# Lefschetz Fibration Structures on Knot Surgery 4-Manifolds 

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## 1. Introduction

Since Seiberg-Witten theory was introduced in 1994, many techniques in 4dimensional topology have been developed to show that a large class of simply connected smooth 4-manifolds admit infinitely many distinct smooth structures. Among them, a knot surgery technique introduced by R. Fintushel and R. Stern turned out to be one of the most powerful tools for changing the smooth structure on a given 4-manifold [4]. The knot surgery construction is as follows. Suppose that $X$ is a simply connected smooth 4 -manifold containing an embedded torus $T$ of square 0 . Then, for any knot $K \subset S^{3}$, one can construct a new 4-manifold, called a knot surgery 4-manifold,

$$
X_{K}=X \sharp_{T=T_{m}}\left(S^{1} \times M_{K}\right)
$$

by taking a fiber sum along a torus $T$ in $X$ and $T_{m}=S^{1} \times m$ in $S^{1} \times M_{K}$, where $M_{K}$ is the 3-manifold obtained by doing 0 -framed surgery along $K$ and $m$ is the meridian of $K$. Then Fintushel and Stern proved that, under a mild condition on $X$ and $T$, the knot surgery 4-manifold $X_{K}$ is homeomorphic, but not diffeomorphic, to a given $X$ [4]. Furthermore, if $X$ is a simply connected elliptic surface $E(2), T$ is the elliptic fiber, and $K$ is a fibered knot, then it is also known that the knot surgery 4-manifold $E(2)_{K}$ admits not only a symplectic structure but also a genus $2 g(K)+1$ Lefschetz fibration structure $[6 ; 23]$. Note that there are only two inequivalent genus 1 fibered knots, but there are infinitely many inequivalent genus $g$ fibered knots for $g \geq 2$. So one may dig out some interesting properties of $E(2)_{K}$ by carefully investigating genus 2 fibered knots and related Lefschetz fibration structures.

On the one hand, Fintushel and Stern [5] conjectured that the set of all knot surgery 4-manifolds of the form $E(2)_{K}$ up to diffeomorphism is in one-to-one correspondence with the set of all knots in $S^{3}$ up to knot equivalence. Some progress related to the conjecture has been made by S. Akbulut [2] and M. Akaho [1]. However, a complete answer to the conjecture for prime knots up to mirror image is not known yet. Furthermore, Fintushel and Stern [6] also questioned whether or not any two in the following 4-manifolds,

[^0]\[

$$
\begin{aligned}
&\left\{Y\left(2 ; K_{1}, K_{2}\right):=E(2)_{K_{1}} \sharp_{\mathrm{id}: \Sigma_{2 g+1} \rightarrow \Sigma_{2 g+1}} E(2)_{K_{2}}\right. \\
&\left.K_{1}, K_{2} \text { are genus } g \text { fibered knots }\right\},
\end{aligned}
$$
\]

are mutually diffeomorphic. The second author obtained a partial result related to this question under the constraint that one of $K_{i}(i=1,2)$ be fixed [23].

In this paper we investigate Lefschetz fibration structures on the knot surgery 4-manifold $E(2)_{K}$, where $K$ ranges over a family of Kanenobu knots. Recall that Kanenobu [13;14] found an interesting family of inequivalent genus 2 fibered prime knots

$$
\left\{K_{p, q} \mid(p, q) \in \mathcal{R}\right\} \quad \text { for } \mathcal{R}=\left\{(p, q) \in \mathbb{Z}^{2} \mid p \in \mathbb{Z}^{+},-p \leq q \leq p\right\}
$$

where no two of the knots are in mirror relation and all of them have the same Alexander polynomials. In Section 3 we consider the following family of simply connected symplectic 4-manifolds that have the same Seiberg-Witten invariants:

$$
\left\{Y\left(2 ; K_{p, q}, K_{r, s}\right):=E(2)_{K_{p, q}} \not \sharp_{\mathrm{id}: \Sigma_{5} \rightarrow \Sigma_{5}} E(2)_{K_{r, s}} \mid(p, q),(r, s) \in \mathcal{R}\right\} .
$$

By investigating the monodromy factorization expression corresponding to the Lefschetz fibration structure on $Y\left(2 ; K_{p, q}, K_{r, s}\right)$, we answer the question raised in [6].

Theorem 1.1. Any two symplectic 4-manifolds in

$$
\left\{Y\left(2 ; K_{p, q}, K_{p+1, q}\right) \mid p, q \in \mathbb{Z}\right\}
$$

are mutually diffeomorphic. Similarly, any two symplectic 4-manifolds in

$$
\left\{Y\left(2 ; K_{p, q}, K_{p, q+1}\right) \mid p, q \in \mathbb{Z}\right\}
$$

are mutually diffeomorphic.
In Section 4 we also study nonisomorphic Lefschetz fibration structures on simply connected symplectic 4-manifolds that share the same Seiberg-Witten invariants.

Let $\xi_{p, q}$ be the monodromy factorization of a genus 5 Lefschetz fibration structure on $E(2)_{K_{p, q}}$ corresponding to the fixed generic fiber (as in Theorem 2.8) and the specified monodromy $\Phi_{K_{p, q}}$ of the fibered knot $K_{p, q}$ (as in Section 3). Then, by investigating the monodromy group $G_{F}\left(\xi_{p, q}\right)$ of $\xi_{p, q}$, we get the following theorem.

