point here is the interpretation of automorphic representations in terms of the moduli spaces of rank n vector bundles.

For simplicity, we will restrict ourselves from now on to the irreducible automorphic representations of $GL_n(\mathbb{A})$ that are unramified at all points of X , in the sense explained in Sect. 2.3. Suppose that we are given such a representation π of $GL_n(\mathbb{A})$. Then the space of $GL_n(\mathcal{O})$ -invariants in π , where $\mathcal{O} = \prod_{x \in X} \mathcal{O}_x$, is one-dimensional, spanned by the

vector

$$v = \bigotimes_{x \in X} v_x \in \bigotimes_{x \in X} \pi_x = \pi,$$

where v_x is defined in Sect. 2.3. Hence v gives rise to a $GL_n(\mathcal{O})$ -invariant function on $GL_n(F) \backslash GL_n(\mathbb{A})$, or equivalently, a function f_π on the double quotient

$$GL_n(F) \backslash GL_n(\mathbb{A}) / GL_n(\mathcal{O}).$$

By construction, this function is an eigenfunction of the spherical Hecke algebras \mathcal{H}_x defined above for all $x \in X$, a property we will discuss in more detail later.

The function f_π completely determines the representation π because other vectors in π may be obtained as linear combinations of the right translates of f_π on $GL_n(F) \backslash GL_n(\mathbb{A})$. Hence instead of considering the set of equivalence classes of irreducible unramified cuspidal automorphic representations of $GL_n(\mathbb{A})$, one may consider the set of unramified automorphic functions on $GL_n(F) \backslash GL_n(\mathbb{A}) / GL_n(\mathcal{O})$ associated to them (each defined up to multiplication by a non-zero scalar).²⁵

The following key observation is due to A. Weil. Let X be a smooth projective curve over any field k and $F = k(X)$ the function field of X . We define the ring \mathbb{A} of adèles and its subring \mathcal{O} of integer adèles in the same way as in the case when $k = \mathbb{F}_q$. Then we have the following:

Lemma 2. *There is a bijection between the set $GL_n(F) \backslash GL_n(\mathbb{A}) / GL_n(\mathcal{O})$ and the set of isomorphism classes of rank n vector bundles on X .*

For simplicity, we consider this statement in the case when X is a complex curve (the proof in general is similar). We note that in the context of conformal field theory this statement has been discussed in [5], Sect. V.

We use the following observation: any rank n vector bundle \mathcal{V} on X can be trivialized over the complement of finitely many points. This is equivalent to the existence of n meromorphic sections of \mathcal{V} whose values are linearly independent away from finitely many points of X . These sections can be constructed as follows: choose a non-zero meromorphic section of \mathcal{V} . Then, over the complement of its zeros and poles, this section spans a line subbundle of \mathcal{V} . The quotient of \mathcal{V} by this line subbundle is a vector bundle \mathcal{V}' of rank $n - 1$. It also has a non-zero meromorphic section. Lifting this section to a section of \mathcal{V} in an arbitrary way, we obtain two sections of \mathcal{V} which are linearly independent away from finitely many points of X . Continuing like this, we construct n meromorphic sections of \mathcal{V} satisfying the above conditions.

Let x_1, \dots, x_N be the set of points such that \mathcal{V} is trivialized over $X \setminus \{x_1, \dots, x_N\}$. The bundle \mathcal{V} can also be trivialized over the small discs D_{x_i} around those points. Thus, we consider the covering of X by the open subsets $X \setminus \{x_1, \dots, x_N\}$ and $D_{x_i}, i = 1, \dots, N$. The overlaps are the punctured discs $D_{x_i}^\times$, and our vector bundle is determined by the transition functions on the overlaps, which are GL_n -valued functions g_i on $D_{x_i}^\times, i = 1, \dots, N$.

The difference between two trivializations of \mathcal{V} on D_{x_i} amounts to a GL_n -valued function h_i on D_{x_i} . If we consider a new trivialization on D_{x_i} that differs from the old one by h_i ,

²⁵note that this is analogous to replacing an automorphic representation of $GL_2(\mathbb{A}_\mathbb{Q})$ by the corresponding modular form, a procedure that we discussed in Sect. 1.6

then the i th transition function g_i will get multiplied on the right by h_i : $g_i \mapsto g_i h_i|_{D_{x_i}^\times}$, whereas the other transition functions will remain the same. Likewise, the difference between two trivializations of \mathcal{V} on $X \setminus \{x_1, \dots, x_N\}$ amounts to a GL_n -valued function h on $X \setminus \{x_1, \dots, x_N\}$. If we consider a new trivialization on $X \setminus \{x_1, \dots, x_N\}$ that differs from the old one by h , then the i th transition function g_i will get multiplied on the left by h : $g_i \mapsto h|_{D_{x_i}^\times} g_i$ for all $i = 1, \dots, N$.

We obtain that the set of isomorphism classes of rank n vector bundles on X which become trivial when restricted to $X \setminus \{x_1, \dots, x_N\}$ is the same as the quotient

$$(3.3) \quad GL_n(X \setminus \{x_1, \dots, x_N\}) \backslash \prod_{i=1}^N GL_n(D_{x_i}^\times) / \prod_{i=1}^N GL_n(D_{x_i}).$$

Here for an open set U we denote by $GL_n(U)$ the group of GL_n -valued function on U , with pointwise multiplication.

If we replace each D_{x_i} by the formal disc at x_i , then $GL_n(D_{x_i}^\times)$ will become $GL_n(F_x)$, where $F_x \simeq \mathbb{C}((t_x))$ is the algebra of formal Laurent series with respect to a local coordinate t_x at x , and $GL_n(D_{x_i})$ will become $GL_n(\mathcal{O}_x)$, where $\mathcal{O}_x \simeq \mathbb{C}[[t_x]]$ is the ring of formal Taylor series. Then, if we also allow the set x_1, \dots, x_N to be an arbitrary finite subset of X , we will obtain instead of (3.3) the double quotient

$$GL_n(F) \backslash \prod'_{x \in X} GL_n(F_x) / \prod_{x \in X} GL_n(\mathcal{O}_x),$$

where $F = \mathbb{C}(X)$ and the prime means the restricted product, defined as in Sect. 2.3.²⁶ But this is exactly the double quotient in the statement of the Lemma. This completes the proof.

3.3. From functions to sheaves. Thus, when X is a curve over \mathbb{F}_q , irreducible unramified automorphic representations π are encoded by the automorphic functions f_π , which are functions on $GL_n(F) \backslash GL_n(\mathbb{A}) / GL_n(\mathcal{O})$. This double quotient makes perfect sense when X is defined over \mathbb{C} and is in fact the set of isomorphism classes of rank n bundles on X . But what should replace the notion of an automorphic function f_π in this case? We will argue that the proper analogue is not a function, as one might naively expect, but a *sheaf* on the corresponding algebro-geometric object: the moduli stack Bun_n of rank n bundles on X .

This certainly requires a leap of faith. The key step is the Grothendieck *fonctions-faisceaux dictionary*. Let V be an algebraic variety over \mathbb{F}_q . Then, according to Grothendieck, the “correct” geometric counterpart of the notion of a ($\overline{\mathbb{Q}}_\ell$ -valued) function on the set of \mathbb{F}_q -points of V is the notion of a *complex of ℓ -adic sheaves* on V . A precise definition of an ℓ -adic sheaf would take us too far afield. Let us just say that the simplest example of an ℓ -adic sheaf is an ℓ -adic local system, which is, roughly speaking, a locally constant $\overline{\mathbb{Q}}_\ell$ -sheaf on V (in the étale topology).²⁷ For a general ℓ -adic sheaf there exists a stratification of V by locally closed subvarieties V_i such that the sheaves $\mathcal{F}|_{V_i}$ are locally constant.

²⁶the passage to the formal discs is justified by an analogue of the “strong approximation theorem” that was mentioned in Sect. 1.6

²⁷The precise definition (see, e.g., [45, 46]) is more subtle: a typical example is a compatible system of locally constant $\mathbb{Z}/\ell^n \mathbb{Z}$ -sheaves for $n > 0$

The important property of the notion of an ℓ -adic sheaf \mathcal{F} on V is that for any morphism $f : V' \rightarrow V$ from another variety V' to V the group of symmetries of this morphism will act on the pull-back of \mathcal{F} to V' . In particular, let x be an \mathbb{F}_q -point of V and \bar{x} the $\overline{\mathbb{F}}_q$ -point corresponding to an inclusion $\mathbb{F}_q \hookrightarrow \overline{\mathbb{F}}_q$. Then the pull-back of \mathcal{F} with respect to the composition $\bar{x} \rightarrow x \rightarrow V$ is a sheaf on \bar{x} , which is nothing but the fiber $\mathcal{F}_{\bar{x}}$ of \mathcal{F} at \bar{x} , a $\overline{\mathbb{Q}}_\ell$ -vector space. But the Galois group $\text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ is the symmetry of the map $\bar{x} \rightarrow x$, and therefore it acts on $\mathcal{F}_{\bar{x}}$. In particular, the (geometric) Frobenius element $\text{Fr}_{\bar{x}}$, which is the generator of this group acts on $\mathcal{F}_{\bar{x}}$. Taking the trace of $\text{Fr}_{\bar{x}}$ on $\mathcal{F}_{\bar{x}}$, we obtain a number $\text{Tr}(\text{Fr}_{\bar{x}}, \mathcal{F}_{\bar{x}}) \in \overline{\mathbb{Q}}_\ell$.

Hence we obtain a function $\mathbf{f}_{\mathcal{F}}$ on the set of \mathbb{F}_q -points of V , whose value at x is

$$\mathbf{f}_{\mathcal{F}}(x) = \text{Tr}(\text{Fr}_{\bar{x}}, \mathcal{F}_{\bar{x}}).$$

More generally, if \mathcal{K} is a complex of ℓ -adic sheaves, one defines a function $\mathbf{f}(\mathcal{K})$ on $V(\mathbb{F}_q)$ by taking the alternating sums of the traces of $\text{Fr}_{\bar{x}}$ on the stalk cohomologies of \mathcal{K} at \bar{x} :

$$\mathbf{f}_{\mathcal{K}}(x) = \sum_i (-1)^i \text{Tr}(\text{Fr}_{\bar{x}}, H_{\bar{x}}^i(\mathcal{K})).$$

The map $\mathcal{K} \rightarrow \mathbf{f}_{\mathcal{K}}$ intertwines the natural operations on complexes of sheaves with natural operations on functions (see [47], Sect. 1.2). For example, pull-back of a sheaf corresponds to the pull-back of a function, and push-forward of a sheaf with compact support corresponds to the fiberwise integration of a function.²⁸

Thus, because of the existence of the Frobenius automorphism in the Galois group $\text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ (which is the group of symmetries of an \mathbb{F}_q -point) we can pass from ℓ -adic sheaves to functions on any algebraic variety over \mathbb{F}_q . This suggests that the proper geometrization of the notion of a function in this setting is the notion of ℓ -adic sheaf.

The passage from complexes of sheaves to functions is given by the alternating sum of cohomologies. Hence what matters is not \mathcal{K} itself, but the corresponding object of the derived category of sheaves. However, the derived category is too big, and there are many objects of the derived category which are non-zero, but whose function is equal to zero. For example, consider a complex of the form $0 \rightarrow \mathcal{F} \rightarrow \mathcal{F} \rightarrow 0$ with the zero differential. It has non-zero cohomologies in degrees 0 and 1, and hence is a non-zero object of the derived category. But the function associated to it is identically zero. That is why it would be useful to identify a natural abelian category \mathcal{C} in the derived category of ℓ -adic sheaves such that the map assigning to an object $\mathcal{K} \in \mathcal{C}$ the function $\mathbf{f}_{\mathcal{K}}$ gives rise to an *injective* map from the Grothendieck group of \mathcal{C} to the space of functions on V .²⁹

The naive category of ℓ -adic sheaves (included into the derived category as the subcategory whose objects are the complexes situated in cohomological degree 0) is not a good choice for various reasons; for instance, it is not stable under the Verdier duality. The correct choice turns out to be the abelian category of *perverse sheaves*.

What is a perverse sheaf? It is not really a sheaf, but a complex of ℓ -adic sheaves on V satisfying certain restrictions on the degrees of their non-zero stalk cohomologies

²⁸this follows from the Grothendieck-Lefschetz trace formula

²⁹more precisely, to do that we need to extend this function to the set of all \mathbb{F}_{q_1} -points of V , where $q_1 = q^m$, $m > 0$

(see [48, 49, 50, 51]).³⁰ Examples are ℓ -adic local systems on a smooth variety V , placed in cohomological degree equal to $-\dim V$. General perverse sheaves are “glued” from such local systems defined on the strata of a particular stratification $V = \bigcup_i V_i$ of V by locally closed subvarieties. Even though perverse sheaves are complexes of sheaves, they form an abelian subcategory inside the derived category of sheaves, so we can work with them like with ordinary sheaves. Unlike the ordinary sheaves though, the perverse sheaves have the following remarkable property: an irreducible perverse sheaf on a variety V is completely determined by its restriction to an arbitrary open dense subset (provided that this restriction is non-zero). For more on this, see Sect. 5.4.

Experience shows that many “interesting” functions on the set $V(\mathbb{F}_q)$ of points of an algebraic variety V over \mathbb{F}_q are of the form $\int \mathcal{K}$ for a perverse sheaf \mathcal{K} on V . Unramified automorphic functions on $GL_n(F)\backslash GL_n(\mathbb{A})/GL_n(\mathcal{O})$ certainly qualify as “interesting” functions. Can we obtain them from perverse sheaves on some algebraic variety underlying the set $GL_n(F)\backslash GL_n(\mathbb{A})/GL_n(\mathcal{O})$?

In order to do that we need to interpret the set $GL_n(F)\backslash GL_n(\mathbb{A})/GL_n(\mathcal{O})$ as the set of \mathbb{F}_q -points of an algebraic variety over \mathbb{F}_q . Lemma 2 gives us a hint as to what this variety should be: the *moduli space* of rank n vector bundles on the curve X .

Unfortunately, for $n > 1$ there is no algebraic variety whose set of \mathbb{F}_q -points is the set of isomorphism classes of *all* rank n bundles on X .³¹ The reason is that bundles have groups of automorphisms, which vary from bundle to bundle. So in order to define the structure of an algebraic variety we need to throw away the so-called unstable bundles, whose groups of automorphisms are too large, and glue together the so-called semi-stable bundles. Only the points corresponding to the so-called stable bundles will survive. But an automorphic function is a priori defined on the set of isomorphism classes of all bundles. Therefore we do not want to throw away any of them.³²

The solution is to consider the *moduli stack* Bun_n of rank n bundles on X . It is not an algebraic variety, but it looks locally like the quotient of an algebraic variety by the action of an algebraic group (these actions are not free, and therefore the quotient is no longer an algebraic variety). For a nice introduction to algebraic stacks, see [52]. Examples of stacks familiar to physicists include the Deligne-Mumford stack of stable curves of a fixed genus and the moduli stacks of stable maps. In these cases the groups of automorphisms are actually *finite*, so these stacks may be viewed as orbifolds. The situation is more complicated for vector bundles, for which the groups of automorphisms are typically continuous. The corresponding moduli stacks are called Artin stacks. For example, even in the case of line bundles, each of them has a continuous groups of automorphisms, namely, the multiplicative group. What saves the day is the fact that the group of automorphisms is the *same* for all line bundles. This is not true for bundles of rank higher than 1.

The technique developed in [53, 15] allows us to define sheaves on algebraic stacks and to operate with these sheaves in ways that we are accustomed to when working with algebraic varieties. So the moduli stack Bun_n will be sufficient for our purposes.

³⁰more precisely, a perverse sheaf is an object of the derived category of sheaves

³¹for $n = 1$, the Picard variety of X may be viewed as the moduli space of line bundles

³²actually, one can show that each cuspidal automorphic function vanishes on a subset of unstable bundles (see [55], Lemma 6.11), and this opens up the possibility that somehow moduli spaces of semi-stable bundles would suffice

Thus, we have now identified the geometric objects which should replace unramified automorphic functions: these should be perverse sheaves on the moduli stack Bun_n of rank n bundles on our curve X . The concept of perverse sheaf makes perfect sense for varieties over \mathbb{C} (see, e.g., [49, 50, 51]), and this allows us to formulate the geometric Langlands conjecture when X (and hence Bun_n) is defined over \mathbb{C} . But over the field of complex numbers there is one more reformulation that we can make, namely, we can pass from perverse sheaves to \mathcal{D} -modules. We now briefly discuss this last reformulation.

3.4. From perverse sheaves to \mathcal{D} -modules. If V is a smooth complex algebraic variety, we can define the sheaf \mathcal{D}_V of algebraic differential operators on V (in Zariski topology). The space of its sections on a Zariski open subset $U \subset V$ is the algebra $\mathcal{D}(U)$ of differential operators on U . For instance, if $U \simeq \mathbb{C}^n$, then this algebra is isomorphic to the Weyl algebra generated by coordinate functions $x_i, i = 1, \dots, n$, and the vector fields $\partial/\partial x_i, i = 1, \dots, n$. A (left) \mathcal{D} -module \mathcal{F} on V is by definition a sheaf of (left) modules over the sheaf \mathcal{D}_V . This means that for each open subset $U \subset V$ we are given a module $\mathcal{F}(U)$ over $\mathcal{D}(U)$, and these modules satisfy the usual compatibilities.

The simplest example of a \mathcal{D}_V -module is the sheaf of holomorphic sections of a holomorphic vector bundle \mathcal{E} on V equipped with a holomorphic (more precisely, algebraic) flat connection. Note that $\mathcal{D}(U)$ is generated by the algebra of holomorphic functions $\mathcal{O}(U)$ on U and the holomorphic vector fields on U . We define the action of the former on $\mathcal{E}(U)$ in the usual way, and the latter by means of the holomorphic connection. In the special case when \mathcal{E} is the trivial bundle with the trivial connection, its sheaf of sections is the sheaf \mathcal{O}_V of holomorphic functions on V .

Another class of examples is obtained as follows. Let $D_V = \Gamma(V, \mathcal{D}_V)$ be the algebra of global differential operators on V . Suppose that this algebra is commutative and is in fact isomorphic to the free polynomial algebra $D_V = \mathbb{C}[D_1, \dots, D_N]$, where D_1, \dots, D_N are some global differential operators on V . We will see below examples of this situation. Let $\lambda : D_V \rightarrow \mathbb{C}$ be an algebra homomorphism, which is completely determined by its values on the operators D_i . Define the (left) \mathcal{D}_V -module Δ_λ by the formula

$$(3.4) \quad \Delta_\lambda = \mathcal{D}_V / (\mathcal{D}_V \cdot \text{Ker } \lambda) = \mathcal{D}_V \otimes_{D_V} \mathbb{C},$$

where the action of D_V on \mathbb{C} is via λ .

Now consider the system of differential equations

$$(3.5) \quad D_i f = \lambda(D_i) f, \quad i = 1, \dots, N.$$

Observe if f_0 is any function on V which is a solution of (3.5), then for any open subset U the restriction $f_0|_U$ is automatically annihilated by $\mathcal{D}(U) \cdot \text{Ker } \lambda$. Therefore we have a natural \mathcal{D}_V -homomorphism from the \mathcal{D} -module Δ_λ defined by formula (3.4) to the sheaf of functions \mathcal{O}_V sending $1 \in \Delta_\lambda$ to f_0 . Conversely, since Δ_λ is generated by 1, any homomorphism $\Delta_\lambda \rightarrow \mathcal{O}_V$ is determined by the image of 1 and hence to be a solution f_0 of (3.5). In this sense, we may say that the \mathcal{D} -module Δ_λ represents the system of differential equations (3.5).

More generally, the f in the system (3.5) could be taking values in other spaces of functions, or distributions, etc. In other words, we could consider f as a section of some sheaf \mathcal{F} . This sheaf has to be a \mathcal{D}_V -module, for otherwise the system (3.5) would not

make sense. But no matter what \mathcal{F} is, an \mathcal{F} -valued solution f_0 of the system (3.5) is the same as a homomorphism $\Delta_\lambda \rightarrow \mathcal{F}$. Thus, Δ_λ is the “universal \mathcal{D}_V -module” for the system (3.5). This \mathcal{D}_V -module is called *holonomic* if the system (3.5) is holonomic, i.e., if $N = \dim_{\mathbb{C}} V$. We will see various examples of such \mathcal{D} -modules below.

As we discussed above, the sheaf of horizontal sections of a holomorphic vector bundle \mathcal{E} with a holomorphic flat connection on V is a locally constant sheaf (in the analytic, not Zariski, topology!), which becomes a perverse sheaf after the shift in cohomological degree by $\dim_{\mathbb{C}} V$. The corresponding functor from the category of bundles with flat connection on V to the category of locally constant sheaves on V may be extended to a functor from the category of holonomic \mathcal{D} -modules to the category of perverse sheaves. *A priori* this functor sends a \mathcal{D} -module to an object of the derived category of sheaves, but one shows that it is actually an object of the *abelian* subcategory of perverse sheaves. This provides another explanation why the category of perverse sheaves is the “right” abelian subcategory of the derived category of sheaves (as opposed to the naive abelian subcategory of complexes concentrated in cohomological degree 0, for example). This functor is called the Riemann-Hilbert correspondence. For instance, this functor assigns to a holonomic \mathcal{D} -module (3.4) on V the sheaf whose sections over an open analytic subset $U \subset V$ is the space of holomorphic functions on T that are solutions of the system (3.5) on U . In the next section we will see how this works in a simple example.

3.5. Example: a \mathcal{D} -module on the line. Consider the differential equation $t\partial_t = \lambda f$ on \mathbb{C} . The corresponding \mathcal{D} -module is

$$\Delta_\lambda = \mathcal{D}/(\mathcal{D} \cdot (t\partial_t - \lambda)).$$

It is sufficient to describe its sections on \mathbb{C} and on $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. We have

$$\Gamma(\mathbb{C}, \Delta_\lambda) = \mathbb{C}[t, \partial_t]/\mathbb{C}[t, \partial_t] \cdot (t\partial_t - \lambda),$$

so it is a space with the basis $\{t^n, \partial_t^m\}_{n>0, m \geq 0}$, and the action of $\mathbb{C}[t, \partial_t]$ is given by the formulas $\partial_t \cdot \partial_t^m = \partial_t^{m+1}$, $m \geq 0$; $\partial_t \cdot t^n = (n+\lambda)t^{n-1}$, $n > 0$, and $t \cdot t^n = t^{n+1}$, $n \geq 0$; $t \cdot \partial_t^m = (m-1+\lambda)\partial_t^{m-1}$, $m > 0$.

On the other hand,

$$\Gamma(\mathbb{C}^\times, \Delta_\lambda) = \mathbb{C}[t^{\pm 1}, \partial_t]/\mathbb{C}[t^{\pm 1}, \partial_t] \cdot (t\partial_t - \lambda),$$

and so it is isomorphic to $\mathbb{C}[t^{\pm 1}]$, but instead of the usual action of $\mathbb{C}[t^{\pm 1}, \partial_t]$ on $\mathbb{C}[t^{\pm 1}]$ we have the action given by the formulas $t \mapsto t, \partial_t \mapsto \partial_t - \lambda t^{-1}$. The restriction map $\Gamma(\mathbb{C}, \Delta_\lambda) \rightarrow \Gamma(\mathbb{C}^\times, \Delta_\lambda)$ sends $t^n \mapsto t^n, \partial_t^n \mapsto \lambda \partial_t^{n-1} \cdot t^{-1} = (-1)^{n-1} (n-1)! \lambda t^{-n}$.

Let \mathcal{P}_λ be the perverse sheaf on \mathbb{C} obtained from Δ_λ via the Riemann-Hilbert correspondence. What does it look like? It is easy to describe the restriction of \mathcal{P}_λ to \mathbb{C}^\times . A general local analytic solution of the equation $t\partial_t = \lambda f$ on \mathbb{C}^\times is $Ct^\lambda, C \in \mathbb{C}$. The restrictions of these functions to open analytic subsets of \mathbb{C}^\times define a rank one local system on \mathbb{C}^\times . This local system \mathcal{L}_λ is the restriction of the perverse sheaf \mathcal{P}_λ to \mathbb{C}^\times .³³ But what about its restriction to \mathbb{C} ? If λ is not a non-negative integer, there are no solutions of our

³³Note that the solutions Ct^λ are not algebraic functions for non-integer λ , and so it is very important that we consider the sheaf \mathcal{P}_λ in the analytic, *not* Zariski, topology! However, the equation defining it, and hence the \mathcal{D} -module Δ_λ , are algebraic for all λ , so we may consider Δ_λ in either analytic or Zariski topology.

equation on \mathbb{C} (or on any open analytic subset of \mathbb{C} containing 0). Therefore the space of sections of \mathcal{P}_λ on \mathbb{C} is 0. Thus, \mathcal{P}_λ is the so-called “!-extension” of the local system \mathcal{L}_λ on \mathbb{C}^\times , denoted by $j_!(\mathcal{L}_\lambda)$, where $j : \mathbb{C}^\times \hookrightarrow \mathbb{C}$.

But if $\lambda \in \mathbb{Z}_+$, then there is a solution on \mathbb{C} : $f = t^\lambda$, and so the space $\Gamma(\mathbb{C}, \mathcal{P}_\lambda)$ is one-dimensional. However, in this case there also appears the first cohomology $H^1(\mathbb{C}, \mathcal{P}_\lambda)$, which is also one-dimensional.

To see that, note that the Riemann-Hilbert correspondence is defined by the functor $\mathcal{F} \mapsto \text{Sol}(\mathcal{F}) = \mathcal{H}om_{\mathcal{D}}(\mathcal{F}, \mathcal{O})$, which is not right exact. Its higher derived functors are given by the formula $\mathcal{F} \mapsto R\text{Sol}(\mathcal{F}) = R\mathcal{H}om_{\mathcal{D}}(\mathcal{F}, \mathcal{O})$. Here we consider the derived $\mathcal{H}om$ functor in the analytic topology. The perverse sheaf \mathcal{P}_λ attached to Δ_λ by the Riemann-Hilbert correspondence is therefore the complex $R\text{Sol}(\Delta_\lambda)$. To compute it explicitly, we replace the \mathcal{D} -module Δ_λ by the free resolution $C^{-1} \rightarrow C^0$ with the terms $C^0 = C^{-1} = \mathcal{D}$ and the differential given by multiplication on the right by $t\partial_t - \lambda$. Then $R\text{Sol}(\mathcal{F})$ is represented by the complex $\mathcal{O} \rightarrow \mathcal{O}$ (in degrees 0 and 1) with the differential $t\partial_t - \lambda$. In particular, its sections over \mathbb{C} are represented by the complex $\mathbb{C}[t] \rightarrow \mathbb{C}[t]$ with the differential $t\partial_t - \lambda$. For $\lambda \in \mathbb{Z}_+$ this map has one-dimensional kernel and cokernel (spanned by t^λ), which means that $\Gamma(\mathbb{C}, \mathcal{P}_\lambda) = H^1(\mathbb{C}, \mathcal{P}_\lambda) = \mathbb{C}$. Thus, \mathcal{P}_λ is not a sheaf, but a complex of sheaves when $\lambda \in \mathbb{Z}_+$. Nevertheless, this complex is a perverse sheaf, i.e., it belongs to the abelian category of perverse sheaves in the corresponding derived category. This complex is called the *-extension of the constant sheaf $\underline{\mathbb{C}}$ on \mathbb{C}^\times , denoted by $j_*(\underline{\mathbb{C}})$.

Thus, we see that if the monodromy of our local system \mathcal{L}_λ on \mathbb{C}^\times is non-trivial, then it has only one extension to \mathbb{C} , denoted above by $j_!(\mathcal{L}_\lambda)$. In this case the *-extension $j_*(\mathcal{L}_\lambda)$ is also well-defined, but it is equal to $j_!(\mathcal{L}_\lambda)$. Placed in cohomological degree -1 , this sheaf becomes an irreducible perverse sheaf on \mathbb{C} .

On the other hand, for $\lambda \in \mathbb{Z}$ the local system \mathcal{L}_λ on \mathbb{C}^\times is trivial, i.e., $\mathcal{L}_\lambda \simeq \underline{\mathbb{C}}, \lambda \in \mathbb{Z}$. In this case we have two different extensions: $j_!(\underline{\mathbb{C}})$, which is realized as $\text{Sol}(\Delta_\lambda)$ for $\lambda \in \mathbb{Z}_{<0}$, and $j_*(\underline{\mathbb{C}})$, which is realized as $\text{Sol}(\Delta_\lambda)$ for $\lambda \in \mathbb{Z}_+$. Both of them are perverse sheaves on \mathbb{C} (even though the latter is actually a complex of sheaves), if we shift their cohomological degrees by 1. But neither of them is an irreducible perverse sheaf. The irreducible perverse extension of the constant sheaf on \mathbb{C}^\times is the constant sheaf on \mathbb{C} (again, placed in cohomological degree -1). We have natural maps $j_!(\underline{\mathbb{C}}) \rightarrow \underline{\mathbb{C}} \rightarrow j_*(\underline{\mathbb{C}})$, so $\underline{\mathbb{C}}$ appears as an extension that is “intermediate” between the !- and the *-extensions. This is the reason why such sheaves are often called “intermediate extensions”.

3.6. More on \mathcal{D} -modules. One of the lessons that we should learn from this elementary example is that when our differential equations (3.5) have regular singularities, as is the case for the equation $(t\partial_t - \lambda)f = 0$, the corresponding \mathcal{D} -module reflects these singularities. Namely, only its restriction to the complement of the singularity divisor is a vector bundle with a connection, but usually it is extended in a non-trivial way to this divisor. This will be one of the salient features of the Hecke eigensheaves that we will discuss below (in the non-abelian case).

The Riemann-Hilbert functor Sol sets up an equivalence between the category of holonomic \mathcal{D} -modules with *regular singularities* on V (such as the \mathcal{D} -module that we considered

above) and the category of perverse sheaves on V . This equivalence is called the Riemann-Hilbert correspondence (see [49, 50, 51, 54]).³⁴ Therefore we may replace perverse sheaves on smooth algebraic varieties (or algebraic stacks, see [15]) over \mathbb{C} by holonomic \mathcal{D} -modules with regular singularities.

Under this equivalence of categories natural operations (functors) on perverse sheaves, such as the standard operations of direct and inverse images, go to certain operations on \mathcal{D} -modules. We will not describe these operations here in detail referring the reader to [49, 50, 51, 54]. But one way to think about them which is consistent with the point of view presented above is as follows. If we think of a \mathcal{D} -module \mathcal{F} as something that encodes a system of differential equations, then applying an operation to \mathcal{F} , such as the inverse or direct image, corresponds to applying the same type of operation (pull-back in the case of inverse image, an integral in the case of direct image) to the *solutions* of the system of differential equations encoded by \mathcal{F} . So the solutions of the system of differential equations encoded by the inverse or direct image of \mathcal{F} are the pull-backs or the integrals of the solutions of the system encoded by \mathcal{F} , respectively.

The fact that natural operations on \mathcal{D} -modules correspond to natural operations on their solutions (which are functions) provides another point of view on the issue why, when moving from a finite field to \mathbb{C} , we decided to replace the notion of a function by the notion of a \mathcal{D} -module. We may think that there is actually a function, or perhaps a vector space of functions, lurking in the background, but these functions may be too complicated to write down - they may be multi-valued and have nasty singularities (for more on this, see Sect. 9.5). For all intents and purposes it might be better to write down the system of differential equations that these functions satisfy, i.e, consider the corresponding \mathcal{D} -module, instead.

Let us summarize: we have seen that an automorphic representation may be encapsulated by an automorphic function on the set of isomorphism classes of rank n vector bundles on the curve X . We then apply the following progression to the notion of “function”

$$\boxed{\text{functions}} \xrightarrow{\text{over } \mathbb{F}_q} \boxed{\ell\text{-adic sheaves}} \implies \boxed{\text{perverse sheaves}} \xrightarrow{\text{over } \mathbb{C}} \boxed{\mathcal{D}\text{-modules}}$$

and end up with the notion of “ \mathcal{D} -module” instead. This leads us to believe that the proper replacement for the notion of automorphic representation in the case of a curve X over \mathbb{C} is the notion of \mathcal{D} -module on the moduli stack Bun_n of rank n vector bundles on X . In order to formulate precisely the geometric Langlands correspondence we need to figure out what properties these \mathcal{D} -modules should satisfy.

3.7. Hecke correspondences. The automorphic function on $GL_n(F) \backslash GL_n(\mathbb{A}) / GL_n(\mathcal{O})$ associated to an irreducible unramified automorphic representation π had an important property: it was a Hecke eigenfunction.

In order to state the geometric Langlands correspondence in a meaningful way we need to formulate the Hecke eigenfunction condition in sheaf-theoretic terms. The key to this is the interpretation of the spherical Hecke algebras \mathcal{H}_x in terms of the *Hecke correspondences*.

³⁴it is often more convenient to use the closely related (covariant) “de Rham functor” $\mathcal{F} \mapsto \omega_V \overset{L}{\otimes}_{\mathcal{D}} \mathcal{F}$

In what follows we will consider instead of vector bundles on X the corresponding sheaves of their holomorphic sections, which are locally free coherent sheaves of \mathcal{O}_X -modules, where \mathcal{O}_X is the sheaf of holomorphic functions on X . By abuse of notation, we will use the same symbol for a vector bundle and for the sheaf of its sections.

We again let X be a smooth projective connected curve over a field k , which could be a finite field or \mathbb{C} .

By definition, the i th Hecke correspondence $\mathcal{H}ecke_i$ is the moduli space of quadruples

$$(\mathcal{M}, \mathcal{M}', x, \beta : \mathcal{M}' \hookrightarrow \mathcal{M}),$$

where $\mathcal{M}', \mathcal{M} \in \text{Bun}_n$, $x \in X$, and β is an embedding of the sheaves of sections³⁵ $\beta : \mathcal{M}' \hookrightarrow \mathcal{M}$ such that \mathcal{M}/\mathcal{M}' is supported at x and is isomorphic to $\mathcal{O}_x^{\oplus i}$, the direct sum of i copies of the *skyscraper sheaf* $\mathcal{O}_x = \mathcal{O}_X/\mathcal{O}_X(-x)$.

We thus have a correspondence

$$\begin{array}{ccc} & \mathcal{H}ecke_i & \\ h^\leftarrow & \swarrow & \text{supp} \times h^\rightarrow \\ \text{Bun}_n & & X \times \text{Bun}_n \end{array}$$

where $h^\leftarrow(x, \mathcal{M}, \mathcal{M}') = \mathcal{M}$, $h^\rightarrow(x, \mathcal{M}, \mathcal{M}') = \mathcal{M}'$, and $\text{supp}(x, \mathcal{M}, \mathcal{M}') = x$.

Let $\mathcal{H}ecke_{i,x} = \text{supp}^{-1}(x)$. This is a correspondence over $\text{Bun}_n \times \text{Bun}_n$:

$$(3.6) \quad \begin{array}{ccc} & \mathcal{H}ecke_{i,x} & \\ h^\leftarrow & \swarrow & h^\rightarrow \\ \text{Bun}_n & & \text{Bun}_n \end{array}$$

What does it look like? Consider the simplest case when $n = 2$ and $i = 1$. Then the points in the fiber of $\mathcal{H}ecke_{i,x}$ over a point \mathcal{M} in the “left” Bun_n (which we view as the sheaf of sections of a rank two vector bundle on X) correspond to all locally free subsheaves $\mathcal{M}' \subset \mathcal{M}$ such that the quotient \mathcal{M}/\mathcal{M}' is the skyscraper sheaf \mathcal{O}_x . Defining \mathcal{M}' is the same as choosing a line \mathcal{L}_x in the dual space \mathcal{M}_x^* to the fiber of \mathcal{M} at x (which is a two-dimensional vector space over k). The sections of the corresponding sheaf \mathcal{M}' are just the sections of \mathcal{M} which vanish along \mathcal{L}_x , i.e., such a section s (over an open set containing x) must satisfy $\langle v, s(x) \rangle = 0$ for any non-zero $v \in \mathcal{L}_x$.

Therefore the fiber of $\mathcal{H}ecke_{1,x}$ over \mathcal{M} is isomorphic to the projectivization of the two-dimensional fiber \mathcal{M}_x of \mathcal{M} at x . Hence $\mathcal{H}ecke_{i,x}$ is a \mathbb{P}_k^1 -fibration over Bun_n . It is also easy to see that $\mathcal{H}ecke_{i,x}$ is a \mathbb{P}_k^1 -fibration over the “right” Bun_n in the diagram (3.6) (whose points are labeled as \mathcal{M}).

Now it should be clear what $\mathcal{H}ecke_{i,x}$ looks like for general n and i : it is a fibration over both Bun_n ’s, with the fibers being isomorphic to the Grassmannian $\text{Gr}(i, n)$ of i -dimensional subspaces in k^n .

To understand the connection with the classical Hecke operators $H_{i,x}$ introduced in Sect. 2.3, we set $k = \mathbb{F}_q$ and look at the sets of \mathbb{F}_q -points of the correspondence (3.6). Recall from Lemma 2 that the set of \mathbb{F}_q -points of Bun_n is $GL_n(F) \backslash GL_n(\mathbb{A}) / GL_n(\mathcal{O})$.

³⁵this is the place where the difference between a vector bundle and its sheaf of sections is essential: an embedding of vector bundles of the same rank is necessarily an isomorphism, but an embedding of their sheaves of sections is not; their quotient can be a torsion sheaf on X

Therefore the correspondence $\mathcal{H}ecke_{i,x}(\mathbb{F}_q)$ defines an operator on the space of functions on $GL_n(F)\backslash GL_n(\mathbb{A})/GL_n(\mathcal{O})$

$$f \mapsto T_{i,x}(f) = h_*^{-\rightarrow}(h^{\leftarrow*}(f)),$$

where $h^{\leftarrow*}$ is the operator of pull-back of a function under h^{\leftarrow} , and $h_*^{-\rightarrow}$ is the operator of integration of a function along the fibers of h^{\rightarrow} .

Now observe that the set of points in the fiber of h^{\rightarrow} over a point

$$(g_y)_{y \in X} \in GL_n(F)\backslash GL_n(\mathbb{A})/GL_n(\mathcal{O})$$

is the set of double cosets of the adèles whose components at each point $y \neq x$ is g_y (the same as before) and the component at x is of the form $g_x h_x$, where $h_x \in M_n^i(\mathcal{O}_x)$, and the set $M_n^i(\mathcal{O}_x)$ is defined by formula (2.4). This means that

$$(3.7) \quad T_{i,x}(f) = H_{i,x} \star f,$$

where $H_{i,x}$ is the characteristic function of $M_n^i(\mathcal{O}_x)$, which is a generator of the spherical Hecke algebra \mathcal{H}_x introduced in Sect. 2.3. It acts on the space of functions on $GL_n(F)\backslash GL_n(\mathbb{A})$ according to formulas (2.2) and (2.6). Therefore we find that $T_{i,x}$ is precisely the i th Hecke operator given by formula (2.6) with $f_x = H_{i,x}$! Thus, we obtain an interpretation of the generators $H_{i,x}$ of the spherical Hecke algebra \mathcal{H}_x in terms of Hecke correspondences.

By construction (see formula (2.7)), the automorphic function f_π on the double quotient $GL_n(F)\backslash GL_n(\mathbb{A})/GL_n(\mathcal{O})$ associated to an irreducible unramified automorphic representation π of $GL_n(\mathbb{A})$ satisfies

$$T_{i,x}(f_\pi) = H_{i,x} \star f_\pi = q_x^{i(n-i)/2} s_i(z_1(\sigma_x), \dots, z_n(\sigma_x)) f_\pi.$$

This is the meaning of the classical Hecke condition.

Now it is clear how to define a geometric analogue of the Hecke condition (for an arbitrary k). This geometric Hecke property will comprise all points of the curve at once. Namely, we use the Hecke correspondences to define the *Hecke functors* H_i from the category of perverse sheaves on Bun_n to the derived category of sheaves on $X \times \text{Bun}_n$ by the formula

$$(3.8) \quad H_i(\mathcal{K}) = (\text{supp} \times h^{\rightarrow})_* h^{\leftarrow*}(\mathcal{K}).$$

Note that when we write $(\text{supp} \times h^{\rightarrow})_*$ we really mean the corresponding derived functor.

3.8. Hecke eigensheaves and the geometric Langlands conjecture. Now let E be a local system E of rank n on X . A perverse sheaf \mathcal{K} on Bun_n is called a *Hecke eigensheaf with eigenvalue E* , if $\mathcal{K} \neq 0$ and we have the following isomorphisms:

$$(3.9) \quad \iota_i : H_i(\mathcal{K}) \xrightarrow{\sim} \wedge^i E \boxtimes \mathcal{K}[-i(n-i)], \quad i = 1, \dots, n,$$

where $\wedge^i E$ is the i th exterior power of E . Here $[-i(n-i)]$ indicates the shift in cohomological degree to the right by $i(n-i)$, which is the complex dimension of the fibers of h^{\rightarrow} .

Let us see that this condition really corresponds to an old condition from Theorem 1 matching the Hecke and Frobenius eigenvalues. So let X be a curve over \mathbb{F}_q and σ an n -dimensional unramified ℓ -adic representation of $\text{Gal}(\overline{F}/F)$. Denote by E the corresponding

ℓ -adic local system on X . Then it follows from the definitions that

$$\mathrm{Tr}(\mathrm{Fr}_{\bar{x}}, E_{\bar{x}}) = \mathrm{Tr}(\sigma(\mathrm{Fr}_x), \overline{\mathbb{Q}}_{\ell}^n) = \sum_{i=1}^n z_i(\sigma_x)$$

(see Sect. 2.2 for the definition of $z_i(\sigma_x)$), and so

$$\mathrm{Tr}(\mathrm{Fr}_{\bar{x}}, \wedge^i E_{\bar{x}}) = s_i(z_1(\sigma_x), \dots, z_n(\sigma_x)),$$

where s_i is the i th elementary symmetric polynomial.

Recall that the passage from complexes of sheaves to functions intertwined the operations of inverse and direct image on sheaves with the operations of pull-back and integration of functions. Therefore we find that the function $\mathbf{f}_q(\mathcal{K})$ on

$$GL_n(F) \backslash GL_n(\mathbb{A}) / GL_n(\mathcal{O}) = \mathrm{Bun}_n(\mathbb{F}_q)$$

associated to a Hecke eigensheaf \mathcal{K} satisfies

$$T_{i,x}(\mathbf{f}_{\mathcal{K}}) = q_x^{i(n-i)} s_i(z_1(\sigma_x), \dots, z_n(\sigma_x)) \mathbf{f}_{\mathcal{K}}$$

(the q_x -factor comes from the cohomological degree shift). In other words, if \mathcal{K} is a Hecke eigensheaf with eigenvalue E , then the function $\mathbf{f}_{\mathcal{K}}$ associated to it via the Grothendieck dictionary is a Hecke eigenfunction whose Hecke eigenvalues are equal to the Frobenius eigenvalues of σ , which is the condition of Theorem 1 (for an irreducible local system E).

The difference between the classical Hecke operators and their geometric counterparts is that the former are defined pointwise while the latter are defined globally on the curve X . In the classical setting therefore it was not clear whether for a given automorphic representation π one could always find a Galois representation (or an ℓ -adic local system) with the same Frobenius eigenvalues as the Hecke eigenvalues of π (part of Theorem 1 is the statement that there is always a unique one). In the geometric setting this question is mute, because the very notion of a Hecke eigensheaf presumes that we know what its eigenvalue E is. That is why the geometric Langlands correspondence in the geometric setting is a map in one direction: from local systems to Hecke eigensheaves.

We are now naturally led to the geometric Langlands conjecture for GL_n , whose formulation is due to Drinfeld and Laumon [57]. This statement makes sense when X is over \mathbb{F}_q or over \mathbb{C} , and it is now a theorem in both cases. Note that Bun_n is a disjoint union of connected components Bun_n^d corresponding to vector bundles of degree d .