Theorem 1.2. $\quad \xi_{p, q}$ is not equivalent to $\xi_{r, s}$ if $(p, q) \not \equiv(r, s)(\bmod 2)$.
Remark 1.3. For any $(p, q) \in \mathbb{Z}^{2}, K_{p, q}$ is equivalent to $K_{q, p}$ and therefore $E(2)_{K_{p, q}}$ is diffeomorphic to $E(2)_{K_{q, p}}$. Since $K_{p, q}$ and $K_{q, p}$ are equivalent fibered knots, their monodromy can be conjugated, which means that we can select a pair of isomorphic Lefschetz fibration structures from $E(2)_{K_{p, q}}$ and $E(2)_{K_{q, p}}$. But this does not imply that the Lefschetz fibration structure on $E(2)_{K_{p, q}} \approx E(2)_{K_{q, p}}$ is unique (see Remark 2.9 for details). In fact, Theorem 1.2 implies that, if $p \not \equiv q$ $(\bmod 2)$, then we can select a pair of inequivalent special monodromy factorizations $\xi_{p, q}$ of $E(2)_{K_{p, q}}$ and $\xi_{q, p}$ of $E(2)_{K_{q, p}}$.

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## 2. Preliminaries

In this section we briefly review some well-known facts about Lefschetz fibrations on 4-manifolds and surface mapping class groups (refer to [8] for details).

Definition 2.1. Let $X$ be a compact, oriented smooth 4-manifold. A Lefschetz fibration is a proper smooth map $\pi: X \rightarrow B$, where $B$ is a compact connected oriented surface and $\pi^{-1}(\partial B)=\partial X$ such that:
(1) the set of critical points $C=\left\{p_{1}, p_{2}, \ldots, p_{n}\right\}$ of $\pi$ is nonempty and lies in $\operatorname{int}(X)$, and $\pi$ is injective on $C$;
(2) for each $p_{i}$ and $b_{i}:=\pi\left(p_{i}\right)$, there are local complex coordinate charts agreeing with the orientations of $X$ and $B$ such that $\pi$ can be expressed as $\pi\left(z_{1}, z_{2}\right)=z_{1}^{2}+z_{2}^{2}$.

Two Lefschetz fibrations $f_{1}: X_{1} \rightarrow B_{1}$ and $f_{2}: X_{2} \rightarrow B_{2}$ are called isomorphic if there are orientation-preserving diffeomorphisms $H: X_{1} \rightarrow X_{2}$ and $h: B_{1} \rightarrow B_{2}$ such that the following diagram commutes:


Monodromy factorization of a Lefschetz fibration is an ordered sequence of righthanded Dehn twists along simple closed curves on the fixed generic fiber $F$ of the Lefschetz fibration whose composition becomes the identity element in the mapping class group of $F$.

Two monodromy factorizations $W_{1}$ and $W_{2}$ are referred to as a Hurwitz equivalence if $W_{1}$ can be changed to $W_{2}$ in finitely many steps of the following two operations:
(1) Hurwitz move: $t_{c_{n}} \cdots t_{c_{i+1}} \cdot t_{c_{i}} \cdots t_{c_{1}} \sim t_{c_{n}} \cdots t_{c_{i+1}}\left(t_{c_{i}}\right) \cdot t_{c_{i+1}} \cdots t_{c_{1}}$;
(2) inverse Hurwitz move: $t_{c_{n}} \cdots t_{c_{i+1}} \cdot t_{c_{i}} \cdots t_{c_{1}} \sim t_{c_{n}} \cdots t_{c_{i}} \cdot t_{c_{i}}^{-1}\left(t_{c_{i+1}}\right) \cdots t_{c_{1}}$.

Here $t_{a}\left(t_{b}\right)=t_{t_{a}(b)}$, and $t_{a}\left(t_{b}\right)=t_{a} \circ t_{b} \circ t_{a}^{-1}$ as an element of mapping class group. This relation comes from the choice of Hurwitz system, a set of mutually disjoint arcs that connect $b_{0}$ to $b_{i}$ but exclude the base point $b_{0}$.

A choice of generic fiber also gives another equivalence relation. Two monodromy factorizations $W_{1}$ and $W_{2}$ are called a simultaneous conjugation equivalence if $W_{2}=f\left(W_{1}\right)$ for some element $f$ of the mapping class group of the chosen generic fiber of the Lefschetz fibration $W_{1}$.

It is well known that monodromy factorizations of two isomorphic Lefschetz fibrations are related by a finite sequence of Hurwitz equivalences and simultaneous conjugation equivalences $[8 ; 15 ; 18]$. Therefore, in this paper we do not distinguish a monodromy factorization from the corresponding Lefschetz fibration up to isomorphism.

Terminology. In order to emphasize that a chosen generic fiber is fixed, we sometimes use the term marked Lefschetz fibration to refer to a Lefschetz fibration whose chosen generic fiber is fixed. Two monodromy factorizations are also called marked equivalent if they are equivalent under a chosen fixed generic fiber.

Notation. We write $W_{1} \cong W_{2}$ if two monodromy factorizations $W_{1}$ and $W_{2}$ give the isomorphic Lefschetz fibration. When two manifolds $X_{1}$ and $X_{2}$ are diffeomorphic, we write this as $X_{1} \approx X_{2}$.

Definition 2.2. Let $\pi: X \rightarrow S^{2}$ be a Lefschetz fibration and let $F$ be a fixed generic fiber of the Lefschetz fibration. Let $W=w_{n} \cdots w_{2} \cdot w_{1}$ be a monodromy factorization of the Lefschetz fibration corresponding to $F$. Then the monodromy group $G_{F}(W)$ is a subgroup of the mapping class group $\mathcal{M}_{F}=\pi_{0}\left(\operatorname{Diff}^{+}(F)\right)$ generated by $w_{1}, w_{2}, \ldots, w_{n}$. We will simply write $G(W)$ when the generic fiber $F$ is clear from the context. The element $w_{n} \circ \cdots \circ w_{2} \circ w_{1}$ in $\mathcal{M}_{F}$ is denoted by $\lambda_{W}$.