Theorem 3. *For each irreducible rank n local system E on X there exists a perverse sheaf Aut_E on Bun_n which is a Hecke eigensheaf with respect to E . Moreover, Aut_E is irreducible on each connected component Bun_n^d ,*

$$\begin{array}{ccc} \boxed{\text{irreducible rank } n} & \longrightarrow & \boxed{\text{Hecke eigensheaves}} \\ \boxed{\text{local systems on } X} & & \boxed{\text{on } \mathrm{Bun}_n} \\ & & \\ & & E \longrightarrow \mathrm{Aut}_E \end{array}$$

This theorem was proved by Deligne for GL_1 (we recall it in the next section) and by Drinfeld in the case of GL_2 [39] (see [28], Sect. 6, for a review). These works motivated the conjecture in the case of GL_n , which has been proved in [55, 56] (these works were

also influenced by [57, 58]). In the case when X is over \mathbb{C} we can replace “perverse sheaf” in the statement of Theorem 3 by “ \mathcal{D} -module”.³⁶

The reader may be wondering what has become of the cuspidality condition, which was imposed in Sect. 2.3. It has a transparent geometric analogue (see [57, 55]). As shown in [55], the geometric cuspidality condition is automatically satisfied for the Hecke eigensheaves Aut_E associated in [55] to irreducible local systems E .

One cannot emphasize enough the importance of the fact that E is an *irreducible* rank n local system on X in the statement Theorem 3. It is only in this case that we expect the Hecke eigensheaf Aut_E to be as nice as described in the theorem. Moreover, in this case we expect that Aut_E is unique up to an isomorphism. If E is not irreducible, then the situation becomes more complicated. For example, Hecke eigensheaves corresponding to local systems that are direct sums of n rank 1 local systems – the so-called geometric Eisenstein series – have been constructed in [59, 60, 61]. The best case scenario is when these rank 1 local systems are pairwise non-isomorphic. The corresponding Hecke eigensheaf is a direct sum of infinitely many irreducible perverse sheaves on Bun_n , labeled by the lattice \mathbb{Z}^n . More general geometric Eisenstein series are complexes of perverse sheaves. Moreover, it is expected that in general there are several non-isomorphic Hecke eigensheaves corresponding to such a local system, so it is appropriate to talk not about a single Hecke eigensheaf Aut_E , but a *category* Aut_E of Hecke eigensheaves with eigenvalue E .

An object of Aut_E is by definition a collection (\mathcal{K}, ι_i) , where \mathcal{K} is a Hecke eigensheaf with eigenvalue E and ι_i are isomorphisms (3.9). In general, we should allow objects to be complexes (not necessarily perverse sheaves), but in principle there are several candidates for Aut_E depending on what kinds of complexes we allow (bounded, unbounded, etc.).

The group of automorphisms of E naturally acts on the category Aut_E . Namely, to an automorphism g of E we assign the functor $\text{Aut}_E \rightarrow \text{Aut}_E$ sending $(\mathcal{F}, \{\iota_i\}_{\lambda \in P_+})$ to $(\mathcal{F}, \{g \circ \iota_i\}_{\lambda \in P_+})$. For example, in the case when E is the direct sum of rank 1 local systems that are pairwise non-isomorphic, the group of automorphisms of E is the n -dimensional torus. Its action on the geometric Eisenstein series sheaf constructed in [59, 61] amounts to a \mathbb{Z}^n -grading on this sheaf, which comes from the construction expressing it as a direct sum of irreducible objects labeled by \mathbb{Z}^n . For non-abelian groups of automorphisms the corresponding action will be more sophisticated.

This means that, contrary to our naive expectations, the most difficult rank n local system on X is the *trivial* local system E_0 . Its group of automorphisms is GL_n which acts non-trivially on the corresponding category Aut_{E_0} . Some interesting Hecke eigensheaves are unbounded complexes in this case, and a precise definition of the corresponding category that would include such complexes is an open problem [62]. Note that for $X = \mathbb{CP}^1$ the trivial local system is the only local system. The corresponding category Aut_{E_0} can probably be described rather explicitly. Some results in this direction are presented in [59], Sect. 5.

³⁶We remark that the proof of the geometric Langlands correspondence, Theorem 3, gives an alternative proof of the classical Langlands correspondence, Theorem 1, in the case when the Galois representation σ is unramified everywhere. A geometric version of the Langlands correspondence for general ramified local systems is much more complicated (see the discussion in Sect. 9.8).

But is it possible to give an elementary example of a Hecke eigensheaf? For $n = 1$ these are rank one local systems on the Picard variety which will be discussed in the next section. They are rather easy to construct. Unfortunately, it seems that for $n > 1$ there are no elementary examples. We will discuss in Part III the construction of Hecke eigensheaves using conformal field theory methods, but these constructions are non-trivial.

However, there is one simple Hecke eigensheaf whose eigenvalue is not a local system on X , but a complex of local systems. This is the constant sheaf $\underline{\mathbb{C}}$ on Bun_n . Let us apply the Hecke functors H_i to the constant sheaf. By definition,

$$H_i(\underline{\mathbb{C}}) = (\text{supp} \times h^\rightarrow)_* h^{\leftarrow*}(\underline{\mathbb{C}}) = (\text{supp} \times h^\rightarrow)_*(\underline{\mathbb{C}}).$$

As we explained above, the fibers of $\text{supp} \times h^\rightarrow$ are isomorphic to $\text{Gr}(i, n)$, and so $H_i(\underline{\mathbb{C}})$ is the constant sheaf on Bun_n with the fiber being the cohomology $H^*(\text{Gr}(i, n), \mathbb{C})$. Let us write

$$H^*(\text{Gr}(i, n), \mathbb{C}) = \wedge^i(\mathbb{C}[0] \oplus \mathbb{C}[-2] \oplus \dots \oplus \mathbb{C}[-2(n-1)])$$

(recall that $V[n]$ means V placed in cohomological degree $-n$). Thus, we find that

$$(3.10) \quad H_i(\underline{\mathbb{C}}) \simeq \wedge^i E'_0 \boxtimes \underline{\mathbb{C}}[-i(n-i)], \quad i = 1, \dots, n,$$

where

$$E'_0 = \underline{\mathbb{C}}_X[-(n-1)] \oplus \underline{\mathbb{C}}_X[-(n-3)] \oplus \dots \oplus \underline{\mathbb{C}}_X[(n-1)]$$

is a “complex of trivial local systems” on X . Remembering the cohomological degree shift in formula (3.9), we see that formula (3.10) may be interpreted as saying that the constant sheaf on Bun_n is a Hecke eigensheaf with eigenvalue E'_0 .

The Hecke *eigenfunction* corresponding to the constant sheaf is just the constant function on $GL_n(F) \backslash GL_n(\mathbb{A}) / GL_n(\mathcal{O})$, which corresponds to the trivial one-dimensional representation of the adèlic group $GL_n(\mathbb{A})$. The fact that the “eigenvalue” E'_0 is not a local system, but a complex, indicates that something funny is going on with the trivial representation. In fact, it has to do with the so-called “Arthur’s SL_2 ” part of the parameter of a general automorphic representation [63]. The precise meaning of this is beyond the scope of the present article, but the idea is as follows. Arthur has conjectured that if we want to consider unitary automorphic representations of $GL_n(\mathbb{A})$ that are not necessarily cuspidal, then the true parameters for those are n -dimensional representations not of $\text{Gal}(\bar{F}/F)$, but of the product $\text{Gal}(\bar{F}/F) \times SL_2$. The homomorphisms whose restriction to the SL_2 factor are trivial correspond to the so-called tempered representations. In the case of GL_n all cuspidal unitary representations are tempered, so the SL_2 factor does not play a role. But what about the trivial representation of $GL_n(\mathbb{A})$? It is unitary, but certainly not tempered (nor cuspidal). According to [63], the corresponding parameter is the n -dimensional representation of $\text{Gal}(\bar{F}/F) \times SL_2$, which is trivial on the first factor and is the irreducible representation of the second factor. One can argue that it is this non-triviality of the action of Arthur’s SL_2 that is observed geometrically in the cohomological grading discussed above.

In any case, this is a useful example to consider.

4. GEOMETRIC ABELIAN CLASS FIELD THEORY

In this section we discuss the geometric Langlands correspondence for $n = 1$, i.e., for rank one local systems. This is a particularly simple case, which is well understood. Still,

it already contains the germs of some of the ideas and constructions that we will use for local systems of higher rank.

Note that because \mathbb{CP}^1 is simply-connected, there is only one (unramified) rank one local system on it, so the (unramified) geometric Langlands correspondence is vacuous in this case. Hence throughout this section we will assume that the genus of X is positive.

4.1. Deligne's proof. We present here Deligne's proof of the $n = 1$ case of Theorem 3, following [57, 59, 26]; it works when X is over \mathbb{F}_q and over \mathbb{C} , but when X is over \mathbb{C} there are additional simplifications which we will discuss below.

For $n = 1$ the moduli stack Bun_n is the Picard variety Pic of X classifying line bundles on X . Recall that Pic has components Pic_d labeled by the integer d which corresponds to the degree of the line bundle. The degree zero component Pic_0 is the Jacobian variety Jac of X , which is a complex g -dimensional torus $H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$.

Conjecture 3 means the following in this case: for each rank one local system E on X there exists a perverse sheaf (or a \mathcal{D} -module, when X is over \mathbb{C}) Aut_E on Pic which satisfies the following Hecke eigensheaf property:

$$(4.1) \quad h^{\leftarrow *}(\text{Aut}_E) \simeq E \boxtimes \text{Aut}_E,$$

where $h^{\leftarrow} : X \times \text{Pic} \rightarrow \text{Pic}$ is given by $(\mathcal{L}, x) \mapsto \mathcal{L}(x)$. In this case the maps h^{\leftarrow} and h^{\rightarrow} are one-to-one, and so the Hecke condition simplifies.

To construct Aut_E , consider the Abel-Jacobi map $\pi_d : S^d X \rightarrow \text{Pic}_d$ sending the divisor D to the line bundle $\mathcal{O}_X(D)$.³⁷ If $d > 2g - 2$, then π_d is a projective bundle, with the fibers $\pi_d^{-1}(\mathcal{L}) = \mathbb{P}H^0(X, \mathcal{L})$ being projective spaces of dimension $d - g$. It is easy to construct a local system $E^{(d)}$ on $\bigcup_{d > 2g-2} S^d X$ satisfying an analogue of the Hecke eigensheaf property

$$(4.2) \quad \tilde{h}^{\leftarrow *}(E^{(d+1)}) \simeq E \boxtimes E^{(d)},$$

where $\tilde{h}^{\leftarrow} : S^d X \times X \rightarrow S^{d+1} X$ is given by $(D, x) \mapsto D + [x]$. Namely, let

$$\text{sym}^d : X^n \rightarrow S^n X$$

be the symmetrization map and set

$$E^{(d)} = (\text{sym}_*^d(E^{\boxtimes n}))^{S_d}.$$

So we have rank one local systems $E^{(d)}$ on $S^d X$, $d > 2g - 2$, which satisfy an analogue (4.2) of the Hecke eigensheaf property, and we need to prove that they descend to Pic_d , $d > 2g - 2$, under the Abel-Jacobi maps π_d . In other words, we need to prove that the restriction of $E^{(d)}$ to each fiber of π_d is a constant sheaf. Since $E^{(d)}$ is a local system, these restrictions are locally constant. But the fibers of π_d are projective spaces, hence simply-connected. Therefore any locally constant sheaf along the fiber is constant! So there exists a local system Aut_E^d on Pic_d such that $E^{(d)} = \pi_d^*(\text{Aut}_E^d)$. Formula (4.2) implies that the sheaves Aut_E^d form a Hecke eigensheaf on $\bigcup_{d > 2g-2} \text{Pic}_d$. We extend them by induction to the remaining components Pic_d , $d \leq 2g - 2$ by using the Hecke eigensheaf property (4.1).

³⁷by definition, the sections of $\mathcal{O}_X(D)$ are meromorphic functions f on X such that for any $x \in X$ we have $-\text{ord}_x f \leq D_x$, the coefficient of $[x]$ in D

To do that, let us observe that for any $x \in X$ and $d > 2g - 1$ we have an isomorphism $\text{Aut}_E^d \simeq E_x^* \otimes h_x^{\leftarrow *}(\text{Aut}_E^d)$, where $h_x^{\leftarrow}(\mathcal{L}) = \mathcal{L}(x)$. This implies that for any N -tuple of points $(x_i), i = 1, \dots, N$ and $d > 2g - 2 + N$ we have a canonical isomorphism

$$(4.3) \quad \text{Aut}_E^d \simeq \bigotimes_{i=1}^N E_{x_i}^* \otimes (h_{x_1}^{\leftarrow *}\circ \dots \circ h_{x_N}^{\leftarrow *}(\text{Aut}_E^{d+N})),$$

and so in particular we have a compatible (i.e., transitive) system of canonical isomorphisms

$$(4.4) \quad \bigotimes_{i=1}^N E_{x_i}^* \otimes (h_{x_1}^{\leftarrow *}\dots h_{x_N}^{\leftarrow *}(\text{Aut}_E^{d+N})) \simeq \bigotimes_{i=1}^N E_{y_i}^* \otimes (h_{y_1}^{\leftarrow *}\circ \dots \circ h_{y_N}^{\leftarrow *}(\text{Aut}_E^{d+N})),$$

for any two N -tuples of points (x_i) and (y_i) of X and $d > 2g - 2$.

We now define Aut_E^d on Pic_d with $d = 2g - 1 - N$ as the right hand side of formula (4.3) using any N -tuple of points $(x_i), i = 1, \dots, N$.³⁸ The resulting sheaf on Pic_d is independent of these choices. To see that, choose a point $x_0 \in X$ and using (4.3) with $d = 2g - 1$ write

$$\text{Aut}_E^{2g-1} = (E_{x_0}^*)^{\otimes N} \otimes (h_{x_0}^{\leftarrow *}\circ \dots \circ h_{x_0}^{\leftarrow *}(\text{Aut}_E^{2g-1+N})).$$

Then the isomorphism (4.4) with $d = 2g - 1 - N$, which we want to establish, is just the isomorphism (4.4) with $d = 2g - 1$, which we already know, to which we apply N times $h_{x_0}^{\leftarrow *}$ and tensor with $(E_{x_0}^*)^{\otimes N}$ on both sides. In the same way we show that the resulting sheaves Aut_E^d on Pic_d with $d = 2g - 1 - N$ satisfy the Hecke property (4.1): it follows from the corresponding property of the sheaves Aut_E^d with $d > 2g - 2$.

Thus, we obtain a Hecke eigensheaf on the entire Pic , and this completes Deligne's proof of the geometric Langlands conjecture for $n = 1$. It is useful to note that the sheaf Aut_E satisfies the following additional property that generalizes the Hecke eigensheaf property (4.1). Consider the natural morphism $m : \text{Pic} \times \text{Pic} \rightarrow \text{Pic}$ taking $(\mathcal{L}, \mathcal{L}')$ to $\mathcal{L} \otimes \mathcal{L}'$. Then we have an isomorphism

$$m^*(\text{Aut}_E) \simeq \text{Aut}_E \boxtimes \text{Aut}_E.$$

The important fact is that each Hecke eigensheaves Aut_E is the simplest possible perverse sheaf on Pic : namely, a rank one local system. When X is over \mathbb{C} , the \mathcal{D} -module corresponding to this local system is a rank one holomorphic vector bundle with a holomorphic connection on Pic . This will not be true when n , the rank of E , is greater than 1.

4.2. Functions vs. sheaves. Let us look more closely at the case when X is defined over a finite field. Then to the sheaf Aut_E we attach a function on $F^\times \backslash \mathbb{A}^\times / \mathcal{O}^\times$, which is the set of \mathbb{F}_q -points of Pic . This function is a Hecke eigenfunction f_σ with respect to a one-dimensional Galois representation σ corresponding to E , i.e., it satisfies the equation $f_\sigma(\mathcal{L}(x)) = \sigma(\text{Fr}_x) f_\sigma(\mathcal{L})$ (since σ is one-dimensional, we do not need to take the trace).

³⁸we could use instead formula (4.3) with $d = d' - N$ with any $d' > 2g - 2$

We could try to construct this function proceeding in the same way as above. Namely, we define first a function f'_σ on the set of all divisors on X by the formula

$$f'_\sigma \left(\sum_i n_i [x_i] \right) = \prod_i \sigma(\text{Fr}_{x_i})^{n_i}.$$

This function clearly satisfies an analogue of the Hecke eigenfunction condition. It remains to show that the function f'_σ descends to $\text{Pic}(\mathbb{F}_q)$, namely, that if two divisors D and D' are rationally equivalent, then $f'_\sigma(D) = f'_\sigma(D')$. This is equivalent to the identity

$$\prod_i \sigma(\text{Fr}_{x_i})^{n_i} = 1, \quad \text{if} \quad \sum_i n_i [x_i] = (g),$$

where g is an arbitrary rational function on X . This identity is a non-trivial reciprocity law which has been proved in the abelian class field theory, by Lang and Rosenlicht (see [64]).

It is instructive to contrast this to Deligne's geometric proof reproduced above. When we replace functions by sheaves we can use additional information which is "invisible" at the level of functions, such as the fact that the sheaf corresponding to the function f'_σ is locally constant and that the fibers of the Abel-Jacobi map are simply-connected. This is one of the main motivations for studying the Langlands correspondence in the geometric setting.

4.3. Another take for curves over \mathbb{C} . In the case when X is a complex curve, there is a more direct construction of the local system Aut_E^0 on the Jacobian $\text{Jac} = \text{Pic}_0$. Namely, we observe that defining a rank one local system E on X is the same as defining a homomorphism $\pi_1(X, x_0) \rightarrow \mathbb{C}^\times$. But since \mathbb{C}^\times is abelian, this homomorphism factors through the quotient of $\pi_1(X, x_0)$ by its commutator subgroup, which is isomorphic to $H_1(X, \mathbb{Z})$. However, it is known that the cup product on $H_1(X, \mathbb{Z})$ is a unimodular bilinear form, so we can identify $H_1(X, \mathbb{Z})$ with $H^1(X, \mathbb{Z})$. But $H^1(X, \mathbb{Z})$ is isomorphic to the fundamental group $\pi_1(\text{Jac})$, because we can realize the Jacobian as the quotient $H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z}) \simeq \mathbb{C}^g/H^1(X, \mathbb{Z})$. Thus, we obtain a homomorphism $\pi_1(\text{Jac}) \rightarrow \mathbb{C}^\times$, which gives us a rank one local system E_{Jac} on Jac . We claim that this is Aut_E^0 . We can then construct Aut_E^d recursively using formula (4.3).

It is not immediately clear why the sheaves Aut_E^d , $d \neq 0$, constructed this way should satisfy the Hecke property (4.1) and why they do not depend on the choices of points on X , which is essentially an equivalent question. To see that, consider the map $j : X \rightarrow \text{Jac}$ sending $x \in X$ to the line bundle $\mathcal{O}_X(x - x_0)$ for some fixed reference point $x_0 \in X$. In more concrete terms this map may be described as follows: choose a basis $\omega_1, \dots, \omega_g$ of the space $H^0(X, \Omega)$ of holomorphic differentials on X . Then

$$j(x) = \left(\int_{x_0}^x \omega_1, \dots, \int_{x_0}^x \omega_g \right)$$

considered as a point in $\mathbb{C}^g/L \simeq \text{Jac}$, where L is the lattice spanned by the integrals of ω_i 's over the integer one-cycles in X .

It is clear from this construction that the homomorphism $H_1(X, \mathbb{Z}) \rightarrow H_1(\text{Jac}, \mathbb{Z})$, induced by the map j is an isomorphism. Viewing it as a homomorphism of the abelian

quotients of the corresponding fundamental groups, we see that the pull-back of E_{Jac} to X under the map j has to be isomorphic to E .

More generally, the homomorphism $H_1(S^d X, \mathbb{Z}) \simeq H_1(X, \mathbb{Z}) \rightarrow H_1(\text{Jac}, \mathbb{Z})$ induced by the map $S^d X \rightarrow \text{Jac}$ sending $(x_i), i = 1, \dots, d$ to the line bundle $\mathcal{O}_X(x_1 + \dots + x_d - dx_0)$ is also an isomorphism. This means that the pull-back of E_{Jac} to $S^d X$ under this map is isomorphic to $E^{(d)}$, for any $d > 0$. Thus, we obtain a different proof of the fact that $E^{(d)}$ is constant along the fibers of the Abel-Jacobi map. By using an argument similar to the recursive algorithm discussed above that extended Aut_E to $\text{Pic}_d, d \leq 2g - 2$, we then identify E_{Jac} with Aut_E^0 . In addition, we also identify the sheaves on the other components Pic_d obtained from E_{Jac} by applying formula (4.3), with Aut_E . The bonus of this argument is that we obtain another geometric insight (in the case when X is a complex curve) into why $E^{(d)}$ is constant along the fibers of the Abel-Jacobi map.

4.4. Connection to the Fourier-Mukai transform. As we saw at the end of the previous section, the construction of the Hecke eigensheaf Aut_E associated to a rank one local system E on a complex curve X (the case $n = 1$) is almost tautological: we use the fact that the fundamental group of Jac is the maximal abelian quotient of the fundamental group of X .

However, one can strengthen the statement of the geometric Langlands conjecture by interpreting it in the framework of the Fourier-Mukai transform. Let Loc_1 be the moduli space of rank one local systems on X . A local system is a pair (\mathcal{F}, ∇) , where \mathcal{F} is a holomorphic line bundle and ∇ is a holomorphic connection on \mathcal{F} . Since \mathcal{F} supports a holomorphic (hence flat) connection, the first Chern class of \mathcal{F} , which is the degree of \mathcal{F} , has to vanish. Therefore \mathcal{F} defines a point of $\text{Pic}_0 = \text{Jac}$. Thus, we obtain a natural map $p : \text{Loc}_1 \rightarrow \text{Jac}$ sending (\mathcal{F}, ∇) to \mathcal{F} . What are the fibers of this map?

The fiber of p over \mathcal{F} is the space of holomorphic connections on \mathcal{F} . Given a connection ∇ on \mathcal{F} , any other connection can be written uniquely as $\nabla' = \nabla + \omega$, where ω is a holomorphic one-form on X . It is clear that any \mathcal{F} supports a holomorphic connection. Therefore the fiber of p over \mathcal{F} is an affine space over the vector space $H^0(X, \Omega)$ of holomorphic one-forms on X . Thus, Loc_1 is an affine bundle over Jac over the trivial vector bundle with the fiber $H^0(X, \Omega)$. This vector bundle is naturally identified with the cotangent bundle $T^* \text{Jac}$. Indeed, the tangent space to Jac at a point corresponding to a line bundle \mathcal{F} is the space of infinitesimal deformations of \mathcal{F} , which is $H^1(X, \text{End } \mathcal{F}) = H^1(X, \mathcal{O}_X)$. Therefore its dual is isomorphic to $H^0(X, \Omega)$ by the Serre duality. Therefore Loc_1 is what is called the *twisted cotangent bundle* to Jac .

As we explained in the previous section, a holomorphic line bundle with a holomorphic connection on X is the same thing as a holomorphic line bundle with a flat holomorphic connection on Jac , $E = (\mathcal{F}, \nabla) \mapsto E_{\text{Jac}} = \text{Aut}_E^0$. Therefore Loc_1 may be interpreted as the moduli space of pairs $(\tilde{\mathcal{F}}, \tilde{\nabla})$, where $\tilde{\mathcal{F}}$ is a holomorphic line bundle on Jac and $\tilde{\nabla}$ is a flat holomorphic connection on $\tilde{\mathcal{F}}$.

Now consider the product $\text{Loc}_1 \times \text{Jac}$. Over it we have the “universal flat holomorphic line bundle” \mathcal{P} , whose restriction to $(\tilde{\mathcal{F}}, \tilde{\nabla}) \times \text{Jac}$ is the line bundle with connection $(\tilde{\mathcal{F}}, \tilde{\nabla})$ on Jac . It has a partial flat connection along Jac , i.e., we can differentiate its sections

along Jac using $\tilde{\nabla}$. Thus, we have the following diagram:

$$\begin{array}{ccc}
 & \mathcal{P} & \\
 & \downarrow & \\
 & \text{Loc}_1 \times \text{Jac} & \\
 p_1 \swarrow & & \searrow p_2 \\
 \text{Loc}_1 & & \text{Jac}
 \end{array}$$

It enables us to define functors F and G between the (bounded) derived category $D^b(\mathcal{O}_{\text{Loc}_1}\text{-mod})$ of (coherent) \mathcal{O} -modules on Loc_1 and the derived category $D^b(\mathcal{D}_{\text{Jac}}\text{-mod})$ of \mathcal{D} -modules on Jac:

$$(4.5) \quad F : \mathcal{M} \mapsto Rp_{1*}p_2^*(\mathcal{M} \otimes \mathcal{P}), \quad G : \mathcal{K} \mapsto Rp_{2*}p_1^*(\mathcal{K} \otimes \mathcal{P}).$$

For example, let $E = (\mathcal{F}, \nabla)$ be a point of Loc_1 and consider the “skyscraper” sheaf \mathcal{S}_E supported at this point. Then by definition $G(\mathcal{S}_E) = (\tilde{\mathcal{F}}, \tilde{\nabla})$, considered as a \mathcal{D} -module on Jac. So the simplest \mathcal{O} -modules on Loc_1 , namely, the skyscraper sheaves supported at points, go to the simplest \mathcal{D} -modules on Jac, namely, flat line bundles, which are the (degree zero components of) the Hecke eigensheaves Aut_E .

We should compare this picture to the picture of Fourier transform. The Fourier transform sends the delta-functions $\delta_x, x \in \mathbb{R}$ (these are the analogues of the skyscraper sheaves) to the exponential functions $e^{ixy}, y \in \mathbb{R}$, which can be viewed as the simplest \mathcal{D} -modules on \mathbb{R} . Indeed, e^{ixy} is the solution of the differential equation $(\partial_y - ix)\Phi(y) = 0$, so it corresponds to the trivial line bundle on \mathbb{R} with the connection $\nabla = \partial_y - ix$. Now, it is quite clear that a general function in x can be thought of as an integral, or superposition, of the delta-functions $\delta_x, x \in \mathbb{R}$. The main theorem of the Fourier analysis is that the Fourier transform is an isomorphism (of the appropriate function spaces). It may be viewed, loosely, as the statement that on the other side of the transform the exponential functions $e^{ixy}, x \in \mathbb{R}$, also form a good “basis” for functions. In other words, other functions can be written as Fourier integrals.

An analogous thing happens in our situation. It has been shown by G. Laumon [65] and M. Rothstein [66] that the functors F and G give rise to mutually inverse (up to a sign and cohomological shift) equivalences of derived categories

$$(4.6) \quad \boxed{\text{derived category of } \mathcal{O}\text{-modules on } \text{Loc}_1} \longleftrightarrow \boxed{\text{derived category of } \mathcal{D}\text{-modules on Jac}}$$

$$\mathcal{S}_E \longleftrightarrow \text{Aut}_E^0$$

Loosely speaking, this means that the Hecke eigensheaves Aut_E^0 on Jac form a “good basis” of the derived category on the right hand side of this diagram. In other words, any object of $D^b(\mathcal{D}_{\text{Jac}}\text{-mod})$ may be represented as a “Fourier integral” of Hecke eigensheaves, just like any object of $D^b(\mathcal{O}_{\text{Loc}_1}\text{-mod})$ may be thought of as an “integral” of the skyscraper sheaves \mathcal{S}_E .

This equivalence reveals the true meaning of the Hecke eigensheaves and identifies them as the building blocks of the derived category of \mathcal{D} -modules on Jac, just like the skyscraper sheaves are the building blocks of the derived category of \mathcal{D} -modules.

This is actually consistent with the picture emerging from the classical Langlands correspondence. In the classical Langlands correspondence (when X is a curve over \mathbb{F}_q) the Hecke eigenfunctions on $GL_n(F)\backslash GL_n(\mathbb{A})/GL_n(\mathcal{O})$ form a basis of the appropriate space of functions on $GL_n(F)\backslash GL_n(\mathbb{A})/GL_n(\mathcal{O})$.³⁹ That is why we should expect that the geometric objects that replace the Hecke eigenfunctions – namely, the Hecke eigensheaves on Bun_n – should give us a kind of “spectral decomposition” of the derived category of \mathcal{D} -modules on Bun_n^0 . The Laumon-Rothstein theorem may be viewed a precise formulation of this statement.

The above equivalence is very closely related to the Fourier-Mukai transform. Let us recall that the Fourier-Mukai transform is an equivalence between the derived categories of coherent sheaves on an abelian variety A and its dual A^\vee , which is the moduli space of line bundles on A (and conversely, A is the moduli space of line bundles on A^\vee). Then we have the universal (also known as the Poincaré) line bundle \mathcal{P} on $A^\vee \times A$ whose restriction to $a^\vee \times a$, where $a^\vee \in A^\vee$, is the line bundle $\mathcal{L}(a^\vee)$ corresponding to a^\vee (and likewise for the restriction to $A^\vee \times a$). Then we have functors between the derived categories of coherent sheaves (of \mathcal{O} -modules) on A and A^\vee defined in the same way as in formula (4.5), which set up an equivalence of categories, called the Fourier-Mukai transform.

Rothstein and Laumon have generalized the Fourier-Mukai transform by replacing A^\vee , which is the moduli space of line bundles on A , by A^\natural , the moduli space of *flat* line bundles on A . They showed that the corresponding functors set up an equivalence between the derived category of coherent sheaves on A^\natural and the derived category of \mathcal{D} -modules on A .

Now, if A is the Jacobian variety Jac of a complex curve X , then $A^\vee \simeq \text{Jac}$ and $A^\natural \simeq \text{Loc}_1$, so we obtain the equivalence discussed above.

A slightly disconcerting feature of this construction, as compared to the original Fourier-Mukai transform, is the apparent asymmetry between the two categories. But it turns out that this equivalence has a deformation in which this asymmetry disappears (see Sect. 6.3).

4.5. A special case of the Fourier-Mukai transform. Recall that the moduli space Loc_1 of flat line bundles on X fibers over $\text{Jac} = \text{Pic}_0$ with the fiber over $\mathcal{F} \in \text{Jac}$ being the space of all (holomorphic) connections on \mathcal{F} , which is an affine space over the space $H^0(X, \Omega)$ of holomorphic one-forms on X . In particular, the fiber $p^{-1}(\mathcal{F}_0)$ over the trivial line bundle \mathcal{F}_0 is just the space of holomorphic differentials on X , $H^0(X, \Omega)$. As we have seen above, each point of Loc_1 gives rise to a Hecke eigensheaf on Pic , which is a line bundle with holomorphic connection. Consider a point in the fiber over \mathcal{F}_0 , i.e., a flat line bundle of the form $(\mathcal{F}_0, d + \omega)$. It turns out that in this case we can describe the corresponding Hecke eigen-line bundle quite explicitly.

We will describe its restriction to Jac . First of all, as a line bundle on Jac , it is trivial (as \mathcal{F}_0 is the trivial line bundle on X), so all we need to do is to specify a connection on the trivial bundle corresponding to $\omega \in H^0(X, \Omega)$. This connection is given by a holomorphic one-form on Jac , which we denote by $\tilde{\omega}$. But now observe that that space of holomorphic one-forms on Jac is isomorphic to the space $H^0(X, \Omega)$ of holomorphic one-forms on X . Hence $\omega \in H^0(X, \Omega)$ gives rise to a holomorphic one-form on Jac , and this is the desired $\tilde{\omega}$.

³⁹actually, this is only true if one restricts to the cuspidal functions; but for $n = 1$ the cuspidality condition is vacuous

One can also say it slightly differently: observe that the tangent bundle to Jac is trivial, with the fiber isomorphic to the g -dimensional complex vector space $H^1(X, \mathcal{O}_X)$. Hence the Lie algebra of global vector fields on Jac is isomorphic to $H^1(X, \mathcal{O}_X)$, and it acts simply transitively on Jac. Therefore to define a connection on the trivial line bundle on Jac we need to attach to each $\xi \in H^1(X, \Omega)$ a holomorphic function f_ξ on Jac, which is necessarily constant as Jac is compact. The corresponding connection operators are then $\nabla_\xi = \xi + f_\xi$. This is the same as the datum of a linear functional $H^1(X, \mathcal{O}_X) \rightarrow \mathbb{C}$. Our $\omega \in H^0(X, \Omega)$ gives us just such a functional by the Serre duality.

We may also express the resulting \mathcal{D} -module on Jac in terms of the general construction outlined in Sect. 3.4 (which could be called “ \mathcal{D} -modules as systems of differential equations”). Consider the algebra D_{Jac} of global differential operators on Jac. From the above description of the Lie algebra of global vector fields on Jac it follows that D_{Jac} is commutative and is isomorphic to $\text{Sym } H^1(X, \mathcal{O}_X) = \text{Fun } H^0(X, \Omega)$, by the Serre duality.⁴⁰ Therefore each point $\omega \in H^0(X, \Omega)$ gives rise to a homomorphism $\lambda_\omega : D_{\text{Jac}} \rightarrow \mathbb{C}$. Define the \mathcal{D} -module $\text{Aut}_{E_\omega}^0$ on Jac by the formula

$$(4.7) \quad \text{Aut}_{E_\omega}^0 = \mathcal{D} / \text{Ker } \lambda_\omega,$$

where \mathcal{D} is the sheaf of differential operators on Jac, considered as a (left) module over itself (compare with formula (3.4)). This is the holonomic \mathcal{D} -module on Jac that is the restriction of the Hecke eigensheaf corresponding to the trivial line bundle on X with the connection $d + \omega$.

The \mathcal{D} -module $\text{Aut}_{E_\omega}^0$ represents the system of differential equations

$$(4.8) \quad D \cdot f = \lambda_\omega(D)f, \quad D \in D_{\text{Jac}}$$

(compare with (3.5)) in the sense that for any homomorphism from $\text{Aut}_{E_\omega}^0$ to another \mathcal{D} -module \mathcal{K} the image of $1 \in \text{Aut}_{E_\omega}^0$ in \mathcal{K} is (locally) a solution of the system (4.8). Of course, the equations (4.8) are just equivalent to the equations $(d + \tilde{\omega})f = 0$ on horizontal sections of the trivial line bundle on Jac with the connection $d + \tilde{\omega}$.

The concept of Fourier-Mukai transform leads us to a slightly different perspective on the above construction. The point of the Fourier-Mukai transform was that not only do we have a correspondence between rank one vector bundles with a flat connection on Jac and points of Loc_1 , but more general \mathcal{D} -modules on Jac correspond to \mathcal{O} -modules on Loc_1 other than the skyscraper sheaves.⁴¹ One such \mathcal{D} -module is the sheaf \mathcal{D} itself, considered as a (left) \mathcal{D} -module. What \mathcal{O} -module on Loc_1 corresponds to it? From the point of view of the above analysis, it is not surprising what the answer is: it is the \mathcal{O} -module $i_*(\mathcal{O}_{p^{-1}(\mathcal{F}_0)})$ (see [66]).

Here $\mathcal{O}_{p^{-1}(\mathcal{F}_0)}$ denotes the structure sheaf of the subspace of connections on the trivial line bundle \mathcal{F}_0 (which is the fiber over \mathcal{F}_0 under the projection $p : \text{Loc}_1 \rightarrow \text{Jac}$), and i is the inclusion $i : p^{-1}(\mathcal{F}_0) \hookrightarrow \text{Loc}_1$.

This observation allows us to represent a special case of the Fourier-Mukai transform in more concrete terms. Namely, amongst all \mathcal{O} -modules on Loc_1 consider those that are supported on $p^{-1}(\mathcal{F}_0)$, in other words, the \mathcal{O} -modules of the form $\mathcal{M} = i_*(M)$, where M

⁴⁰here and below for an affine algebraic variety V we denote by $\text{Fun } V$ the algebra of polynomial functions on V

⁴¹in general, objects of the derived category of \mathcal{O} -modules

is an \mathcal{O} -module on $p^{-1}(\mathcal{F}_0)$, or equivalently, a $\text{Fun } H^0(X, \Omega)$ -module. Then the restriction of the Fourier-Mukai transform to the subcategory of these \mathcal{O} -modules is a functor from the category of $\text{Fun } H^0(X, \Omega)$ -modules to the category of \mathcal{D} -modules on Jac given by

$$(4.9) \quad M \mapsto G(M) = \mathcal{D} \underset{D_{\text{Jac}}}{\otimes} M.$$

Here we use the fact that $\text{Fun } H^0(X, \Omega) \simeq D_{\text{Jac}}$. In particular, if we take as M the one-dimensional module corresponding to a homomorphism λ_ω as above, then $G(M) = \text{Aut}_{E_\omega}^0$. Thus, we obtain a very explicit formula for the Fourier-Mukai functor restricted to the subcategory of \mathcal{O} -modules on Loc_1 supported on $H^0(X, \Omega) \subset \text{Loc}_1$.

We will discuss in Sect. 6.3 and Sect. 9.5 a non-abelian generalization of this construction, due to Beilinson and Drinfeld, in which instead of the moduli space of line bundles on X we consider the moduli space of G -bundles, where G is a simple Lie group. We will see that the role of a trivial line bundle on X with a flat connection will be played by a flat ${}^L G$ -bundle on X (where ${}^L G$ is the Langlands dual group to G introduced in the next section), with an additional structure of an *oper*. But first we need to understand how to formulate the geometric Langlands conjecture for general reductive algebraic groups.

5. FROM GL_n TO OTHER REDUCTIVE GROUPS

One adds a new dimension to the Langlands Program by considering arbitrary reductive groups instead of the group GL_n . This is when some of the most beautiful and mysterious aspects of the Program are revealed, such as the appearance of the Langlands dual group. In this section we will trace the appearance of the dual group in the classical context and then talk about its geometrization/categorification.

5.1. The spherical Hecke algebra for an arbitrary reductive group. Suppose we want to find an analogue of the Langlands correspondence from Theorem 1 where instead of automorphic representations of $GL_n(\mathbb{A})$ we consider automorphic representations of $G(\mathbb{A})$, where G is a connected reductive algebraic group over \mathbb{F}_q . To simplify our discussion, we will assume in what follows that G is also split over \mathbb{F}_q , which means that G contains a split torus T of maximal rank (isomorphic to the direct product of copies of the multiplicative group).⁴²

We wish to relate those representations to some data corresponding to the Galois group $\text{Gal}(\overline{F}/F)$, the way we did for GL_n . In the case of GL_n this relation satisfies an important compatibility condition that the Hecke eigenvalues of an automorphic representation coincide with the Frobenius eigenvalues of the corresponding Galois representation. Now we need to find an analogue of this compatibility condition for general reductive groups. The first step is to understand the structure of the proper analogue of the spherical Hecke algebra \mathcal{H}_x . For $G = GL_n$ we saw that this algebra is isomorphic to the algebra of symmetric Laurent polynomials in n variables. Now we need to give a similar description of the analogue of this algebra \mathcal{H}_x for a general reductive group G .

So let G be a connected reductive group over a finite field k which is split over k , and T a split maximal torus in G . Then we attach to this torus two lattices, P and \tilde{P} , or

⁴²since \mathbb{F}_q is not algebraically closed, this is not necessarily the case; for example, the Lie group $SL_2(\mathbb{R})$ is split over \mathbb{R} , but SU_2 is not

characters and cocharacters, respectively. The elements of the former are homomorphisms $\mu : T(k) \rightarrow k^\times$, and the elements of the latter are homomorphisms $\check{\lambda} : k^\times \rightarrow T(k)$. Both are free abelian groups (lattices), with respect to natural operations, of rank equal to the dimension of T . Note that $T(k) \simeq k^\times \otimes_{\mathbb{Z}} \check{P}$. We have a pairing $\langle \cdot, \cdot \rangle : P \times \check{P} \rightarrow \mathbb{Z}$. The composition $\mu \circ \check{\lambda}$ is a homomorphism $k^\times \rightarrow k^\times$, which are classified by an integer (“winding number”), and $\langle \mu, \check{\lambda} \rangle$ is equal to this number.

The sets P and \check{P} contain subsets Δ and Δ^\vee of roots and coroots of G , respectively (see, e.g., [68] for more details). Let now X be a smooth projective curve over \mathbb{F}_q and let us pick a point $x \in X$. Assume for simplicity that its residue field is \mathbb{F}_q . To simplify notation we will omit the index x from our formulas in this section. Thus, we will write $\mathcal{H}, F, \mathcal{O}$ for $\mathcal{H}_x, F_x, \mathcal{O}_x$, etc. We have $F \simeq \mathbb{F}_q((t)), \mathcal{O} \simeq \mathbb{F}_q[[t]]$, where t is a uniformizer in \mathcal{O} .

The Hecke algebra $\mathcal{H} = \mathcal{H}(G(F), G(\mathcal{O}))$ is by definition the space of \mathbb{C} -valued compactly supported functions on $G(F)$ which are bi-invariant with respect to the maximal compact subgroup $G(\mathcal{O})$. It is equipped with the convolution product

$$(5.1) \quad (f_1 \star f_2)(g) = \int_{G(F)} f_1(gh^{-1})f_2(h) dh,$$

where dh is the Haar measure on $G(F)$ normalized so that the volume of $G(\mathcal{O})$ is equal to 1.⁴³

What is this algebra equal to? The Hecke algebra $\mathcal{H}(T(F), T(\mathcal{O}))$ of the torus T is easy to describe. For each $\check{\lambda} \in \check{P}$ we have an element $\check{\lambda}(t) \in T(F)$. For instance, if $G = GL_n$ and T is the group of diagonal matrices, then $P \simeq \check{P} \simeq \mathbb{Z}^n$. For $\check{\lambda} \in \mathbb{Z}^n$ the element $\check{\lambda}(t) = (\check{\lambda}_1, \dots, \check{\lambda}_n) \in T(F)$ is just the diagonal matrix $\text{diag}(t^{\check{\lambda}_1}, \dots, t^{\check{\lambda}_n})$. Thus, we have (for GL_n and for a general group G)

$$T(\mathcal{O}) \backslash T(F) / T(\mathcal{O}) = T(F) / T(\mathcal{O}) = \{\check{\lambda}(t)\}_{\check{\lambda} \in \check{P}}.$$

The convolution product is given by $\check{\lambda}(t) \star \check{\mu}(t) = (\check{\lambda} + \check{\mu})(t)$. In other words, $\mathcal{H}(T(F), T(\mathcal{O}))$ is isomorphic to the group algebra $\mathbb{C}[\check{P}]$ of \check{P} . This isomorphism takes $\check{\lambda}(t)$ to $e^{\check{\lambda}} \in \mathbb{C}[\check{P}]$.

Note that the algebra $\mathbb{C}[\check{P}]$ is naturally the complexified representation ring $\text{Rep } \check{T}$ of the *dual* torus \check{T} , which is defined in such a way that its lattice of characters is \check{P} and the lattice of cocharacters is P . Under the identification $\mathbb{C}[\check{P}] \simeq \text{Rep } \check{T}$ an element $e^{\check{\lambda}} \in \mathbb{C}[\check{P}]$ is interpreted as the class of the one-dimensional representation of \check{T} corresponding to $\check{\lambda} : \check{T}(\mathbb{F}_q) \rightarrow \mathbb{F}_q^\times$.