Lemma 2.3. If two monodromy factorizations $W_{1}$ and $W_{2}$ give isomorphic Lefschetz fibrations over $S^{2}$ with respect to chosen generic fibers $F_{1}$ and $F_{2}$ (respectively) that are homeomorphic to $F$, then the monodromy groups $G_{F_{1}}\left(W_{1}\right)$ and $G_{F_{2}}\left(W_{2}\right)$ are isomorphic as a subgroup of the mapping class group $\mathcal{M}_{F}$. Moreover, if a generic fiber $F=F_{1}=F_{2}$ is fixed then $G_{F}\left(W_{1}\right)=G_{F}\left(W_{2}\right)$.

Remark 2.4. As mentioned previously, the role of simultaneous conjugation equivalence is in the choice of a generic fiber. If we use the same fixed generic fiber for $W_{1}$ and $W_{2}$ (i.e., if $F_{1}=F=F_{2}$ ), then the global conjugation cannot occur. Therefore we get $G_{F}\left(W_{1}\right)=G_{F}\left(W_{2}\right)$.

A monodromy factorization of a Lefschetz fibration structure on $E(n)_{K}$ was studied by Fintushel and Stern [6]. We were able to find an explicit monodromy factorization of $E(n)_{K}$ [23] with the help of factorizations of the identity element in the mapping class group that were discovered by Y. Matsumoto [18], M. Kork$\operatorname{maz}$ [17], and Y. Gurtas [9].

Definition 2.5. Let $M(n, g)$ be the desingularization of the double cover of $\Sigma_{g} \times S^{2}$ branched over $2 n\left(\{\right.$ point $\left.\} \times S^{2}\right) \cup 2\left(\Sigma_{g} \times\{\right.$ point $\left.\}\right)$.

Lemma $2.6[17 ; 22]$. $\quad M(2, g)$ has a monodromy factorization $\eta_{1, g}^{2}$, where

$$
\eta_{1, g}=t_{B_{0}} \cdot t_{B_{1}} \cdot t_{B_{2}} \cdots t_{B_{2 g}} \cdot t_{B_{2 g+1}} \cdot t_{b_{g+1}}^{2} \cdot t_{b_{g+1}^{\prime}}^{2}
$$

and $\left\{B_{j}, b_{g+1}, b_{g+1}^{\prime}\right\}$ are simple closed curves on $\Sigma_{2 g+1}$ as in Figure 1.


Figure 1 Vanishing cycles of $M(2, g)$ with $g=2$
Remark 2.7. In this paper we assume that we have already fixed a reference generic fiber as in Figure 1 and read the monodromy factorization with respect to the chosen generic fiber. From now on we use the monodromy factorization $\eta_{1, g}^{2}$ in Lemma 2.6 for $M(2, g)$ as a genus $2 g+1$ Lefschetz fibration with respect to the given fixed generic fiber.

Theorem $2.8[6 ; 23]$. Let $K \subset S^{3}$ be a fibered knot of genus $g$. Then $E(2)_{K}$, as a genus $2 g+1$ Lefschetz fibration, has a monodromy factorization of the form

$$
\Phi_{K}\left(\eta_{1, g}\right) \cdot \Phi_{K}\left(\eta_{1, g}\right) \cdot \eta_{1, g} \cdot \eta_{1, g},
$$

where $\eta_{1, g}^{2}$ is a monodromy factorization of $M(2, g)$ and

$$
\Phi_{K}=\varphi_{K} \oplus \mathrm{id} \oplus \mathrm{id}: \Sigma_{g} \sharp \Sigma_{1} \sharp \Sigma_{g} \rightarrow \Sigma_{g} \sharp \Sigma_{1} \sharp \Sigma_{g}
$$

is a diffeomorphism obtained by using a (geometric) monodromy $\varphi_{K}$ of $K$ defined by

$$
S^{3} \backslash v(K)=\left(I \times \Sigma_{g}^{1}\right) /\left((1, x) \sim\left(0, \varphi_{K}(x)\right)\right)
$$

where $\Sigma_{g}^{1}$ is an oriented surface of genus $g$ with one boundary component.
Remark 2.9. If two fibered knots $K_{1}$ and $K_{2}$ are equivalent with fiber surface $\Sigma_{g}^{1}$, then there is a homeomorphism $\phi: \Sigma_{g}^{1} \rightarrow \Sigma_{g}^{1}$ such that

$$
S^{3} \backslash v\left(K_{1}\right)=\left(I \times \Sigma_{g}^{1}\right) / \sim_{\varphi_{K_{1}}} \approx\left(I \times \Sigma_{g}^{1}\right) / \sim_{\phi \circ \varphi_{K_{1}} \circ \phi^{-1}}=S^{3} \backslash v\left(K_{2}\right) .
$$

So if we select a generic fiber $F^{\prime} \approx \Sigma_{2 g+1}$ of $M(2, g)$ such that $\Phi\left(\eta_{1, g}^{2}\right)$ is a monodromy factorization of $M(2, g)$ as a genus $2 g+1$ Lefschetz fibration, then

$$
\begin{aligned}
\Phi\left(\eta_{1, g}^{2}\right) \cdot \Phi_{K_{2}}\left(\Phi\left(\eta_{1, g}^{2}\right)\right) & =\Phi\left(\eta_{1, g}^{2}\right) \cdot\left(\Phi \circ \Phi_{K_{1}} \circ \Phi^{-1}\right)\left(\Phi\left(\eta_{1, g}^{2}\right)\right) \\
& =\Phi\left(\eta_{1, g}^{2} \cdot \Phi_{K_{1}}\left(\eta_{1, g}^{2}\right)\right) \cong \eta_{1, g}^{2} \cdot \Phi_{K_{1}}\left(\eta_{1, g}^{2}\right) ;
\end{aligned}
$$

this implies that we can select a pair of isomorphic Lefschetz fibration structures from $E(2)_{K_{1}}$ and $E(2)_{K_{2}}$.