5.2. Satake isomorphism. We would like to generalize this description to the case of an arbitrary split reductive group G . First of all, let \check{P}_+ be the set of dominant integral weights of ${}^L G$ with respect to a Borel subgroup of ${}^L G$ that we fix once and for all. It is

⁴³Let K be a compact subgroup of $G(F)$. Then one can define the Hecke algebra $\mathcal{H}(G(F), K)$ in a similar way. For example, $\mathcal{H}(G(F), I)$, where I is the Iwahori subgroup, is the famous *affine Hecke algebra*. The remarkable property of the spherical Hecke algebra $\mathcal{H}(G(F), G(\mathcal{O}))$ is that it is *commutative*, and so its irreducible representations are one-dimensional. This enables us to parameterize irreducible unramified representations by the characters of $\mathcal{H}(G(F), G(\mathcal{O}))$ (see Sect. 5.3). In general, the Hecke algebra $\mathcal{H}(G(F), K)$ is commutative if and only if K is a maximal compact subgroup of $G(F)$, such as $G(\mathcal{O})$. For more on this, see Sect. 9.7.

easy to see that the elements $\check{\lambda}(t)$, where $\check{\lambda} \in \check{P}_+$, are representatives of the double cosets of $G(F)$ with respect to $G(\mathcal{O})$. In other words,

$$G(\mathcal{O}) \backslash G(F) / G(\mathcal{O}) \simeq \check{P}_+.$$

Therefore \mathcal{H} has a basis $\{c_{\check{\lambda}}\}_{\check{\lambda} \in \check{P}_+}$, where $c_{\check{\lambda}}$ is the characteristic function of the double coset $G(\mathcal{O})\check{\lambda}(t)G(\mathcal{O}) \subset G(F)$.

An element of $\mathcal{H}(G(F), G(\mathcal{O}))$ is a $G(\mathcal{O})$ bi-invariant function on $G(F)$ and it can be restricted to $T(F)$, which is automatically $T(\mathcal{O})$ bi-invariant. Thus, we obtain a linear map $\mathcal{H}(G(F), G(\mathcal{O})) \rightarrow \mathcal{H}(T(F), T(\mathcal{O}))$ which can be shown to be injective. Unfortunately, this restriction map is not compatible with the convolution product, and hence is not an algebra homomorphism.

However, I. Satake [67] has constructed a different map

$$\mathcal{H}(G(F), G(\mathcal{O})) \rightarrow \mathcal{H}(T(F), T(\mathcal{O})) \simeq \mathbb{C}[\check{P}]$$

which is an algebra homomorphism. Let N be a unipotent subgroup of G . For example, if $G = GL_n$ we may take as N the group of upper triangular matrices with 1's on the diagonal. Satake's homomorphism takes $f \in \mathcal{H}(G(F), G(\mathcal{O}))$ to

$$\hat{f} = \sum_{\check{\lambda} \in \check{P}} \left(q^{\langle \rho, \check{\lambda} \rangle} \int_{N(F)} f(n \cdot \check{\lambda}(t)) dn \right) e^{\check{\lambda}} \in \mathbb{C}[\check{P}].$$

Here and below we denote by ρ the half-sum of positive roots of G , and dn is the Haar measure on $N(F)$ normalized so that the volume of $N(\mathcal{O})$ is equal to 1. The fact that f is compactly supported implies that the sum in the right hand side is finite.

From this formula it is not at all obvious why this map should be a homomorphism of algebras. The proof is based on the usage of matrix elements of a particular class of induced representations of $G(F)$, called the principal series (see [67]).

The following result is referred to as the Satake isomorphism.

Theorem 4. *The algebra homomorphism $\mathcal{H} \rightarrow \mathbb{C}[\check{P}]$ is injective and its image is equal to the subalgebra $\mathbb{C}[\check{P}]^W$ of W -invariants, where W is the Weyl group of G .*

A crucial observation of R. Langlands [1] was that $\mathbb{C}[\check{P}]^W$ is nothing but the representation ring of a complex reductive group. But this group is not $G(\mathbb{C})$! The representation ring of $G(\mathbb{C})$ is $\mathbb{C}[P]^W$, not $\mathbb{C}[\check{P}]^W$. Rather, it is the representation ring of the so-called *Langlands dual group* of G , which is usually denoted by ${}^L G(\mathbb{C})$. By definition, ${}^L G(\mathbb{C})$ is the reductive group over \mathbb{C} with a maximal torus ${}^L T(\mathbb{C})$ that is dual to T , so that the lattices of characters and cocharacters of ${}^L T(\mathbb{C})$ are those of T interchanged. The sets of roots and coroots of ${}^L G(\mathbb{C})$ are by definition those of G , but also interchanged. By the general classification of reductive groups over an algebraically closed field, this defines ${}^L G(\mathbb{C})$ uniquely up to an isomorphism (see [68]). For instance, the dual group of GL_n is again GL_n , SL_n is dual to PGL_n , SO_{2n+1} is dual to Sp_n , and SO_{2n} is self-dual.

At the level of Lie algebras, the Langlands duality changes the types of the simple factors of the Lie algebra of G by taking the transpose of the corresponding Cartan matrices. Thus, only the simple factors of types B and C are affected (they get interchanged). But the duality is more subtle at the level of Lie groups, as there is usually more than one Lie group attached to a given Lie algebra. For instance, if G is a connected simply-connected

simple Lie group, such as SL_n , its Langlands dual group is a connected Lie group with the same Lie algebra, but it is of adjoint type (in this case, PGL_n).

Let $\text{Rep}^L G$ be the Grothendieck ring of the category of finite-dimensional representations of ${}^L G(\mathbb{C})$. The lattice of characters of ${}^L G$ is \check{P} , and so we have the character homomorphism $\text{Rep}^L G \rightarrow \mathbb{C}[\check{P}]$. It is injective and its image is equal to $\mathbb{C}[\check{P}]^W$. Therefore Theorem 4 may be interpreted as saying that $\mathcal{H} \simeq \text{Rep}^L G(\mathbb{C})$. It follows then that the homomorphisms $\mathcal{H} \rightarrow \mathbb{C}$ are nothing but the semi-simple conjugacy classes of ${}^L G(\mathbb{C})$. Indeed, if γ is a semi-simple conjugacy class in ${}^L G(\mathbb{C})$, then we attach to it a one-dimensional representation of $\text{Rep}^L G \simeq \mathcal{H}$ by the formula $[V] \mapsto \text{Tr}(\gamma, V)$. This is the key step towards formulating the Langlands correspondence for arbitrary reductive groups. Let us summarize:

Theorem 5. *The spherical Hecke algebra $\mathcal{H}(G(F), G(\mathcal{O}))$ is isomorphic to the complexified representation ring $\text{Rep}^L G(\mathbb{C})$ where ${}^L G(\mathbb{C})$ is the Langlands dual group to G . There is a bijection between $\text{Spec } \mathcal{H}(G(F), G(\mathcal{O}))$, i.e., the set of homomorphisms $\mathcal{H}(G(F), G(\mathcal{O})) \rightarrow \mathbb{C}$, and the set of semi-simple conjugacy classes in ${}^L G(\mathbb{C})$.*

5.3. The Langlands correspondence for an arbitrary reductive group. Now we can formulate for an arbitrary reductive group G an analogue of the compatibility statement in the Langlands correspondence Theorem 1 for GL_n . Namely, suppose that $\pi = \bigotimes'_{x \in X} \pi_x$ is a cuspidal automorphic representation of $G(\mathbb{A})$. For all but finitely many $x \in X$ the representation π_x of $G(F_x)$ is unramified, i.e., the space of $G(\mathcal{O}_x)$ -invariants in π_x is non-zero. One shows that in this case the space of $G(\mathcal{O}_x)$ -invariants is one-dimensional, generated by a non-zero vector v_x , and \mathcal{H}_x acts on it by the formula

$$f_x \cdot v_x = \phi(f_x)v_x, \quad f_x \in \mathcal{H}_x,$$

where ϕ is a homomorphism $\mathcal{H}_x \rightarrow \mathbb{C}$. By Theorem 5, ϕ corresponds to a semi-simple conjugacy class γ_x in ${}^L G(\mathbb{C})$. Thus, we attach to an automorphic representation a collection $\{\gamma_x\}$ of semi-simple conjugacy classes in ${}^L G(\mathbb{C})$ for almost all points of X .

For example, if $G = GL_n$, then a semi-simple conjugacy class γ_x in ${}^L GL_n(\mathbb{C}) = GL_n(\mathbb{C})$ is the same as an unordered n -tuple of non-zero complex numbers. In Sect. 2.3 we saw that such a collection $(z_1(\pi_x), \dots, z_n(\pi_x))$ indeed encoded the eigenvalues of the Hecke operators. Now we see that for a general group G the eigenvalues of the Hecke algebra \mathcal{H}_x are encoded by a semi-simple conjugacy class γ_x in the Langlands dual group ${}^L G(\mathbb{C})$. Therefore on the other side of the Langlands correspondence we need some sort of Galois data which would also involve such conjugacy classes. Up to now we have worked with complex valued functions on $G(F)$, but when trying to formulate the global Langlands correspondence, we should replace \mathbb{C} by $\overline{\mathbb{Q}}_\ell$, and in particular, consider the Langlands dual group over $\overline{\mathbb{Q}}_\ell$, just as we did before for GL_n (see the discussion after Theorem 1).

One candidate for the Galois parameters of automorphic representations that immediately comes to mind is a homomorphism

$$\sigma : \text{Gal}(\overline{F}/F) \rightarrow {}^L G(\overline{\mathbb{Q}}_\ell),$$

which is almost everywhere unramified. Then we may attach to σ a collection of conjugacy classes $\{\sigma(\text{Fr}_x)\}$ of ${}^L G(\overline{\mathbb{Q}}_\ell)$ at almost all points $x \in X$, and those are precisely the parameters of the irreducible unramified representations of the local factors $G(F_x)$ of

$G(\mathbb{A})$, by the Satake isomorphism. Thus, if σ is everywhere unramified, we obtain for each $x \in X$ an irreducible representation π_x of $G(F_x)$, and their restricted tensor product is an irreducible representation of $G(\mathbb{A})$ attached to σ , which we hope to be automorphic, in the appropriate sense.

So in the first approximation we may formulate the Langlands correspondence for general reductive groups as a correspondence between automorphic representations of $G(\mathbb{A})$ and Galois homomorphisms $\text{Gal}(\overline{F}/F) \rightarrow {}^L G(\mathbb{Q}_\ell)$ which satisfies the following compatibility condition: if π corresponds to σ , then the ${}^L G$ -conjugacy classes attached to π through the action of the Hecke algebra are the same as the Frobenius ${}^L G$ -conjugacy classes attached to σ .

Unfortunately, the situation is not as clear-cut as in the case of GL_n because many of the results which facilitate the Langlands correspondence for GL_n are no longer true in general. For instance, it is not true in general that the collection of the Hecke conjugacy classes determines the automorphic representation uniquely or that the collection of the Frobenius conjugacy classes determines the Galois representation uniquely. For this reason one expects that to a Galois representation corresponds not a single automorphic representation but a finite set of those (an “L-packet” or an “A-packet”). Moreover, the multiplicities of automorphic representations in the space of functions on $G(F)\backslash G(\mathbb{A})$ can now be greater than 1, unlike the case of GL_n . Therefore even the statement of the Langlands conjecture becomes a much more subtle issue for a general reductive group (see [63]). However, the main idea appears to be correct: we expect that there is a relationship, still very mysterious, between automorphic representations of $G(\mathbb{A})$ and homomorphisms from the Galois group $\text{Gal}(\overline{F}/F)$ to the Langlands dual group ${}^L G$.

We are not going to explore in this survey the subtle issues related to a more precise formulation of this relationship.⁴⁴ Rather, in the hope of gaining some insight into this mystery, we would like to formulate a geometric analogue of this relationship. The first step is to develop a geometric version of the Satake isomorphism.

5.4. Categorification of the spherical Hecke algebra. Let us look at the isomorphism of Theorem 4 more closely. It is useful to change our notation at this point and denote the weight lattice of ${}^L G$ by P (that used to be \check{P} before) and the coweight lattice of ${}^L G$ by \check{P} (that used to be P before). Accordingly, we will denote the weights of ${}^L G$ by λ , etc., and not $\check{\lambda}$, etc., as before. We will again suppress the subscript x in our notation.

As we saw in the previous section, the spherical Hecke algebra \mathcal{H} has a basis $\{c_\lambda\}_{\lambda \in P_+}$, where c_λ is the characteristic function of the double coset $G(\mathcal{O})\lambda(t)G(\mathcal{O}) \subset G$. On the other hand, $\text{Rep } {}^L G$ also has a basis labeled by the set P_+ of dominant weights of ${}^L G$. It consists of the classes $[V_\lambda]$, where V_λ is the irreducible representation with highest weight λ . However, under the Satake isomorphism these bases *do not* coincide! Instead, we have

⁴⁴An even more general *functoriality principle* of R. Langlands asserts the existence of a relationship between automorphic representations of two adèlic groups $H(\mathbb{A})$ and $G(\mathbb{A})$, where G is split, but H is not necessarily split over F , for any given homomorphism $\text{Gal}(\overline{F}/F) \times {}^L H \rightarrow {}^L G$ (see the second reference in [21] for more details). The Langlands correspondence that we discuss in this survey is the special case of the functoriality principle, corresponding to $H = \{1\}$; in this case the above homomorphism becomes $\text{Gal}(\overline{F}/F) \rightarrow {}^L G$

the following formula

$$(5.2) \quad H_\lambda = q^{-\langle \check{\rho}, \lambda \rangle} c_\lambda + \sum_{\mu \in P_+; \mu < \lambda} a_{\lambda\mu} c_\mu, \quad a_{\lambda\mu} \in \mathbb{Z}_+[q],$$

where H_λ is the image of $[V_\lambda]$ in \mathcal{H} under the Satake isomorphism.⁴⁵ This formula, which looks perplexing at first glance, actually has a remarkable geometric explanation.

Let us consider \mathcal{H} as the algebra of functions on the quotient $G(F)/G(\mathcal{O})$ which are left invariant with respect to $G(\mathcal{O})$. We have learned in Sect. 3.3 that “interesting” functions often have an interpretation as sheaves, via the Grothendieck fonctions-faisceaux dictionary. So it is natural to ask whether $G(F)/G(\mathcal{O})$ is the set of \mathbb{F}_q -points of an algebraic variety, and if so, whether H_λ is the function corresponding to a perverse sheaf on this variety. It turns out that this is indeed the case.

The quotient $G(F)/G(\mathcal{O})$ is the set of points of an ind-scheme Gr over \mathbb{F}_q called the *affine Grassmannian* associated to G . Let $\mathcal{P}_{G(\mathcal{O})}$ be the category of $G(\mathcal{O})$ -equivariant perverse sheaves on Gr . This means that the restriction of an objects of $\mathcal{P}_{G(\mathcal{O})}$ to each $G(\mathcal{O})$ -orbit in Gr is locally constant. Because these orbits are actually simply-connected, these restrictions will then necessarily be constant. For each $\lambda \in P_+$ we have a finite-dimensional $G(\mathcal{O})$ -orbit $\text{Gr}_\lambda = G(\mathcal{O}) \cdot \lambda(t)G(\mathcal{O})$ in Gr . Let $\overline{\text{Gr}}_\lambda$ be its closure in Gr . This is a finite-dimensional algebraic variety, usually singular, and it is easy to see that it is stratified by the orbits Gr_μ , where $\mu \in P_+$ are such that $\mu \leq \lambda$ with respect to the usual ordering on the set of weights.

As we mentioned in Sect. 3.3, an irreducible perverse sheaf on a variety V is uniquely determined by its restriction to an open dense subset $U \subset V$, if it is non-zero (and in that case it is necessarily an irreducible perverse sheaf on U). Let us take $\overline{\text{Gr}}_\lambda$ as V and Gr_λ as U . Then U is smooth and so the rank one constant sheaf on U , placed in cohomological degree $-\dim_{\mathbb{C}} U = -2\langle \check{\rho}, \lambda \rangle$, is a perverse sheaf. Therefore there exists a unique, up to an isomorphism, irreducible perverse sheaf on $\overline{\text{Gr}}_\lambda$ whose restriction to Gr_λ is this constant sheaf. The sheaf on $\overline{\text{Gr}}_\lambda$ is called the *Goresky-MacPherson* or *intersection cohomology* sheaf on $\overline{\text{Gr}}_\lambda$. We will denote it by IC_λ .

This is quite a remarkable complex of sheaves on $\overline{\text{Gr}}_\lambda$. The cohomology of $\overline{\text{Gr}}_\lambda$ with coefficients in IC_λ , the so-called *intersection cohomology* of $\overline{\text{Gr}}_\lambda$, satisfies the Poincaré duality: $H^i(\overline{\text{Gr}}_\lambda, \text{IC}_\lambda) \simeq H^{-i}(\overline{\text{Gr}}_\lambda, \text{IC}_\lambda)$.⁴⁶ If $\overline{\text{Gr}}_\lambda$ were a smooth variety, then IC_λ would be just the constant sheaf placed in cohomological degree $-\dim_{\mathbb{C}} \overline{\text{Gr}}_\lambda$, and so its cohomology would just be the ordinary cohomology of $\overline{\text{Gr}}_\lambda$, shifted by $\dim_{\mathbb{C}} \overline{\text{Gr}}_\lambda$.

A beautiful result (due to Goresky and MacPherson when V is defined over a field of characteristic zero and to Beilinson, Bernstein and Deligne when V is defined over a finite field) is that a complex of sheaves satisfying the Poincaré duality property always exists on singular varieties, and it is unique (up to an isomorphism) if we require in addition that its restriction to any smooth open subset (such as Gr_λ in our case) is a rank one constant sheaf.

⁴⁵ $\mu \leq \lambda$ means that $\lambda - \mu$ can be written as a linear combination of simple roots with non-negative integer coefficients

⁴⁶the unusual normalization is due to the fact that we have shifted the cohomological degrees by $\dim_{\mathbb{C}} \overline{\text{Gr}}_\lambda = \frac{1}{2} \dim_{\mathbb{R}} \overline{\text{Gr}}_\lambda$

The perverse sheaves IC_λ are in fact all the irreducible objects of the category $\mathcal{P}_{G(\mathcal{O})}$, up to an isomorphism.⁴⁷

Assigning to a perverse sheaf its “trace of Frobenius” function, as explained in Sect. 3.3, we obtain an identification between the Grothendieck group of $\mathcal{P}_{G(\mathcal{O})}$ and the algebra of $G(\mathcal{O})$ -invariant functions on $G(F)/G(\mathcal{O})$, i.e., the spherical Hecke algebra \mathcal{H} . In that sense, $\mathcal{P}_{G(\mathcal{O})}$ is a *categorification* of the Hecke algebra. A remarkable fact is that the function H_λ in formula (5.2) is precisely equal to the function associated to the perverse sheaf IC_λ , up to a sign $(-1)^{2\langle \check{\rho}, \lambda \rangle}$.

Now we can truly appreciate formula (5.2). Under the Satake isomorphism the classes of irreducible representations V_λ of ${}^L G$ do not go to the characteristic functions c_λ of the orbits, as one could naively expect. The reason is that those functions correspond to the constant sheaves on Gr_λ . The constant sheaf on Gr_λ (extended by zero to $\overline{\mathrm{Gr}}_\lambda$) is the wrong sheaf. The correct substitute for it, from the geometric perspective, is the irreducible perverse sheaf IC_λ . The corresponding function is then $(-1)^{2\langle \check{\rho}, \lambda \rangle} H_\lambda$, where H_λ is given by formula (5.2), and this is precisely the function that corresponds to V_λ under the Satake correspondence.

The coefficients $a_{\lambda\mu}$ appearing in H_λ also have a transparent geometric meaning: they measure the dimensions of the stalk cohomologies of IC_λ at various strata Gr_μ , $\mu \leq \lambda$ that lie in the closure of Gr_λ ; more precisely, $a_{\lambda\mu} = \sum_i a_{\lambda\mu}^{(i)} q^{i/2}$, where $a_{\lambda\mu}^{(i)}$ is the dimension of the i th stalk cohomology of IC_λ on Gr_μ .⁴⁸

We have $H_\lambda = q^{-\langle \check{\rho}, \lambda \rangle} c_\lambda$ only if the orbit Gr_λ is already closed. This is equivalent to the weight λ being *miniscule*, i.e., the only dominant integral weight occurring in the weight decomposition of V_λ is λ itself. This is a very rare occurrence. A notable exception is the case of $G = GL_n$, when all fundamental weights ω_i , $i = 1, \dots, n-1$, are miniscule. The corresponding $G(\mathcal{O})$ -orbit is the (ordinary) Grassmannian $\mathrm{Gr}(i, n)$ of i -dimensional subspaces of the n -dimensional vector space. Whenever we have the equality $H_\lambda = q^{-\langle \check{\rho}, \lambda \rangle} c_\lambda$ the definition of the Hecke operators, both at the level of functions and at the level of sheaves, simplifies dramatically.

5.5. Example: the affine Grassmannian of PGL_2 . Let us look more closely at the affine Grassmannian $\mathrm{Gr} = PGL_2((t))/PGL_2[[t]]$ associated to $PGL_2(\mathbb{C})$. Since the fundamental group of PGL_2 is \mathbb{Z}_2 , the loop group $PGL_2((t))$ has two connected components, and so does its Grassmannian. We will denote them by $\mathrm{Gr}^{(0)}$ and $\mathrm{Gr}^{(1)}$. The component $\mathrm{Gr}^{(0)}$ is in fact isomorphic to the Grassmannian $SL_2((t))/SL_2[[t]]$ of SL_2 .

The $PGL_2[[t]]$ -orbits in Gr are parameterized by set of dominant integral weights of the dual group of PGL_2 , which is SL_2 . We identify it with the set \mathbb{Z}_+ of non-negative integers. The orbit Gr_n corresponding to $n \in \mathbb{Z}_+$ is equal to

$$\mathrm{Gr}_n = PGL_2[[t]] \begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix} PGL_2[[t]].$$

⁴⁷in general, we would also have to include the perverse sheaves obtained by extensions of non-trivial (irreducible) local systems on the smooth strata, such as our Gr_λ ; but since these strata are simply-connected in our case, there are no non-trivial local systems supported on them

⁴⁸to achieve this, we need to restrict ourselves to the so-called pure perverse sheaves; otherwise, H_λ could in principle be multiplied by an arbitrary overall scalar

It has complex dimension $2n$. If $n = 2k$ is even, then it belongs to $\mathrm{Gr}^{(0)} = \mathrm{Gr}_{SL_2}$ and may be realized as

$$\mathrm{Gr}_{2k} = SL_2[[t]] \begin{pmatrix} t^k & 0 \\ 0 & t^{-k} \end{pmatrix} SL_2[[t]].$$

The smallest of those is Gr_0 , which is a point.

If n is odd, then Gr_n belongs to $\mathrm{Gr}^{(1)}$. The smallest is Gr_1 , which is isomorphic to \mathbb{CP}^1 .

The closure $\overline{\mathrm{Gr}}_n$ of Gr_n is the disjoint union of Gr_m , where $m \leq n$ and m has the same parity as n . The irreducible perverse sheaf IC_n is actually a constant sheaf in this case (placed in cohomological dimension $-2n$), even though $\overline{\mathrm{Gr}}_n$ is a singular algebraic variety. This variety has a nice description in terms of the $N((t))$ -orbits in Gr (where N is the subgroup of upper triangular unipotent matrices). These are

$$S_m = N((t)) \begin{pmatrix} t^m & 0 \\ 0 & 1 \end{pmatrix} PGL_2[[t]], \quad m \in \mathbb{Z}.$$

Then $\overline{\mathrm{Gr}}_n$ is the disjoint union of the intersections $\overline{\mathrm{Gr}}_n \cap S_m$ where $|m| \leq n$ and m has the same parity as n , and in this case

$$\overline{\mathrm{Gr}}_n \cap S_m = \left\{ \begin{pmatrix} 1 & \sum_{i=(n-m)/2}^{n-1} a_i t^i \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix}, a_i \in \mathbb{C} \right\} \simeq \mathbb{C}^{(n+m)/2}$$

(otherwise $\overline{\mathrm{Gr}}_n \cap S_m = \emptyset$).

5.6. The geometric Satake equivalence. We have seen above that the Satake isomorphism may be interpreted as an isomorphism between the Grothendieck group of the category $\mathcal{P}_{G(\mathfrak{o})}$ and the Grothendieck group of the category $\mathcal{R}ep^L G$ of finite-dimensional representations of the Langlands dual group ${}^L G$. Under this isomorphism the irreducible perverse sheaf IC_λ goes to the irreducible representation V_λ . This suggests that perhaps the Satake isomorphism may be elevated from the level of Grothendieck groups to the level of categories. This is indeed true.

In fact, it is possible to define the structure of tensor category on $\mathcal{P}_{G(\mathfrak{o})}$ with the tensor product given by a convolution functor corresponding to the convolution product (5.1) at the level of functions. The definition of this tensor product, which is due to Beilinson and Drinfeld (see [69]), is reminiscent of the fusion product arising in conformal field theory. It uses a remarkable geometric object, the Beilinson-Drinfeld Grassmannian $\mathrm{Gr}^{(2)}$, which may be defined for any curve X . This $\mathrm{Gr}^{(2)}$ fibers over X^2 , but its fiber over $(x, y) \in X^2$, where $x \neq y$, is isomorphic to $\mathrm{Gr} \times \mathrm{Gr}$, whereas the fiber over $(x, x) \in X^2$ is isomorphic to a single copy of Gr (see [20], Sect. 20.3, for a review of this construction). One can define in terms of $\mathrm{Gr}^{(2)}$ the other ingredients necessary for the structure of tensor category on $\mathcal{P}_{G(\mathfrak{o})}$, namely, the commutativity and associativity constraints (see [69]).

Then we have the following beautiful result. It has been conjectured by V. Drinfeld and proved in the most general setting by I. Mirković and K. Vilonen [69] (some important results in this direction were obtained earlier by V. Ginzburg [70] and G. Lusztig [71]).

Theorem 6. *The tensor category $\mathcal{P}_{G(\mathfrak{o})}$ is equivalent to the tensor category $\mathcal{R}ep^L G$.*

Moreover, the fiber functor from $\mathcal{P}_{G(\mathfrak{o})}$ to the category of vector spaces, corresponding to the forgetful functor on $\mathcal{R}ep^L G$, is just the global cohomology functor $\mathcal{F} \mapsto \bigoplus_i H^i(\mathrm{Gr}, \mathcal{F})$.

The second assertion allows one to reconstruct the Langlands dual group ${}^L G$ by means of the standard Tannakian formalism.

For instance, let us consider the irreducible perverse sheaves IC_{ω_i} corresponding to the closed $GL_n(\mathcal{O})$ -orbits Gr_{ω_i} in the Grassmannian, attached to the miniscule fundamental weights ω_i of the dual GL_n . As we saw above, Gr_{ω_i} is the Grassmannian $\mathrm{Gr}(i, n)$, and IC_{ω_i} is the constant sheaf on it placed in the cohomological degree $-\dim_{\mathbb{C}} \mathrm{Gr}(i, n) = -i(n-i)$. Therefore the fiber functor takes IC_{ω_i} to $\bigoplus_i H^i(\mathrm{Gr}(i, n-i), \mathbb{C})$, which is isomorphic to $\wedge^i \mathbb{C}^n$. This space is indeed isomorphic to the i th fundamental representation V_{ω_i} of the dual GL_n .⁴⁹

In particular, the Langlands dual group of GL_n can be *defined* as the group of automorphisms of the total cohomology space $H^*(\mathrm{Gr}_{\omega_1}, \mathbb{C})$ of $\mathrm{Gr}_{\omega_1} \simeq \mathbb{P}^{n-1}$, which is the projectivization of the n -dimensional defining representation of the original group GL_n . It just happens that the dual group is isomorphic to GL_n again, but this construction makes it clear that it is a *different* GL_n !

So we get a completely new perspective on the nature of the Langlands dual group (as compared to the Satake construction). This is a good illustration of why geometry is useful in the Langlands Program.

The above theorem should be viewed as a categorification of the Satake isomorphism of Theorem 4. We will now use it to define the notion of a Hecke eigensheaf for an arbitrary reductive group and to formulate a geometric version of the Langlands correspondence.

6. THE GEOMETRIC LANGLANDS CONJECTURE OVER \mathbb{C}

From now on we will work exclusively with curves over \mathbb{C} , even though the definition of the Hecke eigensheaves, for example, can be made for curves over the finite field as well. In this section we will formulate the geometric Langlands conjecture for an arbitrary reductive group G over \mathbb{C} . Once we do that, we will be able to use methods of conformal field theory to try and establish this correspondence.

6.1. Hecke eigensheaves. Let us recall from the previous section that we have the affine Grassmannian Gr (over \mathbb{C}) and the category $\mathcal{P}_{G(\mathcal{O})}$ of $G(\mathcal{O})$ -equivariant perverse sheaves (of \mathbb{C} -vector spaces) on Gr . This category is equivalent, as a tensor category, to the category of finite-dimensional representations of the Langlands dual group ${}^L G(\mathbb{C})$. Under this equivalence, the irreducible representation of ${}^L G$ with highest weight $\lambda \in P_+$ corresponds to the irreducible perverse sheaf IC_{λ} .

Now we can define the analogues of the GL_n Hecke functors introduced in Sect. 3.7 for a general reductive group G . Let Bun_G be the moduli stack of G -bundles on X . Consider the stack $\mathcal{H}ecke$ which classifies quadruples $(\mathcal{M}, \mathcal{M}', x, \beta)$, where \mathcal{M} and \mathcal{M}' are G -bundles on X , $x \in X$, and β is an isomorphism between the restrictions of \mathcal{M} and \mathcal{M}' to $X \setminus x$. We have natural morphisms

$$\begin{array}{ccc} & \mathcal{H}ecke & \\ h^{\leftarrow} \swarrow & & \searrow h^{\rightarrow} \\ \mathrm{Bun}_G & & X \times \mathrm{Bun}_G \end{array}$$

⁴⁹note that this space comes with a cohomological gradation, which we have already encountered in Sect. 3.8

where $h^\leftarrow(\mathcal{M}, \mathcal{M}', x, \beta) = \mathcal{M}$ and $h^\rightarrow(\mathcal{M}, \mathcal{M}', x, \beta) = (x, \mathcal{M}')$.

Note that the fiber of $\mathcal{H}ecke$ over (x, \mathcal{M}') is the moduli space of pairs (\mathcal{M}, β) , where \mathcal{M} is a G -bundles on X , and $\beta : \mathcal{M}'|_{X \setminus x} \xrightarrow{\sim} \mathcal{M}|_{X \setminus x}$. It is known that this moduli space is isomorphic to a twist of $\text{Gr}_x = G(F_x)/G(\mathcal{O}_x)$ by the $G(\mathcal{O})_x$ -torsor $\mathcal{M}'(\mathcal{O}_x)$ of sections of \mathcal{M}' over $\text{Spec } \mathcal{O}_x$:

$$(h^\rightarrow)^{-1}(x, \mathcal{M}') = \mathcal{M}'(\mathcal{O}_x) \times_{G(\mathcal{O}_x)} \text{Gr}_x.$$

Therefore we have a stratification of each fiber, and hence of the entire $\mathcal{H}ecke$, by the substacks $\mathcal{H}ecke_\lambda, \lambda \in P_+$, which correspond to the $G(\mathcal{O})$ -orbits Gr_λ in Gr . Consider the irreducible perverse sheaf on $\mathcal{H}ecke$, which is the Goresky-MacPherson extension of the constant sheaf on $\mathcal{H}ecke_\lambda$. Its restriction to each fiber is isomorphic to IC_λ , and by abuse of notation we will denote this entire sheaf also by IC_λ .

Define the Hecke functor H_λ from the derived category of perverse sheaves on Bun_G to the derived category of perverse sheaves on $X \times \text{Bun}_G$ by the formula

$$(6.1) \quad H_\lambda(\mathcal{F}) = h_*^\rightarrow(h^{\leftarrow*}(\mathcal{F}) \otimes \text{IC}_\lambda).$$

Let E be a ${}^L G$ -local system on X . Then for each irreducible representation V_λ of ${}^L G$ we have a local system $V_\lambda^E = E \times_{{}^L G} V_\lambda$.

Now we define Hecke eigensheaves as follows. A perverse sheaf (or, more generally, a complex of sheaves) on Bun_G is called a *Hecke eigensheaf with eigenvalue E* if we are given isomorphisms

$$(6.2) \quad \iota_\lambda : H_\lambda(\mathcal{F}) \xrightarrow{\sim} V_\lambda^E \boxtimes \mathcal{F}, \quad \lambda \in P_+,$$

which are compatible with the tensor product structure on the category of representations of ${}^L G$.

In the case when $G = GL_n$ this definition is equivalent to equations (3.9). This is because the fundamental representations $V_{\omega_i}, i = 1, \dots, n-1$, and the one-dimensional determinant representation generate the tensor category of representations of GL_n . Hence it is sufficient to have the isomorphisms (6.2) just for those representations. These conditions are equivalent to the Hecke conditions (3.9).

Now we wish state the geometric Langlands conjecture which generalizes the geometric Langlands correspondence for $G = GL_n$ (see Theorem 3). One subtle point is what should take place of the irreducibility condition of a local system E for a general group G . As we saw in Sect. 3.8, this condition is very important. It seems that there is no consensus on this question at present, so in what follows we will use a provisional definition: ${}^L G$ -local system is called irreducible if it cannot be reduced to a proper parabolic subgroup of ${}^L G$.

Conjecture 1. *Let E be an irreducible ${}^L G$ -local system on X . Then there exists a non-zero Hecke eigensheaf Aut_E on Bun_G with the eigenvalue E whose restriction to each connected component of Bun_G is an irreducible perverse sheaf.*

$$\boxed{\begin{array}{c} \text{irreducible} \\ {}^L G\text{-local systems on } X \end{array}} \longrightarrow \boxed{\begin{array}{c} \text{Hecke eigensheaves} \\ \text{on } \text{Bun}_G \end{array}}$$

$$E \longrightarrow \text{Aut}_E$$

As explained in Sect. 3.4, when working over \mathbb{C} we may switch from perverse sheaves to \mathcal{D} -modules, using the Riemann-Hilbert correspondence (see [49, 50, 51, 54]). Therefore we may replace in the above conjecture perverse sheaves by \mathcal{D} -modules. In what follows we will consider this \mathcal{D} -module version of the geometric Langlands conjecture.

The Hecke eigensheaves corresponding to a fixed ${}^L G$ -local system E give rise to a category Aut_E whose objects are collections $(\mathcal{F}, \{\iota_\lambda\}_{\lambda \in P_+})$, where \mathcal{F} is an object of the derived category of sheaves on Bun_G , and ι_λ are the isomorphisms entering the definition of Hecke eigensheaves (6.2) which are compatible with the tensor product structure on the category of representations of ${}^L G$. Just as in the case of $G = GL_n$ (see Sect. 3.8), it is important to realize that the structure of this category changes dramatically depending on whether E is irreducible (in the above sense) or not.

If E is irreducible, then we expect that this category contains a unique, up to an isomorphism, perverse sheaf (or a \mathcal{D} -module) that is irreducible on each component of Bun_G . But this is not true for a reducible local system: it may have non-isomorphic objects, and the objects may not be perverse sheaves, but complexes of perverse sheaves. For example, in [61] Hecke eigensheaves corresponding to ${}^L G$ -local systems that are reducible to the maximal torus ${}^L T \subset {}^L G$ were constructed. These are the geometric Eisenstein series generalizing those discussed in Sect. 3.8. In the best case scenario these are direct sums of infinitely many irreducible perverse sheaves on Bun_G , but in general these are complicated complexes of perverse sheaves.

The group of automorphisms of E naturally acts on the category Aut_E as follows. Given an automorphism g of E , we obtain a compatible system of automorphisms of the local systems V_λ^E , which we also denote by g . The corresponding functor $\mathit{Aut}_E \rightarrow \mathit{Aut}_E$ sends $(\mathcal{F}, \{\iota_\lambda\}_{\lambda \in P_+})$ to $(\mathcal{F}, \{g \circ \iota_\lambda\}_{\lambda \in P_+})$. For a generic E the group of automorphisms is the center $Z({}^L G)$ of ${}^L G$, which is naturally identified with the group of characters of the fundamental group $\pi_1(G)$ of G . The latter group labels connected components of $\mathit{Bun}_G = \bigcap_{\gamma \in \pi_1(G)} \mathit{Bun}_G^\gamma$. So given $z \in Z({}^L G)$, we obtain a character $\chi_z : \pi_1(G) \rightarrow \mathbb{C}^\times$. The action of z on Aut_E then amounts to multiplying $\mathcal{F}|_{\mathit{Bun}_G^\gamma}$ by $\chi_z(\gamma)$. On the other hand, the group of automorphisms of the trivial local system E_0 is ${}^L G$ itself, and the corresponding action of ${}^L G$ on the category Aut_{E_0} is more sophisticated.

As we discussed in the case of GL_n (see Sect. 3.8), we do not know any elementary examples of Hecke eigensheaves for reductive groups other than the tori. However, just as in the case of GL_n , the constant sheaf $\underline{\mathbb{C}}$ on Bun_G may be viewed as a Hecke eigensheaf, except that its eigenvalue is not a local system on X but a complex of local systems.

Indeed, by definition, for a dominant integral weight $\lambda \in P_+$ of ${}^L G$, $H_\lambda(\underline{\mathbb{C}})$ is the constant sheaf on Bun_n with the fiber being the cohomology $\bigoplus_i H^i(\mathrm{Gr}_\lambda, \mathrm{IC}_\lambda)$, which is isomorphic to V_λ , according to Theorem 6, as a vector space. But it is “spread out” in cohomological degrees, and so one cannot say that $\underline{\mathbb{C}}$ is a Hecke eigensheaf with the eigenvalue being a local system on X . Rather, its “eigenvalue” is something like a complex of local systems. As in the case of GL_n discussed in Sect. 3.8, the non-triviality of cohomological grading fits nicely with the concept of Arthur’s SL_2 (see [63]).

6.2. Non-abelian Fourier-Mukai transform? In Sect. 4.4 we explained the connection between the geometric Langlands correspondence for the abelian group GL_1 and the Fourier-Mukai transform (4.6) (in the context of \mathcal{D} -modules, as proposed by Laumon and

Rothstein). In fact, the Fourier-Mukai transform may be viewed as a stronger version of the geometric Langlands correspondence in the abelian case in that it assigns \mathcal{D} -modules (more precisely, objects of the corresponding derived category) not just to individual rank one local systems on X (viewed as skyscraper sheaves on the moduli space Loc_1 of such local systems), but also to more arbitrary \mathcal{O} -modules on Loc_1 . Moreover, this assignment is an equivalence of derived categories, which may be viewed as a “spectral decomposition” of the derived category of \mathcal{D} -modules on Jac . It is therefore natural to look for a similar stronger version of the geometric Langlands correspondence for other reductive groups - a kind of non-abelian Fourier-Mukai transform. The discussion below follows the ideas of Beilinson and Drinfeld.

Naively, we expect a non-abelian Fourier-Mukai transform to be an equivalence of derived categories

$$(6.3) \quad \boxed{\text{derived category of } \mathcal{O}\text{-modules on } \mathrm{Loc}_{L_G}} \longleftrightarrow \boxed{\text{derived category of } \mathcal{D}\text{-modules on } \mathrm{Bun}_G^\circ}$$

where Loc_{L_G} is the moduli stack of L_G -local systems on X and Bun_G° is the connected component of Bun_G . This equivalence should send the skyscraper sheaf on Loc_{L_G} supported at the local system E to the restriction to Bun_G° of the Hecke eigensheaf Aut_E . If this were true, it would mean that Hecke eigensheaves provide a good “basis” in the category of \mathcal{D} -modules on Bun_G° , just as flat line bundles provide a good “basis” in the category of \mathcal{D} -modules on Jac .

Unfortunately, a precise formulation of such a correspondence, even as a conjecture, is not so clear because of various subtleties involved. One difficulty is the structure of Loc_{L_G} . Unlike the case of $L_G = GL_1$, when all local systems have the same groups of automorphisms (namely, \mathbb{C}^\times), for a general group L_G the groups of automorphisms are different for different local systems, and so Loc_{L_G} is a complicated stack. For example, if L_G is a simple Lie group of adjoint type, then a generic local system has no automorphisms, while the group of automorphisms of the trivial local system E_0 is isomorphic to L_G . This has to be reflected somehow in the structure of the corresponding Hecke eigensheaves. For a generic local system E we expect that there is only one, up to an isomorphism, irreducible Hecke eigensheaf with the eigenvalue E , and the category Aut_E of Hecke eigensheaves with this eigenvalue is equivalent to the category of vector spaces. But the category Aut_{E_0} of Hecke eigensheaves with the eigenvalue E_0 is non-trivial, and it carries an action of the group L_G of symmetries of E_0 . Some examples of Hecke eigensheaves with eigenvalue E_0 that have been constructed are unbounded complexes of perverse sheaves (i.e., their cohomological degrees are unbounded). The non-abelian Fourier-Mukai transform has to reflect both the stack structure of Loc_{L_G} and the complicated structure of the categories of Hecke eigensheaves such as these. In particular, it should presumably involve unbounded complexes and so the precise definition of the categories appearing in (6.3) is unclear [62].

We may choose a slightly different perspective on the equivalence of categories (6.3) and ask about the existence of an analogue of the Poincaré line bundle \mathcal{P} (see Sect. 4.4) in the non-abelian case. This would be a “universal” Hecke eigensheaf \mathcal{P}_G on $\mathrm{Loc}_{L_G} \times \mathrm{Bun}_G$ which comprises the Hecke eigensheaves for individual local systems. One can use such a sheaf as the “kernel” of the “integral transform” functors between the two categories

(6.3), the way \mathcal{P} was used in the abelian case. If Conjecture 1 were true, then it probably would not be difficult to construct such a sheaf on $\mathrm{Loc}_{LG}^{\mathrm{irr}} \times \mathrm{Bun}_G$, where $\mathrm{Loc}_{LG}^{\mathrm{irr}}$ is the locus of irreducible LG -local systems. The main problem is how to extend it to the entire $\mathrm{Loc}_{LG} \times \mathrm{Bun}_G$ [62].

While it is not known whether a non-abelian Fourier-Mukai transform exists, A. Beilinson and V. Drinfeld have constructed an important special case of this transform. Let us assume that G is a connected and simply-connected simple Lie group. Then this transform may be viewed as a generalization of the construction in the abelian case that was presented in Sect. 4.5. Namely, it is a functor from the category of \mathcal{O} -modules supported on a certain affine subvariety $i : \mathrm{Op}_{LG}(X) \hookrightarrow \mathrm{Loc}_{LG}$, called the space of $L\mathfrak{g}$ -opers on X , to the category of \mathcal{D} -modules on Bun_G (in this case it has only one component). Actually, $\mathrm{Op}_{L\mathfrak{g}}(X)$ may be identified with the fiber $p^{-1}(\mathcal{F}_{LG})$ of the forgetful map $p : \mathrm{Loc}_{LG} \rightarrow \mathrm{Bun}_{LG}$ over a particular LG -bundle described in Sect. 8.3, which plays the role that the trivial line bundle plays in the abelian case. The locus of $L\mathfrak{g}$ -opers in Loc_{LG} is particularly nice because local systems underlying opers are irreducible and their groups of automorphisms are trivial.

We will review the Beilinson-Drinfeld construction within the framework of two-dimensional conformal field theory in Sect. 9. Their results may be interpreted as saying that the non-abelian Fourier-Mukai transform sends the \mathcal{O} -module $i_*(\mathcal{O}_{\mathrm{Op}_{L\mathfrak{g}}(X)})$ on Loc_{LG} to the sheaf \mathcal{D} of differential operators on Bun_G , considered as a left \mathcal{D} -module (see the end of Sect. 9.5).

Additional evidence for the existence of the non-abelian Fourier transform comes from certain orthogonality relations between natural sheaves on both moduli spaces that have been established in [72, 73].

In the next section we speculate about a possible two-parameter deformation of the naive non-abelian Fourier-Mukai transform, loosely viewed as an equivalence between the derived categories of \mathcal{D} -modules on Bun_G and \mathcal{O} -modules on Loc_{LG} .

6.3. A two-parameter deformation. This deformation is made possible by the realization that the above two categories are actually not that far away from each other. Indeed, first of all, observe that Loc_{LG} is the twisted cotangent bundle to Bun_{LG}° , a point that we already noted in the abelian case in Sect. 4.4. Indeed, a LG -local system on X is a pair (\mathcal{F}, ∇) , where \mathcal{F} is a (holomorphic) LG -bundle on X and ∇ is a (holomorphic) connection on \mathcal{F} . Thus, we have a forgetful map $\mathrm{Loc}_{LG} \rightarrow \mathrm{Bun}_{LG}^\circ$ taking (\mathcal{F}, ∇) to \mathcal{F} . The fiber of this map over \mathcal{F} is the space of all connections on \mathcal{F} , which is either empty or an affine space modeled on the vector space $H^0(X, {}^L\mathfrak{g}_{\mathcal{F}} \otimes \Omega)$, where $\mathfrak{g}_{\mathcal{F}} = \mathcal{F} \times_G {}^L\mathfrak{g}$. Indeed, we can add a one-form $\omega \in H^0(X, {}^L\mathfrak{g}_{\mathcal{F}} \otimes \Omega)$ to any given connection on \mathcal{F} , and all connections on \mathcal{F} can be obtained this way.⁵⁰

But now observe that $H^0(X, {}^L\mathfrak{g}_{\mathcal{F}} \otimes \Omega)$ is isomorphic to the cotangent space to \mathcal{F} in Bun_{LG}° . Indeed, the tangent space to \mathcal{F} is the space of infinitesimal deformations of \mathcal{F} , which is $H^1(X, {}^L\mathfrak{g}_{\mathcal{F}})$. Therefore, by the Serre duality, the cotangent space is isomorphic to $H^0(X, {}^L\mathfrak{g}_{\mathcal{F}}^* \otimes \Omega)$. We may identify \mathfrak{g}^* with \mathfrak{g} using a non-degenerate inner product on \mathfrak{g} , and

⁵⁰These fibers could be empty; this is the case for GL_n bundles which are direct sums of subbundles of non-zero degrees, for example. Nevertheless, one can still view Loc_{LG} as a twisted cotangent bundle to Bun_{LG}° in the appropriate sense. I thank D. Ben-Zvi for a discussion of this point.

therefore identify $H^0(X, {}^L\mathfrak{g}_{\mathcal{F}}^* \otimes \Omega)$ with $H^0(X, {}^L\mathfrak{g}_{\mathcal{F}} \otimes \Omega)$. Thus, we find that Loc_{LG} is an affine bundle over Bun_{LG}° which is modeled on the cotangent bundle $T^* \text{Bun}_{LG}^\circ$. Thus, if we denote the projections $T^* \text{Bun}_{LG}^\circ \rightarrow \text{Bun}_{LG}^\circ$ and $\text{Loc}_{LG} \rightarrow \text{Bun}_{LG}^\circ$ by \check{p} and \check{p}' , respectively, then we see that the sheaf $\check{p}'_*(\mathcal{O}_{\text{Loc}_{LG}})$ on Bun_{LG}° locally looks like $\check{p}'_*(\mathcal{O}_{T^* \text{Bun}_{LG}^\circ})$. Since the fibers of \check{p}' are affine spaces, a sheaf of $\mathcal{O}_{\text{Loc}_{LG}}$ -modules on Loc_{LG} is the same as a sheaf of $\check{p}'_*(\mathcal{O}_{\text{Loc}_{LG}})$ -modules on Bun_{LG}° .

On the other hand, consider the corresponding map for the group G , $p : T^* \text{Bun}_G^\circ \rightarrow \text{Bun}_G^\circ$. The corresponding sheaf $p_*(\mathcal{O}_{T^* \text{Bun}_G^\circ})$ is the sheaf of *symbols* of differential operators on Bun_G° . This means the following. The sheaf $\mathcal{D}_{\text{Bun}_G^\circ}$ carries a filtration $\mathcal{D}_{\leq i}, i \geq 0$, by the subsheaves of differential operators of order less than or equal to i . The corresponding associated graded sheaf $\bigoplus_{i \geq 0} \mathcal{D}_{\leq (i+1)} / \mathcal{D}_{\leq i}$ is the sheaf of symbols of differential operators on Bun_G° , and it is canonically isomorphic to $p_*(\mathcal{O}_{T^* \text{Bun}_G^\circ})$.

Thus, $\check{p}'_*(\mathcal{O}_{\text{Loc}_{LG}})$ is a commutative deformation of $\check{p}'_*(\mathcal{O}_{T^* \text{Bun}_{LG}^\circ})$, while $\mathcal{D}_{\text{Bun}_{LG}^\circ}$ is a non-commutative deformation of $p_*(\mathcal{O}_{T^* \text{Bun}_G^\circ})$. Moreover, one can include $\mathcal{D}_{\text{Bun}_{LG}^\circ}$ and $\check{p}'_*(\mathcal{O}_{\text{Loc}_{LG}})$, where $p' : \text{Loc}_G \rightarrow \text{Bun}_G^\circ$, into a two-parameter family of sheaves of associative algebras. This will enable us to speculate about a deformation of the non-abelian Fourier-Mukai transform which will make it look more “symmetric”.

The construction of this two-parameter deformation is explained in [75] and is in fact applicable in a rather general situation. Here we will only consider the specific case of Bun_G° and Bun_{LG}° following [74].

Recall that we have used a non-degenerate invariant inner product $\check{\kappa}_0$ on ${}^L\mathfrak{g}$ in order to identify ${}^L\mathfrak{g}$ with ${}^L\mathfrak{g}^*$. This inner product automatically induces a non-degenerate invariant inner product κ_0 on \mathfrak{g} . This is because we can identify a Cartan subalgebra of \mathfrak{g} with the dual of the Cartan subalgebra of ${}^L\mathfrak{g}$, and the invariant inner products are completely determined by their restrictions to the Cartan subalgebras. We will fix these inner products once and for all. Now, a suitable multiple $k\kappa_0$ of the inner product κ_0 induces, in the standard way, which will be recalled in Sect. 7.5, a line bundle on Bun_G° which we will denote by $\mathcal{L}^{\otimes k}$. The meaning of this notation is that we would like to think of \mathcal{L} as the line bundle corresponding to κ_0 , even though it may not actually exist. But this will not be important to us, because we will not be interested in the line bundle itself, but in the sheaf of differential operators acting on the sections of this line bundle. The point is that if \mathcal{L}' is an honest line bundle, one can make sense of the sheaf of differential operators acting on $\mathcal{L}'^{\otimes s}$ for any complex number s (see [75] and Sect. 7.4 below). This is an example of the sheaf of *twisted* differential operators on Bun_G° .

So we denote the sheaf of differential operators acting on $\mathcal{L}^{\otimes k}$, where $k \in \mathbb{C}$, by $\mathcal{D}(\mathcal{L}^{\otimes k})$. Thus, we now have a one-parameter family of sheaves of associative algebras depending on $k \in \mathbb{C}$. These sheaves are filtered by the subsheaves $\mathcal{D}_{\leq i}(\mathcal{L}^{\otimes k})$ of differential operators of order less than or equal to i . The first term of the filtration, $\mathcal{D}_{\leq 1}(\mathcal{L}^{\otimes k})$ is a Lie algebra (and a Lie algebroid), which is an extension

$$0 \rightarrow \mathcal{O}_{\text{Bun}_G^\circ} \rightarrow \mathcal{D}_{\leq 1}(\mathcal{L}^{\otimes k}) \rightarrow \Theta_{\text{Bun}_G^\circ} \rightarrow 0,$$

where $\Theta_{\text{Bun}_G^\circ}$ is the tangent sheaf on Bun_G° . The sheaf $\mathcal{D}(\mathcal{L}^{\otimes k})$ itself is nothing but the quotient of the universal enveloping algebra sheaf of the Lie algebra sheaf $\mathcal{D}_{\leq 1}(\mathcal{L}^{\otimes k})$ by the relation identifying the unit with $1 \in \mathcal{O}_{\text{Bun}_G^\circ}$.

We now introduce a second deformation parameter $\lambda \in \mathbb{C}$ as follows: let $\mathcal{D}_{\leq 1}^\lambda(\mathcal{L}^{\otimes k})$ be the Lie algebra $\mathcal{D}_{\leq 1}(\mathcal{L}^{\otimes k})$ in which the Lie bracket is equal to the Lie bracket on $\mathcal{D}_{\leq 1}(\mathcal{L}^{\otimes k})$ multiplied by λ . Then $\mathcal{D}^\lambda(\mathcal{L}^{\otimes k})$ is defined as the quotient of the universal enveloping algebra of $\mathcal{D}_{\leq 1}^\lambda(\mathcal{L}^{\otimes k})$ by the relation identifying the unit with $1 \in \mathcal{O}_{\text{Bun}_G^\circ}$. This sheaf of algebras is isomorphic to $\mathcal{D}(\mathcal{L}^{\otimes k})$ for $\lambda \neq 0$, and $\mathcal{D}^0(\mathcal{L}^{\otimes k})$ is isomorphic to the sheaf of symbols $\check{p}_*(\mathcal{O}_{T^*\text{Bun}_{L_G}^\circ})$.

Thus, we obtain a family of algebras parameterized by $\mathbb{C} \times \mathbb{C}$. We now further extend it to $\mathbb{CP}^1 \times \mathbb{C}$ by defining the limit as $k \rightarrow \infty$. In order to do this, we need to rescale the operators of order less than or equal to i by $(\frac{\lambda}{k})^i$, so that the relations are well-defined in the limit $k \rightarrow \infty$. So we set

$$\mathcal{D}^{k,\lambda} = \bigoplus_{i \geq 0} \left(\frac{\lambda}{k} \right)^i \cdot \mathcal{D}_{\leq i}^\lambda(\mathcal{L}^{\otimes k}).$$

Then by definition

$$\mathcal{D}^{\infty,\lambda} = \mathcal{D}^{k,\lambda}/k^{-1} \cdot \mathcal{D}^{k,\lambda}.$$

Therefore we obtain a family of sheaves of algebras parameterized by $\mathbb{CP}^1 \times \mathbb{C}$.

Moreover, when $k = \infty$ the algebra becomes commutative, and we can actually identify it with $p'_*(\mathcal{O}_{\text{Loc}_G^\lambda})$. Here Loc_G^λ is by definition the moduli space of pairs $(\mathcal{F}, \nabla_\lambda)$, where \mathcal{F} is a holomorphic G -bundle on X and ∇_λ is a holomorphic λ -connection on \mathcal{F} . A λ -connection is defined in the same way as a connection, except that locally it looks like $\nabla_\lambda = \lambda d + \omega$. Thus, if $\lambda \neq 0$ a λ -connection is the same thing as a connection, and so $\text{Loc}_G^\lambda \simeq \text{Loc}_G$, whereas for $\lambda = 0$ a λ -connection is the same as a \mathfrak{g}_G -valued one-form, and so $\text{Loc}_G^0 \simeq T^*\text{Bun}_G^\circ$.

Let us summarize: we have a nice family of sheaves $\mathcal{D}^{k,\lambda}$ of associative algebras on Bun_G° parameterized by $(k, \lambda) \in \mathbb{CP}^1 \times \mathbb{C}$. For $\lambda \neq 0$ and $k \neq \infty$ this is the sheaf of differential operators acting on $\mathcal{L}^{\otimes k}$. For $\lambda \neq 0$ and $k = \infty$ this is $p'_*(\mathcal{O}_{\text{Loc}_G})$, and for $\lambda = 0$ and arbitrary k this is $p_*(\mathcal{O}_{T^*\text{Bun}_G^\circ})$. Thus, $\mathcal{D}^{k,\lambda}$ “smoothly” interpolates between these three kinds of sheaves on Bun_G° .

Likewise, we have a sheaf of differential operators acting on the “line bundle” $\check{\mathcal{L}}^{\otimes \check{k}}$ (where $\check{\mathcal{L}}$ corresponds to the inner product $\check{\kappa}_0$) on $\text{Bun}_{L_G}^\circ$, and we define in the same way the family of sheaves $\check{\mathcal{D}}^{\check{k},\check{\lambda}}$ of algebras on $\text{Bun}_{L_G}^\circ$ parameterized by $(\check{k}, \check{\lambda}) \in \mathbb{CP}^1 \times \mathbb{C}$.

Now, as we explained above, the naive non-abelian Fourier-Mukai transform should be viewed as an equivalence between the derived categories of $\mathcal{D}^{0,1}$ -modules on Bun_G° and $\check{\mathcal{D}}^{\infty,1}$ -modules on $\text{Bun}_{L_G}^\circ$. It is tempting to speculate that such an equivalence (if exists) may be extended to an equivalence⁵¹

⁵¹as we will see in Sect. 8.6, there is a “quantum correction” to this equivalence: namely, k and \check{k} should be shifted by the dual Coxeter numbers of G and L_G

$$(6.4) \quad \boxed{\text{derived category of } \check{\mathcal{D}}^{\check{k},\lambda}\text{-modules on } \text{Bun}_{L_G}^\circ} \longleftrightarrow \boxed{\text{derived category of } \mathcal{D}^{k,\lambda}\text{-modules on } \text{Bun}_G^\circ} \quad k = \check{k}^{-1}$$

In fact, in the abelian case, where the Fourier-Mukai transform exists, such a deformation also exists and has been constructed in [76].

While the original Langlands correspondence (6.3) looks quite asymmetric: it relates flat L_G -bundles on X and \mathcal{D} -modules on Bun_G° , the Fourier-Mukai perspective allows us to think of it as a special case of a much more symmetric picture.

Another special case of this picture is $\lambda = 0$. In this case $\mathcal{D}^{k,\lambda} = p_*(\mathcal{O}_{T^*\text{Bun}_G^\circ})$ and $\check{\mathcal{D}}^{k^{-1},\lambda} = \check{p}_*(\mathcal{O}_{T^*\text{Bun}_{L_G}^\circ})$, so we are talking about the equivalence between the derived categories of \mathcal{O} -modules on the cotangent bundles $T^*\text{Bun}_G^\circ$ and $T^*\text{Bun}_{L_G}^\circ$. If G is abelian, this equivalence follows from the original Fourier-Mukai transform. For example, if $G = L_G = GL_1$, we have $T^*\text{Bun}_G^\circ = T^*\text{Bun}_{L_G}^\circ = \text{Jac} \times H^0(X, \Omega)$, and we just apply the Fourier-Mukai transform along the first factor Jac.

The above decomposition of $T^*\text{Bun}_G^\circ$ in the abelian case has an analogue in the non-abelian case as well: this is the Hitchin fibration $T^*\text{Bun}_G^\circ \rightarrow H_G$, where H_G is a vector space (see Sect. 9.5). The generic fibers of this map are abelian varieties (generalized Prym varieties of the so-called spectral curves of X). We will discuss it in more detail in Sect. 9.5 below. The point is that there is an isomorphism of vector space $H_G \simeq H_{L_G}$. Roughly speaking, the corresponding equivalence of the categories of \mathcal{O} -modules on $T^*\text{Bun}_G^\circ$ and $T^*\text{Bun}_{L_G}^\circ$ should be achieved by applying a fiberwise Fourier-Mukai transform along the fibers of the Hitchin fibration. However, the singular fibers complicate matters (not to mention the “empty fibers”), and as far as we know, such an equivalence has not yet been established.⁵² In [78] some results concerning this equivalence in the formal neighborhood of the point $\lambda = 0$ are obtained.

6.4. \mathcal{D} -modules are D-branes? Derived categories of coherent \mathcal{O} -modules on algebraic varieties have recently become staples of string theory, where objects of these categories are viewed as examples of “D-branes”. Moreover, various equivalences involving these categories have been interpreted by physicists in terms of some sort of dualities of quantum field theories. For example, homological mirror symmetry proposed by Kontsevich has been interpreted as an equivalence of the categories of D-branes in two topological string theories, type A and type B, associated to a pair of mirror dual Calabi-Yau manifolds.

However, in the Langlands correspondence, and in particular in the Fourier-Mukai picture outlined in the previous section, we see the appearance of the categories of \mathcal{D} -modules instead of (or alongside) categories of \mathcal{O} -modules. Could \mathcal{D} -modules also be interpreted as D-branes of some kind? An affirmative answer to this question is an essential part of Witten’s proposal [3] relating S-duality in four-dimensional gauge theories and the Langlands correspondence that was mentioned in the Introduction. Examples of “non-commutative” D-branes related to \mathcal{D} -modules have also been considered in [79], and in fact they are

⁵²These dual Hitchin fibrations (restricted to the open subsets of stable Higgs pairs in $T^*\text{Bun}_G^\circ$ and $T^*\text{Bun}_{L_G}^\circ$) have been shown by T. Hausel and M. Thaddeus [77] to be an example of the Strominger-Yau-Zaslow duality.

closely related to the deformed Fourier-Mukai equivalence in the abelian case that we mentioned above.

We close this section with the following remark. We have looked above at the cotangent bundle $T^* \text{Bun}_G^\circ$ to Bun_G° and the twisted cotangent bundle to Bun_G viewed as moduli space Loc_G of flat holomorphic bundles on X . Both are algebraic stacks. But they contain large open dense subsets which are algebraic varieties. For example, in the case when $G = GL_n$, these are the moduli space of stable Higgs pairs of rank n and degree 0 and the moduli space of irreducible rank flat vector bundles of rank n . Both are smooth (quasi-projective) algebraic varieties. Though they are different as algebraic (or complex) varieties, the underlying real manifolds are diffeomorphic to each other. This is the so-called non-abelian Hodge theory diffeomorphism [80]. In fact, the underlying real manifold is hyperkähler, and the above two incarnations correspond to two particular choices of the complex structure. It is natural to ask what, if anything, this hyperkähler structure has to do with the Langlands correspondence, in which both of these algebraic varieties play such a prominent role. The answer to this question is presently unknown.

Part III. Conformal field theory approach

We have now come to point where we can relate the geometric Langlands correspondence to two-dimensional conformal field theory and reveal some of the secrets of the Langlands correspondence. The reason why conformal field theory is useful in our enterprise is actually very simple: the problem that we are trying to solve is how to attach to a flat ${}^L G$ -bundle E on X a \mathcal{D} -module Aut_E on the moduli stack Bun_G of G -bundles on X , which is a Hecke eigensheaf with the eigenvalue E . Setting the Hecke condition aside for a moment, we ask: how can we possibly construct \mathcal{D} -modules on Bun_G ? The point is that conformal field theories with affine Lie algebra (or Kac-Moody) symmetry corresponding to the group G give us precisely what we need - \mathcal{D} -modules on Bun_G (more precisely, twisted \mathcal{D} -modules, as explained below). These \mathcal{D} -modules encode chiral correlation functions of the model and it turns out that Hecke eigensheaves may be obtained this way.

In this part of the survey I will recall this formalism and then apply it to a particular class of conformal field theories: namely, those where the affine Kac-Moody algebra has *critical level*. As the result we will obtain the Beilinson-Drinfeld construction [15] of Hecke eigensheaves on Bun_G associated to special ${}^L G$ -local systems on X called *opers*. Moreover, we will see that the Hecke operators may be interpreted in terms of the insertion of certain vertex operators in the correlation functions of this conformal field theory.

7. CONFORMAL FIELD THEORY WITH KAC-MOODY SYMMETRY

The \mathcal{D} -modules on Bun_G arise in conformal field theories as the sheaves of *conformal blocks*, or the sheaves of *coinvariants* (the dual spaces to the spaces of conformal blocks), as I will now explain. Throughout Part III of these notes, unless specified otherwise, G will denote a connected simply-connected simple Lie group over \mathbb{C} .

7.1. Conformal blocks. The construction of the sheaves of conformal blocks (or coinvariants) is well-known in conformal field theory. For example, consider the *WZW model* [10] corresponding to a connected and simply-connected simple compact Lie group U and a positive integral level k . Let G be the corresponding complex Lie group and \mathfrak{g} its Lie algebra. The affine Kac-Moody algebra corresponding to \mathfrak{g} is defined as the central extension

$$(7.1) \quad 0 \rightarrow \mathbb{C}\mathbf{1} \rightarrow \widehat{\mathfrak{g}} \rightarrow \mathfrak{g} \otimes \mathbb{C}((t)) \rightarrow 0$$

with the commutation relations

$$(7.2) \quad [A \otimes f(t), B \otimes g(t)] = [A, B] \otimes fg - \kappa_0(A, B) \int fdg \cdot \mathbf{1}.$$

Here κ_0 denotes a non-degenerate invariant inner product on \mathfrak{g} . It is unique up to a non-zero scalar, and we normalize it in the standard way so that the square of length of the maximal root is equal to 2 [100]. So, for instance, if $\mathfrak{g} = \mathfrak{sl}_N$, we have $\kappa_0(A, B) =$

$\text{Tr}_{\mathbb{C}^N}(AB)$. We will say that a representation M of $\widehat{\mathfrak{g}}$ has level $k \in \mathbb{C}$ if $\mathbf{1}$ acts on M by multiplication by k .

The Hilbert space of the WZW theory of level k is the direct sum [81]

$$\mathbf{H}_k = \bigoplus_{\lambda \in \widehat{P}_+^k} L_\lambda \otimes \bar{L}_\lambda,$$

Here L_λ and \bar{L}_λ are two copies of the irreducible integrable representation of the corresponding affine Lie algebra $\widehat{\mathfrak{g}}$ of level k and highest weight λ , and the set \widehat{P}_+^k labels the highest weights of level k (see [100]). Thus, \mathbf{H}_k is a representation of the direct sum of two copies of $\widehat{\mathfrak{g}}$, corresponding to the chiral and anti-chiral symmetries of the theory.

Let X be a smooth projective curve over \mathbb{C} and x_1, \dots, x_n an n -tuple of points of X with local coordinates t_1, \dots, t_n . We attach to this points integrable representations $L_{\lambda_1}, \dots, L_{\lambda_n}$ of $\widehat{\mathfrak{g}}$ of level k . The diagonal central extension of the direct sum $\bigoplus_{i=1}^n \mathfrak{g} \otimes \mathbb{C}((t_i))$ acts on the tensor product $\bigotimes_{i=1}^n L_{\lambda_i}$. Consider the Lie algebra

$$\mathfrak{g}_{\text{out}} = \mathfrak{g} \otimes \mathbb{C}[X \setminus \{x_1, \dots, x_n\}]$$

of \mathfrak{g} -valued meromorphic functions on X with poles allowed only at the points x_1, \dots, x_n . We have an embedding

$$\mathfrak{g}_{\text{out}} \hookrightarrow \bigoplus_{i=1}^n \mathfrak{g} \otimes \mathbb{C}((t_i)).$$

It follows from the above commutation relations in $\widehat{\mathfrak{g}}$ and the residue theorem that this embedding lifts to the diagonal central extension of $\bigoplus_{i=1}^n \mathfrak{g} \otimes \mathbb{C}((t_i))$. Hence the Lie algebra $\mathfrak{g}_{\text{out}}$ acts on $\bigotimes_{i=1}^n L_{\lambda_i}$.

By definition, the corresponding space of *conformal blocks* is the space $C_{\mathfrak{g}}(L_{\lambda_1}, \dots, \lambda_n)$ of linear functionals

$$\varphi : \bigotimes_{i=1}^n L_{\lambda_i} \rightarrow \mathbb{C}$$

invariant under $\mathfrak{g}_{\text{out}}$, i.e., such that

$$(7.3) \quad \varphi(\eta \cdot v) = 0, \quad \forall v \in \bigotimes_{i=1}^n L_{\lambda_i}, \quad \eta \in \mathfrak{g} \otimes \mathbb{C}[X \setminus \{x_1, \dots, x_n\}].$$

Its dual space

$$(7.4) \quad H_{\mathfrak{g}}(L_{\lambda_1}, \dots, \lambda_n) = \bigotimes_{i=1}^n L_{\lambda_i} / \mathfrak{g}_{\text{out}} \cdot \bigotimes_{i=1}^n L_{\lambda_i}$$

is called the *space of coinvariants*.

The relevance of the space of conformal blocks to the WZW model is well-known. Pick states $\Phi_i = v_i \otimes \bar{v}_i \in L_{\lambda_i} \otimes \bar{L}_{\lambda_i} \subset \mathbf{H}$, $i = 1, \dots, n$. Let $\Phi_i(x_i)$ be the corresponding operators of the WZW model inserted at the points $x_i \in X$. The correlation function $\langle \Phi_1(x_1) \dots \Phi_n(x_n) \rangle$ satisfies the equations (7.3) with respect to the action of $\mathfrak{g}_{\text{out}}$ on the left factors; these are precisely the chiral *Ward identities*. It also satisfies the anti-chiral Ward identities with respect to the action of $\mathfrak{g}_{\text{out}}$ on the right factors. The same property holds for other conformal field theories with chiral and anti-chiral symmetries of $\widehat{\mathfrak{g}}$ level k .

Thus, we see that a possible strategy to find the correlation functions in the WZW model, or a more general model with Kac-Moody symmetry [9], is to consider the vector space of *all* functionals on $\mathbf{H}^{\otimes n}$ which satisfy the identities (7.3) and their anti-chiral analogues. If we further restrict ourselves to the insertion of operators corresponding to $L_{\lambda_i} \otimes \bar{L}_{\lambda_i}$ at the point x_i , then we find that this space is just the tensor product of $C_{\mathfrak{g}}(L_{\lambda_1}, \dots, \lambda_n)$ and its complex conjugate space.

A collection of states $\Phi_i \in L_{\lambda_i} \otimes \bar{L}_{\lambda_i}$ then determines a vector ϕ in the dual vector space, which is the tensor product of the space of coinvariants $H_{\mathfrak{g}}(L_{\lambda_1}, \dots, \lambda_n)$ and its complex conjugate space. The corresponding correlation function $\langle \Phi_1(x_1) \dots \Phi(x_n) \rangle$ may be expressed as the square $\|\phi\|^2$ of length of ϕ with respect to a particular hermitean inner product on $H_{\mathfrak{g}}(L_{\lambda_1}, \dots, \lambda_n)$. Once we determine this inner product on the space of coinvariants, we find all correlation functions. In a rational conformal field theory, such as the WZW model, the spaces of conformal blocks are finite-dimensional, and so this really looks like a good strategy.

7.2. Sheaves of conformal blocks as \mathcal{D} -modules on the moduli spaces of curves.

In the above definition of conformal blocks the curve X as well as the points x_1, \dots, x_n appear as parameters. The correlation functions of the model depend on these parameters. Hence we wish to consider the spaces of conformal blocks as these parameters vary along the appropriate moduli space $\mathfrak{M}_{g,n}$, the moduli space of n -pointed complex curves of genus g .⁵³ This way we obtain the holomorphic *vector bundles* of conformal blocks and coinvariants on $\mathfrak{M}_{g,n}$, which we denote by $\mathcal{C}_{\mathfrak{g}}(L_{\lambda_1}, \dots, L_{\lambda_n})$ and $\Delta_{\mathfrak{g}}(L_{\lambda_1}, \dots, L_{\lambda_n})$, respectively.

A collection of states $\Phi_i \in L_{\lambda_i} \otimes \bar{L}_{\lambda_i}$ now determines a holomorphic section $\phi(X, (x_i))$ of the vector bundle $\Delta_{\mathfrak{g}}(L_{\lambda_1}, \dots, L_{\lambda_n})$. The correlation function $\langle \Phi_1(x_1) \dots \Phi(x_n) \rangle$ with varying complex structure on X and varying points is the square $\|\phi(X, (x_i))\|^2$ of length of $\phi(X, (x_i))$ with respect to a “natural” hermitean inner product which is constructed in [82, 83] (see also [84]).⁵⁴ There is a unique unitary connection compatible with the holomorphic structure on $\Delta_{\mathfrak{g}}(L_{\lambda_1}, \dots, L_{\lambda_n})$ and this hermitean metric. This connection is projectively flat.⁵⁵ It follows from the construction that the correlation functions, considered as sections of the bundle $\mathcal{C}_{\mathfrak{g}}(L_{\lambda_1}, \dots, L_{\lambda_n}) \otimes \bar{\mathcal{C}}_{\mathfrak{g}}(L_{\lambda_1}, \dots, L_{\lambda_n})$, are horizontal with respect to the dual connection acting along the first factor (and its complex conjugate acting along the second factor).

For a more general rational conformal field theory, we also have a holomorphic bundle of conformal blocks on $\mathfrak{M}_{g,n}$ (for each choice of an n -tuple of representations of the corresponding chiral algebra, assuming that the theory is “diagonal”), and it is expected to carry a hermitean metric, such that the corresponding unitary connection is projectively flat. As was first shown by D. Friedan and S. Shenker [7], the holomorphic part of this projectively flat connection comes from the insertion in the correlation functions of the stress

⁵³and even more generally, its Deligne-Mumford compactification $\bar{\mathfrak{M}}_{g,n}$

⁵⁴for a given curve X , this inner product depends on the choice of a metric in the conformal class determined by the complex structure on X , and this is the source of the conformal anomaly of the correlation functions

⁵⁵i.e., its curvature is proportional to the identity operator on the vector bundle; this curvature is due to the conformal anomaly

tensor $T(z)$. Concretely, an infinitesimal deformation of the pointed curve $(X, (x_i))$ represented by a Beltrami differential μ , which is a $(-1, 1)$ -form on X with zeroes at the points of insertion. The variation of the (unnormalized) correlation function $\langle \Phi_1(x_1) \dots \Phi(x_n) \rangle$ under this deformation is given by the formula

$$(7.5) \quad \delta_\mu \langle \Phi_1(x_1) \dots \Phi(x_n) \rangle = \int_X \mu \langle T(z) \Phi_1(x_1) \dots \Phi(x_n) \rangle.$$

The way it is written, this formula seems to define a holomorphic connection on the bundle of conformal blocks and at the same time it states that the correlation functions are horizontal sections with respect to this connection. However, there is a small caveat here: the right hand side of this formula is not well-defined, because $T(z)$ transforms not as a quadratic differential, but as a projective connection (with the Schwarzian derivative term proportional to the central charge c of the model). Because of that, formula (7.5) only defines a *projectively* flat connection on the bundle of conformal blocks. The curvature of this connection is proportional to the curvature of the determinant line bundle on $\mathfrak{M}_{g,n}$, with the coefficient of proportionality being the central charge c . This is, of course, just the usual statement of conformal anomaly.

Another way to define this connection is to use the ‘‘Virasoro uniformization’’ of the moduli space $\mathfrak{M}_{g,n}$ (see [20], Sect. 17.3, and references therein). Namely, we identify the tangent space to a point $(X, (x_i))$ of $\mathfrak{M}_{g,n}$ with the quotient

$$T_{(X, (x_i))} \mathfrak{M}_{g,n} = \Gamma(X \setminus \{x_1, \dots, x_n\}, \Theta_X) \setminus \bigoplus_{i=1}^n \mathbb{C}((t_i)) \partial_{t_i} / \bigoplus_{i=1}^n \mathbb{C}[[t_i]] \partial_{t_i},$$

where Θ_X is the tangent sheaf of X . Let $\xi_i = f_i(t_i) \partial_{t_i} \in \mathbb{C}((t_i)) \partial_{t_i}$ be a vector field on the punctured disc near x_i , and μ_i be the corresponding element of $T_{(X, (x_i))} \mathfrak{M}_{g,n}$, viewed as an infinitesimal deformation of $(X, (x_i))$. Then the variation of the correlation function under this deformation is given by the formula

$$(7.6) \quad \delta_{\mu_i} \langle \Phi_1(x_1) \dots \Phi(x_n) \rangle = \left\langle \Phi_1(x_1) \dots \int f_i(t_i) T(t_i) dt_i \cdot \Phi_i(x_i) \dots \Phi(x_n) \right\rangle,$$

where the contour of integration is a small loop around the point x_i .

Here it is important to note that the invariance of the correlation function under $\mathfrak{g}_{\text{out}}^{\mathbb{P}}$ (see formula (7.3)) implies its invariance under $\Gamma(X \setminus \{x_1, \dots, x_n\}, \Theta_X)$, and so the above formula gives rise to a well-defined connection. This guarantees that the right hand side of formula (7.6) depends only on μ_i and not on ξ_i . Since $T(z)$ transforms as a projective connection on X , this connection is projectively flat (see [20], Ch. 17, for more details). This is the same connection as the one given by formula (7.5).

The projectively flat connection on the bundle of conformal blocks of the WZW theory has been constructed by various methods in [85, 86, 87, 88, 89].

For a general conformal field theory the notion of conformal blocks is spelled out in [20], Sect. 9.2. Consider the case of a rational conformal field theory. Then the chiral algebra A has finitely many isomorphism classes of irreducible modules (and the corresponding category is semi-simple). Given a collection M_1, \dots, M_n of irreducible modules over the chiral algebra, the corresponding space of conformal blocks $C_A(M_1, \dots, M_n)$ is defined as the space of linear functionals on the tensor product $\bigotimes_{i=1}^n M_i$ which are invariant under

the analogue of the Lie algebra $\mathfrak{g}_{\text{out}}$ corresponding to *all* chiral fields in the chiral algebra A (in the sense of [20]).⁵⁶ This invariance condition corresponds to the Ward identities of the theory.

If A is generated by some fields $J^a(z)$ (as is the case in the WZW model), then it is sufficient to impose the Ward identities corresponding to those fields only. That is why in the case of WZW model we defined the space of conformal blocks as the space of $\mathfrak{g}_{\text{out}}$ -invariant functionals. These functionals automatically satisfy the Ward identities with respect to all other fields from the chiral algebra. For example, they satisfy the Ward identities for the stress tensor $T(z)$ (given by the Segal-Sugawara formula (8.3)), which we have used above in verifying that the connection defined by formula (7.6) is well-defined.

In a rational conformal field theory the spaces $C_A(M_1, \dots, M_n)$ are expected to be finite-dimensional (see, e.g., [90]), and as we vary $(X, (x_i))$, they glue into a vector bundle $\mathcal{C}_A(M_1, \dots, M_n)$ on the moduli space $\mathfrak{M}_{g,n}$. It is equipped with a projectively flat connection defined as above (see [20] for more details). So the structure is very similar to that of the WZW models.

Let us summarize: the correlation functions in a rational conformal field theory are interpreted as the squares of holomorphic sections of a vector bundle (of coinvariants) on $\mathfrak{M}_{g,n}$, equipped with a projectively flat connection. The sheaf of sections of this bundle may be viewed as the simplest example of a twisted \mathcal{D} -module on $\mathfrak{M}_{g,n}$.⁵⁷

If our conformal field theory is not rational, we can still define the spaces of conformal blocks $C_A(M_1, \dots, M_n)$ and coinvariants $H_A(M_1, \dots, M_n)$, but they may not be finite-dimensional. In the general case it is better to work with the spaces of coinvariants $H_A(M_1, \dots, M_n)$, because the quotient of $\bigotimes_{i=1}^n M_i$ (see formula (7.4)), it has discrete topology even if it is infinite-dimensional, unlike its dual space of conformal blocks. These spaces form a sheaf of coinvariants on $\mathfrak{M}_{g,n}$, which has the structure of a twisted \mathcal{D} -module, even though in general it is not a vector bundle. This is explained in detail in [20].

Thus, the chiral sector of conformal field theory may be viewed as a *factory for producing twisted \mathcal{D} -modules on the moduli spaces of pointed curves*. These are the \mathcal{D} -modules that physicists are usually interested in.

But the point is that a very similar construction also gives us \mathcal{D} -modules on the moduli spaces of bundles Bun_G for conformal field theories with Kac-Moody symmetry corresponding to the group G .⁵⁸ So from this point of view, the chiral sector of conformal field theory with Kac-Moody symmetry is a *factory for producing twisted \mathcal{D} -modules on the moduli spaces of G -bundles*. Since our goal is to find some way to construct Hecke eigensheaves, which are \mathcal{D} -modules on Bun_G , it is natural to try to utilize the output of this factory.

7.3. Sheaves of conformal blocks on Bun_G . The construction of twisted \mathcal{D} -modules on Bun_G is completely analogous to the corresponding construction on $\mathfrak{M}_{g,n}$ outlined above. We now briefly recall it (see [9, 91, 93, 94, 95, 83, 20]).

⁵⁶the spaces $C_A(M_1, \dots, M_n)$ give rise to what is known as the modular functor of conformal field theory [8]

⁵⁷it is a twisted \mathcal{D} -module because the connection is not flat, but only projectively flat

⁵⁸and more generally, one can construct twisted \mathcal{D} -modules on the combined moduli spaces of curves and bundles

Consider first the case of WZW model. Suppose we are given a G -bundle \mathcal{P} on X . Let $\mathfrak{g}_{\mathcal{P}} = \mathcal{P} \times_G \mathfrak{g}$ be the associated vector bundle of Lie algebras on X . Define the Lie algebra

$$(7.7) \quad \mathfrak{g}_{\text{out}}^{\mathcal{P}} = \Gamma(X \setminus \{x_1, \dots, x_n\}, \mathfrak{g}_{\mathcal{P}}).$$

Choosing local trivializations of \mathcal{P} near the points x_i , we obtain an embedding of $\mathfrak{g}_{\text{out}}^{\mathcal{P}}$ into $\bigoplus_{i=1}^n \mathfrak{g} \otimes \mathbb{C}((t_i))$ which, by residue theorem, lifts to its diagonal central extension. Therefore we can define the space $C_{\mathfrak{g}}^{\mathcal{P}}(L_{\lambda_1}, \dots, \lambda_n)$ of \mathcal{P} -twisted conformal blocks as the space of $\mathfrak{g}_{\text{out}}^{\mathcal{P}}$ -invariant functionals on $\bigotimes_{i=1}^n L_{\lambda_i}$.

These spaces now depend on \mathcal{P} . As we vary the G -bundle \mathcal{P} , these spaces combine into a vector bundle over Bun_G . We define a projectively flat connection on it in the same way as above. The idea is the same as in the case of the moduli space of curves: instead of $T(z)$ we use the action of the currents $J^a(z)$ of the chiral algebra associated to $\widehat{\mathfrak{g}}$, corresponding to a basis $\{J^a\}$ of \mathfrak{g} . Insertion of these currents into the correlation function gives us the variation of the correlation function under infinitesimal deformations of our bundles [9, 91, 93].

To implement this idea, we have to realize deformations of the G -bundle in terms of our theory. This can be done in several ways. One way is to consider the gauged WZW model, as explained in [82, 83, 84]. Then we couple the theory to a $(0, 1)$ -connection $A_{\bar{z}} d\bar{z}$ on the trivial bundle⁵⁹ on X into the action and consider the correlation function as a holomorphic function of $A_{\bar{z}}$. The caveat is that it is not invariant under the gauge transformations, but rather defines a section of a line bundle on the quotient of the space of all $(0, 1)$ -connections by the (complex group G -valued) gauge transformations. This space is precisely the moduli space of holomorphic structures on our (topologically trivial) G -bundle, and hence it is just our moduli space Bun_G . From this point of view, the projectively flat connection on the bundle of conformal blocks comes from the formula for the variation of the correlation function of the gauged WZW model under the action of infinitesimal gauge transformations on the space of anti-holomorphic connections. This is explained in detail in [83, 84].

For us it will be more convenient to define this connection from a slightly different point of view. Just as the moduli space of curves is (infinitesimally) uniformized by the Virasoro algebra, the moduli space Bun_G of G -bundles on X is locally (or infinitesimally) uniformized by the affine Kac-Moody algebra. In fact, it is uniformized even globally by the corresponding Lie group, as we will see presently. Using this uniformization, we will write the connection operators as in formula (7.6), except that we will replace the stress tensor $T(z)$ by the currents $J^a(z)$ of the affine Lie algebra. This derivation will be more convenient for us because it also works for general conformal field theories with Kac-Moody symmetry, not only for the WZW models.

In what follows we will restrict ourselves to the simplest case when there is only one insertion point $x \in X$. The case of an arbitrary number of insertions may be analyzed similarly. We will follow closely the discussion of [20], Ch. 18.

To explain the Kac-Moody uniformization of Bun_G , we recall the Weil realization of the set of \mathbb{C} -points of Bun_n (i.e., isomorphism classes of rank n bundles on X) given in

⁵⁹since we assumed our group G to be connected and simply-connected, any G -bundle on X is topologically trivial; for other groups one has to include non-trivial bundles as well, see [95]

Lemma 2 of Sect. 3.2 as the double quotient $GL_n(F)\backslash GL_n(\mathbb{A})/GL_n(\mathcal{O})$. Likewise, for a general reductive group G the set of \mathbb{C} -points of Bun_G is realized as the double quotient $G(F)\backslash G(\mathbb{A})/G(\mathcal{O})$. The proof is the same as in Lemma 2: any G -bundle on X may be trivialized on the complement of finitely many points. It can also be trivialized on the formal discs around those points, and the corresponding transition functions give us an element of the adèlic group $G(\mathbb{A})$ defined up to the right action of $G(\mathcal{O})$ and left action of $G(F)$.

For a general reductive Lie group G and a general G -bundle \mathcal{P} the restriction of \mathcal{P} to the complement of a single point x in X may be non-trivial. But if G is a semi-simple Lie group, then it is trivial, according to a theorem of Harder. Hence we can trivialize \mathcal{P} on $X\setminus x$ and on the disc around x . Therefore our G -bundle \mathcal{P} may be represented by a single transition function on the punctured disc D_x around x . This transition function is an element of the loop group $G((t))$, where, as before, t is a local coordinate at x . If we change our trivialization on D_x , this function will get multiplied on the right by an element of $G[[t]]$, and if we change our trivialization on $X\setminus x$, it will get multiplied on the left by an element of $G_{\text{out}} = \{(X\setminus x) \rightarrow G\}$.

Thus, we find that the set of isomorphism classes of G -bundles on X is in bijection with the double quotient $G_{\text{out}}\backslash G((t))/G[[t]]$. This is a ‘‘one-point’’ version of the Weil type adèlic uniformization given in Lemma 2. Furthermore, it follows from the results of [96, 97] that this identification is not only an isomorphism of the sets of points, but we actually have an isomorphism of algebraic stacks

$$(7.8) \quad \text{Bun}_G \simeq G_{\text{out}}\backslash G((t))/G[[t]],$$

where G_{out} is the group of algebraic maps $X\setminus x \rightarrow G$.⁶⁰ This is what we mean by the global Kac-Moody uniformization of Bun_G .

The local (or infinitesimal) Kac-Moody uniformization of Bun_G is obtained from the global one. It is the statement that the tangent space $T_{\mathcal{P}}\text{Bun}_G$ to the point of Bun_G corresponding to a G -bundle \mathcal{P} is isomorphic to the double quotient $\mathfrak{g}_{\text{out}}^{\mathcal{P}}\backslash \mathfrak{g}((t))/\mathfrak{g}[[t]]$. Thus, any element $\eta(t) = J^a\eta_a(t)$ of the loop algebra $\mathfrak{g}((t))$ gives rise to a tangent vector ν in $T_{\mathcal{P}}\text{Bun}_G$. This is completely analogous to the Virasoro uniformization of the moduli spaces of curves considered above. The analogue of formula (7.6) for the variation of the one-point correlation function of our theory with respect to the infinitesimal deformation of the G -bundle \mathcal{P} corresponding to ν is then

$$(7.9) \quad \delta_{\nu} \langle \Phi(x) \rangle = \left\langle \int \eta_a(t) J^a(t) dt \cdot \Phi(x) \right\rangle,$$

where the contour of integration is a small loop around the point x . The formula is well-defined because of the Ward identity expressing the invariance of the correlation function under the action of the Lie algebra $\mathfrak{g}_{\text{out}}^{\mathcal{P}}$. This formula also has an obvious multi-point generalization.

⁶⁰for this one needs to show that this uniformization is true for any family of G -bundles on X , and this is proved in [96, 97]

Thus, we obtain a connection on the bundle of conformal blocks over Bun_G , or, more generally, the structure of a \mathcal{D} -module on the sheaf of conformal blocks,⁶¹ and the correlation functions of our model are sections of this sheaf that are horizontal with respect to this connection. The conformal anomaly that we observed in the analysis of the sheaves of conformal blocks on the moduli spaces of curves has an analogue here as well: it is expressed in the fact that the above formulas do not define a flat connection on the sheaf of conformal blocks, but only a projectively flat connection (unless the level of $\widehat{\mathfrak{g}}$ is 0). In other words, we obtain the structure of a twisted \mathcal{D} -module. The basic reason for this is that we consider the spaces of conformal blocks for *projective* representations of the loop algebra $\mathfrak{g}((t))$, i.e., representations of its central extension $\widehat{\mathfrak{g}}$ of non-zero level k , as we will see in the next section.