On the other hand, for a given fibered knot $K$ and its fiber surface $\Sigma_{K}^{1}$, we identify $\Sigma_{K}^{1}$ and $\Sigma_{g}^{1}=\Sigma_{g}-\operatorname{int}\left(D^{2}\right) \subset \Sigma_{g} \sharp \Sigma_{1} \sharp \Sigma_{g}$ by a fixed homeomorphism. Even though we fix a generic fiber $\Sigma_{2 g+1}$ of $M(2, g)$ and fix an identification between $\Sigma_{K}^{1}$ and $\Sigma_{g}^{1}$, there is still some ambiguity regarding the choice of monodromy factorization. For a given homeomorphism $\phi: \Sigma_{g}^{1} \rightarrow \Sigma_{g}^{1}$ that fixes $\partial \Sigma_{g}^{1}$ pointwise, there is a fiber-preserving homeomorphism

$$
\left(I \times \Sigma_{g}^{1}\right) / \sim_{\varphi_{K}} \rightarrow\left(I \times \Sigma_{g}^{1}\right) / \sim_{\phi \circ \varphi_{K} \circ \phi^{-1}}
$$

[3, 5.B]. Hence we do not change the fixed generic fiber and corresponding monodromy factorization $\eta_{1, g}^{2}$ of $M(2, g)$, but the gluing map is changed to $\Phi \circ \Phi_{K} \circ \Phi^{-1}$, where $\Phi$ is the extension of the homeomorphism $\phi$ to $\Sigma_{2 g+1}$. We can interpret this phenomenon as a change of chosen generic fiber in $M(2, g)$ so that the monodromy factorization becomes $\Phi^{-1}\left(\eta_{1, g}^{2}\right)$. But it does not mean that $\Phi_{K}\left(\eta_{1, g}^{2}\right) \cdot \eta_{1, g}^{2}$ is isomorphic to $\left(\Phi \circ \Phi_{K} \circ \Phi^{-1}\right)\left(\eta_{1, g}^{2}\right) \cdot \eta_{1, g}^{2}$ as a marked Lefschetz fibration. We will consider this phenomenon in Section 4.

## 3. Isomorphic Lefschetz Fibrations

In this section we construct examples of simply connected isomorphic symplectic Lefschetz fibrations with the same generic fiber but coming from a pair of inequivalent fibered knots. In [6], Fintushel and Stern constructed families of simply connected symplectic 4-manifolds with the same Seiberg-Witten invariants. Among them, they considered a set of the following symplectic 4-manifolds,

$$
\begin{aligned}
&\left\{Y\left(2 ; K_{1}, K_{2}\right):=E(2)_{K_{1}} \nVdash_{\mathrm{id}: \Sigma_{2 g+1} \rightarrow \Sigma_{2 g+1}} E(2)_{K_{2}}\right. \\
&\left.K_{1}, K_{2} \text { are genus } g \text { fibered knots }\right\}
\end{aligned}
$$

and they showed that

$$
\mathcal{S} \mathcal{W}_{Y\left(2 ; K_{1}, K_{2}\right)}=t_{L}+t_{L}^{-1}
$$

because the only basic classes of $Y\left(2 ; K_{1}, K_{2}\right)$ are $\pm L$, where $L$ is the canonical class of $Y\left(2 ; K_{1}, K_{2}\right)$. In [23] we found examples such that $Y\left(2 ; K, K_{1}\right)$ and $Y\left(2 ; K, K_{2}\right)$ are diffeomorphic even though $K_{1}$ is not equivalent to $K_{2}$. In this section we will generalize such a construction. That is, we will construct infinitely many pairs ( $K, K^{\prime}$ ) of inequivalent genus 2 fibered knots such that all the $Y\left(2 ; K, K^{\prime}\right)$ are mutually diffeomorphic.

A family of inequivalent knots with the same Alexander polynomials has been constructed by several authors. Among them, Kinoshita and Terasaka [16] constructed a nontrivial knot with the trivial Alexander polynomial by using a knot union operation. Thereafter, Kanenobu constructed infinitely many inequivalent knots $K_{p, q}(p, q \in \mathbb{Z})$ with the same Alexander polynomials [13; 14]. These examples were constructed from the ribbon fibered knot $4_{1} \#\left(-4_{1}^{*}\right)$ by repeatedly applying the Stallings' twist [21] at two different locations where $K^{*}$ is the mirror image of $K$.

The following lemma was cited by Kanenobu.
Lemma 3.1 [13]. Let $K_{p, q}$ be a Kanenobu knot as in Figure 2. Then
(1) $K_{0,0}=4_{1} \#\left(-4_{1}^{*}\right)$,
(2) the Alexander matrix of $K_{p, q}$ is $\left(\begin{array}{cc}t^{2}-3 t+1 & (p-q) t \\ 0 & t^{2}-3 t+1\end{array}\right)$,
(3) $\Delta_{K_{p, q}}(t) \doteq\left(t-3+t^{-1}\right)^{2}$,
(4) $K_{p, q}$ is a fibered ribbon knot,
(5) $K_{p, q} \sim K_{r, s}$ if and only if $(p, q)=(r, s)$ or $(s, r)$,
(6) $K_{p, q}^{*} \sim K_{-q,-p}$, and
(7) $K_{p, q}$ is a prime knot if $(p, q) \neq(0,0)$.


Figure 2 A Kanenobu $\operatorname{knot} K_{p, q}$
It is not hard to see [10] that the monodromy map $\Phi_{K_{p, q}}$ of a Kanenobu knot $K_{p, q}$ is

$$
\Phi_{K_{p, q}}=t_{d}^{q} \circ t_{c_{2}}^{p} \circ t_{a_{2}} \circ t_{b_{2}}^{-1} \circ t_{a_{1}}^{-1} \circ t_{b_{1}}
$$

where $\left\{a_{i}, b_{i}, c_{i}, d\right\}$ are the simple closed curves shown in Figure 3. The reason is that we first perform Hopf plumbings of right-handed Hopf bands along the arc $b_{1}$ and of left-handed Hopf bands along $b_{2}$ and then perform Hopf plumbings of left-handed Hopf bands along arcs $a_{1}$ and of right-handed Hopf bands along $a_{2}$; see Figure 4. After that, we repeatedly perform Stallings' twists along simple closed curves $c_{2}$ and $d$ as in Figure 4. The result is a monodromy of the fibered knot $K_{p, q}$ corresponding to the fiber surface, as in the right-hand side of Figure 4. We can naturally identify the simple closed curves $a_{1}, b_{1}, a_{2}, b_{2}, c_{2}$, and $d$ in Figure 4 with the same lettered curves on the surface $\Sigma_{5}$ in Figure 3. We will read the monodromy factorization $\xi_{p, q}$ of $E(2)_{K_{p, q}}$ as a genus 5 Lefschetz fibration by using this identification.