In the rest of this section we describe this above construction of the \mathcal{D} -modules on Bun_G in more detail from the point of view of the mathematical theory of “localization functors”.

7.4. Construction of twisted \mathcal{D} -modules. Let us consider a more general situation. Let $\mathfrak{k} \subset \mathfrak{g}$ be a pair consisting of a Lie algebra and its Lie subalgebra. Let K be the Lie group with the Lie algebra \mathfrak{k} . The pair (\mathfrak{g}, K) is called a *Harish-Chandra pair*.⁶² Let Z be a variety over \mathbb{C} . A (\mathfrak{g}, K) -action on Z is the data of an action of \mathfrak{g} on Z (that is, a homomorphism α from \mathfrak{g} to the tangent sheaf Θ_Z), together with an action of K on Z satisfying natural compatibility conditions. The homomorphism α gives rise to a homomorphism of \mathcal{O}_Z -modules

$$a : \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_Z \rightarrow \Theta_Z.$$

This map makes $\mathfrak{g} \otimes \mathcal{O}_Z$ into a *Lie algebroid* (see [75] and [20], Sect. A.3.2). The action is called *transitive* if the map a (the “anchor map”) is surjective. In this case Θ_Z may be realized as the quotient $\mathfrak{g} \otimes \mathcal{O}_Z / \text{Ker } a$.

For instance, let Z be the quotient $H \backslash G$, where G is a Lie group with the Lie algebra \mathfrak{g} and H is a subgroup of G . Then G acts transitively on $H \backslash G$ on the right, and hence we obtain a transitive (\mathfrak{g}, K) -action on $H \backslash G$. Now let V be a (\mathfrak{g}, K) -module, which means that it is a representation of the Lie algebra \mathfrak{g} and, moreover, the action of \mathfrak{k} may be exponentiated to an action of K . Then the Lie algebroid $\mathfrak{g} \otimes \mathcal{O}_{H \backslash G}$ acts on the sheaf $V \otimes_{\mathbb{C}} \mathcal{O}_{H \backslash G}$ of sections of the trivial vector bundle on $H \backslash G$ with the fiber V .

The sheaf $V \otimes_{\mathbb{C}} \mathcal{O}_{H \backslash G}$ is naturally an $\mathcal{O}_{H \backslash G}$ -module. Suppose we want to make $V \otimes_{\mathbb{C}} \mathcal{O}_{H \backslash G}$ into a $\mathcal{D}_{H \backslash G}$ -module. Then we need to learn how to act on it by $\Theta_{H \backslash G}$. But we know that $\Theta_{H \backslash G} = \mathfrak{g} \otimes \mathcal{O}_{H \backslash G} / \text{Ker } a$. Therefore $\Theta_{H \backslash G}$ acts naturally on the quotient

$$\widetilde{\Delta}(V) = (V \otimes_{\mathbb{C}} \mathcal{O}_{H \backslash G}) / \text{Ker } a \cdot (V \otimes_{\mathbb{C}} \mathcal{O}_{H \backslash G}).$$

Thus, $\widetilde{\Delta}(V)$ is a $\mathcal{D}_{H \backslash G}$ -module. The fiber of $\widetilde{\Delta}(V)$ (considered as a $\mathcal{O}_{H \backslash G}$ -module) at a point $p \in H \backslash G$ is the quotient $V / \text{Stab}_p \cdot V$, where Stab_p is the stabilizer of \mathfrak{g} at p .

⁶¹as in the case of the moduli of curves, it is often more convenient to work with the sheaf of coinvariants instead

⁶²note that we have already encountered a Harish-Chandra pair (\mathfrak{gl}_2, O_2) when discussing automorphic representations of $GL_2(\mathbb{A}_{\mathbb{Q}})$ in Sect. 1.6

Thus, we may think of $\tilde{\Delta}(V)$ as the *sheaf of coinvariants*: it glues together the spaces of coinvariants $V/\text{Stab}_p \cdot V$ for all $p \in H \backslash G$.

The $\mathcal{D}_{H \backslash G}$ -module $\tilde{\Delta}(V)$ is the sheaf of sections of a vector bundle with a flat connection if and only if the spaces of coinvariants have the same dimension for all $p \in H \backslash G$. But different points have different stabilizers, and so the dimensions of these spaces may be different for different points p . So $\tilde{\Delta}(V)$ can be a rather complicated \mathcal{D} -module in general.

By our assumption, the action of \mathfrak{k} on V can be exponentiated to an action of the Lie group K . This means that the \mathcal{D} -module $\tilde{\Delta}(V)$ is K -equivariant, in other words, it is the pull-back of a \mathcal{D} -module on the double quotient $H \backslash G / K$, which we denote by $\Delta(V)$. Thus, we have defined for any (\mathfrak{g}, K) -module V a \mathcal{D} -module of coinvariants $\Delta(V)$ on $H \backslash G / K$.

Now suppose that V is a projective representation of \mathfrak{g} , i.e., a representation of a central extension $\hat{\mathfrak{g}}$ of \mathfrak{g} :

$$(7.10) \quad 0 \rightarrow \mathbb{C}\mathbf{1} \rightarrow \hat{\mathfrak{g}} \rightarrow \mathfrak{g} \rightarrow 0$$

We will assume that it splits over \mathfrak{k} and \mathfrak{h} . Then $(\hat{\mathfrak{g}}, K)$ is also a Harish-Chandra pair which acts on $H \backslash G$ via the projection $\hat{\mathfrak{g}} \rightarrow \mathfrak{g}$. But since the central element $\mathbf{1}$ is mapped to the zero vector field on $H \backslash G$, we obtain that if $\mathbf{1}$ acts as a non-zero scalar on V , the corresponding \mathcal{D} -module $\Delta(V)$ is equal to zero.

It is clear that what we should do in this case is to replace G by its central extension corresponding to $\hat{\mathfrak{g}}$ and take into account the \mathbb{C}^\times -bundle $H \backslash \hat{G}$ over $H \backslash G$.

This can be phrased as follows. Consider the $\mathcal{O}_{H \backslash G}$ -extension

$$(7.11) \quad 0 \rightarrow \mathcal{O}_{H \backslash G} \cdot \mathbf{1} \rightarrow \hat{\mathfrak{g}} \otimes \mathcal{O}_{H \backslash G} \rightarrow \mathfrak{g} \otimes \mathcal{O}_{H \backslash G} \rightarrow 0$$

obtained by taking the tensor product of (7.10) with $\mathcal{O}_{H \backslash G}$. By our assumption, the central extension (7.10) splits over the Lie algebra \mathfrak{h} . Therefore (7.11) splits over the kernel of the anchor map $a : \mathfrak{g} \otimes \mathcal{O}_{H \backslash G} \rightarrow \Theta_{H \backslash G}$. Therefore we have a Lie algebra embedding $\text{Ker } a \hookrightarrow \hat{\mathfrak{g}} \otimes \mathcal{O}_{H \backslash G}$. The quotient \mathcal{T} of $\hat{\mathfrak{g}} \otimes \mathcal{O}_{H \backslash G}$ by $\text{Ker } a$ is now an extension

$$0 \rightarrow \mathcal{O}_{H \backslash G} \rightarrow \mathcal{T} \rightarrow \Theta_{H \backslash G} \rightarrow 0,$$

and it carries a natural Lie algebroid structure.

We now modify the above construction as follows: we take the coinvariants of $V \otimes \mathcal{O}_{H \backslash G}$ only with respect to $\text{Ker } a \hookrightarrow \hat{\mathfrak{g}} \otimes \mathcal{O}_{H \backslash G}$. Thus we define the sheaf

$$\tilde{\Delta}(V) = \mathcal{O}_{H \backslash G} \otimes V / \text{Ker } a \cdot (\mathcal{O}_{H \backslash G} \otimes V).$$

The sheaf $\tilde{\Delta}(V)$ is an $\mathcal{O}_{H \backslash G}$ -module whose fibers are the spaces of coinvariants as above. But it is no longer a $\mathcal{D}_{H \backslash G}$ -module, since it carries an action of the Lie algebroid \mathcal{T} , not of $\Theta_{H \backslash G}$. But suppose that the central element $\mathbf{1}$ acts on V as the identity. Then the quotient of the enveloping algebra $U(\mathcal{T})$ of \mathcal{T} by the relation identifying $1 \in \mathcal{O}_{H \backslash G} \subset \mathcal{T}$ with the unit element of $U(\mathcal{T})$ acts on $\tilde{\Delta}(V)$. This quotient, which we denote by $\mathcal{D}'_{H \backslash G}$ is a sheaf of *twisted differential operators* on $H \backslash G$. Furthermore, the Lie algebroid \mathcal{T} is identified with the subsheaf of differential operators of order less than or equal to 1 inside $\mathcal{D}'_{H \backslash G}$.

But what if $\mathbf{1}$ acts on V as $k \cdot \text{Id}$, where $k \in \mathbb{C}$? Then on $\tilde{\Delta}(V)$ we have an action of the quotient of the enveloping algebra $U(\mathcal{T})$ of \mathcal{T} by the relation identifying $1 \in \mathcal{O}_{H \backslash G} \subset \mathcal{T}$ with

k times the unit element of $U(\mathcal{T})$. We denote this quotient by $\widetilde{\mathcal{D}}'_k$. Suppose that the central extension (7.10) can be exponentiated to a central extension \widehat{G} of the corresponding Lie group G . Then we obtain a \mathbb{C}^\times -bundle $H\backslash\widehat{G}$ over $H\backslash G$. Let $\widetilde{\mathcal{L}}$ be the corresponding line bundle. For integer values of k the sheaf $\widetilde{\mathcal{D}}'_k$ may be identified with the sheaf of differential operators acting on $\widetilde{\mathcal{L}}^{\otimes k}$. However, $\widetilde{\mathcal{D}}'_k$ is also well-defined for an arbitrary complex value of k , whereas $\mathcal{L}^{\otimes k}$ is not.

Finally, suppose that the action of the Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$ on V (it acts on V because we have assumed the central extension (7.10) to be split over it) exponentiates to an action of the corresponding Lie group K . Then the $\widetilde{\mathcal{D}}'_k$ -module $\widetilde{\Delta}(V)$ is the pull-back of a sheaf $\Delta(V)$ on $H\backslash G/K$. This sheaf is a module over the sheaf \mathcal{D}'_k of twisted differential operators on $H\backslash G/K$ that we can define using $\widetilde{\mathcal{D}}'_k$ (for instance, for integer values of k , \mathcal{D}'_k is the sheaf of differential operators acting on $\mathcal{L}^{\otimes k}$, where \mathcal{L} is the line bundle on $H\backslash G/K$ which is the quotient of $\widetilde{\mathcal{L}}$ by K).

As the result of this construction we obtain a localization functor

$$\Delta : (\widehat{\mathfrak{g}}, K)\text{-mod}_k \longrightarrow \mathcal{D}'_k\text{-mod}$$

sending a $(\widehat{\mathfrak{g}}, K)$ -module V of level k to the sheaf of coinvariants $\Delta(V)$.⁶³

7.5. Twisted \mathcal{D} -modules on Bun_G . Let us now return to the subject of our interest: \mathcal{D} -modules on Bun_G obtained from conformal field theories with Kac-Moody symmetry. The point is that this is a special case of the above construction. Namely, we take the loop group $G((t))$ as G , G_{out} as H and $G[[t]]$ as K . Then the double quotient $H\backslash G/K$ is Bun_G according to the isomorphism (7.8).⁶⁴ In this case we find that the localization functor Δ sends a $(\widehat{\mathfrak{g}}, G[[t]])$ -module V to a \mathcal{D}'_k -module $\Delta(V)$ on Bun_G .

The twisted \mathcal{D} -module $\Delta(V)$ is precisely the sheaf of coinvariants arising from conformal field theory! Indeed, in this case the stabilizer subalgebra $\text{Stab}_{\mathcal{P}}$, corresponding to a G -bundle \mathcal{P} on X , is just the Lie algebra $\mathfrak{g}_{\text{out}}^{\mathcal{P}}$ defined by formula (7.7). Therefore the fiber of $\Delta(V)$ is the space of coinvariants $V/\mathfrak{g}_{\text{out}}^{\mathcal{P}} \cdot V$, i.e., the dual space to the space of conformal blocks on V .⁶⁵ Moreover, it is easy to see that the action of the Lie algebroid \mathcal{T} is exactly the same as the one described in Sect. 7.3 (see formula (7.9)).

The idea that the sheaves of coinvariants arising in conformal field theory may be obtained via a localization functor goes back to [98, 99].

For integer values of k the sheaf \mathcal{D}'_k is the sheaf of differential operators on a line bundle over Bun_G that is constructed in the following way. Note that the quotient $G((t))/G[[t]]$ appearing in formula (7.8) is the affine Grassmannian Gr that we discussed in Sect. 5.4. The loop group $G((t))$ has a universal central extension, the affine Kac-Moody group \widehat{G} . It contains $G[[t]]$ as a subgroup, and the quotient $\widehat{G}/G[[t]]$ is a \mathbb{C}^\times -bundle on the

⁶³the reason for the terminology “localization functor” is explained in [20], Sect. 17.2.7

⁶⁴ Bun_G is not an algebraic variety, but an algebraic stack, but it was shown in [15], Sect. 1, that the localization functor can be applied in this case as well

⁶⁵Strictly speaking, this quotient is the true space of coinvariants of our conformal field theory only if the chiral algebra of our conformal field theory is generated by the affine Kac-Moody algebra, as in the case of WZW model. In general, we need to modify this construction and also take the quotient by the additional Ward identities corresponding to other fields in the chiral algebra (see [20], Ch. 17, for details).

Grassmannian Gr. Let $\tilde{\mathcal{L}}$ be the corresponding line bundle on Gr. The group \widehat{G} acts on $\tilde{\mathcal{L}}$, and in particular any subgroup of $\widehat{\mathfrak{g}}((t))$ on which the central extension is trivial also acts on $\tilde{\mathcal{L}}$. The subgroup G_{out} is such a subgroup, hence it acts on $\tilde{\mathcal{L}}$. Taking the quotient of $\tilde{\mathcal{L}}$ by G_{out} , we obtain a line bundle \mathcal{L} on Bun_G (see (7.8)). This is the non-abelian version of the *theta line bundle*, the generator of the Picard group of Bun_G .⁶⁶ Then \mathcal{D}'_k be the sheaf of differential operators acting on $\mathcal{L}^{\otimes k}$. The above general construction gives us a description of the sheaf \mathcal{D}'_k in terms of the local Kac-Moody uniformization of Bun_G .

Again, we note that while $\mathcal{L}^{\otimes k}$ exists as a line bundle only for integer values of k , the sheaf \mathcal{D}'_k is well-defined for an arbitrary complex k .

Up to now we have considered the case of one insertion point. It is easy to generalize this construction to the case of multiple insertion points. We then obtain a functor assigning to n -tuples of highest weight $\widehat{\mathfrak{g}}$ -modules (inserted at the points x_1, \dots, x_n of a curve X) to the moduli space of G -bundles on X with parabolic structures at the points x_1, \dots, x_n (see [20], Sect. 18.1.3).⁶⁷

Thus, we see that the conformal field theory “factory” producing \mathcal{D} -modules on Bun_G is neatly expressed by the mathematical formalism of “localization functors” from representations of $\widehat{\mathfrak{g}}$ to \mathcal{D} -modules on Bun_G .

7.6. Example: the WZW \mathcal{D} -module. Let us see what the \mathcal{D} -modules of coinvariants look like in the most familiar case of the WZW model corresponding to a compact group U and a positive integer level k (we will be under the assumptions of Sect. 7.1). Let $L_{0,k}$ be the vacuum irreducible integrable representation of $\widehat{\mathfrak{g}}$ of level k (it has highest weight 0). Then the corresponding sheaf of coinvariants is just the \mathcal{D}'_k -module $\Delta(L_{0,k})$. Because $L_{0,k}$ is an integrable module, so not only the action of the Lie subalgebra $\mathfrak{g}[[t]]$ exponentiates, but the action of the entire Lie algebra $\widehat{\mathfrak{g}}$ exponentiates to an action of the corresponding group \widehat{G} , the spaces of coinvariants $L_{0,k}/\mathfrak{g}_{\text{out}}^{\mathcal{P}}$ are isomorphic to each other for different bundles \mathcal{P} . Hence $\Delta(L_{0,k})$ is a vector bundle with a projectively flat connection in this case. We will consider the dual bundle of conformal blocks $\mathcal{C}_{\mathfrak{g}}(L_{0,k})$.

The fiber $C_{\mathfrak{g}}(L_{0,k})$ of this bundle at the trivial G -bundle is just the space of $\mathfrak{g}_{\text{out}}$ -invariant functionals on $L_{0,k}$. One can show that it coincides with the space of G_{out} -invariant functionals on $L_{0,k}$. By an analogue of the Borel-Weil-Bott theorem, the dual space to the vacuum representation $L_{0,k}$ is realized as the space of sections of a line bundle $\tilde{\mathcal{L}}^{\otimes k}$ on the quotient LU/U , which is nothing but the affine Grassmannian $\text{Gr} = G((t))/G[[t]]$ discussed above, where G is the complexification of U . Therefore the space of conformal blocks $C_{\mathfrak{g}}(L_{0,k})$ is the space of global sections of the corresponding line bundle $\mathcal{L}^{\otimes k}$ on Bun_G , realized as the quotient (7.8) of Gr. We obtain that the space of conformal blocks corresponding to the vacuum representation is realized as the space $\Gamma(\text{Bun}_G, \mathcal{L}^{\otimes k})$ of global sections of $\mathcal{L}^{\otimes k}$ over Bun_G .

⁶⁶various integral powers of \mathcal{L} may be constructed as determinant line bundles corresponding to representations of G , see [20], Sect. 18.1.2 and references therein for more details

⁶⁷The reason for the appearance of parabolic structures (i.e., reductions of the fibers of the G -bundle at the marked points to a Borel subgroup B of G) is that a general highest weight module is not a $(\widehat{\mathfrak{g}}, G[[t]])$ -module, but a $(\widehat{\mathfrak{g}}, I)$ -module, where I is the Iwahori subgroup of $G((t))$, the preimage of B in $G[[t]]$ under the homomorphism $G[[t]] \rightarrow G$. For more on this, see Sect. 9.7.

It is not hard to derive from this fact that the bundle $\mathcal{C}_{\mathfrak{g}}(L_{0,k})$ of conformal blocks over Bun_G is just the tensor product of the vector space $\Gamma(\text{Bun}_G, \mathcal{L}^{\otimes k})$ and the line bundle $\mathcal{L}^{\otimes(-k)}$. Thus, the dual bundle $\Delta(L_{0,k})$ of coinvariants is $\Gamma(\text{Bun}_G, \mathcal{L}^{\otimes k})^* \otimes \mathcal{L}^{\otimes k}$. It has a canonical section ϕ whose values are the projections of the vacuum vector in $L_{0,k}$ onto the spaces of coinvariants. This is the chiral partition function of the WZW model. The partition function is the square of length of this section $\|\phi\|^2$ with respect to a hermitean inner product on $\Delta(L_{0,k})$.

Since the bundle $\Delta(L_{0,k})$ of coinvariants in the WZW model is the tensor product $\mathcal{L}^{\otimes k} \otimes V$, where \mathcal{L} is the determinant line bundle on Bun_G and V is a vector space, we find that the dependence of $\Delta(L_{0,k})$ on the Bun_G moduli is only through the determinant line bundle $\mathcal{L}^{\otimes k}$. However, despite this decoupling, it is still very useful to take into account the dependence of the correlation functions in the WZW model on the moduli of bundles. More precisely, we should combine the above two constructions and consider the sheaf of coinvariants on the *combined* moduli space of curves and bundles. Then the variation along the moduli of curves is given in terms of the Segal-Sugawara stress tensor, which is quadratic in the Kac-Moody currents. Therefore we find that the correlation functions satisfy a non-abelian version of the heat equation. These are the Knizhnik-Zamolodchikov-Bernard equations [9, 93].⁶⁸ In addition, the bundle of conformal blocks over Bun_G may be used to define the hermitean inner product on the space of conformal blocks, see [83].

However, it would be misleading to think that $\mathcal{L}^{\otimes k} \otimes V$ is the only possible twisted \mathcal{D} -module that can arise from the data of a conformal field theory with Kac-Moody symmetry. There are more complicated examples of such \mathcal{D} -modules which arise from other (perhaps, more esoteric) conformal field theories, some of which we will consider in the next section. We believe that this is an important point that up to now has not been fully appreciated in the physics literature.

It is instructive to illustrate this by an analogy with the Borel-Weil-Bott theorem. This theorem says that an irreducible finite-dimensional representation of highest weight λ of a compact group U may be realized as the space of global holomorphic sections of a holomorphic line bundle $\mathcal{O}(\lambda)$ on the flag variety U/T , where T is the maximal torus of U . Any representation of U is a direct sum of such irreducible representations, so based on that, one may conclude that the only interesting twisted \mathcal{D} -modules on U/T are the sheaves of sections of the line bundles $\mathcal{O}(\lambda)$. But in fact, the space of global sections of *any* twisted \mathcal{D} -module on the flag variety has a natural structure of a representation of the corresponding (complexified) Lie algebra \mathfrak{g} . Moreover, according to a theorem of A. Beilinson and J. Bernstein, the category of $\mathcal{D}_{\mathcal{O}(\lambda)}$ -modules corresponding to a non-degenerate weight λ is equivalent to the category of \mathfrak{g} -modules with a fixed central character determined by λ . So if one is interested in representations of the Lie algebra \mathfrak{g} , then there are a lot more interesting \mathcal{D} -modules to go around. For example, the Verma modules, with respect to a particular Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ come from the \mathcal{D} -modules of “delta-functions” supported at the point of the flag variety stabilized by \mathfrak{b} .

Likewise, we have a Borel-Weil-Bott type theorem for the loop group LU of U : all irreducible representations of the central extension of LU of positive energy may be realized

⁶⁸for an interpretation of these equations in the framework of the above construction of twisted \mathcal{D} -modules see [74]

as the duals of the spaces of global holomorphic sections of line bundles on the quotient LU/T , which is the affine analogue of U/T . This quotient is isomorphic to the quotient $G((t))/I$, where I is the Iwahori subgroup. The vacuum irreducible representation of a given level k is realized as the dual space to the space of sections of a line bundle $\tilde{\mathcal{L}}^{\otimes k}$ on the smaller quotient $\text{Gr} = LU/U$. This is the reason why the space of conformal blocks in the corresponding WZW theory (with one insertion) is the space of global sections of a line bundle on Bun_G , as we saw above.

But again, just as in the finite-dimensional case, it would be misleading to think that these line bundles on the affine Grassmannian and on Bun_G tell us the whole story about twisted \mathcal{D} -modules in this context. Indeed, the infinitesimal symmetries of our conformal field theories are generated by the corresponding Lie algebra, that is the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ (just as the Virasoro algebra generates the infinitesimal conformal transformations). The sheaves of coinvariants corresponding to representations of $\widehat{\mathfrak{g}}$ that are not necessarily integrable to the corresponding group \widehat{G} (but only integrable to its subgroup $G[[t]]$) give rise to more sophisticated \mathcal{D} -modules on Bun_G , and this is one of the main points we wish to underscore in this survey. In the next section we will see that this way we can actually construct the sought-after Hecke eigensheaves.

8. CONFORMAL FIELD THEORY AT THE CRITICAL LEVEL

In this section we apply the construction of the sheaves of coinvariants from conformal field theory to a particular class of representations of the affine Kac-Moody algebra of *critical level*. The critical level is $k = -h^\vee$, where h^\vee is the *dual Coxeter number* of \mathfrak{g} (see [100]). Thus, we may think about these sheaves as encoding a chiral conformal field theory with Kac-Moody symmetry of critical level. This conformal field theory is peculiar because it lacks the stress tensor (the Segal-Sugawara current becomes commutative at $k = -h^\vee$). As bizarre as this may sound, this cannot prevent us from constructing the corresponding sheaves of coinvariants on Bun_G . Indeed, as we explained in the previous section, all we need to construct them is an action of $\widehat{\mathfrak{g}}$. The stress tensor (and the action of the Virasoro algebra it generates) is needed in order to construct sheaves of coinvariants on the moduli spaces of punctured curves (or on the combined moduli of curves and bundles), and this we will not be able to do. But the Hecke eigensheaves that we wish to construct in the geometric Langlands correspondence are supposed to live on Bun_G , so this will be sufficient for our purposes.⁶⁹

Before explaining all of this, we wish to indicate a simple reason why one should expect Hecke eigensheaves to have something to do with the critical level. The Hecke eigensheaves that we will construct in this section, following Beilinson and Drinfeld, will be of the type discussed in Sect. 3.4: they will correspond to systems of differential equations on Bun_G obtained from a large algebra of global commuting differential operators on it. However, one can show that there are no global commuting differential operators on Bun_G , except for the constant functions. Hence we look at twisted global differential operators acting on the line bundle $\mathcal{L}^{\otimes k}$ introduced in the previous section. Suppose we find that for some value of k there is a large commutative algebra of differential operators acting on $\mathcal{L}^{\otimes k}$. Then the adjoint differential operators will be acting on the Serre dual line bundle

⁶⁹affine algebras at the critical level have also been considered recently by physicists, see [101, 102]

$K \otimes \mathcal{L}^{\otimes(-k)}$, where K is the canonical line bundle. It is natural to guess that k should be such that the two line bundles are actually isomorphic to each other. But one can show that $K \simeq \mathcal{L}^{\otimes-2h^\vee}$. Therefore we find that if such global differential operators were to exist, they would most likely be found for $k = -h^\vee$, when $\mathcal{L}^{\otimes k} \simeq K^{1/2}$. This is indeed the case. In fact, these global commuting differential operators come from the Segal-Sugawara current and its higher order generalizations which at level $-h^\vee$ become commutative, and moreover central, in the chiral algebra generated by $\widehat{\mathfrak{g}}$, as we shall see presently.

8.1. The chiral algebra. We start with the description of the chiral vertex algebra associated to $\widehat{\mathfrak{g}}$ at the level $-h^\vee$. We recall that a representation of $\widehat{\mathfrak{g}}$ defined as the extension (7.1) with the commutation relations (7.2), where κ_0 is the standard normalized invariant inner product on \mathfrak{g} , is called a representation of level k if the central element $\mathbf{1}$ acts as k times the identity. Representation of $\widehat{\mathfrak{g}}$ of the critical level $-h^\vee$ may be described as representations of $\widehat{\mathfrak{g}}$ with the relations (7.2), where κ_0 is replaced by the critical inner product $-\frac{1}{2}\kappa_{\text{Kil}}$, such that $\mathbf{1}$ acts as the identity. Here $\kappa_{\text{Kil}}(A, B) = \text{Tr}_{\mathfrak{g}}(\text{ad } A \text{ ad } B)$ is the Killing form.

In conformal field theory we have state-field correspondence. So we may think of elements of chiral algebras in two different ways: as the space of states and the space of fields. In what follows we will freely switch between these two pictures.

Viewed as the space of states, the chiral algebra at level $k \in \mathbb{C}$ is just the vacuum Verma module

$$V_k(\mathfrak{g}) = \text{Ind}_{\widehat{\mathfrak{g}}[[t]] \oplus \mathbb{C}K}^{\widehat{\mathfrak{g}}} \mathbb{C}_k = U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{g}}[[t]] \oplus \mathbb{C}\mathbf{1})} \mathbb{C}_k,$$

where \mathbb{C}_k is the one-dimensional representation of $\widehat{\mathfrak{g}}[[t]] \oplus \mathbb{C}\mathbf{1}$ on which $\widehat{\mathfrak{g}}[[t]]$ acts by 0 and $\mathbf{1}$ acts as multiplication by k . As a vector space,

$$V_k(\mathfrak{g}) \simeq U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]).$$

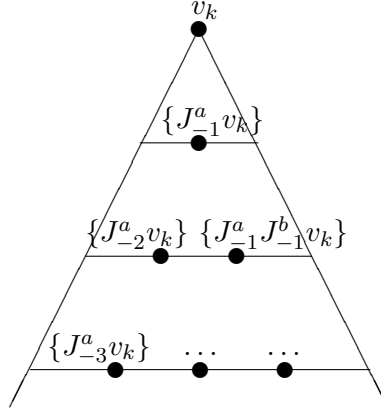
Let $\{J^a\}_{a=1, \dots, \dim \mathfrak{g}}$ be a basis of \mathfrak{g} . For any $A \in \mathfrak{g}$ and $n \in \mathbb{Z}$, we denote $A_n = A \otimes t^n \in L\mathfrak{g}$. Then the elements $J_n^a, n \in \mathbb{Z}$, and $\mathbf{1}$ form a (topological) basis for $\widehat{\mathfrak{g}}$. The commutation relations read

$$(8.1) \quad [J_n^a, J_m^b] = [J^a, J^b]_{n+m} + n(J^a, J^b)\delta_{n,-m}\mathbf{1}.$$

Denote by v_k the vacuum vector in $V_k(\mathfrak{g})$, the image of $1 \otimes 1 \in U\widehat{\mathfrak{g}} \otimes \mathbb{C}_k$ in V_k . We define a \mathbb{Z} -grading on $\widehat{\mathfrak{g}}$ and on $V_k(\mathfrak{g})$ by the formula $\deg J_n^a = -n, \deg v_k = 0$. By the Poincaré-Birkhoff-Witt theorem, $V_k(\mathfrak{g})$ has a basis of lexicographically ordered monomials of the form

$$J_{n_1}^{a_1} \dots J_{n_m}^{a_m} v_k,$$

where $n_1 \leq n_2 \leq \dots \leq n_m < 0$, and if $n_i = n_{i+1}$, then $a_i \leq a_{i+1}$. Here is the picture of the first few ‘‘layers’’ (i.e., homogeneous components) of $V_k(\mathfrak{g})$:



The state-field correspondence is given by the following assignment of fields to vectors in $V_k(\mathfrak{g})$:

$$v_k \mapsto \text{Id},$$

$$J_{-1}^a v_k \mapsto J^a(z) = \sum_{n \in \mathbb{Z}} J_n^a z^{-n-1},$$

$$J_{n_1}^{a_1} \dots J_{n_m}^{a_m} v_k \mapsto \frac{1}{(-n_1 - 1)! \dots (-n_m - 1)!} : \partial_z^{-n_1 - 1} J^{a_1}(z) \dots \partial_z^{-n_m - 1} J^{a_m}(z) :$$

(the normal ordering is understood as nested from right to left).

In addition, we have the translation operator ∂ on $V_k(\mathfrak{g})$, defined by the formulas $\partial v_k = 0$, $[\partial, J_n^a] = -n J_{n-1}^a$. It is defined so that the field $(\partial A)(z)$ is $\partial_z A(z)$. These data combine into what mathematicians call the structure of a (chiral) vertex algebra. In particular, the space of fields is closed under the operator product expansion (OPE), see [20] for more details.

Let $\{J_a\}$ be the basis of \mathfrak{g} dual to $\{J^a\}$ with respect to the inner product κ_0 . Consider the following vector in $V_k(\mathfrak{g})$:

$$(8.2) \quad S = \frac{1}{2} J_{a,-1} J_{-1}^a v_k$$

(summation over repeating indices is understood). The corresponding field is the *Segal-Sugawara current*

$$(8.3) \quad S(z) = \frac{1}{2} : J_a(z) J^a(z) : = \sum_{n \in \mathbb{Z}} S_n z^{-n-2}.$$

We have the following OPEs:

$$S(z) J^a(w) = (k + h^\vee) \frac{J^a(w)}{z - w} + \text{reg.},$$

$$S(z) S(w) = (k + h^\vee) \left(\frac{k \dim \mathfrak{g} / 2}{(z - w)^4} + \frac{2S(w)}{(z - w)^2} + \frac{\partial_w S(w)}{z - w} \right) + \text{reg.},$$

which imply the following commutation relations:

$$\begin{aligned} [S_n, J_m^a] &= -(k + h^\vee)mJ_{n+m}^a, \\ [S_n, S_m] &= (k + h^\vee) \left((n - m)S_{n+m} + \frac{1}{12}k \dim \mathfrak{g} \delta_{n,-m} \right). \end{aligned}$$

Thus, if $k \neq -h^\vee$, the second set of relations shows that the rescaled operators $L_n = (k + h)^{-1}S_n$ generate the Virasoro algebra with central charge $c_k = k \dim \mathfrak{g}/(k + h)$. The commutation relations

$$(8.4) \quad [L_n, J_m^a] = -mJ_{n+m}^a$$

show that the action of this Virasoro algebra on $\widehat{\mathfrak{g}}$ coincides with the natural action of infinitesimal diffeomorphisms of the punctured disc.

But if $k = h^\vee$, then the operators S_n commute with $\widehat{\mathfrak{g}}$ and therefore belong to the center of the completed enveloping algebra of $\widehat{\mathfrak{g}}$ at $k = -h^\vee$. In fact, one can easily show that the chiral algebra at this level does not contain any elements which generate an action of the Virasoro algebra and have commutation relations (8.4) with $\widehat{\mathfrak{g}}$. In other words, the Lie algebra of infinitesimal diffeomorphisms of the punctured disc acting on $\widehat{\mathfrak{g}}$ cannot be realized as an ‘‘internal symmetry’’ of the chiral algebra $V_{-h^\vee}(\mathfrak{g})$. This is the reason why the level $k = -h^\vee$ is called the critical level.⁷⁰

8.2. The center of the chiral algebra. It is natural to ask what is the center of the completed enveloping algebra of $\widehat{\mathfrak{g}}$ at level k . This may be reformulated as the question of finding the fields in the chiral algebra $V_k(\mathfrak{g})$ which have *regular* OPEs with the currents $J^a(z)$. If this is the case, then the Fourier coefficients of these fields commute with $\widehat{\mathfrak{g}}$ and hence lie in the center of the enveloping algebra. Such fields are in one-to-one correspondence with the vectors in $V_k(\mathfrak{g})$ which are annihilated by the Lie subalgebra $\mathfrak{g}[[t]]$. We denote the subspace of $\mathfrak{g}[[t]]$ -invariants in $V_k(\mathfrak{g})$ by $\mathfrak{z}_k(\mathfrak{g})$. This is a commutative chiral subalgebra of $V_k(\mathfrak{g})$, and hence it forms an ordinary commutative algebra. According to the above formulas, $S \in \mathfrak{z}_{-h^\vee}(\mathfrak{g})$. Since the translation operator T commutes with $\mathfrak{g}[[t]]$, we find that $\partial^m S = m!S_{-m-2}v_k$, $m \geq 0$ is also in $\mathfrak{z}_{-h^\vee}(\mathfrak{g})$. Therefore the commutative algebra $\mathbb{C}[\partial^m S]_{m \geq 0} = \mathbb{C}[S_n]_{n \leq -2}$ is a commutative chiral subalgebra of $\mathfrak{z}(\mathfrak{g})$.

Consider first the case when $\mathfrak{g} = \mathfrak{sl}_2$. In this case the critical level is $k = -2$.

Theorem 7. (1) $\mathfrak{z}_k(\mathfrak{sl}_2) = \mathbb{C}v_k$, if $k \neq -2$.

(2) $\mathfrak{z}_{-2}(\mathfrak{sl}_2) = \mathbb{C}[S_n]_{n \leq -2}$.

Thus, the center of $V_{-2}(\mathfrak{sl}_2)$ is generated by the Segal-Sugawara current $S(z)$ and its derivatives. In order to get a better understanding of the structure of the center, we need to understand how $S(z)$ transforms under coordinate changes. For $k \neq -2$, the stress tensor $T(z) = (k + 2)^{-1}S(z)$ transforms in the usual way under the coordinate change $w = \varphi(z)$:

$$T(w) \mapsto T(\varphi(z))\varphi'(z)^2 - \frac{c_k}{12}\{\varphi, z\},$$

⁷⁰This terminology is somewhat unfortunate because of the allusion to the ‘‘critical central charge’’ $c = 26$ in string theory. In fact, the analogue of the critical central charge for $\widehat{\mathfrak{g}}$ is level $-2h^\vee$, because, as we noted above, it corresponds to the canonical line bundle on Bun_G , whereas the critical level $-h^\vee$ corresponds to the square root of the canonical line bundle.

where

$$\{\varphi, z\} = \frac{\varphi'''}{\varphi'} - \frac{3}{2} \left(\frac{\varphi''}{\varphi'} \right)^2$$

is the Schwarzian derivative and $c_k = 3k/(k+2)$ is the central charge (see, e.g., [20], Sect. 8.2, for a derivation). This gives us the following transformation formula for $S(z)$ at $k = -2$:

$$S(w) \mapsto S(\varphi(z))\varphi'(z)^2 - \frac{1}{2}\{\varphi, z\}.$$

It coincides with the transformation formula for self-adjoint differential operators $\partial_z^2 - v(z)$ acting from $\Omega^{-1/2}$ to $\Omega^{3/2}$, where Ω is the canonical line bundle. Such operators are called *projective connections*.⁷¹

Thus, we find that while $S(z)$ has no intrinsic meaning, the second order operator $\partial_z^2 - S(z)$ acting from $\Omega^{-1/2}$ to $\Omega^{3/2}$ has intrinsic coordinate-independent meaning. Therefore the isomorphism of Theorem 7,(2) may be rephrased in a coordinate-independent fashion by saying that

$$(8.5) \quad \mathfrak{z}_{-2}(\mathfrak{sl}_2) \simeq \text{Fun Proj}(D),$$

where $\text{Fun Proj}(D)$ is the algebra of polynomial functions on the space $\text{Proj}(D)$ of projective connections on the (formal) disc D . If we choose a coordinate z on the disc, then we may identify $\text{Proj}(D)$ with the space of operators $\partial_z^2 - v(z)$, where $v(z) = \sum_{n \leq -2} v_n z^{-n-2}$, and $\text{Fun Proj}(D)$ with $\mathbb{C}[v_n]_{n \leq -2}$. Then the isomorphism (8.5) sends $S_n \in \mathfrak{z}_{-2}(\mathfrak{sl}_2)$ to $v_n \in \text{Fun Proj}(D)$. But the important fact is that in the formulation (8.5) the isomorphism is coordinate-independent: if we choose a different coordinate w on D , then the generators of the two algebras will transform in the same way, and the isomorphism will stay the same.

We now look for a similar coordinate-independent realization of the center $\mathfrak{z}_{-h^\vee}(\mathfrak{g})$ of $V_{-h^\vee}(\mathfrak{g})$ for a general simple Lie algebra \mathfrak{g} .

It is instructive to look first at the center of the universal enveloping algebra $U(\mathfrak{g})$. It is a free polynomial algebra with generators P_i of degrees $d_i + 1$, $i = 1, \dots, \ell = \text{rank } \mathfrak{g}$, where d_1, \dots, d_ℓ are called the exponents of \mathfrak{g} . In particular, $P_1 = \frac{1}{2} J_a J^a$. It is natural to try to imitate formula (8.3) for $S(z)$ by taking other generators P_i , $i > 1$, and replacing each J^a by $J^a(z)$. Unfortunately, the normal ordering that is necessary to regularize these fields distorts the commutation relation between them. We already see that for $S(z)$ where h^\vee appears due to double contractions in the OPE. Thus, $S(z)$ becomes central not for $k = 0$, as one might expect, but for $k = -h^\vee$. For higher order fields the distortion is more severe, and because of that explicit formulas for higher order Segal-Sugawara currents are unknown in general.

However, if we consider the symbols instead, then normal ordering is not needed, and we indeed produce commuting ‘‘currents’’ $\bar{S}_i(z) = P_i(\bar{J}^a(z))$ in the Poisson version of the chiral algebra $V_k(\mathfrak{g})$ generated by the quasi-classical ‘‘fields’’ $\bar{J}^a(z)$. We then ask whether each $\bar{S}_i(z)$ can be quantized to give a field $S_i(z) \in V_{-h^\vee}(\mathfrak{g})$ which belongs to the center.

⁷¹in order to define them, one needs to choose the square root of Ω , but the resulting space of projective connections is independent of this choice

The following generalization of Theorem 7 was obtained by B. Feigin and the author [11, 12] and gives the affirmative answer to this question.

Theorem 8. (1) $\mathfrak{z}_k(\mathfrak{g}) = \mathbb{C}v_k$, if $k \neq -h^\vee$.

(2) There exist elements $S_1, \dots, S_\ell \in \mathfrak{z}(\mathfrak{g})$, such that $\deg S_i = d_i + 1$, and $\mathfrak{z}(\mathfrak{g}) \simeq \mathbb{C}[\partial^n S_i]_{i=1, \dots, \ell; n \geq 0}$. In particular, S_1 is the Segal-Sugawara element (8.2).

As in the \mathfrak{sl}_2 case, we would like to give an intrinsic coordinate-independent interpretation of the isomorphism in part (2). It turns out that projective connections have analogues for arbitrary simple Lie algebras, called *opers*, and $\mathfrak{z}(\mathfrak{g})$ is isomorphic to the space of opers on the disc, associated to the Langlands dual Lie algebra ${}^L\mathfrak{g}$. It is this appearance of the Langlands dual Lie algebra that will ultimately allow us to make contact with the geometric Langlands correspondence.

8.3. Opers. But first we need to explain what opers are. In the case of \mathfrak{sl}_2 these are projective connections, i.e., second order operators of the form $\partial_t^2 - v(t)$ acting from $\Omega^{-1/2}$ to $\Omega^{3/2}$. This has an obvious generalization to the case of \mathfrak{sl}_n . An \mathfrak{sl}_n -oper on X is an n th order differential operator acting from $\Omega^{-(n-1)/2}$ to $\Omega^{(n+1)/2}$ whose principal symbol is equal to 1 and subprincipal symbol is equal to 0.⁷² If we choose a coordinate z , we write this operator as

$$(8.6) \quad \partial_t^n - u_1(t)\partial_t^{n-2} + \dots + u_{n-2}(t)\partial_t - (-1)^n u_{n-1}(t).$$

Such operators are familiar from the theory of n -KdV equations. In order to define similar soliton equations for other Lie algebras, V. Drinfeld and V. Sokolov [13] have introduced the analogues of operators (8.6) for a general simple Lie algebra \mathfrak{g} . Their idea was to replace the operator (8.6) by the first order matrix differential operator

$$(8.7) \quad \partial_t + \begin{pmatrix} 0 & u_1 & u_2 & \cdots & u_{n-1} \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Now consider the space of more general operators of the form

$$(8.8) \quad \partial_t + \begin{pmatrix} * & * & * & \cdots & * \\ + & * & * & \cdots & * \\ 0 & + & * & \cdots & * \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & + & * \end{pmatrix}$$

where $*$ indicates an arbitrary function and $+$ indicates a nowhere vanishing function. The group of upper triangular matrices acts on this space by gauge transformations

$$\partial_t + A(t) \mapsto \partial_t + gA(t)g^{-1} - \partial_t g(t) \cdot g(t)^{-1}.$$

⁷²note that for these conditions to be coordinate-independent, this operator must act from $\Omega^{-(n-1)/2}$ to $\Omega^{(n+1)/2}$

It is not difficult to show that this action is free and each orbit contains a unique operator of the form (8.7). Therefore the space of \mathfrak{sl}_n -opers may be identified with the space of equivalence classes of the space of operators of the form (8.8) with respect to the gauge action of the group of upper triangular matrices.

This definition has a straightforward generalization to an arbitrary simple Lie algebra \mathfrak{g} . We will work over the formal disc, so all functions that appear in our formulas will be formal powers series in the variable t . But the same definition also works for any (analytic or Zariski) open subset on a smooth complex curve, equipped with a coordinate t .