Figure 3 Standard simple closed curves


Figure 4 A fiber surface of $K_{p, q}$

Then we get that $Y\left(2 ; K_{p, q}, K_{r, s}\right)$ has a monodromy factorization of the form

$$
\Phi_{K_{r, s}}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2} \cdot \Phi_{K_{p, q}}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2}
$$

Lemma 3.2. For any $p, q \in \mathbb{Z}$ and $\Phi_{K_{p, q}}=t_{d}^{q} \circ t_{c_{2}}^{p} \circ t_{a_{2}} \circ t_{b_{2}}^{-1} \circ t_{a_{1}}^{-1} \circ t_{b_{1}}$, we have

$$
t_{c_{2}} \in G_{F}\left(\eta_{1,2}^{2} \cdot t_{c_{2}}\left(\eta_{1,2}^{2}\right)\right), \quad t_{d} \in G_{F}\left(\eta_{1,2}^{2} \cdot t_{d}\left(\eta_{1,2}^{2}\right)\right)
$$

and

$$
t_{c_{2}} \in G_{F}\left(\Phi_{K_{p+1, q}}\left(\eta_{1,2}^{2}\right) \cdot \Phi_{K_{p, q}}\left(\eta_{1,2}^{2}\right)\right), \quad t_{d} \in G_{F}\left(\Phi_{K_{p, q+1}}\left(\eta_{1,2}^{2}\right) \cdot \Phi_{K_{p, q}}\left(\eta_{1,2}^{2}\right)\right)
$$

Proof. Since $B_{2}$ and $c_{2}$ meet at one point on $\Sigma_{5}$, by the braid relation we get

$$
t_{c_{2}} \circ t_{B_{2}} \circ t_{c_{2}}=t_{B_{2}} \circ t_{c_{2}} \circ t_{B_{2}}
$$

This implies that

$$
t_{c_{2}}=t_{B_{2}} \circ t_{c_{2}} \circ t_{B_{2}} \circ t_{c_{2}}^{-1} \circ t_{B_{2}}^{-1}=t_{B_{2}} \circ t_{c_{2}}\left(t_{B_{2}}\right) \circ t_{B_{2}}^{-1}
$$

Since $t_{B_{2}}, t_{c_{2}}\left(t_{B_{2}}\right) \in G_{F}\left(\eta_{1,2}^{2} \cdot t_{c_{2}}\left(\eta_{1,2}^{2}\right)\right)$, we get

$$
t_{c_{2}} \in G_{F}\left(\eta_{1,2}^{2} \cdot t_{c_{2}}\left(\eta_{1,2}^{2}\right)\right)
$$

Each of $B_{1}, B_{2}, B_{3}, B_{4}$ meets at one point with the simple closed curve $d$. So by the braid relation we get

$$
t_{d} \circ t_{B_{i}} \circ t_{d}=t_{B_{i}} \circ t_{d} \circ t_{B_{i}}, \quad i=1,2,3,4,
$$

which implies

$$
t_{d}=t_{B_{i}} \circ t_{d}\left(t_{B_{i}}\right) \circ t_{B_{i}}^{-1}, \quad i=1,2,3,4 .
$$

Since $t_{B_{i}}, t_{d}\left(t_{B_{i}}\right) \in G_{F}\left(\eta_{1,2}^{2} \cdot t_{d}\left(\eta_{1,2}^{2}\right)\right)$, we get

$$
t_{d} \in G_{F}\left(\eta_{1,2}^{2} \cdot t_{d}\left(\eta_{1,2}^{2}\right)\right)
$$

Observe that $\Phi_{K_{0,0}}\left(B_{3}\right)$ meets with $c_{2}$ at one point and $\Phi_{K_{0,0}}\left(B_{4}\right)$ meets with $d$ at one point. Therefore,

$$
\begin{aligned}
t_{\Phi_{K_{0,0}}\left(B_{3}\right)} \circ t_{c_{2}} \circ t_{\Phi_{K_{0,0}}\left(B_{3}\right)} & =t_{c_{2}} \circ t_{\Phi_{K_{0,0}}\left(B_{3}\right)} \circ t_{c_{2}} \\
t_{\Phi_{K_{0,0}}\left(B_{4}\right)} \circ t_{d} \circ t_{\Phi_{K_{0,0}}\left(B_{4}\right)} & =t_{d} \circ t_{\Phi_{K_{0,0}}\left(B_{4}\right)} \circ t_{d} .
\end{aligned}
$$

This implies that

$$
\begin{aligned}
t_{c_{2}}= & t_{d}^{q} \circ t_{c_{2}}^{p} \circ t_{c_{2}} \circ t_{c_{2}}^{-p} \circ t_{d}^{-q} \\
= & t_{d}^{q} \circ t_{c_{2}}^{p} \circ\left(t_{\Phi_{K_{0,0}}\left(B_{3}\right)}^{-q} \circ t_{c_{2}} \circ t_{\Phi_{K_{0,0}}\left(B_{3}\right)} \circ t_{c_{2}}^{-1} \circ t_{\left.\Phi_{K_{0,0}\left(B_{3}\right)}^{-1}\right)}\right) \circ t_{c_{2}}^{-p} \circ t_{d}^{-q} \\
= & t_{d}^{q} \circ t_{c_{2}}^{p} \circ\left(\Phi_{K_{0,0}} \circ t_{B_{3}} \circ \Phi_{K_{0,0}}^{-1}\right) \circ t_{c_{2}} \circ\left(\Phi_{K_{0,0}} \circ t_{B_{3}} \circ \Phi_{K_{0,0}}^{-1}\right) \\
& \circ t_{c_{2}}^{-1} \circ\left(\Phi_{K_{0,0}} \circ t_{B_{3}}^{-1} \circ \Phi_{K_{0,0}}^{-1}\right) \circ t_{c_{2}}^{-p} \circ t_{d}^{-q} \\
= & t_{d}^{q} \circ t_{c_{2}}^{p} \circ\left(\Phi_{K_{0,0}} \circ t_{B_{3}} \circ \Phi_{K_{0,0}}^{-1}\right) \circ\left(t_{c_{2}}^{-p} \circ t_{d}^{-q} \circ t_{c_{2}}^{p+1} \circ t_{d}^{q}\right) \circ\left(\Phi_{K_{0,0}} \circ t_{B_{3}} \circ \Phi_{K_{0,0}}^{-1}\right) \\
& \circ\left(t_{c_{2}}^{-p-1} \circ t_{d}^{-q} \circ t_{c_{2}}^{p} \circ t_{d}^{q}\right) \circ\left(\Phi_{K_{0,0}} \circ t_{B_{3}}^{-1} \circ \Phi_{K_{0,0}}^{-1}\right) \circ t_{c_{2}}^{-p} \circ t_{d}^{-q} \\
= & \left(t_{d}^{q} \circ t_{c_{2}}^{p} \circ \Phi_{K_{0,0}}\right) \circ t_{B_{3}} \circ\left(\Phi_{K_{0,0}}^{-1} \circ t_{c_{2}}^{-p} \circ t_{d}^{-q}\right) \circ\left(t_{c_{2}}^{p+1} \circ t_{d}^{q} \circ \Phi_{K_{0,0}}\right) \circ t_{B_{3}} \\
& \circ\left(\Phi_{K_{0,0}}^{-1} \circ t_{c_{2}}^{-p-1} \circ t_{d}^{-q}\right) \circ\left(t_{c_{2}}^{p} \circ t_{d}^{q} \circ \Phi_{K_{0,0}}\right) \circ t_{B_{3}}^{-q} \circ\left(\Phi_{K_{0,0}}^{-1} \circ t_{c_{2}}^{-p} \circ t_{d}^{-q}\right) \\
= & t_{\Phi_{K_{p, q}}\left(B_{3}\right)} \circ t_{\Phi_{K_{p+1, q}}\left(B_{3}\right)} \circ t_{\Phi_{K_{p, q}}\left(B_{3}\right)}^{-1} .
\end{aligned}
$$

By the same method we also get

$$
t_{d}=t_{\Phi_{K_{p, q}}\left(B_{4}\right)} \circ t_{\Phi_{K_{p, q+1}}\left(B_{4}\right)} \circ t_{\Phi_{K_{p, q}\left(B_{4}\right)}}^{-1}
$$

Since

$$
\Phi_{K_{p, q}}\left(t_{B_{3}}\right), \Phi_{K_{p+1, q}}\left(t_{B_{3}}\right) \in G_{F}\left(\Phi_{K_{p+1, q}}\left(\eta_{1,2}^{2}\right) \cdot \Phi_{K_{p, q}}\left(\eta_{1,2}^{2}\right)\right)
$$

and

$$
\Phi_{K_{p, q}}\left(t_{B_{4}}\right), \Phi_{K_{p, q+1}}\left(t_{B_{4}}\right) \in G_{F}\left(\Phi_{K_{p, q+1}}\left(\eta_{1,2}^{2}\right) \cdot \Phi_{K_{p, q}}\left(\eta_{1,2}^{2}\right)\right),
$$

we obtain the conclusion
$t_{c_{2}} \in G_{F}\left(\Phi_{K_{p+1, q}}\left(\eta_{1,2}^{2}\right) \cdot \Phi_{K_{p, q}}\left(\eta_{1,2}^{2}\right)\right), \quad t_{d} \in G_{F}\left(\Phi_{K_{p, q+1}}\left(\eta_{1,2}^{2}\right) \cdot \Phi_{K_{p, q}}\left(\eta_{1,2}^{2}\right)\right)$.
Lemma 3.3 [23]. Let $W_{i}=w_{i, n_{i}} \cdots w_{i, 2} \cdot w_{i, 1}$ be a sequence of right-handed Dehn twists along a simple closed curves on $\Sigma_{g}$ such that $\lambda_{W_{i}}:=w_{i, n_{i}} \circ \cdots \circ w_{i, 1}=$ id in $\mathcal{M}_{F}$ for $i=1,2$. Then

$$
W_{1} \cdot W_{2} \sim W_{2} \cdot W_{1}
$$

Suppose $f \in G\left(W_{2}\right)$; then

$$
f\left(W_{1}\right) \cdot W_{2} \sim W_{1} \cdot W_{2}
$$

Theorem 3.4. For each pair $p, q \in \mathbb{Z}$, we get diffeomorphisms

$$
Y\left(2 ; K_{p, q}, K_{p+1, q}\right) \approx Y\left(2 ; K_{p+1, q}, K_{p+2, q}\right)
$$

and

$$
Y\left(2 ; K_{p, q}, K_{p, q+1}\right) \approx Y\left(2 ; K_{p, q+1}, K_{p, q+2}\right)
$$