Let $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be the Cartan decomposition of \mathfrak{g} and e_i, h_i and $f_i, i = 1, \dots, \ell$, be the Chevalley generators of $\mathfrak{n}_+, \mathfrak{h}$ and \mathfrak{n}_- , respectively. We denote by \mathfrak{b}_+ the Borel subalgebra $\mathfrak{h} \oplus \mathfrak{n}_+$; it is the Lie algebra of upper triangular matrices in the case of \mathfrak{sl}_n . Then the analogue of the space of operators of the form (8.8) is the space of operators

$$(8.9) \quad \partial_t + \sum_{i=1}^{\ell} \psi_i(t) f_i + \mathbf{v}(t), \quad \mathbf{v}(t) \in \mathfrak{b}_+,$$

where each $\psi_i(t)$ is a nowhere vanishing function. This space is preserved by the action of the group of B_+ -valued gauge transformations, where B_+ is the Lie group corresponding to \mathfrak{n}_+ .

Following [13], we define a \mathfrak{g} -oper (on the formal disc or on a coordinatized open subset of a general curve) as an equivalence class of operators of the form (8.9) with respect to the N_+ -valued gauge transformations.

It is proved in [13] that these gauge transformations act freely. Moreover, one defines canonical representatives of each orbit as follows. Set

$$p_{-1} = \sum_{i=1}^{\ell} f_i \in \mathfrak{n}_-.$$

This element may be included into a unique \mathfrak{sl}_2 triple $\{p_{-1}, p_0, p_1\}$, where $p_0 \in \mathfrak{h}$ and $p_1 \in \mathfrak{n}_+$ satisfying the standard relations of \mathfrak{sl}_2 :

$$[p_1, p_{-1}] = 2p_0, \quad [p_0, p_{\pm 1}] = \pm p_{\pm 1}.$$

The element $\text{ad } p_0$ determines the so-called principal grading on \mathfrak{g} , such that the e_i 's have degree 1, and the f_i 's have degree -1 .

Let V_{can} be the subspace of $\text{ad } p_1$ -invariants in \mathfrak{n}_+ . This space is ℓ -dimensional, and it has a decomposition into homogeneous subspaces

$$V_{\text{can}} = \bigoplus_{i \in E} V_{\text{can}, i},$$

where the set E is precisely the set of exponents of \mathfrak{g} . For all $i \in E$ we have $\dim V_{\text{can}, i} = 1$, except when $\mathfrak{g} = \mathfrak{so}_{2n}$ and $i = 2n$, in which case it is equal to 2. In the former case we will choose a linear generator p_j of V_{can, d_j} , and in the latter case we will choose two linearly independent vectors in $V_{\text{can}, 2n}$, denoted by p_n and p_{n+1} (in other words, we will set $d_n = d_{n+1} = 2n$).

In particular, $V_{\text{can}, 1}$ is generated by p_1 and we will choose it as the corresponding generator. Then canonical representatives of the N_+ gauge orbits in the space of operators

of the form (8.9) are the operators

$$(8.10) \quad \partial_t + p_{-1} + \sum_{j=1}^{\ell} v_j(t) \cdot p_j.$$

Thus, a \mathfrak{g} -oper is uniquely determined by a collection of ℓ functions $v_i(t), i = 1, \dots, \ell$. However, these functions transform in a non-trivial way under changes of coordinates.

Namely, under a coordinate transformation $t = \varphi(s)$ the operator (8.10) becomes

$$\partial_s + \varphi'(s) \sum_{i=1}^{\ell} f_i + \varphi'(s) \sum_{j=1}^{\ell} v_j(\varphi(s)) \cdot p_j.$$

Now we apply a gauge transformation

$$(8.11) \quad g = \exp \left(\frac{1}{2} \frac{\varphi''}{\varphi'} \cdot p_1 \right) \check{\rho}(\varphi')$$

to bring it back to the form

$$\partial_s + p_{-1} + \sum_{j=1}^{\ell} \bar{v}_j(s) \cdot p_j,$$

where

$$\bar{v}(s) = v_1(\varphi(s)) (\varphi'(s))^2 - \frac{1}{2} \{\varphi, s\},$$

$$\bar{v}_j(s) = v_j(\varphi(s)) (\varphi'(s))^{d_j+1}, \quad j > 1$$

(see [12]). Thus, we see that v_1 transforms as a projective connection, and $v_j, j > 1$, transforms as a $(d_j + 1)$ -differential.

Denote by $\text{Op}_{\mathfrak{g}}(D)$ the space of \mathfrak{g} -opers on the formal disc D . Then we have an isomorphism

$$(8.12) \quad \text{Op}_{\mathfrak{g}}(D) \simeq \text{Proj}(D) \times \bigoplus_{j=2}^{\ell} \Omega^{\otimes(d_j+1)}(D).$$

The drawback of the above definition of opers is that we can work with operators (8.9) only on open subsets of algebraic curves equipped with a coordinate t . It is desirable to have an alternative definition that does not use coordinates and hence makes sense on any curve. Such a definition has been given by Beilinson and Drinfeld (see [14] and [15], Sect. 3). The basic idea is that operators (8.9) may be viewed as connections on a G -bundle.⁷³ The fact that we consider gauge equivalence classes with respect to the gauge action of the subgroup B_+ means that this G -bundle comes with a reduction to B_+ . However, we should also make sure that our connection has a special form as prescribed in formula (8.9).

So let G be the Lie group of adjoint type corresponding to \mathfrak{g} (for example, for \mathfrak{sl}_n it is PGL_n), and B_+ its Borel subgroup. A \mathfrak{g} -oper is by definition a triple $(\mathcal{F}, \nabla, \mathcal{F}_{B_+})$, where \mathcal{F} is a principal G -bundle on X , ∇ is a connection on \mathcal{F} and \mathcal{F}_{B_+} is a B_+ -reduction of \mathcal{F} ,

⁷³as we discussed before, all of our bundles are holomorphic and all of our connections are holomorphic, hence automatically flat as they are defined on curves

such that for any open subset U of X with a coordinate t and any trivialization of \mathcal{F}_{B_+} on U the connection operator $\nabla_{\partial/\partial t}$ has the form (8.9). We denote the space of G -opers on X by $\text{Op}_{\mathfrak{g}}(X)$.

The identification (8.12) is still valid for any smooth curve X :

$$(8.13) \quad \text{Op}_{\mathfrak{g}}(X) \simeq \text{Proj}(X) \times \bigoplus_{j=2}^{\ell} H^0(X, \Omega^{\otimes(d_j+1)}).$$

In particular, we find that if X is a compact curve of genus $g > 1$ then the dimension of $\text{Op}_{\mathfrak{g}}(X)$ is equal to $\sum_{i=1}^{\ell} (2d_i + 1)(g - 1) = \dim_{\mathbb{C}} G(g - 1)$.

It turns out that if X is compact, then the above conditions completely determine the underlying G -bundle \mathcal{F} . Consider first the case when $G = PGL_2$. We will describe the PGL_2 -bundle \mathcal{F} as the projectivization of rank 2 degree 0 vector bundle \mathcal{F}_0 on X . Let us choose a square root $\Omega_X^{1/2}$ of the canonical line bundle Ω_X . Then there is a unique (up to an isomorphism) extension

$$0 \rightarrow \Omega_X^{1/2} \rightarrow \mathcal{F}_0 \rightarrow \Omega_X^{-1/2} \rightarrow 0.$$

This PGL_2 -bundle \mathcal{F}_{PGL_2} is the projectivization of this bundle, and it does not depend on the choice of $\Omega_X^{1/2}$. This bundle underlies all \mathfrak{sl}_2 -opers on a compact curve X .

To define \mathcal{F} for a general simple Lie group G of adjoint type, we use the \mathfrak{sl}_2 triple $\{p_{-1}, p_0, p_1\}$ defined above. It gives us an embedding $PGL_2 \rightarrow G$. Then \mathcal{F} is the G -bundle induced from \mathcal{F}_{PGL_2} under this embedding (note that this follows from formula (8.11)). We call this \mathcal{F} the *oper G -bundle*. For $G = PGL_n$ it may be described as the projectivization of the rank n vector bundle on X obtained by taking successive non-trivial extensions of $\Omega_X^i, i = -(n-1)/2, -(n-3)/2, \dots, (n-1)/2$. It has the dubious honor of being the most unstable indecomposable rank n bundle of degree 0.

One can show that *any* connection on the oper G -bundle \mathcal{F}_G supports a unique structure of a G -oper. Thus, we obtain an identification between $\text{Op}_{\mathfrak{g}}(X)$ and the space of all connections on the oper G -bundle, which is the fiber of the forgetful map $\text{Loc}_G(X) \rightarrow \text{Bun}_G$ over the oper G -bundle.

8.4. Back to the center. Using opers, we can reformulate Theorem 8 in a coordinate-independent fashion. From now on we will denote the center of $V_{-h^\vee}(\mathfrak{g})$ simply by $\mathfrak{z}(\mathfrak{g})$. Let ${}^L\mathfrak{g}$ be the Langlands dual Lie algebra to \mathfrak{g} . Recall that the Cartan matrix of ${}^L\mathfrak{g}$ is the transpose of that of \mathfrak{g} . The following result is proved by B. Feigin and the author [11, 12].

Theorem 9. *The center $\mathfrak{z}(\mathfrak{g})$ is canonically isomorphic to the algebra $\text{Fun Op}_{L\mathfrak{g}}(D)$ of ${}^L\mathfrak{g}$ -opers on the formal disc D .*

Theorem 8 follows from this because once we choose a coordinate t on the disc we can bring any ${}^L\mathfrak{g}$ -oper to the canonical form (8.10), in which it determines ℓ formal power series

$$v_i(t) = \sum_{n \leq -d_i - 1} v_{i,n} t^{-n - d_i - 1}, \quad i = 1, \dots, \ell.$$

The shift of the labeling of the Fourier components by $d_i + 1$ is made so as to have $\deg v_{i,n} = -n$. Note that the exponents of \mathfrak{g} and ${}^L\mathfrak{g}$ coincide. Then we obtain

$$\text{Fun Op}_{L\mathfrak{g}}(D) = \mathbb{C}[v_{i,n_i}]_{i=1,\dots,\ell; n_i \leq -d_i-1}.$$

Under the isomorphism of Theorem 9 the generator $v_{i,-d_i-1}$ goes to some $S_i \in \mathfrak{z}(\mathfrak{g})$ of degree $d_i + 1$. This implies that v_{i,n_i} goes to $\frac{1}{(-n-d_i-1)!} \partial^{-n_i-d_i-1} S_i$, and so we recover the isomorphism of Theorem 8.

By construction, the Fourier coefficients $S_{i,n}$ of the fields $S_i(z) = \sum_{n \in \mathbb{Z}} S_{i,n} z^{-n-d_i-1}$ generating the center $\mathfrak{z}(\mathfrak{g})$ of the chiral algebra $V_{-h^\vee}(\mathfrak{g})$ are central elements of the completed enveloping algebra $\widetilde{U}_{-h^\vee}(\widehat{\mathfrak{g}})$ of $\widehat{\mathfrak{g}}$ at level $k = -h^\vee$. One can show that the center $Z(\widehat{\mathfrak{g}})$ of $\widetilde{U}_{-h^\vee}(\widehat{\mathfrak{g}})$ is topologically generated by these elements, and so we have

$$(8.14) \quad Z(\widehat{\mathfrak{g}}) \simeq \text{Fun Op}_{L\mathfrak{g}}(D^\times)$$

(see [12] for more details). The isomorphism (8.14) is in fact not only an isomorphism of commutative algebras, but also of Poisson algebras, with the Poisson structures on both sides defined in the following way.

Let $\widetilde{U}_k(\widehat{\mathfrak{g}})$ be the completed enveloping algebra of $\widehat{\mathfrak{g}}$ at level k . Given two elements, $A, B \in Z(\widehat{\mathfrak{g}})$, we consider their arbitrary ϵ -deformations, $A(\epsilon), B(\epsilon) \in \widetilde{U}_{k+\epsilon}(\widehat{\mathfrak{g}})$. Then the ϵ -expansion of the commutator $[A(\epsilon), B(\epsilon)]$ will not contain a constant term, and its ϵ -linear term, specialized at $\epsilon = 0$, will again be in $Z(\widehat{\mathfrak{g}})$ and will be independent of the deformations of A and B . Thus, we obtain a bilinear operation on $Z(\widehat{\mathfrak{g}})$, and one checks that it satisfies all properties of a Poisson bracket.

On the other hand, according to [13], the above definition of the space $\text{Op}_{L\mathfrak{g}}(D^\times)$ may be interpreted as the hamiltonian reduction of the space of all operators of the form $\partial_t + A(t)$, $A(t) \in {}^L\mathfrak{g}((t))$. The latter space may be identified with a hyperplane in the dual space to the affine Lie algebra $\widehat{{}^L\mathfrak{g}}$, which consists of all linear functionals taking value 1 on the central element $\mathbf{1}$. It carries the Kirillov-Kostant Poisson structure, and may in fact be realized as the $k \rightarrow \infty$ quasi-classical limit of the completed enveloping algebra $\widetilde{U}_k(\widehat{\mathfrak{g}})$.

Applying the Drinfeld-Sokolov reduction, we obtain a Poisson structure on the algebra $\text{Fun Op}_{L\mathfrak{g}}(D^\times)$ of functions on $\text{Op}_{L\mathfrak{g}}(D^\times)$. This Poisson algebra is called the *classical \mathcal{W} -algebra* associated to ${}^L\mathfrak{g}$. For example, in the case when $\mathfrak{g} = {}^L\mathfrak{g} = \mathfrak{sl}_n$, this Poisson structure is the (second) Adler-Gelfand-Dickey Poisson structure. Actually, it is included in a two-parameter family of Poisson structures on $\text{Op}_{L\mathfrak{g}}(D^\times)$ with respect to which the flows of the ${}^L\mathfrak{g}$ -KdV hierarchy are hamiltonian, as shown in [13].

Now, the theorem of [11, 12] is that (8.14) is an isomorphism of *Poisson algebras*. As shown in [15], this determines it uniquely, up to an automorphism of the Dynkin diagram of \mathfrak{g} .⁷⁴

How can the center of the chiral algebra $V_{-h^\vee}(\mathfrak{g})$ be identified with an the classical \mathcal{W} -algebra, and why does the Langlands dual Lie algebra appear here? To answer this question, we need to explain the main idea of the proof of Theorem 9 from [11, 12]. We will see that the crucial observation that leads to the appearance of the Langlands

⁷⁴Likewise, both sides of the isomorphism of Theorem 9 are Poisson algebras in the category of chiral algebras, and this isomorphism preserves these structures. In particular, $\text{Fun Op}_{L\mathfrak{g}}(D)$ is a quasi-classical limit of the \mathcal{W} -algebra associated to ${}^L\mathfrak{g}$ considered as a chiral algebra.

dual Lie algebra is closely related to the T-duality in free bosonic conformal field theory compactified on a torus.

8.5. Free field realization. The idea of the proof [11, 12] of Theorem 9 is to realize the center $\mathfrak{z}(\mathfrak{g})$ inside the Poisson version of the chiral algebra of free bosonic field with values in the dual space to the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. For that we use the free field realization of $\widehat{\mathfrak{g}}$, which was constructed by M. Wakimoto [103] for $\mathfrak{g} = \mathfrak{sl}_2$ and by B. Feigin and the author [104] for an arbitrary simple Lie algebra \mathfrak{g} .

We first recall the free field realization in the case of \mathfrak{sl}_2 . In this case we need a chiral bosonic $\beta\gamma$ system generated by the fields $\beta(z), \gamma(z)$ and a free chiral bosonic field $\phi(z)$. These fields have the following OPEs:

$$(8.15) \quad \begin{aligned} \beta(z)\gamma(w) &= -\frac{1}{z-w} + \text{reg.}, \\ \phi(z)\phi(w) &= -2\log(z-w) + \text{reg.} \end{aligned}$$

We have the following expansion of these fields:

$$\beta(z) = \sum_{n \in \mathbb{Z}} \beta_n z^{-n-1}, \quad \gamma(z) = \sum_{n \in \mathbb{Z}} \gamma_n z^{-n-1}, \quad \partial_z \phi(z) = \sum_{n \in \mathbb{Z}} b_n z^{-n-1}.$$

The Fourier coefficients satisfy the commutation relations

$$[\beta_n, \gamma_m] = -\delta_{n,-m}, \quad [b_n, b_m] = -2n\delta_{n,-m}.$$

Let \mathcal{F} be the chiral algebra of the $\beta\gamma$ system. Realized as the space of states, it is a Fock representation of the Heisenberg algebra generated by $\beta_n, \gamma_n, n \in \mathbb{Z}$, with the vacuum vector $|0\rangle$ annihilated by $\beta_n, n \geq 0, \gamma_m, m > 0$. The state-field correspondence is defined in such a way that $\beta_{-1}|0\rangle \mapsto \beta(z), \gamma_0|0\rangle \mapsto \gamma(z)$, etc.

Let π_0 be the chiral algebra of the boson $\phi(z)$. It is the Fock representation of the Heisenberg algebra generated by $b_n, n \in \mathbb{Z}$, with the vacuum vector annihilated by $b_n, n \geq 0$. The state-field correspondence sends $b_{-1}|0\rangle \mapsto b(z)$, etc. We also denote by π_λ the Fock representation of this algebra with the highest weight vector $|\lambda\rangle$ such that $b_n|\lambda\rangle = 0, n > 0$ and $ib_0|\lambda\rangle = \lambda|\lambda\rangle$.

The Lie algebra \mathfrak{sl}_2 has the standard basis elements J^\pm, J^0 satisfying the relations

$$[J^+, J^-] = 2J^0, \quad [J^0, J^\pm] = \pm J^\pm.$$

The free field realization of $\widehat{\mathfrak{sl}}_2$ at level $k \neq -2$ is a homomorphism (actually, injective) of chiral algebras $V_k(\mathfrak{sl}_2) \rightarrow \mathcal{F} \otimes \pi_0$. It is defined by the following maps of the generating fields of $V_k(\mathfrak{sl}_2)$:

$$(8.16) \quad \begin{aligned} J^+(z) &\mapsto \beta(z), \\ J^0(z) &\mapsto :\beta(z)\gamma(z): + \frac{\nu i}{2} \partial_z \phi(z), \\ J^-(z) &\mapsto -:\beta(z)\gamma(z)^2: - k\partial_z \gamma(z) - \nu i \gamma(z) \partial_z \phi(z), \end{aligned}$$

where $\nu = \sqrt{k+2}$. The origin of this free field realization is in the action of the Lie algebra $\mathfrak{sl}_2((t))$ on the loop space of \mathbb{CP}^1 . This is discussed in detail in [20], Ch. 11-12. It

is closely related to the sheaf of chiral differential operators introduced in [105] and [17], Sect. 2.9 (this is explained in [20], Sect. 18.5.7).⁷⁵

We would like to use this free field realization at the critical level $k = -2$ (i.e., $\nu = 0$). Unfortunately, if we set $k = -2$ in the above formulas, the field $\phi(z)$ will completely decouple and we will be left with a homomorphism $V_{-2}(\mathfrak{g}) \rightarrow \mathcal{F}$. This homomorphism is not injective. In fact, its kernel contains $\mathfrak{z}(\mathfrak{sl}_2)$, and so it is not very useful for elucidating the structure of $\mathfrak{z}(\mathfrak{sl}_2)$.

The solution is to rescale $\partial_z \phi(z)$ and replace it by a new field

$$\tilde{b}(z) = \nu i \partial_z \phi(z) = \sum_{n \in \mathbb{Z}} \tilde{b}_n z^{-n-1}.$$

The above formulas will now depend on $\tilde{b}(z)$ even when $k = -2$. But the chiral algebra π_0 will degenerate into a commutative chiral algebra $\tilde{\pi}_0 = \mathbb{C}[\tilde{b}_n]_{n < 0}$ at $k = -2$. Thus, we obtain a rescaled version of the free field homomorphism: $V_{-2}(\mathfrak{sl}_2) \rightarrow \mathcal{F} \otimes \tilde{\pi}_0$. This map is injective, and moreover, one can show that the image of the center $\mathfrak{z}(\mathfrak{sl}_2)$ of $V_{-2}(\mathfrak{sl}_2)$ is entirely contained in the commutative part $|0\rangle \otimes \tilde{\pi}_0$ of $\mathcal{F} \otimes \tilde{\pi}_0$. Thus, the rescaled free field realization at the critical level gives us an embedding $\mathfrak{z}(\mathfrak{sl}_2) \hookrightarrow \tilde{\pi}_0$ of the center of $V_{-2}(\mathfrak{sl}_2)$ into a commutative degeneration of the chiral algebra of the free bosonic field.

It is easy to write explicit formulas for this embedding. Recall that $\mathfrak{z}(\mathfrak{sl}_2)$ is generated by the Sugawara current $S(z)$ given by formula (8.3), hence this embedding is determined by the image of $S(z)$ in $\tilde{\pi}_0$. We find after a short calculation that

$$(8.17) \quad S(z) \mapsto \frac{1}{4} \tilde{b}(z)^2 - \frac{1}{2} \partial_z \tilde{b}(z).$$

This formula is known as the *Miura transformation*. In fact, $\tilde{\pi}$ may be interpreted as the algebra $\text{Fun Conn}(D)$ on the space $\text{Conn}(D)$ of connections $\partial_z + u(z)$ on the line bundle $\Omega^{-1/2}$ on the disc D . The Miura transformation is a map $\text{Conn}(D) \rightarrow \text{Proj}(D)$ sending $\partial_z + b(z)$ to the projective connection

$$\partial_z^2 - v(z) = \left(\partial_z - \frac{1}{2} u(z) \right) \left(\partial_z + \frac{1}{2} u(z) \right).$$

Under the isomorphism between $\mathfrak{z}(\mathfrak{sl}_2)$ and $\text{Proj}(D)$, this becomes formula (8.17).

However, for a general Lie algebra \mathfrak{g} we do not know explicit formulas for the generators of $\mathfrak{z}(\mathfrak{g})$. Therefore we cannot rely on a formula like (8.17) to describe $\mathfrak{z}(\mathfrak{g})$ in general. So we seek a different strategy.

The idea is to characterize the image of $\mathfrak{z}(\mathfrak{sl}_2)$ in $\tilde{\pi}_0$ as the kernel of a certain operator. This operator is actually defined not only for $k = -2$, but also for other values of k , and for $k \neq -2$ it is the residue of a standard vertex operator of the free field theory,

$$(8.18) \quad V_{-1/\nu}(z) = :e^{-\frac{i}{\nu} \phi(z)}: = T_{-1/\nu} \exp \left(\frac{1}{\nu} \sum_{n < 0} \frac{i b_n}{n} z^{-n} \right) \exp \left(\frac{1}{\nu} \sum_{n > 0} \frac{i b_n}{n} z^{-n} \right)$$

acting from π_0 to $\pi_{-1/\nu}$ (here $T_{-1/\nu}$ denotes the operator sending $|0\rangle$ to $|-1/\nu\rangle$ and commuting with $b_n, n \neq 0$).

⁷⁵see also [106, 107] for a recent discussion of the curved $\beta\gamma$ systems from the point of view of sigma models

So we consider the following *screening operator*:

$$(8.19) \quad \int V_{-1/\nu}(z)dz : \pi_0 \rightarrow \pi_{-1/\nu}.$$

It diverges when $\nu \rightarrow 0$, which corresponds to $k \rightarrow -2$. But it can be regularized and becomes a well-defined operator \tilde{V} on $\tilde{\pi}_0$. Moreover, the image of $\mathfrak{z}(\mathfrak{sl}_2)$ in $\tilde{\pi}_0$ coincides with the kernel of \tilde{V} (see [11]).

The reason is the following. One checks explicitly that the operator

$$G = \int \beta(z)V_{-1/\nu}(z)dz$$

commutes with the $\widehat{\mathfrak{sl}}_2$ currents (8.16). This means that the image of $V_k(\mathfrak{g})$ in $\mathcal{F} \otimes \pi_0$ is contained in the kernel of G (in fact, the image is equal to the kernel of G for irrational values of k). This remains true for the appropriately renormalized limit \tilde{G} of this operator at $k = -2$. But the image of $\mathfrak{z}(\mathfrak{sl}_2)$ belongs to the subspace $\tilde{\pi}_0 \subset \mathcal{F} \otimes \tilde{\pi}_0$. The restriction of \tilde{G} to $\tilde{\pi}_0$ is equal to \tilde{V} , and so we find that the image of $\mathfrak{z}(\mathfrak{sl}_2)$ in $\tilde{\pi}_0$ belongs to the kernel of \tilde{V} . One then checks that actually it is equal to the kernel of \tilde{V} .

We will now use this realization of $\mathfrak{z}(\mathfrak{sl}_2)$ as $\text{Ker}_{\tilde{\pi}_0} \tilde{V}$ to relate $\mathfrak{z}(\mathfrak{sl}_2)$ to $\text{Fun Proj}(D)$, which will appear as the quasi-classical limit of the Virasoro algebra.

For that we look at the kernel of $\int V_{-1/\nu}(z)dz$ for generic ν . It is a chiral subalgebra of the free bosonic chiral algebra π_0 , which contains the stress tensor

$$(8.20) \quad T_\nu(z) = -\frac{1}{4}:(\partial_z \phi(z))^2: + \frac{1}{2} \left(\nu - \frac{1}{\nu} \right) i \partial_z^2 \phi(z)$$

generating the Virasoro algebra of central charge

$$c_\nu = 1 - 3\left(\nu - \frac{1}{\nu}\right)^2 = 1 - 6(k+1)^2/(k+2).$$

The vertex operator $V_{-1/\nu}(z)$ has conformal dimension 1 with respect to $T_\nu(z)$, and this is the reason why $T_\nu(z)$ commutes with $\int V_{-1/\nu}(z)dz$.

The crucial observation is that there is *one more* vertex operator which has conformal dimension 1 with respect to $T_\nu(z)$, namely,⁷⁶

$$V_\nu(z) = :e^{i\nu\phi(z)}:.$$

Now, if ν^2 is irrational, then the kernels of the operators $\int V_{-1/\nu}(z)dz$ and $\int V_\nu(z)dz$ in π_0 *coincide* and are equal to the chiral algebra generated by $T_\nu(z)$ [11]. Moreover, this duality remains true in the limit $\nu \rightarrow 0$. In this limit $\int V_{-1/\nu}(z)dz$ becomes our renormalized operator \tilde{V} , whose kernel is $\mathfrak{z}(\mathfrak{sl}_2)$. On the other hand, the kernel of the $\nu \rightarrow 0$ limit of the operator $\int V_\nu(z)dz$ is nothing but the quasi-classical limit of the chiral Virasoro algebra generated by $\nu^2 T_\nu(z)$. This *classical Virasoro algebra* is nothing but the algebra $\text{Fun Proj}(D)$. This way we obtain the sought-after isomorphism $\mathfrak{z}(\mathfrak{sl}_2) \simeq \text{Fun Proj}(D)$.

⁷⁶The operators, $\int V_{-1/\nu}(z)dz$ and $\int V_\nu(z)dz$ were introduced by V. Dotsenko and V. Fateev in their work [108] on the free field realization of the correlation functions in the minimal models, and the terminology “screening operators” originates from that work. The parameters ν and $-1/\nu$ correspond to α_+ and α_- of [108]

8.6. T-duality and the appearance of the dual group. The crucial property that enabled us to make this identification is the fact that the kernels of two screening operators coincide (for irrational values of the parameter). This has a nice interpretation from the point of view of the T-duality. Consider the free bosonic theory compactified on the circle of radius $1/\nu$ (here we assume that ν is real and positive). The Hilbert space of this theory is the following module over the tensor product of the chiral algebra π_0 and its anti-chiral counterpart $\bar{\pi}_0$:

$$\bigoplus_{n,m \in \mathbb{Z}} \pi_{n\nu - m/\nu} \otimes \bar{\pi}_{n\nu + m/\nu}.$$

We denote by $\phi(z, \bar{z})$ the “full” bosonic field (the sum of the chiral and anti-chiral components) and by $\widehat{\phi}(z, \bar{z})$ its T-dual field (the difference of the two components of $\phi(z, \bar{z})$). Then the “electric” vertex operator corresponding to unit momentum and zero winding ($n = 1, m = 0$) is

$$(8.21) \quad :e^{i\nu\phi(z, \bar{z})}: = V_\nu(z) \bar{V}_\nu(\bar{z}),$$

whereas the “magnetic” vertex operator corresponding to zero momentum and unit winding ($n = 0, m = 1$) is

$$(8.22) \quad :e^{\frac{i}{\nu}\widehat{\phi}(z, \bar{z})}: = V_{-1/\nu}(z) \bar{V}_{1/\nu}(\bar{z}).$$

The T-dual theory is, by definition, the same theory, but compactified on the circle of radius ν . The T-duality is the statement that the two theories, compactified on the circles of radii ν and $1/\nu$, are equivalent. Under T-duality the electric and magnetic vertex operators are interchanged (see, e.g., [109], Sect. 11.2, for more details).

Now consider the deformation of this free field theory by the magnetic vertex operator (8.22). This operator is marginal (has dimension $(1, 1)$) with respect to the stress tensor $T_\nu(z)$ given by formula (8.20) and its anti-chiral counterpart. According to the general prescription of [110], the chiral algebra of the deformed theory (in the first order of perturbation theory) is the kernel of the operator $\int V_\nu(z) dz$ on the chiral algebra of the free theory, which for irrational ν^2 is π_0 . As we saw above, this chiral algebra is the Virasoro chiral algebra generated by $T_\nu(z)$.

On the other hand, consider the deformation of the T-dual theory by its magnetic operator. Under T-duality it becomes the electric vertex operator of the original theory which is given by formula (8.21). Therefore the corresponding chiral algebra is the kernel of the operator $\int V_{-1/\nu}(z) dz$ on π_0 (for irrational ν^2). The isomorphism between the kernels of the two operators obtained above means that the chiral algebras of the two deformed theories are the same. Thus, we obtain an interpretation of this isomorphism from the point of view of the T-duality. It is this duality which in the limit $\nu \rightarrow 0$ gives us an isomorphism of the center $\mathfrak{z}(\mathfrak{sl}_2)$ and the classical Virasoro algebra $\text{Fun Proj}(D)$.

We now generalize this duality to the case of an arbitrary simple Lie algebra \mathfrak{g} following [11, 12]. We start again with the free field realization of $\widehat{\mathfrak{g}}$. It is now given in terms of the tensor product $\mathcal{F}_\mathfrak{g}$ of copies of the chiral $\beta\gamma$ system labeled by the positive roots of \mathfrak{g} and the chiral algebra $\pi_0(\mathfrak{g})$ of the free bosonic field $\phi(z)$ with values in the dual space \mathfrak{h}^* to the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. More precisely, $\pi_0(\mathfrak{g})$ is generated by the fields $\lambda \cdot \phi(z)$ for

$\check{\lambda} \in \mathfrak{h}$, which satisfy the following OPEs

$$\check{\lambda} \cdot \phi(z) \check{\mu} \cdot \phi(w) = -\kappa_0(\check{\lambda}, \check{\mu}) \log(z-w) + \text{reg}.$$

In particular, the Fourier coefficients of the fields $\check{\lambda} \cdot \partial_z \phi(z)$ generate the Heisenberg Lie algebra $\widehat{\mathfrak{h}}$ and π_0 is its irreducible Fock representation.

The free field realization of $\widehat{\mathfrak{g}}$ is an embedding of chiral algebras $V_k(\mathfrak{g}) \rightarrow \mathcal{F}_{\mathfrak{g}} \otimes \pi_0(\mathfrak{g})$ defined in [104, 12]. This embedding comes from the action of $\mathfrak{g}((t))$ on the loop space of the flag manifold G/B and is closely related to the sheaf of chiral differential operators on the flag manifold (see [104, 12, 105] and [20], Sect. 18.5.7).

As in the case of \mathfrak{sl}_2 , discussed above, in the limit $\nu \rightarrow 0$ the chiral algebra $\pi_0(\mathfrak{g})$ degenerates into a commutative chiral algebra $\widetilde{\pi}_0(\mathfrak{g})$ generated by the rescaled \mathfrak{h}^* -valued field $\check{\mathbf{b}}(z) = \nu i \partial_z \phi(z)$, where $\nu = \sqrt{k + \check{h}^\vee}$. The corresponding map $V_{-h^\vee}(\mathfrak{g}) \rightarrow \mathcal{F}_{\mathfrak{g}} \otimes \pi_0(\mathfrak{g})$ is injective and the image of $\mathfrak{z}(\mathfrak{g})$ under this map is contained in $\pi_0(\mathfrak{g})$. Moreover, it is equal to the intersection of the kernels of the operators $\check{V}_j, j = 1, \dots, \ell$, which are obtained as the appropriately regularized limits of the screening operators as $\nu \rightarrow 0$. They are defined as follows. We identify \mathfrak{h}^* with \mathfrak{h} using the normalized inner product κ_0 , so in particular the fields $\alpha_j \cdot \phi(z)$ make sense. Then the screening operators are the residues of the vertex operators, corresponding to the simple roots of \mathfrak{g} :

$$(8.23) \quad V_{-\alpha_j/\nu}(z) = :e^{-\frac{i}{\nu} \alpha_j \cdot \phi(z)}: \quad j = 1, \dots, \ell.$$

These are the vertex operators of “magnetic” type. We also have a second set of screening operators corresponding to the vertex operators of “electric” type. These are labeled by the simple coroots of \mathfrak{g} :

$$(8.24) \quad V_{\nu \check{\alpha}_j}(z) = :e^{i\nu \check{\alpha}_j \cdot \phi(z)}: \quad j = 1, \dots, \ell.$$

The operator $\int V_{-\alpha_j/\nu}(z) dz$ commutes with the bosonic fields orthogonal to α_j . Therefore its kernel is the tensor product of the kernel “along the α_j direction” and the chiral subalgebra of $\pi_0(\mathfrak{g})$ orthogonal to this direction. But the former may be found in the same way as in the case of \mathfrak{sl}_2 . Thus, we obtain that for irrational ν^2 we have

$$(8.25) \quad \text{Ker}_{\pi_0(\mathfrak{g})} \int V_{-\alpha_j/\nu}(z) dz = \text{Ker}_{\pi_0(\mathfrak{g})} \int V_{\nu \check{\alpha}_j}(z) dz,$$

since $\langle \check{\alpha}_j, \alpha_j \rangle = 2$ as for \mathfrak{sl}_2 (see formula (8.15)).

Following [11] (see also [111] for $\mathfrak{g} = \mathfrak{sl}_n$), introduce the chiral \mathcal{W} -algebra $\mathcal{W}_k(\mathfrak{g})$ by the formula

$$\mathcal{W}_k(\mathfrak{g}) = \bigcap_{j=1, \dots, \ell} \text{Ker}_{\pi_0(\mathfrak{g})} \int V_{-\alpha_j/\nu}(z) dz$$

for generic k , and then analytically continue to all $k \neq -\check{h}^\vee$.

Now let ${}^L\mathfrak{g}$ be the Langlands dual Lie algebra to \mathfrak{g} and ${}^L\mathfrak{h}$ its Cartan subalgebra. Then we have the \mathcal{W} -algebra

$$\mathcal{W}_k({}^L\mathfrak{g}) = \bigcap_{j=1, \dots, \ell} \text{Ker}_{\pi_0({}^L\mathfrak{g})} \int V_{-L\alpha_j/\check{\nu}}(z) dz,$$

where $\check{\nu} = \sqrt{\check{k} + \check{h}^\vee}$, \check{h}^\vee is the dual Coxeter number of ${}^L\mathfrak{g}$, and $L\alpha_j$ is the j th simple root of ${}^L\mathfrak{g}$ realized as an element of ${}^L\mathfrak{h}$ using the normalized inner product $\check{\kappa}_0$.

We have a canonical identification $\mathfrak{h} = {}^L\mathfrak{h}^*$ sending $\check{\alpha}_j \mapsto {}^L\alpha_j$. However, under this identification the inner product κ_0 on \mathfrak{h} corresponds not to the inner product $\check{\kappa}_0^{-1}$ on ${}^L\mathfrak{h}^*$ (the dual of the inner product $\check{\kappa}_0$ on ${}^L\mathfrak{h}$), but to $r^\vee \check{\kappa}_0^{-1}$, where r^\vee is the *lacing number* of \mathfrak{g} (it is equal to the maximal number of edges connecting two vertices of the Dynkin diagram of \mathfrak{g} , see [100]). This means that the isomorphism (8.25) may be rewritten as

$$\mathrm{Ker}_{\pi_0(\mathfrak{g})} \int V_{-\alpha_i/\nu}(z) dz \simeq \mathrm{Ker}_{\pi_0({}^L\mathfrak{g})} \int V_{-{}^L\alpha_i/\check{\nu}}(z) dz,$$

where $\check{\nu} = -(\sqrt{r^\vee}\nu)^{-1}$. Therefore we obtain the following duality isomorphism of \mathcal{W} -algebras [11]:⁷⁷

$$(8.26) \quad \mathcal{W}_k(\mathfrak{g}) \simeq \mathcal{W}_{\check{k}}({}^L\mathfrak{g}), \quad \text{if } (k + h^\vee)r^\vee = (\check{k} + \check{h}^\vee)^{-1}.$$

In the limit $k \rightarrow -h^\vee$, $\check{k} \rightarrow \infty$ the \mathcal{W} -algebra $\mathcal{W}_k(\mathfrak{g})$ becomes the center $\mathfrak{z}(\mathfrak{g})$ of $V_{-h^\vee}(\mathfrak{g})$, whereas the $\mathcal{W}_{\check{k}}({}^L\mathfrak{g})$ degenerates into the quasi-classical version which is nothing but the algebra $\mathrm{Fun} \mathrm{Op}_{L\mathfrak{g}}(D)$ of functions on the space of ${}^L\mathfrak{g}$ -opers on the disc. Thus, we recover the isomorphism of Theorem 9 as the limit of the \mathcal{W} -algebra duality isomorphism (8.26).

This duality isomorphism may be interpreted in terms of the T-duality in the same way as in the case of \mathfrak{sl}_2 . Namely, we consider the free bosonic field theory with the target $\mathfrak{h}_{\mathbb{R}}^*/\frac{1}{\nu}P$, where P is the weight lattice of \mathfrak{g} and the metric induced by κ_0 . Then the Hilbert space of the theory is a direct sum of tensor products of Fock representations over the lattice P and the dual lattice \check{P} of coweights of \mathfrak{g} . The operators (8.23) appear as the chiral magnetic vertex operators corresponding to the simple roots, whereas the operators (8.24) are the chiral electric vertex operators corresponding to the simple coroots (considered as elements of \check{P}). The T-dual theory is the free bosonic theory with the target ${}^L\mathfrak{h}_{\mathbb{R}}^*/\sqrt{r^\vee}\nu\check{P}$ and the metric induced by $\check{\kappa}_0^{-1}$.

Under the T-duality the magnetic operators of the theory on ${}^L\mathfrak{h}_{\mathbb{R}}^*/\sqrt{r^\vee}\nu\check{P}$ become the electric operators of the theory on $\mathfrak{h}_{\mathbb{R}}^*/\frac{1}{\nu}P$. Therefore the isomorphism (8.26) means that the chiral algebras of the two T-dual theories deformed by the magnetic operators corresponding to simple roots of \mathfrak{g} and ${}^L\mathfrak{g}$ are isomorphic (for irrational ν^2). In the “infinite volume” limit one obtains the isomorphism of $\mathfrak{z}(\mathfrak{g})$ and $\mathrm{Fun} \mathrm{Op}_{L\mathfrak{g}}(D)$.

Thus, we see that T-duality is ultimately responsible for the appearance of the Langlands dual Lie algebra in the description of the center at the critical level.

The existence of the duality (8.26) indicates that \mathcal{W} -algebras should play a prominent role in a deformation of the “non-abelian Fourier-Mukai transform” discussed in Sect. 6.3. It also shows that we need to make an adjustment to the formulation (6.4) and replace the relation $k = \check{k}^{-1}$ by the relation that appears in formula (8.26).⁷⁸

9. CONSTRUCTING HECKE EIGENSHEAVES

Having described the center of the chiral algebra $V_{-h^\vee}(\mathfrak{g})$ in terms of ${}^L\mathfrak{g}$ -opers, we now set out to construct the corresponding twisted \mathcal{D} -modules on Bun_G , using the ${}^L\mathfrak{g}$ -opers as parameters. We will see, following Beilinson and Drinfeld [15], that these \mathcal{D} -modules turn

⁷⁷a reformulation that does not use r^\vee is given in [20], Sect. 15.4.7

⁷⁸here we assume that G is a simple Lie group and the inner products κ_0 and $\check{\kappa}_0$ on \mathfrak{g} and ${}^L\mathfrak{g}$ used in Sect. 6.3 are the standard normalized inner products

out to be the sought-after Hecke eigensheaves, whose eigenvalues are global $L\mathfrak{g}$ -opers on our curve.

We are ready to apply the machinery of localization functors developed in Sect. 7.4 to representations of $\widehat{\mathfrak{g}}$ of critical level. So let X be a smooth projective curve over \mathbb{C} . Recall that for any $(\widehat{\mathfrak{g}}, G[[t]])$ -module M of level k we construct a \mathcal{D}'_k -module $\Delta(M)$ on Bun_G , the moduli stack of G -bundles on X . As a warm-up, let us apply this construction to $M = V_k(\mathfrak{g})$, the vacuum module of level k introduced in Sect. 8.1. We claim that $\Delta(V_k(\mathfrak{g}))$ is the sheaf \mathcal{D}'_k considered as a left module over itself.

In order to see that, we observe that $\Delta(M)$ may be defined as follows. In the notation of Sect. 7.4, we have a $\widetilde{\mathcal{D}}'_k$ -module $\widetilde{\Delta}(M) = \widetilde{\mathcal{D}}'_k \otimes_{U_k(\widehat{\mathfrak{g}})} M$ on $G_{\text{out}} \backslash G((t))$, and $\Delta(M) = (\pi_*(\widetilde{\Delta}(M)))^{G[[t]]}$, where π is the projection

$$G_{\text{out}} \backslash G((t)) \rightarrow G_{\text{out}} \backslash G((t))/G[[t]] = \text{Bun}_G.$$

Now, since $V_k(\mathfrak{g}) = U_k(\widehat{\mathfrak{g}})/U_k(\mathfrak{g}) \cdot \mathfrak{g}[[t]]$, we obtain that $\widetilde{\Delta}(V_k(\mathfrak{g})) = \widetilde{\mathcal{D}}'_k/\widetilde{\mathcal{D}}'_k \cdot \mathfrak{g}[[t]]$ and so

$$\Delta(V_k(\mathfrak{g})) = \left(\pi_*(\widetilde{\mathcal{D}}'_k/\widetilde{\mathcal{D}}'_k \cdot \mathfrak{g}[[t]]) \right)^{G[[t]]} = \mathcal{D}'_k.$$

Here we use the general fact that if Z is a variety with an action of a group K and $S = Z/K$, then

$$\mathcal{D}_S \simeq (\pi_*(\mathcal{D}_Z/\mathcal{D}_Z \cdot \mathfrak{k}))^K,$$

where $\pi : Z \rightarrow S$ is the natural projection. The same is true for twisted \mathcal{D} -modules. Incidentally, this shows that the sheaf of differential operators on a quotient Z/K may be obtained via quantized hamiltonian reduction (also known as the ‘‘BRST reduction’’) of the sheaf of differential operators on Z . The corresponding quasi-classical statement is well-known: the algebras of symbols of differential operators on Z and S are the algebras of functions on the cotangent bundles T^*Z and T^*S , respectively, and the latter may be obtained from the former via the usual hamiltonian (or Poisson) reduction.

Thus, we see that the twisted \mathcal{D} -module corresponding to $V_k(\mathfrak{g})$ is the sheaf \mathcal{D}'_k . This \mathcal{D} -module is ‘‘too big’’. We obtain interesting \mathcal{D} -modules from quotients of $V_k(\mathfrak{g})$ by their ‘‘null-vectors’’. For example, if $k \in \mathbb{Z}_+$, then $V_k(\mathfrak{g})$ has as a quotient the vacuum integrable module $L_{0,k}$. The corresponding \mathcal{D}'_k -module is much smaller. As discussed in Sect. 7.6, it is isomorphic to $H^0(\text{Bun}, \mathcal{L}^{\otimes k})^* \otimes \mathcal{L}^{\otimes k}$.