Proof. $Y\left(2 ; K_{p, q}, K_{p+1, q}\right)$ has a monodromy factorization of the form

$$
\Phi_{K_{p+1, q}}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2} \cdot \Phi_{K_{p, q}}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2}
$$

where $\Phi_{K_{p, q}}=t_{d}^{q} \circ t_{c_{2}}^{p} \circ t_{a_{2}} \circ t_{b_{2}}^{-1} \circ t_{a_{1}}^{-1} \circ t_{b_{1}}$.
By Lemma 3.2, we have

$$
\begin{aligned}
& t_{c_{2}} \in G_{F}\left(\Phi_{K_{p+1, q}}\left(\eta_{1,2}^{2}\right) \cdot \Phi_{K_{p, q}}\left(\eta_{1,2}^{2}\right)\right), \\
& t_{c_{2}} \in G_{F}\left(t_{c_{2}}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2}\right) .
\end{aligned}
$$

Therefore,

$$
\begin{align*}
\Phi_{K_{p+1, q}} & \left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2} \cdot \Phi_{K_{p, q}}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2}  \tag{3.1}\\
& \sim \eta_{1,2}^{2} \cdot \Phi_{K_{p+1, q}}\left(\eta_{1,2}^{2}\right) \cdot \Phi_{K_{p, q}}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2}  \tag{3.2}\\
& \sim t_{c_{2}}\left(\eta_{1,2}^{2}\right) \cdot \Phi_{K_{p+1, q}}\left(\eta_{1,2}^{2}\right) \cdot \Phi_{K_{p, q}}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2}  \tag{3.3}\\
& \sim \Phi_{K_{p+1, q}}\left(\eta_{1,2}^{2}\right) \cdot \Phi_{K_{p, q}}\left(\eta_{1,2}^{2}\right) \cdot t_{c_{2}}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2}  \tag{3.4}\\
& \sim \Phi_{K_{p, q}}\left(\eta_{1,2}^{2}\right) \cdot \Phi_{K_{p-1, q}}\left(\eta_{1,2}^{2}\right) \cdot t_{c_{2}}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2}  \tag{3.5}\\
& \sim t_{c_{2}}\left(\eta_{1,2}^{2}\right) \cdot \Phi_{K_{p, q}}\left(\eta_{1,2}^{2}\right) \cdot \Phi_{K_{p-1, q}}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2}  \tag{3.6}\\
& \sim \eta_{1,2}^{2} \cdot \Phi_{K_{p, q}}\left(\eta_{1,2}^{2}\right) \cdot \Phi_{K_{p-1, q}}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2}  \tag{3.7}\\
& \sim \Phi_{K_{p, q}}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2} \cdot \Phi_{K_{p-1, q}}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2} . \tag{3.8}
\end{align*}
$$

In particular:

- since $\lambda_{\eta_{1,2}^{2}}=$ id, we get (3.1) to (3.2), (3.3) to (3.4), (3.5) to (3.6), and (3.7) to (3.8);
- Lemma 3.2 and Lemma 3.3 together imply (3.2) to (3.3), (3.4) to (3.5), and (3.6) to (3.7).
This implies that, for each fixed $q, Y\left(2 ; K_{p, q}, K_{p+1, q}\right)$ and $Y\left(2 ; K_{p-1, q}, K_{p, q}\right)$ have isomorphic Lefschetz fibration structure; hence they are diffeomorphic.

Similarly, by using

$$
\begin{aligned}
& t_{d} \in G_{F}\left(\Phi_{K_{p, q+1}}\left(\eta_{1,2}^{2}\right) \cdot \Phi_{K_{p, q}}\left(\eta_{1,2}^{2}\right)\right), \\
& t_{d} \in G_{F}\left(t_{d}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2}\right)
\end{aligned}
$$

in Lemma 3.2 we obtain

$$
Y\left(2 ; K_{p, q}, K_{p, q+1}\right) \approx Y\left(2 ; K_{p, q+1}, K_{p, q+2}\right)
$$

## 4. Nonisomorphic Lefschetz Fibrations

In this section we investigate some algebraic and graph-theoretic properties of $\xi_{p, q}=\Phi_{K_{p, q}}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2}$ and its monodromy group $G_{\Sigma_{5}}\left(\xi_{p, q}\right)$ corresponding to the fixed generic fiber $\Sigma_{5}$. In [11], Humphries showed that the minimal number of Dehn twist generators of the mapping class group $\mathcal{M}_{g}$ or $\mathcal{M}_{g}^{1}$ is $2 g+1$; he did this by using symplectic transvection and the Euler number (mod 2) of a graph.

Definition 4.1 [11]. Let $\left\{\gamma_{1}, \gamma_{2}, \ldots, \gamma_{2 g}\right\}$ be a set of simple closed curves on $\Sigma_{g}$ that generate $H_{1}\left(\Sigma_{g} ; \mathbb{Z}_{2}\right)$. Let $\Gamma\left(\gamma_{1}, \gamma_{2}, \ldots, \gamma_{2 g}\right)$ be a graph defined by:

- a vertex for each homology class $\left[\gamma_{i}\right]$ of simple closed curves $\gamma_{i}, i=1,2, \ldots, 2 g$;
- an edge between $\gamma_{i}$ and $\gamma_{j}$ if $i_{2}\left(\gamma_{i}, \gamma_{j}\right)=1$, where $i_{2}\left(\gamma_{i}, \gamma_{j}\right)$ is the modulo 2 algebraic intersection between $\left[\gamma_{i}\right]$ and $\left[\gamma_{j}\right]$; and
- no intersections between any two edges.

Let $\gamma$ be a simple closed curve on $\Sigma_{g}$ such that $[\gamma]=\sum_{i=1}^{2 g} \varepsilon_{i}\left[\gamma_{i}\right]\left(\varepsilon_{i}=0\right.$ or 1$)$ as an element of $H_{1}\left(\Sigma_{g} ; \mathbb{Z}_{2}\right)$. We define $\bar{\gamma}:=\bigcup_{\varepsilon_{i}=1} \overline{\gamma_{i}}$, where $\overline{\gamma_{i}}$ is the union of all closures of half-edges with one end vertex $\gamma_{i}$. We define $\chi_{\Gamma}(\gamma):=\chi(\bar{\gamma})(\bmod 2)$, where $\chi(\bar{\gamma})$ is the Euler number of the graph $\bar{\gamma}$.