9.1. Representations parameterized by opers. Now consider the vacuum module of critical level $V_{-h^\vee}(\mathfrak{g})$. Each element A of the center $\mathfrak{z}(\mathfrak{g}) \subset V_{-h^\vee}(\mathfrak{g})$ gives rise to the non-trivial endomorphism of $V_{-h^\vee}(\mathfrak{g})$, commuting with $\widehat{\mathfrak{g}}$, sending the vacuum vector v_{-h^\vee} to A . Conversely, any endomorphism of $V_{-h^\vee}(\mathfrak{g})$ that commutes with $\widehat{\mathfrak{g}}$ is uniquely determined by the image of v_{-h^\vee} . Since v_{-h^\vee} is annihilated by $\mathfrak{g}[[t]]$, this image necessarily belongs to the space of $\mathfrak{g}[[t]]$ -invariants in $V_{-h^\vee}(\mathfrak{g})$ which is the space $\mathfrak{z}(\mathfrak{g})$. Thus, we obtain an identification $\mathfrak{z}(\mathfrak{g}) = \text{End}_{\widehat{\mathfrak{g}}}(V_{-h^\vee}(\mathfrak{g}))$ which gives $\mathfrak{z}(\mathfrak{g})$ an algebra structure. This is a commutative algebra structure which coincides with the structure induced from the commutative chiral algebra structure on $\mathfrak{z}(\mathfrak{g})$.

Thus, we obtain from Theorem 9 that

$$(9.1) \quad \mathfrak{z}(\mathfrak{g}) = \text{End}_{\widehat{\mathfrak{g}}}(V_{-h^\vee}(\mathfrak{g})) \simeq \text{Fun Op}_{L\mathfrak{g}}(D).$$

Now each $L\mathfrak{g}$ -oper $\chi \in \text{Op}_{L\mathfrak{g}}(D)$ gives rise to an algebra homomorphism $\text{Fun Op}_{L\mathfrak{g}}(D) \rightarrow \mathbb{C}$ taking a function f to its value $f(\chi)$ at χ . Hence we obtain an algebra homomorphism $\text{End}_{\widehat{\mathfrak{g}}}(V_{-h^\vee}(\mathfrak{g})) \rightarrow \mathbb{C}$ which we denote by $\widetilde{\chi}$. We then set

$$(9.2) \quad V_\chi = V_{-h^\vee}(\mathfrak{g}) / \text{Ker } \widetilde{\chi} \cdot V_{-h^\vee}(\mathfrak{g}).$$

For instance, if $\mathfrak{g} = \mathfrak{sl}_2$, then $\text{Op}_{L\mathfrak{g}}(D) = \text{Proj}(D)$, hence χ is described by a second order operator $\partial_t^2 - v(t)$, where

$$v(t) = \sum_{n \leq -2} v_n t^{-n-2}, \quad v_n \in \mathbb{C}.$$

The algebra $\text{End}_{\widehat{\mathfrak{sl}_2}}(V_{-2}(\mathfrak{sl}_2))$ is the free polynomial algebra generated by $S_n, n \leq -2$, where each S_n is the Segal-Sugawara operator given by formula (8.2), considered as an endomorphism of $V_{-2}(\mathfrak{sl}_2)$. The corresponding quotient V_χ is obtained by setting S_n equal to $v_n \in \mathbb{C}$ for all $n \leq -2$ (note that $S_n \equiv 0$ on $V_{-2}(\mathfrak{sl}_2)$ for $n > -2$). We can also think about this as follows: the space of null-vectors in $V_{-2}(\mathfrak{sl}_2)$ is spanned by the monomials $S_{n_1} \dots S_{n_m} v_{-2}$, where $n_1 \leq \dots \leq n_m \leq -2$. We take the quotient of $V_{-2}(\mathfrak{g})$ by identifying each monomial of this form with a multiple of the vacuum vector $v_{n_1} \dots v_{n_m} v_k$ and taking into account all consequences of these identifications. This means, for instance, that the vector $J_{-1}^a S_{n_1} \dots S_{n_m} v_{-2}$ is identified with $v_{n_1} \dots v_{n_m} J_{-1}^a v_k$.

For example, if all v_n 's are equal to zero, this means that we just mod out by the $\widehat{\mathfrak{sl}_2}$ -submodule of $V_{-2}(\mathfrak{sl}_2)$ generated by all null-vectors. But the condition $v(t) = 0$ depends on the choice of coordinate t on the disc. As we have seen, $v(t)$ transforms as a projective connection. Therefore if we apply a general coordinate transformation, the new $v(t)$ will not be equal to zero. That is why there is no intrinsically defined “zero projective connection” on the disc D , and we are forced to consider *all* projective connections on D as the data for our quotients. Of course, these quotients will no longer be \mathbb{Z} -graded. But the \mathbb{Z} -grading has no intrinsic meaning either, because, as we have seen, the action of infinitesimal changes of coordinates (in particular, the vector field $-t\partial_t$) cannot be realized as an “internal symmetry” of $V_{-2}(\mathfrak{sl}_2)$.

Yet another way to think of the module V_χ is as follows. The Sugawara field $S(z)$ defined by formula (8.2) is now central, and so in particular it is regular at $z = 0$. Nothing can prevent us from setting it to be equal to a “ c -number” power series $v(z) \in \mathbb{C}[[z]]$ as long as this $v(z)$ transforms in the same way as $S(z)$ under changes of coordinates, so as not to break any symmetries of our theory. Since $S(z)$ transforms as a projective connection, $v(z)$ has to be a c -number projective connection on D , and then we set $S(z) = v(z)$. Of course, we should also take into account all corollaries of this identification, so, for example, the field $\partial_z S(z)$ should be identified with $\partial_z v(z)$ and the field $A(z)S(z)$ should be identified with $A(z)v(z)$. This gives us a new chiral algebra. As an $\widehat{\mathfrak{sl}_2}$ -module, this is precisely V_χ .

Though we will not use it in this paper, it is possible to realize the $\widehat{\mathfrak{sl}_2}$ -modules V_χ in terms of the $\beta\gamma$ -system introduced in Sect. 8.5. We have seen that at the critical level the bosonic system describing the free field realization of $\widehat{\mathfrak{g}}$ of level k becomes degenerate. Instead of the bosonic field $\partial_z \phi(z)$ we have the commutative field $\widetilde{b}(z)$ which appears as the limit of $\nu i \partial_z \phi(z)$ as $\nu = \sqrt{k+2} \rightarrow 0$. The corresponding commutative chiral algebra

is $\tilde{\pi}_0 \simeq \mathbb{C}[\tilde{b}_n]_{n<0}$. Given a numeric series

$$u(z) = \sum_{n<0} u_n z^{-n-1} \in \mathbb{C}[[z]],$$

we define a one-dimensional quotient of $\tilde{\pi}_0$ by setting $\tilde{b}_n = u_n, n < 0$. Then the free field realization (8.16) becomes

$$(9.3) \quad \begin{aligned} J^+(z) &\mapsto \beta(z), \\ J^0(z) &\mapsto :\beta(z)\gamma(z): + \frac{1}{2}u(z), \\ J^-(z) &\mapsto -:\beta(z)\gamma(z)^2: + 2\partial_z\gamma(z) - \gamma(z)u(z). \end{aligned}$$

It realizes the chiral algebra of $\widehat{\mathfrak{sl}}_2$ of critical level in the chiral algebra \mathcal{F} of the $\beta\gamma$ system (really, in the chiral differential operators of \mathbb{CP}^1), but this realization now depends on a parameter $u(z) \in \mathbb{C}[[z]]$.

It is tempting to set $u(z) = 0$. However, as we indicated in Sect. 8.5, $u(z)$ does not transform as function, but rather as a connection on the line bundle $\Omega^{-1/2}$ on the disc.⁷⁹ So there is no intrinsically defined “zero connection”, just like there is no “zero projective connection”, and we are forced to consider the realizations (9.3) for all possible connections $\partial_z + u(z)$ on $\Omega^{-1/2}$ (they are often referred to as “affine connections” or “affine structures”, see [20], Sect. 8.1). If we fix such a connection, then in the realization (9.3) the current $S(z)$ will act as

$$S(z) \mapsto \frac{1}{4}u(z)^2 - \frac{1}{2}\partial_z u(z).$$

(see formula (8.17)). In other words, it acts via a character corresponding to the projective connection $\chi = \partial_z^2 - v(z)$, where $v(z)$ is given by the right hand side of this formula. Therefore the $\widehat{\mathfrak{sl}}_2$ -module generated in the chiral algebra \mathcal{F} of the $\beta\gamma$ system from the vacuum vector is precisely the module V_χ considered above. Actually, it is equal to the space of global sections of a particular sheaf of chiral differential operators on \mathbb{CP}^1 , as those are also parameterized by affine connections $\partial_z + u(z)$. This gives us a concrete realization of the modules V_χ in terms of free fields.

Now consider an arbitrary simple Lie algebra \mathfrak{g} . Then we have an action of the center $\mathfrak{z}(\mathfrak{g})$ on the module $V_{-h^\vee}(\mathfrak{g})$. The algebra $\mathfrak{z}(\mathfrak{g})$ is generated by the currents $S_i(z), i = 1, \dots, \ell$. Therefore we wish to define a quotient of $V_{-h^\vee}(\mathfrak{g})$ by setting the generating field $S_i(z)$ to be equal to a numeric series $v_i(z) \in \mathbb{C}[[z]], i = 1, \dots, \ell$. But since the $S_i(z)$'s are the components of an operator-valued $L\mathfrak{g}$ -oper on the disc, for this identification to be consistent and coordinate-independent, these $v_i(z)$'s have to be components of a numeric $L\mathfrak{g}$ -oper on the disc, as in formula (8.10). Therefore choosing such $v_i(z), i = 1, \dots, \ell$, amounts to picking a $L\mathfrak{g}$ -oper χ on the disc. The resulting quotient is the $\widehat{\mathfrak{g}}$ -module V_χ given by formula (9.2). These modules may also realized in terms of the $\beta\gamma$ system (see [12]).

⁷⁹This is clear from the second formula in (9.3): the current $J^0(z)$ is a one-form, but the current $:\beta(z)\gamma(z):$ is anomalous. To compensate for this, we must make $u(z)$ transform with the opposite anomalous term, which precisely means that it should transform as a connection on $\Omega^{-1/2}$.

It is instructive to think of the vacuum module $V_{-h\nu}(\mathfrak{g})$ as a vector bundle over the infinite-dimensional affine space $\mathrm{Op}_{L\mathfrak{g}}(D)$. We know that the algebra of functions on $\mathrm{Op}_{L\mathfrak{g}}(D)$ acts on $V_{-h\nu}(\mathfrak{g})$, and we have the usual correspondence between modules over the algebra $\mathrm{Fun} Z$, where Z is an affine algebraic variety, and quasicoherent sheaves over Z . In our case $V_{-h\nu}(\mathfrak{g})$ is a free module over $\mathrm{Fun} \mathrm{Op}_{L\mathfrak{g}}(D)$, and so the corresponding quasicoherent sheaf is the sheaf of sections of a vector bundle on $\mathrm{Op}_{L\mathfrak{g}}(D)$. From this point of view, V_χ is nothing but the fiber of this vector bundle at $\chi \in \mathrm{Op}_{L\mathfrak{g}}(D)$. This more geometrically oriented point of view on $V_{-h\nu}(\mathfrak{g})$ is useful because we can see more clearly various actions on $V_{-h\nu}(\mathfrak{g})$. For example, the action of Lie algebra $\widehat{\mathfrak{g}}$ on $V_{-h\nu}(\mathfrak{g})$ comes from its fiberwise action on this bundle. It is also interesting to consider the Lie group $\mathrm{Aut} \mathcal{O}$ of automorphisms of $\mathbb{C}[[t]]$, which is the formal version of the group of diffeomorphisms of the disc. Its Lie algebra is $\mathrm{Der} \mathcal{O} = \mathbb{C}[[t]]\partial_t$.

The group $\mathrm{Aut} \mathcal{O}$ acts naturally on $\mathfrak{g}((t))$ and hence on $\widehat{\mathfrak{g}}$. Moreover, it preserves the Lie subalgebra $\mathfrak{g}[[t]] \subset \widehat{\mathfrak{g}}$ and therefore acts on $V_{-h\nu}(\mathfrak{g})$. What does its action on $V_{-h\nu}(\mathfrak{g})$ look like when we realize $V_{-h\nu}(\mathfrak{g})$ as a vector bundle over $\mathrm{Op}_{L\mathfrak{g}}(D)$? In contrast to the $\widehat{\mathfrak{g}}$ -action, the action of $\mathrm{Aut} \mathcal{O}$ does not preserve the fibers V_χ ! Instead, it acts along the fibers *and* along the base of this bundle. The base is the space of $L\mathfrak{g}$ -opers on the disc D and $\mathrm{Aut} \mathcal{O}$ acts naturally on it by changes of coordinate (see Sect. 8.3). Thus, we encounter a new phenomenon that the action of the group of formal diffeomorphisms of the disc D does not preserve a given $\widehat{\mathfrak{g}}$ -module V_χ . Instead, $\phi \in \mathrm{Aut} \mathcal{O}$ maps V_χ to another module $V_{\phi(\chi)}$.

Away from the critical level we take it for granted that on any (positive energy) $\widehat{\mathfrak{g}}$ -module the action of $\widehat{\mathfrak{g}}$ automatically extends to an action of the semi-direct product of the Virasoro algebra and $\widehat{\mathfrak{g}}$. The action of the Lie subalgebra $\mathrm{Der} \mathcal{O}$ of the Virasoro algebra may then be exponentiated to an action of the group $\mathrm{Aut} \mathcal{O}$. The reason is that away from the critical level we have the Segal-Sugawara current (8.3) which defines the action of the Virasoro algebra. But at the critical level this is no longer the case. So while the Lie algebra $\mathrm{Der} \mathcal{O}$ and the group $\mathrm{Aut} \mathcal{O}$ still act by symmetries on $\widehat{\mathfrak{g}}$, these actions do not necessarily give rise to actions on any given $\widehat{\mathfrak{g}}$ -module. This is the main difference between the categories of representations of $\widehat{\mathfrak{g}}$ at the critical level and away from it.

9.2. Twisted \mathcal{D} -modules attached to opers. Now to V_χ we wish associate a $\mathcal{D}'_{-h\nu}$ -module $\Delta(V_\chi)$ on Bun_G . What does this twisted \mathcal{D} -module look like?

At this point we need to modify slightly the construction of the localization functor Δ that we have used so far. In our construction we realized Bun_G as the double quotient (7.8). This realization depends on the choice of a point $x \in X$ and a local coordinate t at x . We now would like to rephrase this in a way that does not require us to choose t . Let F_x be the completion of the field F of rational functions on X at the point x , and let $\mathcal{O}_x \subset F_x$ be the corresponding completed local ring. If we choose a coordinate t at x , we may identify F_x with $\mathbb{C}((t))$ and \mathcal{O}_x with $\mathbb{C}[[t]]$, but F_x and \mathcal{O}_x are well-defined without any choices. So are the groups $G(\mathcal{O}_x) \subset G(F_x)$. Moreover, we have a natural embedding $\mathbb{C}[X \setminus x] \hookrightarrow F_x$ and hence the embedding $G_{\mathrm{out}} = G(\mathbb{C}[X \setminus x]) \hookrightarrow G(F_x)$. We now realize Bun_G in a coordinate-independent way as

$$(9.4) \quad \mathrm{Bun}_G = G_{\mathrm{out}} \backslash G(F_x) / G(\mathcal{O}_x).$$

With respect to this realization, the localization functor, which we will denote by Δ_x , assigns twisted \mathcal{D} -modules on Bun_G to $(\widehat{\mathfrak{g}}_x, G(\mathcal{O}_x))$ -modules. Here $\widehat{\mathfrak{g}}_x$ is the central extension of $\mathfrak{g}(F_x)$ defined as in Sect. 7.1. Note that the central extension is defined using the residue of one-form which is coordinate-independent operation. We define the $\widehat{\mathfrak{g}}_x$ -module $V_k(\mathfrak{g})_x$ as $\text{Ind}_{\widehat{\mathfrak{g}}(\mathcal{O}_x) \oplus \mathbb{C}\mathbf{1}}^{\widehat{\mathfrak{g}}_x} \mathbb{C}_k$ and $\mathfrak{z}(\mathfrak{g})_x$ as the algebra of $\widehat{\mathfrak{g}}_x$ -endomorphisms of $V_{-h^\vee}(\mathfrak{g})_x$. As a vector space, it is identified with the subspace of $\mathfrak{g}(\mathcal{O}_x)$ -invariants in $V_{-h^\vee}(\mathfrak{g})_x$. Now, since the isomorphism (9.1) is natural and coordinate-independent, we obtain from it a canonical isomorphism

$$(9.5) \quad \mathfrak{z}(\mathfrak{g})_x \simeq \text{Fun Op}_{L\mathfrak{g}}(D_x),$$

where D_x is the formal disc at $x \in X$ (in the algebro-geometric jargon, $D_x = \text{Spec } \mathcal{O}_x$). Therefore, as before, for any $L\mathfrak{g}$ -oper χ_x on D_x we have a homomorphism $\tilde{\chi}_x : \mathfrak{z}(\mathfrak{g})_x \rightarrow \mathbb{C}$ and so we define a $\widehat{\mathfrak{g}}_x$ module

$$V_{\chi_x} = V_{-h^\vee}(\mathfrak{g})_x / \text{Ker } \tilde{\chi}_x \cdot V_{-h^\vee}(\mathfrak{g})_x.$$

We would like to understand the structure of the \mathcal{D}'_{-h^\vee} -module $\Delta_x(V_{\chi_x})$. This is the twisted \mathcal{D} -module on Bun_G encoding the spaces of conformal blocks of a ‘‘conformal field theory’’ of critical level associated to the $L\mathfrak{g}$ -oper χ_x .

Finally, all of our hard work will pay off: the \mathcal{D} -modules $\Delta_x(V_{\chi_x})$ turn out to be the sought-after Hecke eigensheaves! This is neatly summarized in the following theorem of A. Beilinson and V. Drinfeld, which shows that \mathcal{D} -modules of coinvariants coming from the general machinery of CFT indeed produce Hecke eigensheaves.

Before stating it, we need to make a few remarks. First of all, we recall that our assumption is that G is a connected and simply-connected simple Lie group, and so $L\mathfrak{g}$ is a Lie group of adjoint type. Second, as we mentioned at the beginning of Sect. 8, the line bundle $\mathcal{L}^{\otimes(-h^\vee)}$ is isomorphic to the square root $K^{1/2}$ of the canonical line bundle on Bun_G . This square root exists and is unique under our assumption on G (see [15]). Thus, given a \mathcal{D}'_{-h^\vee} -module \mathcal{F} , the tensor product $\mathcal{F} \otimes_{\mathcal{O}} K^{-1/2}$ is an ordinary (untwisted) \mathcal{D} -module on Bun_G . Finally, as explained at the end of Sect. 8.3, $\text{Op}_{L\mathfrak{g}}(X)$ is naturally identified with the space of all connections on the oper bundle $\mathcal{F}_{L\mathfrak{g}}$ on X . For a $L\mathfrak{g}$ -oper χ on X we denote by E_χ the corresponding $L\mathfrak{g}$ -bundle with connection.

Theorem 10. (1) *The \mathcal{D}'_{-h^\vee} -module $\Delta_x(V_{\chi_x})$ is non-zero if and only if there exists a global $L\mathfrak{g}$ -oper on X , $\chi \in \text{Op}_{L\mathfrak{g}}(X)$ such that $\chi_x \in \text{Op}_{L\mathfrak{g}}(D_x)$ is the restriction of χ to D_x .*

(2) *If this holds, $\Delta_x(V_{\chi_x})$ depends only on χ and is independent of x in the sense that for any other point $y \in X$, if $\chi_y = \chi|_{D_y}$, then $\Delta_x(V_{\chi_x}) \simeq \Delta_y(V_{\chi_y})$.*

(3) *For any $\chi \in \text{Op}_{L\mathfrak{g}}(X)$ the \mathcal{D} -module $\Delta_x(V_{\chi_x}) \otimes K^{-1/2}$ is holonomic and it is a Hecke eigensheaf with the eigenvalue E_χ .*

Thus, for a $L\mathfrak{g}$ -local system E on X that admits the structure of an oper χ , we now have a Hecke eigensheaf Aut_E whose existence was predicted in Conjecture 1: namely, $\text{Aut}_E = \Delta_x(V_{\chi_x}) \otimes K^{-1/2}$.

In the rest of this section we will give an informal explanation of this beautiful result and discuss its generalizations.

9.3. How do conformal blocks know about the global curve? We start with the first statement of Theorem 10. Let us show that if χ_x does not extend to a regular oper χ defined globally on the entire curve X , then $\Delta_x(V_{\chi_x}) = 0$. For that it is sufficient to show that all fibers of $\Delta_x(V_{\chi_x})$ are zero. But these fibers are just the spaces of coinvariants $V_{\chi_x}/\mathfrak{g}_{\text{out}}^{\mathcal{P}} \cdot V_{\chi_x}$, where $\mathfrak{g}_{\text{out}}^{\mathcal{P}} = \Gamma(X \setminus x, \mathcal{P} \times_G \mathfrak{g})$. The key to proving that these spaces are all equal to zero unless χ_x extends globally lies in the fact that chiral correlation functions are global objects.

To explain what we mean by this, let us look at the case when \mathcal{P} is the trivial G -bundle. Then the space of coinvariants is $H_{\mathfrak{g}}(V_{\chi_x}) = V_{\chi_x}/\mathfrak{g}_{\text{out}}$. Let φ be an element of the corresponding space of conformal blocks, which we interpret as a linear functional on the space $H_{\mathfrak{g}}(V_{\chi_x})$. Then φ satisfies the Ward identity (compare with (7.3))

$$(9.6) \quad \varphi(\eta \cdot v) = 0, \quad \forall v \in V_{\chi_x}, \quad \eta \in \mathfrak{g}_{\text{out}}.$$

Now observe that if we choose a local coordinate z at x , and write $\eta = \eta_a(z)J^a$ near x , then

$$(9.7) \quad \varphi(\eta \cdot v) = \int \eta_a(z) \varphi(J^a(z) \cdot v) dz,$$

where the contour of integration is a small loop around the point x .

Consider the expression $\varphi(J^a(z) \cdot v) dz$. Transformation properties of the current $J^a(z)$ imply that this is an intrinsically defined (i.e., coordinate-independent) meromorphic one-form $\omega^a(v)$ on the punctured disc D_x^\times at x . The right hand side of (9.7) is just the residue of the one-form $\omega^a(v)\eta_a$ at x . Therefore the Ward identities (9.6) assert that the residue of $\omega^a(v)\eta_a$ for *any* $\eta_a \in \mathbb{C}[X \setminus x]$ is equal to zero. By (the strong version of) the residue theorem, this is equivalent to saying that $\omega^a(v)$, which is *a priori* a one-form defined on D_x^\times actually extends *holomorphically* to a one-form on $X \setminus x$ (see [20], Sect. 9.2.9). In general, this one-form will have a pole at x (which corresponds to $z = 0$) which is determined by the vector v . But if we choose as v the vacuum vector $v_{-h\nu}$, then $J^a(z)v_{-h\nu}$ is regular, and so we find that this one-form $\varphi(J^a(z) \cdot v_{-h\nu}) dz$ is actually regular everywhere on X .

This one-form is actually nothing but the chiral one-point function corresponding to φ and the insertion of the current $J^a(z) dz$.⁸⁰ It is usually denoted by physicists as $\langle J^a(z) \rangle_\varphi dz$ (we use the subscript φ to indicate which conformal block we are using to compute this correlation function). It is of course a well-known fact that in a conformal field theory with Kac-Moody symmetry this one-point function is a holomorphic one-form on X , and we have just sketched a derivation of this fact from the Ward identities.

Now the point is that the same holomorphy property is satisfied by *any* current of any chiral algebra in place of $J^a(z)$. For example, consider the stress tensor $T(z)$ in a conformal field theory with central charge c (see [20], Sect. 9.2). If $c = 0$, then $T(z)$ transforms as an operator-valued quadratic differential, and so the corresponding one-point function $\langle T(z) \rangle_\varphi (dz)^2$, which is *a priori* defined only on D_x , is in fact the restriction to D_x of a holomorphic (c -number) quadratic differential on the entire curve X , for any conformal block φ of the theory. If $c \neq 0$, then, as we discussed above, the intrinsic object is the operator-valued projective connection $\partial_z^2 - \frac{c}{c} T(z)$. Hence we find that for a conformal

⁸⁰this notation only makes sense on an open subset of X where the coordinate z is well-defined, but the one-form is defined everywhere on X

block φ normalized so that its value on the vacuum vector is 1 (such φ can always be found if the space of conformal blocks is non-zero) the expression $\partial_z^2 - \frac{6}{c}\langle T(z) \rangle_\varphi$, which is *a priori* a projective connection on D_x , is the restriction of a holomorphic projective connection on the entire X .

Now let us consider the Segal-Sugawara current $S(z)$, which is a certain degeneration of the stress tensor of the chiral algebra $V_k(\mathfrak{g})$ as $k \rightarrow -h^\vee$. We have seen that $\partial_z^2 - S(z)$ transforms as a projective connection on D_x^\times . Suppose that the space of conformal blocks $C_{\mathfrak{g}}(V_{\chi_x})$ is non-zero and let φ be a non-zero element of $C_{\mathfrak{g}}(V_{\chi_x})$. Then there exists a vector $A \in V_{\chi_x}$ such that $\varphi(A) = 1$. Since $S(z)$ is central, $S(z)v$ is regular at $z = 0$ for any $A \in V_{\chi_x}$. Therefore we have a projective connection on the disc D_x (with a local coordinate z)

$$\partial_z^2 - \varphi(S(z) \cdot A) = \partial_z^2 - \langle S(z)A(x) \rangle_\varphi,$$

and, as before, this projective connection is necessarily the restriction of a holomorphic projective connection on the *entire* X .⁸¹

Suppose that $\mathfrak{g} = \mathfrak{sl}_2$. Then by definition of V_{χ_x} , where χ_x is a (c -number) projective connection $\partial_z^2 - v(z)$ on D_x , $S(z)$ acts on V_{χ_x} by multiplication by $v(z)$. Therefore if the space of conformal blocks $C_{\mathfrak{sl}_2}(V_{\chi_x})$ is non-zero and we choose $\varphi \in C_{\mathfrak{sl}_2}(V_{\chi_x})$ as above, then

$$\partial_z^2 - \varphi(S(z) \cdot A) = \partial_z^2 - \varphi(v(z)A) = \partial_z^2 - v(z),$$

and so we find that $\partial_z^2 - v(z)$ extends to a projective connection on X ! Therefore the space of conformal blocks $C_{\mathfrak{sl}_2}(V_{\chi_x})$, or equivalently, the space of coinvariants, is non-zero only if the parameter of the module V_{χ_x} extends from the disc D_x to the entire curve X . The argument is exactly the same for a general SL_2 -bundle \mathcal{P} on X . The point is that $S(z)$ commutes with the $\widehat{\mathfrak{sl}}_2$, and therefore twisting by a SL_2 -bundle does not affect it. We conclude that for $\mathfrak{g} = \mathfrak{sl}_2$ we have $\Delta_x(V_{\chi_x}) = 0$ unless the projective connection χ_x extends globally.

Likewise, for a general \mathfrak{g} we have an operator-valued ${}^L\mathfrak{g}$ -oper on the disc D_x , which is written as

$$\partial_z + p_{-1} + \sum_{i=1}^{\ell} S_i(z)p_i$$

in terms of the coordinate z . By definition, it acts on the $\widehat{\mathfrak{g}}_x$ -module V_{χ_x} as the numeric ${}^L\mathfrak{g}$ -oper χ_x given by the formula

$$\partial_z + p_{-1} + \sum_{i=1}^{\ell} v_i(z)p_i, \quad v_i(z) \in \mathbb{C}[[z]]$$

in terms of the coordinate z . If $\varphi \in C_{\mathfrak{g}}^{\mathcal{P}}(V_{\chi_x})$ is a non-zero conformal block and $A \in V_{\chi_x}$ is such that $\varphi(A) = 1$, then in the same way as above it follows from the Ward identities that the ${}^L\mathfrak{g}$ -oper

$$\partial_z + p_{-1} + \sum_{i=1}^{\ell} \varphi(S_i(z) \cdot A)p_i$$

⁸¹this relies on the fact, proved in [20], Sect. 9.3, that the Ward identities (9.6) for the currents $J^a(z)$ automatically imply the Ward identities for all other currents of $V_{-h^\vee}(\mathfrak{g})$, such as $S(z)$

extends from D_x to the curve X . But this oper on D_x is nothing but χ_x ! Therefore, if the space of conformal blocks $C_{\mathfrak{g}}^{\mathcal{P}}(V_{\chi_x})$ is non-zero, then χ_x extends to X .

Thus, we obtain the “only if” part of Theorem 10,(1). The “if” part will follow from the explicit construction of $\Delta_x(V_{\chi_x})$ in the case when χ_x does extend to X , obtained from the quantization of the Hitchin system (see Sect. 9.5 below).

9.4. The Hecke property. We discuss next parts (2) and (3) of Theorem 10. In particular, we will see that the Hecke operators correspond to the insertion in the correlation function of certain vertex operators. We will assume throughout this section that we are given a ${}^L\mathfrak{g}$ -oper defined globally on the curve X , and χ_x is its restriction to the disc D_x .

Up to now, in constructing the localization functor, we have used the realization of Bun_G as the double quotient (9.4). This realization utilizes a single point of X . However, we know from the Weil construction (see Lemma 2) that actually we can utilize *all* points of X instead. In other words, we have an isomorphism

$$\text{Bun}_G \simeq G(F) \backslash G(\mathbb{A}) / G(\mathcal{O}),$$

which is actually how Bun_G appeared in the theory of automorphic representations in the first place. (Here we use our standard notation that F is the field of rational functions on X , $\mathbb{A} = \prod'_{x \in X} F_x$ and $\mathcal{O} = \prod_{x \in X} \mathcal{O}_x$.) This allows us to construct sheaves of coinvariants by utilizing all points of X . We just insert the vacuum representation of our chiral algebra (or its quotient) at all points of X other than the finitely many points with non-trivial insertions. The analogy with automorphic representations has in fact been used by E. Witten [5] in his adèlic formulation of conformal field theory.

More precisely, we define a localization functor Δ_X assigning to a collection $(M_x)_{x \in X}$ of $(\widehat{\mathfrak{g}}_x, G(\mathcal{O}_x))$ -modules of level k a \mathcal{D}'_k -module $\Delta_X((M_x)_{x \in X})$ on Bun_G . This functor is well-defined if M_x is the quotient of the vacuum module $V_k(\mathfrak{g})_x$ for $x \in X \setminus S$, where S is a finite subset of X . If we set $M_x = V_k(\mathfrak{g})_x$ for all $x \in X \setminus S$, then this \mathcal{D}'_k -module may be constructed by utilizing the set of points S as follows. We realize Bun_G as the double quotient

$$\text{Bun}_G \simeq G_{\text{out}} \backslash \prod_{x \in S} G(F_x) / \prod_{x \in S} G(\mathcal{O}_x),$$

where $G_{\text{out}} = G(\mathbb{C}[X \setminus S])$. We then have the localization functor Δ_S

$$(M_x)_{x \in S} \mapsto \Delta_S((M_x)_{x \in S}).$$

If we have $M_x = V_k(\mathfrak{g})_x$ for all $x \in X \setminus S$, then we have an isomorphism

$$\Delta_X((M_x)_{x \in X}) \simeq \Delta_S((M_x)_{x \in S}).$$

Likewise, we have

$$(9.8) \quad \Delta_{S \cup y}((M_x)_{x \in S}, V_k(\mathfrak{g})_y) \simeq \Delta_S((M_x)_{x \in S}).$$

In other words, inserting the vacuum module at additional points does not change the sheaf of coinvariants.

We apply this in our setting. Let us take $S = \{x\}$ and set $M_x = V_{\chi_x}$ and $M_y = V_{-h^\vee}(\mathfrak{g})_y$ for all $y \neq x$. Then we have

$$\Delta_X(V_{\chi_x}, (V_{-h^\vee}(\mathfrak{g})_y)_{y \in X \setminus x}) \simeq \Delta_x(V_{\chi_x}).$$

Using the Ward identities from the previous section, it is not difficult to show that the \mathcal{D} -module in the left hand side will remain the same if we replace each $V_{-h^\vee}(\mathfrak{g})_y$ by its quotient V_{χ_y} where $\chi_y = \chi|_{D_y}$. Thus, we find that

$$\Delta_x(V_{\chi_x}) \simeq \Delta_X((V_{\chi_y})_{y \in X}).$$

The object on the right hand side of this formula does not depend on x , but only on χ . This proves independence of $\Delta_x(V_{\chi_x})$ from the point $x \in X$ stated in part (2) of Theorem 10.

We use a similar idea to show the Hecke property stated in part (3) of Theorem 10. Recall the definition of the Hecke functors from Sect. 6.1. We need to show the existence of a compatible collection of isomorphisms

$$(9.9) \quad \iota_\lambda : \mathbb{H}_\lambda(\Delta_x(V_{\chi_x})) \xrightarrow{\sim} V_\lambda^{E_x} \boxtimes \Delta_x(V_{\chi_x}), \quad \lambda \in P_+,$$

where \mathbb{H}_λ are the Hecke functors defined in formula (6.1). This property will then imply the Hecke property of the untwisted \mathcal{D} -module $\Delta_x(V_{\chi_x}) \otimes K^{-1/2}$.

Let us simplify this problem and consider the Hecke property for a fixed point $y \in X$. Then we consider the correspondence

$$\begin{array}{ccc} & \mathcal{H}ecke_y & \\ & \begin{array}{c} h_y^\leftarrow \\ \swarrow \\ \text{Bun}_G \end{array} & \begin{array}{c} h_y^\rightarrow \\ \searrow \\ \text{Bun}_G \end{array} \end{array}$$

where $\mathcal{H}ecke_y$ classifies triples $(\mathcal{M}, \mathcal{M}', \beta)$, where \mathcal{M} and \mathcal{M}' are G -bundles on X and β is an isomorphism between the restrictions of \mathcal{M} and \mathcal{M}' to $X \setminus y$. As explained in Sect. 6.1, the fibers of h_y^\rightarrow are isomorphic to the affine Grassmannian $\text{Gr}_y = G(F_y)/G(\mathcal{O}_y)$ and hence we have the irreducible \mathcal{D} -modules IC_λ on $\mathcal{H}ecke_y$. This allows us to define the Hecke functors \mathbb{H}_λ on the derived category of twisted \mathcal{D} -modules on Bun_G by the formula

$$\mathbb{H}_{\lambda,y}(\mathcal{F}) = h_{y*}^\rightarrow(h_{y*}^\leftarrow(\mathcal{F}) \otimes \text{IC}_\lambda).$$

The functors \mathbb{H}_λ are obtained by “gluing” together $\mathbb{H}_{\lambda,y}$ for $y \in X$.

Now the specialization of the Hecke property (9.9) to $y \in X$ amounts to the existence of a compatible collection of isomorphisms

$$(9.10) \quad \iota_\lambda : \mathbb{H}_{\lambda,y}(\Delta_x(V_{\chi_x})) \xrightarrow{\sim} V_\lambda \otimes_{\mathbb{C}} \Delta_x(V_{\chi_x}), \quad \lambda \in P_+,$$

where V_λ is the irreducible representation of ${}^L G$ of highest weight λ . We will now explain how Beilinson and Drinfeld derive (9.9). Let us consider a “two-point” realization of the localization functor, namely, we choose as our set of points $S \subset X$ the set $\{x, y\}$ where $x \neq y$. Applying the isomorphism (9.8) in this case, we find that

$$(9.11) \quad \Delta_x(V_{\chi_x}) \simeq \Delta_{x,y}(V_{\chi_x}, V_{-h^\vee}(\mathfrak{g})_y).$$

Consider the Grassmannian Gr_y . Choosing a coordinate t at y , we identify it with $\text{Gr} = G((t))/G[[t]]$. Recall that we have a line bundle $\tilde{\mathcal{L}}^{\otimes(-h^\vee)}$ on Gr . Let again IC_λ be the irreducible \mathcal{D} -module on Gr corresponding to the $G[[t]]$ -orbit Gr_λ . The tensor product $\text{IC}_\lambda \otimes \tilde{\mathcal{L}}^{\otimes(-h^\vee)}$ is a \mathcal{D}_{-h^\vee} -module on Gr , where \mathcal{D}_{-h^\vee} is the sheaf of differential operators acting on $\tilde{\mathcal{L}}^{\otimes(-h^\vee)}$. By construction, the Lie algebra $\hat{\mathfrak{g}}$ maps to \mathcal{D}_{-h^\vee} in such a

way that the central element $\mathbf{1}$ is mapped to $-h^\vee$. Therefore the space of global sections $\Gamma(\mathrm{Gr}, \mathrm{IC}_\lambda \otimes \tilde{\mathcal{L}}^{\otimes(-h^\vee)})$ is a $\widehat{\mathfrak{g}}$ -module of the critical level, which we denote by W_λ .

For example, if $\lambda = 0$, then the corresponding $G[[t]]$ -orbit consists of one point of Gr , the image of $1 \in G((t))$. It is easy to see that the corresponding $\widehat{\mathfrak{g}}$ -module W_0 is nothing but the vacuum module $V_{-h^\vee}(\mathfrak{g})$. What is much more surprising is that for any $\lambda \in P_+$ there is an isomorphism⁸²

$$(9.12) \quad W_\lambda = \Gamma(\mathrm{Gr}, \mathrm{IC}_\lambda \otimes \tilde{\mathcal{L}}^{\otimes(-h^\vee)}) \simeq V_\lambda \otimes_{\mathbb{C}} V_{-h^\vee}(\mathfrak{g})$$

and in addition

$$(9.13) \quad H^i(\mathrm{Gr}, \mathrm{IC}_\lambda \otimes \tilde{\mathcal{L}}^{\otimes(-h^\vee)}) = 0 \quad i > 0.$$

The unexpected isomorphism (9.12), proved by Beilinson and Drinfeld, is the key to establishing the Hecke property of the sheaves $\Delta_x(V_{\chi_x})$.

Indeed, it follows from the definitions that the cohomological components of the image of the Hecke functor $H_{\lambda,y}$ are

$$(9.14) \quad R^i H_{\lambda,y}(\Delta_{x,y}(V_{\chi_x}, V_{-h^\vee}(\mathfrak{g})_y)) \simeq \Delta_{x,y}(V_{\chi_x}, H^i(\mathrm{Gr}, \mathrm{IC}_\lambda \otimes \tilde{\mathcal{L}}^{\otimes(-h^\vee)})).$$

In other words, applying the Hecke correspondence $H_{\lambda,y}$ at the point y to the sheaf of coinvariants corresponding to the insertion of V_{χ_x} at the point $x \in X$ is again a sheaf of coinvariants, but corresponding to the insertion of V_{χ_x} at the point $x \in X$ and the insertion of $H^i(\mathrm{Gr}, \mathrm{IC}_\lambda \otimes \tilde{\mathcal{L}}^{\otimes(-h^\vee)})$ at the point $y \in X$. Thus, from the point of view of conformal field theory the Hecke functors at y correspond simply to the insertion in the correlation function of particular vertex operators at the point y . These vertex operators come from the $\widehat{\mathfrak{g}}$ -module $W_\lambda = \Gamma(\mathrm{Gr}, \mathrm{IC}_\lambda \otimes \tilde{\mathcal{L}}^{\otimes(-h^\vee)})$ (in view of (9.13)).

The identification (9.14), together with (9.12), (9.13) and (9.11), imply the Hecke property (9.10).

How does one prove (9.12)? The proof in [15] is based on the usage of the “renormalized enveloping algebra” U^\natural at the critical level. To illustrate the construction of U^\natural , consider the Segal-Sugawara operators S_n as elements of the completed enveloping algebra $\tilde{U}_{-h^\vee}(\widehat{\mathfrak{g}})$ at the critical level. The homomorphism of Lie algebras $\widehat{\mathfrak{g}} \rightarrow D_{-h^\vee}$, where D_{-h^\vee} is the algebra of global differential operators acting on $\tilde{\mathcal{L}}^{\otimes(-h^\vee)}$, gives rise to a homomorphism of algebras $\tilde{U}_{-h^\vee}(\widehat{\mathfrak{g}}) \rightarrow D_{-h^\vee}$. It is not difficult to see that under this homomorphism $S_n, n > -2$, go to 0. On the other hand, away from the critical level S_n goes to a non-zero differential operator corresponding to the action of the vector field $-(k + h^\vee)t^n \partial_t$. The limit of this differential operator divided by $k + h^\vee$ as $k \rightarrow -h^\vee$ is well-defined in D_{-h^\vee} . Hence we try to adjoin to $\tilde{U}_{-h^\vee}(\widehat{\mathfrak{g}})$ the elements $\bar{L}_n = \lim_{k \rightarrow -h^\vee} \frac{1}{k + h^\vee} S_n, n > -2$.

It turns out that this can be done not only for the Segal-Sugawara operators but also for the “positive modes” of the other generating fields $S_i(z)$ of the center $\mathfrak{z}(\widehat{\mathfrak{g}})$. The result is an associative algebra U^\natural equipped with an injective homomorphism $U^\natural \rightarrow D_{-h^\vee}$. It follows that U^\natural acts on any $\widehat{\mathfrak{g}}$ -module of the form $\Gamma(\mathrm{Gr}, \mathcal{F})$, where \mathcal{F} is a \mathcal{D}_{-h^\vee} -module on Gr , in particular, it acts on W_λ . Using this action and the fact that $V_{-h^\vee}(\mathfrak{g})$ is an irreducible U^\natural -module, Beilinson and Drinfeld prove that W_λ is isomorphic to a direct

⁸²as a $\widehat{\mathfrak{g}}$ -module, the object on the right hand side is just the direct sum of $\dim V_\lambda$ copies of $V_{-h^\vee}(\mathfrak{g})$

sum of copies of $V_{-h^\vee}(\mathfrak{g})$.⁸³ The Tannakian formalism and the Satake equivalence (see Theorem 6) then imply the Hecke property (9.12). A small modification of this argument gives the full Hecke property (9.9).

9.5. Quantization of the Hitchin system. As the result of Theorem 10 we now have at our disposal the Hecke eigensheaves Aut_E on Bun_G associated to the ${}^L G$ -local systems on X admitting an oper structure (such a structure, if exists, is unique). What do these \mathcal{D} -modules on Bun_G look like?

Beilinson and Drinfeld have given a beautiful realization of these \mathcal{D} -modules as the \mathcal{D} -modules associated to systems of differential equations on Bun_G (along the lines of Sect. 3.4). These \mathcal{D} -modules can be viewed as generalizations of the Hecke eigensheaves constructed in Sect. 4.5 in the abelian case. In the abelian case the role of the oper bundle on X is played by the trivial line bundle, and so abelian analogues of opers are connections on the trivial line bundle. For such rank one local systems the construction of the Hecke eigensheaves can be phrased in particularly simple terms. This is the construction which Beilinson and Drinfeld have generalized to the non-abelian case.

Namely, let $D'_{-h^\vee} = \Gamma(\text{Bun}_G, \mathcal{D}'_{-h^\vee})$ be the algebra of global differential operators on the line bundle $K^{1/2} = \mathcal{L}^{\otimes(-h^\vee)}$ over Bun_G . Beilinson and Drinfeld show that

$$(9.15) \quad \text{Fun Op}_{L\mathfrak{g}}(X) \xrightarrow{\sim} D'_{-h^\vee}.$$

To prove this identification, they first construct a map in one direction. This is done as follows. Consider the completed universal enveloping algebra $\tilde{U}_{-h^\vee}(\widehat{\mathfrak{g}})$. As discussed above, the action of $\widehat{\mathfrak{g}}$ on the line bundle $\tilde{\mathcal{L}}^{\otimes(-h^\vee)}$ on Gr gives rise to a homomorphism of algebras $\tilde{U}_{-h^\vee}(\widehat{\mathfrak{g}}) \rightarrow D_{-h^\vee}$, where D_{-h^\vee} is the algebra of global differential operators on $\tilde{\mathcal{L}}^{\otimes(-h^\vee)}$. In particular, the center $Z(\widehat{\mathfrak{g}})$ maps to D_{-h^\vee} . As we discussed above, the “positive modes” from $Z(\widehat{\mathfrak{g}})$ go to zero. In other words, the map $Z(\widehat{\mathfrak{g}}) \rightarrow D_{-h^\vee}$ factors through $Z(\widehat{\mathfrak{g}}) \twoheadrightarrow \mathfrak{z}(\mathfrak{g}) \rightarrow \mathcal{D}_{-h^\vee}$. But central elements commute with the action of G_{out} and hence descend to global differential operators on the line bundle $\mathcal{L}^{\otimes(-h^\vee)}$ on Bun_G . Hence we obtain a map

$$\text{Fun Op}_{L\mathfrak{g}}(D_x) \rightarrow D'_{-h^\vee}.$$

Finally, we use an argument similar to the one outlined in Sect. 9.3 to show that this map factors as follows:

$$\text{Fun Op}_{L\mathfrak{g}}(D_x) \twoheadrightarrow \text{Fun Op}_{L\mathfrak{g}}(X) \rightarrow D'_{-h^\vee}.$$

Thus we obtain the desired homomorphism (9.15).

To show that it is actually an isomorphism, Beilinson and Drinfeld recast it as a quantization of the Hitchin integrable system on the cotangent bundle $T^* \text{Bun}_G$. Let us recall the definition of the Hitchin system.

Observe that the tangent space to Bun_G at $\mathcal{P} \in \text{Bun}_G$ is isomorphic to $H^1(X, \mathfrak{g}_{\mathcal{P}})$, where $\mathfrak{g}_{\mathcal{P}} = \mathcal{P} \times_G \mathfrak{g}$. Hence the cotangent space at \mathcal{P} is isomorphic to $H^0(X, \mathfrak{g}_{\mathcal{P}}^* \otimes \Omega)$ by the Serre

⁸³as for the vanishing of higher cohomologies, expressed by formula (9.13), we note that according to [112], the functor of global sections on the category of all critically twisted \mathcal{D} -modules is exact (so all higher cohomologies are identically zero)

duality. We construct the Hitchin map $p : T^* \text{Bun}_G \rightarrow H_G$, where H_G is the *Hitchin space*

$$H_G(X) = \bigoplus_{i=1}^{\ell} H^0(X, \Omega^{\otimes(d_i+1)}).$$

Recall that the algebra of invariant functions on \mathfrak{g}^* is isomorphic to the graded polynomial algebra $\mathbb{C}[P_1, \dots, P_\ell]$, where $\deg P_i = d_i + 1$. For $\eta \in H^0(X, \mathfrak{g}_\mathfrak{p}^* \otimes \Omega)$, $P_i(\eta)$ is well-defined as an element of $H^0(X, \Omega^{\otimes(d_i+1)})$.

By definition, the Hitchin map p takes $(\mathcal{P}, \eta) \in T^* \text{Bun}_G$, where $\eta \in H^0(X, \mathfrak{g}_\mathfrak{p}^* \otimes \Omega)$ to $(P_1(\eta), \dots, P_\ell(\eta)) \in H_G$. It has been proved in [113, 88] that over an open dense subset of H_G the morphism p is smooth and its fibers are proper. Therefore we obtain an isomorphism

$$(9.16) \quad \text{Fun } T^* \text{Bun}_G \simeq \text{Fun } H_G.$$

Now observe that both $\text{Fun Op}_{L_\mathfrak{g}}(X)$ and D'_{-h^\vee} are filtered algebras. The filtration on $\text{Fun Op}_{L_\mathfrak{g}}(X)$ comes from its realization given in formula (8.13). Since $\text{Proj}(X)$ is an affine space over $H^0(X, \Omega^{\otimes 2})$, we find that $\text{Op}_{L_\mathfrak{g}}(X)$ is an affine space modeled precisely on the Hitchin space H_G . Therefore the associated graded algebra of $\text{Fun Op}_{L_\mathfrak{g}}(X)$ is $\text{Fun } H_G$. The filtration on D'_{-h^\vee} is the usual filtration by the order of differential operator. It is easy to show that the homomorphism (9.15) preserves filtrations. Therefore it induces a map from $\text{Fun } H_G$ the algebra of symbols, which is $\text{Fun } T^* \text{Bun}_G$. It follows from the description given in Sect. 8.2 of the symbols of the central elements that we used to construct (9.15) that this map is just the Hitchin isomorphism (9.16). This immediately implies that the map (9.15) is also an isomorphism.

More concretely, let $\overline{D}_1, \dots, \overline{D}_N$, where $N = \sum_{i=1}^{\ell} (2d_i + 1)(g - 1) = \dim G(g - 1)$ (for $g > 1$), be a set of generators of the algebra of functions on $T^* \text{Bun}_G$ which according to (9.16) is isomorphic to $\text{Fun } H_G$. As shown in [113], the functions \overline{D}_i commute with each other with respect to the natural Poisson structure on $T^* \text{Bun}_G$ (so that p gives rise to an algebraic completely integrable system). According to the above discussion, each of these functions can be “quantized”, i.e., there exists a global differential operator D_i on the line bundle $K^{1/2}$ on Bun_G , whose symbol is \overline{D}_i . Moreover, the algebra D'_{-h^\vee} of global differential operators acting on $K^{1/2}$ is a free polynomial algebra in $D_i, i = 1, \dots, N$.

Now, given an $L_\mathfrak{g}$ -oper χ on X , we have a homomorphism $\text{Fun Op}_{L_G}(X) \rightarrow \mathbb{C}$ and hence a homomorphism $\tilde{\chi} : D'_{-h^\vee} \rightarrow \mathbb{C}$. As in Sect. 3.4, we assign to it a D'_{-h^\vee} -module

$$\Delta_{\tilde{\chi}} = D'_{-h^\vee} / \text{Ker } \tilde{\chi} \cdot D'_{-h^\vee}$$

This \mathcal{D} -module “represents” the system of differential equations

$$(9.17) \quad D_i f = \tilde{\chi}(D_i) f, \quad i = 1, \dots, N.$$

in the sense explained in Sect. 3.4 (compare with formulas (3.4) and (3.5)). The simplest examples of these systems in genus 0 and 1 are closely related to the Gaudin and Calogero systems, respectively (see [28] for more details).

The claim is that $\Delta_{\tilde{\chi}}$ is precisely the D'_{-h^\vee} -module $\Delta_x(V_{\chi_x})$ constructed above by means of the localization functor (for any choice of $x \in X$). Thus, we obtain a more concrete realization of the Hecke eigensheaf $\Delta_x(V_{\chi_x})$ as the \mathcal{D} -module representing a system of differential equations (9.17). Moreover, since $\dim \text{Bun}_G = \dim G(g - 1) = N$, we find that

this Hecke eigensheaf is *holonomic*, so in particular it corresponds to a perverse sheaf on Bun_G under the Riemann-Hilbert correspondence (see Sect. 3.4).

It is important to note that the system (9.17) has singularities. We have analyzed a toy example of a system of differential equations with singularities in Sect. 3.5 and we saw that solutions of such systems in general have monodromies around the singular locus. This is precisely what happens here. In fact, one finds from the construction that the “singular support” of the \mathcal{D} -module $\Delta_{\tilde{\chi}}$ is equal to the zero locus $p^{-1}(0)$ of the Hitchin map p , which is called the *global nilpotent cone* [57, 114, 59, 15]. This means, roughly, that the singular locus of the system (9.17) is the subset of Bun_G that consists of those bundles \mathcal{P} which admit a Higgs field $\eta \in H^0(X, \mathfrak{g}_{\mathcal{M}}^* \otimes \Omega)$ that is everywhere nilpotent. For $G = GL_n$ Drinfeld called the G -bundles in the complement of this locus “very stable” (see [114]). Thus, over the open subset of Bun_G of “very stable” G -bundles the system (9.17) describes a vector bundle (whose rank is as predicted in [59], Sect. 6) with a projectively flat connection. But horizontal sections of this connection have non-trivial monodromies around the singular locus.⁸⁴ These horizontal sections may be viewed as the “automorphic functions” on Bun_G corresponding to the oper χ . However, since they are multivalued and transcendental, we find it more convenient to describe the algebraic system of differential equations that these functions satisfy rather than the functions themselves. This system is nothing but the \mathcal{D} -module $\Delta_{\tilde{\chi}}$.

From the point of view of the conformal field theory definition of $\Delta_{\tilde{\chi}}$, as the sheaf of coinvariants $\Delta_x(V_{\chi_x})$, the singular locus in Bun_G is distinguished by the property that the dimensions of the fibers of $\Delta_x(V_{\chi_x})$ drop along this locus. As we saw above, these fibers are just the spaces of coinvariants $H_{\mathcal{P}}(V_{\chi_x})$. Thus, from this point of view the non-trivial nature of the \mathcal{D} -module $\Delta_{\tilde{\chi}}$ is explained by fact that the dimension of the space of coinvariants (or, equivalently, conformal blocks) depends on the underlying G -bundle \mathcal{P} . This is the main difference between conformal field theory at the critical level that gives us Hecke eigensheaves and the more traditional rational conformal field theories with Kac-Moody symmetry, such as the WZW models discussed in Sect. 7.6, for which the dimension of the spaces of conformal blocks is constant over the entire moduli space Bun_G . The reason is that the $\widehat{\mathfrak{g}}$ -modules that we use in WZW models are integrable, i.e., may be exponentiated to the Kac-Moody group \widehat{G} , whereas the $\widehat{\mathfrak{g}}$ -modules of critical level that we used may only be exponentiated to its subgroup $G[[t]]$.

The assignment $\chi \in \text{Op}_{L_{\mathfrak{g}}}(X) \mapsto \Delta_{\tilde{\chi}}$ extends to a functor from the category of modules over $\text{Fun Op}_{L_{\mathfrak{g}}}(X)$ to the category of $\mathcal{D}'_{-h\nu}$ -modules on Bun_G :

$$M \mapsto \mathcal{D}_{-h\nu} \otimes_{\mathcal{D}'_{-h\nu}} M.$$

Here we use the isomorphism (9.15). This functor is a non-abelian analogue of the functor (4.9) which was the special case of the abelian Fourier-Mukai transform. Therefore we may think of it as a special case of a non-abelian generalization of the Fourier-Mukai transform discussed in Sect. 6.2 (twisted by $K^{1/2}$ along Bun_G).

9.6. Generalization to other local systems. Theorem 10 gives us an explicit construction of Hecke eigensheaves on Bun_G as the sheaves of coinvariants corresponding to a

⁸⁴conjecturally, the connection has regular singularities on the singular locus

“conformal field theory” at the critical level. The caveat is that these Hecke eigensheaves are assigned to ${}^L G$ -local systems of special kind, namely, ${}^L \mathfrak{g}$ -opers on the curve X . Those form a half-dimensional subspace in the moduli stack $\mathrm{Loc}_{{}^L G}$ of all ${}^L G$ -local systems on X , namely, the space of all connections on a particular ${}^L G$ -bundle. Thus, this construction establishes the geometric Langlands correspondence only partially. What about other ${}^L G$ -local systems?

It turns out that the construction can be generalized to accommodate other local systems, with the downside being that this generalization introduces some unwanted parameters (basically, certain divisors on X) into the picture and so at the end of the day one needs to check that the resulting Hecke eigensheaf is independent of those parameters. In what follows we briefly describe this construction, following Beilinson and Drinfeld (unpublished). We recall that throughout this section we are under assumption that G is a connected and simply-connected Lie group and so ${}^L G$ is a group of adjoint type.

From the point of view of conformal field theory this generalization is a very natural one: we simply consider sheaves of coinvariants with insertions of more general vertex operators which are labeled by finite-dimensional representations of \mathfrak{g} .

Let (\mathcal{F}, ∇) be a general flat ${}^L G$ -bundle on a smooth projective complex curve X (equivalently, a ${}^L G$ -local system on X). In Sect. 8.3 we introduced the oper bundle $\mathcal{F}_{{}^L G}$ on X . The space $\mathrm{Op}_{{}^L G}(X)$ is identified with the (affine) space of all connections on $\mathcal{F}_{{}^L G}$, and for such pairs $(\mathcal{F}_{{}^L G}, \nabla)$ the construction presented above gives us the desired Hecke eigensheaf with the eigenvalue $(\mathcal{F}_{{}^L G}, \nabla)$.

Now suppose that we have an arbitrary ${}^L G$ -bundle \mathcal{F} on X with a connection ∇ . This connection does not admit a reduction $\mathcal{F}_{{}^L B_+}$ to the Borel subalgebra ${}^L B_+ \subset {}^L G$ on X that satisfies the oper condition formulated in Sect. 8.3. But one can find such a reduction on the complement to a finite subset S of X . Moreover, it turns out that the degeneration of the oper condition at each point of S corresponds to a dominant integral weight of \mathfrak{g} .

To explain this, recall that \mathcal{F} may be trivialized over $X \setminus x$. Let us choose such a trivialization. Then a ${}^L B_+$ -reduction of $\mathcal{F}|_{X \setminus x}$ is the same as a map $(X \setminus x) \rightarrow {}^L G / {}^L B_+$. A reduction will satisfy the oper condition if its differential with respect to ∇ takes values in an open dense subset of a certain ℓ -dimensional distribution in the tangent bundle to ${}^L G / {}^L B_+$ (see, e.g., [115]). Such a reduction can certainly be found for the restriction of (\mathcal{F}, ∇) to the formal disc at any point $y \in X \setminus x$. This implies that we can find such a reduction on the complement of finitely many points in $X \setminus x$.

For example, if $G = SL_2$, then ${}^L G / {}^L B_+ \simeq \mathbb{C}P^1$. Suppose that (\mathcal{F}, ∇) is the trivial local system on $X \setminus x$. Then a ${}^L B_+$ -reduction is just a map $(X \setminus x) \rightarrow \mathbb{C}P^1$, i.e., a meromorphic function, and the oper condition means that its differential is nowhere vanishing. Clearly, any non-constant meromorphic function on X satisfies this condition away from finitely many points of X .

Thus, we obtain a ${}^L B_+$ -reduction of \mathcal{F} away from a finite subset S of X , which satisfies the oper condition. Since the flag manifold ${}^L G / {}^L B_+$ is proper, this reduction extends to a ${}^L B_+$ -reduction of \mathcal{F} over the entire X . On the disc D_x near a point $x \in S$ the connection ∇ will have the form

$$(9.18) \quad \nabla = \partial_t + \sum_{i=1}^{\ell} \psi_i(t) f_i + \mathbf{v}(t), \quad \mathbf{v}(t) \in {}^L \mathfrak{b}_+[[t]],$$

where

$$\psi_i(t) = t^{\langle \alpha_i, \check{\lambda} \rangle} (\kappa_i + t(\dots)) \in \mathbb{C}[[t]], \quad \kappa_i \neq 0,$$

and $\check{\lambda}$ is a dominant integral weight of \mathfrak{g} (we denote them this way to distinguish them from the weights of ${}^L\mathfrak{g}$). The quotient of the space of operators (9.18) by the gauge action of ${}^L B_+[[t]]$ is the space $\mathrm{Op}_{L\mathfrak{g}}(D_x)_{\check{\lambda}}$ of *opers on D_x with degeneration of type $\check{\lambda}$ at x* . They were introduced by Beilinson and Drinfeld (see [116], Sect. 2.3, and [44]). Opers from $\mathrm{Op}_{L\mathfrak{g}}(D_x)_{\check{\lambda}}$ may be viewed as ${}^L\mathfrak{g}$ -opers on the punctured disc D_x^\times . When brought to the canonical form (8.13), they will acquire poles at $t = 0$. But these singularities are the artifact of a particular gauge, as the connection (9.18) is clearly regular at $t = 0$. In particular, it has trivial monodromy around x .

For example, for $\mathfrak{g} = \mathfrak{sl}_2$, viewing $\check{\lambda}$ as a non-negative integer, the space $\mathrm{Op}_{\mathfrak{sl}_2}(D_x)_{\check{\lambda}}$ is the space of projective connections on D_x^\times of the form

$$(9.19) \quad \partial_t^2 - \frac{\check{\lambda}(\check{\lambda} + 2)}{4} t^{-2} - \sum_{n \leq -1} v_n t^{-n-1}$$

The triviality of monodromy imposes a polynomial equation on the coefficients v_n (see [28], Sect. 3.9).

Thus, we now have a ${}^L B_+$ -reduction on \mathcal{F} such that the restriction of (\mathcal{F}, ∇) to $X \setminus S$, where $S = \{x_1, \dots, x_n\}$ satisfies the oper condition, and so (\mathcal{F}, ∇) is represented by an oper. Furthermore, the restriction of this oper to $D_{x_i}^\times$ is $\chi_{x_i} \in \mathrm{Op}_{L\mathfrak{g}}(D_{x_i})_{\check{\lambda}_i}$ for all $i = 1, \dots, n$. Now we wish to attach to (\mathcal{F}, ∇) a \mathcal{D}'_{-h^\vee} -module on Bun_G . This is done as follows.

Let $L_{\check{\lambda}}$ be the irreducible finite-dimensional representation of \mathfrak{g} of highest weight $\check{\lambda}$. Consider the corresponding induced $\widehat{\mathfrak{g}}_x$ -module of critical level

$$\mathbb{L}_{\check{\lambda}, x} = \mathrm{Ind}_{\mathfrak{g}(0_x) \oplus \mathbb{C}\mathbf{1}}^{\widehat{\mathfrak{g}}_x} L_{\check{\lambda}},$$

where $\mathbf{1}$ acts on $L_{\check{\lambda}}$ by multiplication by $-h^\vee$. Note that $\mathbb{L}_{0, x} = V_{-h^\vee}(\mathfrak{g})_x$. Let $\mathfrak{z}(\mathfrak{g})_{\check{\lambda}, x}$ be the algebra of endomorphisms of $\mathbb{L}_{\check{\lambda}, x}$ which commute with $\widehat{\mathfrak{g}}_x$. We have the following description of $\mathfrak{z}(\mathfrak{g})_{\check{\lambda}, x}$ which generalizes (9.5):

$$(9.20) \quad \mathfrak{z}(\mathfrak{g})_{\check{\lambda}, x} \simeq \mathrm{Fun} \mathrm{Op}_{L\mathfrak{g}}(D_x)_{\check{\lambda}}$$

(see [115, 116] for more details).

For example, for $\mathfrak{g} = \mathfrak{sl}_2$ the operator S_0 acts on $\mathbb{L}_{\check{\lambda}, x}$ by multiplication by $\check{\lambda}(\check{\lambda} + 2)/4$. This is the reason why the most singular coefficient in the projective connection (9.19) is equal to $\check{\lambda}(\check{\lambda} + 2)/4$.

It is now clear what we should do: the restriction of (\mathcal{F}, ∇) to $D_{x_i}^\times$ defines $\chi_{x_i} \in \mathrm{Op}_{L\mathfrak{g}}(D_{x_i})_{\check{\lambda}_i}$, which in turn gives rise to a homomorphism $\tilde{\chi}_{x_i} : \mathfrak{z}(\mathfrak{g})_{\check{\lambda}, x_i} \rightarrow \mathbb{C}$, for all $i = 1, \dots, n$. We then define $\widehat{\mathfrak{g}}_{x_i}$ -modules

$$\mathbb{L}_{\check{\lambda}, \chi_{x_i}} = \mathbb{L}_{\check{\lambda}, x_i} / \mathrm{Ker} \tilde{\chi}_{x_i} \cdot \mathbb{L}_{\check{\lambda}, x_i}, \quad i = 1, \dots, n.$$

Finally, we define the corresponding \mathcal{D}'_{-h^\vee} -module on Bun_G as $\Delta_S((\mathbb{L}_{\check{\lambda}, \chi_{x_i}})_{i=1, \dots, n})$, where Δ_S is the multi-point version of the localization functor introduced in Sect. 9.4. In words, this is the sheaf of coinvariants corresponding to the insertion of the modules $\mathbb{L}_{\check{\lambda}, x_i}$ at the points $x_i, i = 1, \dots, n$.

According to Beilinson and Drinfeld, we then have an analogue of Theorem 10.(3): the \mathcal{D}'_{-h^\vee} -module $\Delta_S((\mathbb{L}_{\tilde{\lambda}_i, \chi_{x_i}})_{i=1, \dots, n}) \otimes K^{-1/2}$ is a Hecke eigensheaf with the eigenvalue being the original local system (\mathcal{F}, ∇) . Thus, we construct Hecke eigensheaves for arbitrary ${}^L G$ -local systems on X , by realizing them as opers with singularities.

The drawback of this construction is that *a priori* it depends on the choice of the Borel reduction $\mathcal{F}_{L_{B^+}}$ satisfying the oper condition away from finitely many points of X . A general local system admits many such reductions (unlike connections on the oper bundle \mathcal{F}_{LG} , which admit a unique reduction that satisfies the oper condition everywhere). We expect that for a generic local system (\mathcal{F}, ∇) all of the resulting \mathcal{D}'_{-h^\vee} -modules on Bun_G are isomorphic to each other, but this has not been proved so far.

9.7. Ramification and parabolic structures. Up to now we have exclusively considered Hecke eigensheaves on Bun_G with the eigenvalues being *unramified* ${}^L G$ -local systems on X . One may wonder whether the conformal field theory approach that we have used to construct the Hecke eigensheaves might be pushed further to help us understand what the geometric Langlands correspondence should look like for ${}^L G$ -local systems that are ramified at finitely many points of X . This is indeed the case as we will now explain, following the ideas of [44].

Let us first revisit the classical setting of the Langlands correspondence. Recall that a representation π_x of $G(F_x)$ is called unramified if it contains a vector invariant under the subgroup $G(\mathcal{O}_x)$. The spherical Hecke algebra $\mathcal{H}(G(F_x), G(\mathcal{O}_x))$ acts on the space of $G(\mathcal{O}_x)$ -invariant vectors in π_x . The important fact is that $\mathcal{H}(G(F_x), G(\mathcal{O}_x))$ is a *commutative* algebra. Therefore its irreducible representations are one-dimensional. That is why an irreducible unramified representation has a one-dimensional space of $G(\mathcal{O}_x)$ -invariants which affords an irreducible representation of $\mathcal{H}(G(F_x), G(\mathcal{O}_x))$, or equivalently, a homomorphism $\mathcal{H}(G(F_x), G(\mathcal{O}_x)) \rightarrow \mathbb{C}$. Such homomorphisms are referred to as *characters* of $\mathcal{H}(G(F_x), G(\mathcal{O}_x))$. According to Theorem 5, these characters are parameterized by semi-simple conjugacy classes in ${}^L G$. As the result, we obtain the Satake correspondence which sets up a bijection between irreducible unramified representations of $G(F_x)$ and semi-simple conjugacy classes in ${}^L G$ for each $x \in X$.

Now, given a collection $(\gamma_x)_{x \in X}$ of semi-simple conjugacy classes in ${}^L G$, we obtain a collection of irreducible unramified representations π_x of $G(F_x)$ for all $x \in X$. Taking their tensor product, we obtain an irreducible unramified representation $\pi = \bigotimes'_{x \in X} \pi_x$ of the adèlic group $G(\mathbb{A})$. We then ask whether this representation is automorphic, i.e., whether it occurs in the appropriate space of functions on the quotient $G(F) \backslash G(\mathbb{A})$ (on which $G(\mathbb{A})$ acts from the right). The Langlands conjecture predicts (roughly) that this happens when the conjugacy classes γ_x are the images of the Frobenius conjugacy classes Fr_x in the Galois group $\text{Gal}(\overline{F}/F)$, under an unramified homomorphism $\text{Gal}(\overline{F}/F) \rightarrow {}^L G$. Suppose that this is the case. Then, according to the Langlands conjecture, π is realized in the space of functions on $G(F) \backslash G(\mathbb{A})$. But π contains a unique, up to a scalar, *spherical vector* that is invariant under $G(\mathcal{O}) = \prod_{x \in X} G(\mathcal{O}_x)$. The spherical vector gives rise to a function f_π on

$$(9.21) \quad G(F) \backslash G(\mathbb{A}) / G(\mathcal{O}),$$

which is a Hecke eigenfunction. This function contains all information about π and so we replace π by f_π . We then realize that (9.21) is the set of points of Bun_G . This allows us to reformulate the Langlands correspondence geometrically by replacing f_π with a Hecke eigensheaf on Bun_G .

This is what happens for the unramified homomorphisms $\sigma : \text{Gal}(\overline{F}/F) \rightarrow {}^L G$. Now suppose that we are given a homomorphism σ that is ramified at finitely many points y_1, \dots, y_n of X . Suppose that $G = GL_n$ and σ is irreducible, in which case the Langlands correspondence is proved for unramified as well as ramified Galois representations (see Theorem 1). Then to such σ we can also attach an automorphic representation $\bigotimes'_{x \in X} \pi_x$, where π_x is still unramified for $x \in X \setminus \{y_1, \dots, y_n\}$, but is *not* unramified at y_1, \dots, y_n , i.e., the space of $G(\mathcal{O}_{y_i})$ -invariant vectors in π_{y_i} is zero. What is this π_{y_i} ?

The equivalence class of each π_x is determined by the *local Langlands correspondence*, which, roughly speaking, relates equivalence classes of n -dimensional representations of the local Galois group $\text{Gal}(\overline{F}_x/F_x)$ and equivalence classes of irreducible *admissible* representations of $G(F_x)$.⁸⁵ The point is that the local Galois group $\text{Gal}(\overline{F}_x/F_x)$ may be realized as a subgroup of the global one $\text{Gal}(\overline{F}/F)$, up to conjugation, and so a representation σ of $\text{Gal}(\overline{F}/F)$ gives rise to an equivalence class of representations σ_x of $\text{Gal}(\overline{F}_x/F_x)$. To this σ_x the local Langlands correspondence attaches an admissible irreducible representation π_x of $G(F_x)$. Schematically, this is represented by the following diagram:

$$\begin{array}{ccc} \sigma & \xleftrightarrow{\text{global}} & \pi = \bigotimes'_{x \in X} \pi_x \\ & & \downarrow \\ \sigma_x & \xleftrightarrow{\text{local}} & \pi_x. \end{array}$$

So π_{y_i} is a bona fide irreducible representation of $G(F_{y_i})$ attached to σ_{y_i} . But because σ_{y_i} is ramified as a representation of the local Galois group $\text{Gal}(\overline{F}_{y_i}/F_{y_i})$, we find that π_{y_i} is ramified too, that is to say it has no non-zero $G(\mathcal{O}_{y_i})$ -invariant vectors. Therefore our representation π does not have a spherical vector. Hence we cannot attach to π a function on $G(F) \backslash G(\mathbb{A})/G(\mathcal{O})$ as we did before. What should we do?

Suppose for simplicity that σ is ramified at a single point $y \in X$. The irreducible representation π_y attached to y is ramified, but it is still *admissible*, in the sense that the subspace of K -invariants in π_y is finite-dimensional for any open compact subgroup K . An example of such a subgroup is the maximal compact subgroup $G(\mathcal{O}_y)$, but by our assumption $\pi_y^{G(\mathcal{O}_y)} = 0$. Another example is the Iwahori subgroup I_y : the preimage of a Borel subgroup $B \subset G$ in $G(\mathcal{O}_y)$ under the homomorphism $G(\mathcal{O}_y) \rightarrow G$. Suppose that the subspace of invariant vectors under the Iwahori subgroup I_y in π_y is non-zero. Such π_y correspond to the so-called tamely ramified representations of the local Galois group $\text{Gal}(\overline{F}_y/F_y)$. The space $\pi_y^{I_y}$ of I_y -invariant vectors in π_y is necessarily finite-dimensional as π_y is admissible. This space carries the action of the *affine Hecke algebra* $\mathcal{H}(G(F_y), I_y)$ of I_y bi-invariant compactly supported functions on $G(F_y)$, and because π_y is irreducible, the $\mathcal{H}(G(F_y), I_y)$ -module $\pi_y^{I_y}$ is also irreducible.

⁸⁵this generalizes the Satake correspondence which deals with unramified Galois representations; these are parameterized by semi-simple conjugacy classes in ${}^L G = GL_n$ and to each of them corresponds an unramified irreducible representation of $G(F_x)$

The problem is that $\mathcal{H}(G(F_y), I_y)$ is *non-commutative*, and so its representations generically have dimension greater than 1.⁸⁶

If π is automorphic, then the finite-dimensional space $\pi_y^{I_y}$, tensored with the one-dimensional space of $\prod_{x \neq y} G(\mathcal{O}_x)$ -invariants in $\bigotimes_{x \neq y} \pi_x$ embeds into the space of functions on the double quotient

$$(9.22) \quad G(F) \backslash G(\mathbb{A}) / I_y \times \prod_{x \neq y} G(\mathcal{O}_x).$$

This space consists of eigenfunctions with respect to the (commutative) spherical Hecke algebras $\mathcal{H}(G(F_x), G(\mathcal{O}_x))$ for $x \neq y$ (with eigenvalues determined by the Satake correspondence), and it carries an action of the (non-commutative) affine Hecke algebra $\mathcal{H}(G(F_y), I_y)$. In other words, there is not a unique (up to a scalar) automorphic function associated to π , but there is a whole finite-dimensional vector space of such functions, and it is realized not on the double quotient (9.21), but on (9.22).

Now let us see how this plays out in the geometric setting. For an unramified ${}^L G$ -local system E on X , the idea is to replace a single cuspidal spherical function f_π on (9.21) corresponding to an unramified Galois representation σ by a single irreducible (on each component) perverse Hecke eigensheaf on Bun_G with eigenvalue E . Since f_π was unique up to a scalar, our expectation is that such Hecke eigensheaf is also unique, up to isomorphism. Thus, we expect that the category of Hecke eigensheaves whose eigenvalue is an irreducible unramified local system which admits no automorphisms is equivalent to the category of vector spaces.

We are ready to consider the ramified case in the geometric setting. The analogue of a Galois representation tamely ramified at a point $y \in X$ in the context of complex curves is a local system $E = (\mathcal{F}, \nabla)$, where \mathcal{F} a ${}^L G$ -bundle \mathcal{F} on X with a connection ∇ that has regular singularity at y and unipotent monodromy around y . What should the geometric Langlands correspondence attach to such E ? It is clear that we need to find a replacement for the finite-dimensional representation of $\mathcal{H}(G(F_y), I_y)$ realized in the space of functions on (9.22). While (9.21) is the set of points of the moduli stack Bun_G of G -bundles, the double quotient (9.22) is the set of points of the moduli space $\text{Bun}_{G,y}$ of G -bundles with the *parabolic structure* at y ; this is a reduction of the fiber of a G -bundle at y to $B \subset G$. Therefore a proper replacement is the *category* of Hecke eigensheaves on $\text{Bun}_{G,y}$. Since our ${}^L G$ -local system E is now ramified at the point y , the definition of the Hecke functors and Hecke property given in Sect. 6.1 should be modified to account for this fact. Namely, the Hecke functors are now defined using the Hecke correspondences over $X \setminus y$ (and not over X as before), and the Hecke condition (6.2) now involves not E , but $E|_{X \setminus y}$ which is unramified.

We expect that there are as many irreducible Hecke eigensheaves on $\text{Bun}_{G,y}$ with the eigenvalue $E|_{X \setminus y}$ as the dimension of the corresponding representation of $\mathcal{H}(G(F_y), I_y)$ arising in the classical context. So we no longer speak of a particular irreducible Hecke eigensheaf (as we did in the unramified case), but of a category $\mathcal{A}ut_E$ of such sheaves.

⁸⁶in the case of GL_n , for any irreducible smooth representation π_y of $GL_n(F_y)$ there exists a particular open compact subgroup K such that $\dim \pi_i^K = 1$, but the significance of this fact for the geometric theory is presently unknown

This category may be viewed as a “categorification” of the corresponding representation of the affine Hecke algebra $\mathcal{H}(G(F_y), I_y)$.

In fact, just like the spherical Hecke algebra, the affine Hecke algebra has a categorical version (discussed in Sect. 5.4), namely, the derived category of I_y -equivariant perverse sheaves (or \mathcal{D} -modules) on the affine flag variety $G(F_y)/I_y$. This category, which we denote by \mathcal{P}_{I_y} , is equipped with a convolution tensor product which is a categorical version of the convolution product of I_y bi-invariant functions on $G(F_y)$. However, in contrast to the categorification $\mathcal{P}_{G(\mathfrak{o})}$ of the spherical Hecke algebra (see Sect. 5.4), this convolution product is not exact, so we are forced to work with the derived category $D^b(\mathcal{P}_{I_y})$. Nevertheless, this category “acts”, in the appropriate sense, on the derived category of the category of Hecke eigensheaves $\mathcal{A}ut_E$. It is this “action” that replaces the action of the affine Hecke algebra on the corresponding space of functions on (9.22).

Finally, we want to mention one special case when the representation of the affine Hecke algebra on $\pi_y^{I_y}$ is one-dimensional. In the geometric setting this corresponds to connections that have regular singularity at y with the monodromy being in the regular unipotent conjugacy class in ${}^L G$. According to [44], we expect that there is a unique irreducible Hecke eigensheaf whose eigenvalue is a local system of this type.⁸⁷ For $G = GL_n$ these eigensheaves have been constructed in [117, 118].

9.8. Hecke eigensheaves for ramified local systems. All this fits very nicely in the formalism of localization functors at the critical level. We explain this briefly following [44] where we refer the reader for more details.

Let us revisit once again how it worked in the unramified case. Suppose first that E is an unramified ${}^L G$ -local system that admits the structure of a ${}^L \mathfrak{g}$ -oper χ on X without singularities. Let χ_y be the restriction of this oper to the disc D_y . According to the isomorphism (9.5), we may view χ_y as a character of $\mathfrak{z}(\mathfrak{g})_y$ and hence of the center $Z(\widehat{\mathfrak{g}}_y)$ of the completed enveloping algebra of $\widehat{\mathfrak{g}}_y$ at the critical level. Let $\mathcal{C}_{G(\mathfrak{o}_y), \chi_y}$ be the category of $(\widehat{\mathfrak{g}}_y, G(\mathfrak{o}_y))$ -modules such that $Z(\widehat{\mathfrak{g}}_y)$ acts according to the character χ_y . Then the localization functor Δ_y may be viewed as a functor from the category $\mathcal{C}_{G(\mathfrak{o}_y), \chi_y}$ to the category of Hecke eigensheaves on Bun_G with the eigenvalue E .

In fact, it follows from the results of [112] that $\mathcal{C}_{G(\mathfrak{o}_y), \chi_y}$ is equivalent to the category of vector spaces. It has a unique up to isomorphism irreducible object, namely, the $\widehat{\mathfrak{g}}_y$ -module V_{χ_y} , and all other objects are isomorphic to the direct sum of copies of V_{χ_y} . The localization functor sends this module to the Hecke eigensheaf $\Delta_y(V_{\chi_y})$, discussed extensively above. Moreover, we expect that Δ_y sets up an equivalence between the categories $\mathcal{C}_{G(\mathfrak{o}_y), \chi_y}$ and $\mathcal{A}ut_E$.

More generally, in Sect. 9.6 we discussed the case when E is unramified and is represented by a ${}^L \mathfrak{g}$ -oper χ with degenerations of types $\check{\lambda}_i$ at points $x_i, i = 1, \dots, n$, but with trivial monodromy around those points. Then we also have a localization functor from the cartesian product of the categories $\mathcal{C}_{G(\mathfrak{o}_{x_i}), \chi_{x_i}}$ to the category $\mathcal{A}ut_E$ of Hecke eigensheaves on Bun_G with eigenvalue E . In this case we expect (although this has not been proved yet) that $\mathcal{C}_{G(\mathfrak{o}_{x_i}), \chi_{x_i}}$ is again equivalent to the category of vector spaces, with the unique

⁸⁷however, we expect that this eigensheaf has non-trivial self-extensions, so the corresponding category is non-trivial

up to isomorphism irreducible object being the $\widehat{\mathfrak{g}}_{x_i}$ -module $\mathbb{L}_{\lambda_i, \chi_{x_i}}$. We also expect that the localization functor $\Delta_{\{x_1, \dots, x_n\}}$ sets up an equivalence between the cartesian product of the categories $\mathcal{C}_{G(\mathcal{O}_{x_i}), \chi_{x_i}}$ and $\mathcal{A}ut_E$ when E is generic.

Now we consider the Iwahori case. Then instead of unramified ${}^L G$ -local systems on X we consider pairs (\mathcal{F}, ∇) , where \mathcal{F} is a ${}^L G$ -bundle and ∇ is a connection with regular singularity at $y \in X$ and unipotent monodromy around y . Suppose that this local system may be represented by a ${}^L \mathfrak{g}$ -oper χ on $X \setminus y$ whose restriction χ_y to the punctured disc D_y^\times belongs to the space $\text{nOp}_{L\mathfrak{g}}(D_y)$ of *nilpotent* ${}^L \mathfrak{g}$ -opers introduced in [44].

The moduli space $\text{Bun}_{G,y}$ has a realization utilizing only the point y :

$$\text{Bun}_{G,y} = G_{\text{out}} \backslash G(F_y) / I_y.$$

Therefore the formalism developed in Sect. 7.5 may be applied and it gives us a localization functor Δ_{I_y} from the category $(\widehat{\mathfrak{g}}_y, I_y)$ -modules of critical level to the category of $\mathcal{D}_{-h^\vee}^{I_y}$ -modules, where $\mathcal{D}_{-h^\vee}^{I_y}$ is the sheaf of differential operators acting on the appropriate critical line bundle on $\text{Bun}_{G,y}$.⁸⁸ Here, as before, by a $(\widehat{\mathfrak{g}}_y, I_y)$ -module we understand a $\widehat{\mathfrak{g}}_y$ -module on which the action of the Iwahori Lie algebra exponentiates to the action of the Iwahori group. For instance, any $\widehat{\mathfrak{g}}_y$ -module generated by a highest weight vector corresponding to an integral weight (not necessarily dominant), such as a Verma module, is a $(\widehat{\mathfrak{g}}_y, I_y)$ -module. Thus, we see that the category of $(\widehat{\mathfrak{g}}_y, I_y)$ -modules is much larger than that of $(\widehat{\mathfrak{g}}_y, G(\mathcal{O}_y))$ -modules.

Let $\mathcal{C}_{I_y, \chi_y}$ be the category $(\widehat{\mathfrak{g}}_y, I_y)$ -modules on which the center $Z(\widehat{\mathfrak{g}}_y)$ acts according to the character $\chi_y \in \text{nOp}_{L\mathfrak{g}}(D_y)$ introduced above.⁸⁹ One shows, in the same way as in the unramified case, that for any object M of this category the corresponding $\mathcal{D}_{-h^\vee}^{I_y}$ -module on $\text{Bun}_{G,y}$ is a Hecke eigensheaf with eigenvalue E . Thus, we obtain a functor from $\mathcal{C}_{I_y, \chi_y}$ to $\mathcal{A}ut_E$, and we expect that it is an equivalence of categories (see [44]).

This construction may be generalized to allow singularities of this type at finitely many points y_1, \dots, y_n . The corresponding Hecke eigensheaves are then \mathcal{D} -modules on the moduli space of G -bundles on X with parabolic structures at y_1, \dots, y_n . Non-trivial examples of these Hecke eigensheaves arise already in genus zero. These sheaves were constructed explicitly in [28] (see also [115, 116]), and they are closely related to the Gaudin integrable system (see [119] for a similar analysis in genus one).

In the language of conformal field theory this construction may be summarized as follows: we realize Hecke eigensheaves corresponding to local systems with ramification by considering chiral correlation functions at the critical level with the insertion at the ramification points of “vertex operators” corresponding to some representations of $\widehat{\mathfrak{g}}$. The type of ramification has to do with the type of highest weight condition that these vertex operators satisfy: no ramification means that they are annihilated by $\mathfrak{g}[[t]]$ (or, at least, $\mathfrak{g}[[t]]$ acts on them through a finite-dimensional representation), “tame” ramification, in the sense described above, means that they are highest weight vectors of $\widehat{\mathfrak{g}}_y$ in the usual

⁸⁸actually, there are now many such line bundles – they are parameterized by integral weights of G , but since at the end of the day we are going to “untwist” our \mathcal{D} -modules anyway, we will ignore this issue

⁸⁹recall that $Z(\widehat{\mathfrak{g}}_y)$ is isomorphic to $\text{Fun Op}_{L\mathfrak{g}}(D_y^\times)$, so any $\chi_y \in \text{nOp}_{L\mathfrak{g}}(D_y) \subset \text{Op}_{L\mathfrak{g}}(D_y^\times)$ determines a character of $Z(\widehat{\mathfrak{g}}_y)$

sense, and so on. The idea of inserting vertex operators at the points of ramification of our local system is of course very natural from the point of view of CFT. For local systems with irregular singularities we should presumably insert vertex operators corresponding to even more complicated representations of $\widehat{\mathfrak{g}}_y$.

What can we learn from this story?

The first lesson is that in the context of general local systems the geometric Langlands correspondence is inherently categorical: we are dealing not with individual Hecke eigensheaves, but with categories of Hecke eigensheaves on moduli spaces of G -bundles on X with parabolic structures (or more general “level structures”). The second lesson is that the emphasis now shifts to the study of local categories of $\widehat{\mathfrak{g}}_y$ -modules, such as the categories $\mathcal{C}_{G(\mathcal{O}_y), \chi_y}$ and $\mathcal{C}_{I_y, \chi_y}$. The localization functor gives us a direct link between these local categories and the global categories of Hecke eigensheaves, and we can infer a lot of information about the global categories by studying the local ones. This is a new phenomenon which does not have an analogue in the classical Langlands correspondence.

This point of view actually changes our whole perspective on representation theory of the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$. Initially, it would be quite tempting for us to believe that $\widehat{\mathfrak{g}}$ should be viewed as a kind of a replacement for the local group $G(F)$, where $F = \mathbb{F}_q((t))$, in the sense that in the geometric situation representations of $G(F)$ should be replaced by representations of $\widehat{\mathfrak{g}}$. Then the tensor product of representations π_x of $G(F_x)$ over $x \in X$ (or a subset of X) should be replaced by the tensor product of representations of $\widehat{\mathfrak{g}}_x$, and so on. But now we see that a single *representation* of $G(F)$ should be replaced in the geometric context by a whole *category* of representations of $\widehat{\mathfrak{g}}$. So a particular representation of $\widehat{\mathfrak{g}}$, such as a module V_χ considered above, which is an object of this category, corresponds not to a representation of $G(F)$, but to a *vector* in such a representation. For instance, V_χ corresponds to the spherical vector as we have seen above. Likewise, the category $\mathcal{C}_{I_y, \chi_y}$ appears to be the correct replacement for the vector subspace of I_y -invariants in a representation π_y of $G(F_y)$.

In retrospect, this does not look so outlandish, because the category of $\widehat{\mathfrak{g}}$ -modules itself may be viewed as a “representation” of the loop group $G((t))$. Indeed, we have the adjoint action of the group $G((t))$ on $\widehat{\mathfrak{g}}$, and this action gives rise to an “action” of $G((t))$ on the category of $\widehat{\mathfrak{g}}$ -modules. So it is the loop group $G((t))$ that replaces $G(F)$ in the geometric context, while the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ of critical level appears as a tool for building categories equipped with an action of $G((t))$! This point of view has been developed in [44], where various conjectures and results concerning these categories may be found. Thus, representation theory of affine Kac-Moody algebras and conformal field theory give us a rare glimpse into the magic world of geometric Langlands correspondence.

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