Lemma 4.2 [11]. Let $\Gamma\left(\gamma_{1}, \ldots, \gamma_{2 g}\right)$ be the graph of simple closed curves $\left\{\gamma_{1}, \ldots, \gamma_{2 g}\right\}$ that generate the $\mathbb{Z}_{2}$ vector space $H_{1}\left(\Sigma_{g} ; \mathbb{Z}_{2}\right)$. Let $G_{\Gamma, g}$ be the subgroup of $\mathcal{M}_{g}$ that is generated by
$\left\{t_{\alpha} \mid \alpha\right.$ is a nonseparating simple closed curve on $\Sigma_{g}$ such that $\left.\chi_{\Gamma}(\alpha)=1\right\}$.
Then $G_{\Gamma, g}$ is a nontrivial proper subgroup of $\mathcal{M}_{g}$. Moreover, if $\beta$ is a nonseparating simple closed curve on $\Sigma_{g}$ with $\chi_{\Gamma}(\beta)=0$, then $t_{\beta} \notin G_{\Gamma, g}$.

Proof. Let us prove that $G_{\Gamma, g}$ is a nontrivial proper subgroup of $\mathcal{M}_{g}$.
The mapping class group $\mathcal{M}_{g}$ acts transitively on $H_{1}\left(\Sigma_{g} ; \mathbb{Z}_{2}\right) \backslash\{0\}$. The action is defined by

$$
t_{c}: H_{1}\left(\Sigma_{g} ; \mathbb{Z}_{2}\right) \rightarrow H_{1}\left(\Sigma_{g} ; \mathbb{Z}_{2}\right), \quad t_{c}(x)=i_{2}(c, x)[c]+x,
$$

where $c$ is a simple closed curve on $\Sigma_{g}, x \in H_{1}\left(\Sigma_{g} ; \mathbb{Z}_{2}\right)$, and $i_{2}(c, x)$ is the modulo 2 algebraic intersection number between $[c]$ and $x$.

If $c$ is a nonseparating simple closed curve on $\Sigma_{g}$ such that $\chi_{\Gamma}(c)=1$, then in $H_{1}\left(\Sigma_{g} ; \mathbb{Z}_{2}\right)$ we have

$$
t_{c}([\gamma])= \begin{cases}{[\gamma]} & \text { if } i_{2}(c, \gamma)=0 \\ {[c]+[\gamma]} & \text { if } i_{2}(c, \gamma)=1\end{cases}
$$

For the $i_{2}(c, \gamma)=0$ case, it is clear that $\chi_{\Gamma}\left(t_{c}(\gamma)\right)=\chi_{\Gamma}(\gamma)$. For the $i_{2}(c, \gamma)=1$ case, if $[c]=\sum_{i=1}^{2 g} \varepsilon_{c, i}\left[\gamma_{i}\right]$ and $[\gamma]=\sum_{i=1}^{2 g} \varepsilon_{\gamma, i}\left[\gamma_{i}\right]$ in $H_{1}\left(\Sigma_{g} ; \mathbb{Z}_{2}\right)$, then

$$
\overline{t_{c}(\gamma)}=\bigcup_{\varepsilon_{c, i}+\varepsilon_{\gamma, i}=1} \overline{\gamma_{i}}
$$

Let

$$
\begin{aligned}
& A=\sum_{\varepsilon_{c, i}=1, \varepsilon_{\gamma, i}=1}\left[\gamma_{i}\right], \\
& B=\sum_{\varepsilon_{c, i}=1, \varepsilon_{\gamma, i}=0}\left[\gamma_{i}\right], \\
& C=\sum_{\varepsilon_{c, i}=0, \varepsilon_{\gamma, i}=1}\left[\gamma_{i}\right] .
\end{aligned}
$$

Then

$$
\begin{aligned}
& \chi(\bar{c})=\chi(\bar{A} \cup \bar{B})=\chi(\bar{A})+\chi(\bar{B})+i_{2}(A, B)(\bmod 2), \\
& \chi(\bar{\gamma})=\chi(\bar{A} \cup \bar{C})=\chi(\bar{A})+\chi(\bar{C})+i_{2}(A, C)(\bmod 2), \\
& \chi\left(\overline{t_{c}(\gamma)}\right)=\chi(\bar{B} \cup \bar{C})=\chi(\bar{B})+\chi(\bar{C})+i_{2}(B, C)(\bmod 2),
\end{aligned}
$$

and $i_{2}(c, \gamma)=i_{2}(A+B, A+C)=i_{2}(A, A)+i_{2}(A, B)+i_{2}(A, C)+i_{2}(B, C)=$ $i_{2}(A, B)+i_{2}(A, C)+i_{2}(B, C)(\bmod 2)$ because $i_{2}(A, A)=0$. Therefore,

$$
\chi_{\Gamma}\left(t_{c}(\gamma)\right)=\chi\left(\overline{t_{c}(\gamma)}\right)=\chi(\bar{c})+\chi(\bar{\gamma})+i_{2}(c, \gamma)=\chi(\bar{\gamma})=\chi_{\Gamma}(\gamma)(\bmod 2)
$$

For any $f \in G_{\Gamma, g}, f$ is of the form $t_{c_{k}}^{\varepsilon_{k}} \circ t_{c_{k-1}}^{\varepsilon_{k-1}} \circ \cdots \circ t_{c_{2}}^{\varepsilon_{2}} \circ t_{c_{1}}^{\varepsilon_{1}}$, where each $c_{i}$ is a nonseparating simple closed curve with $\chi_{\Gamma}\left(c_{i}\right)=1$ and $\varepsilon_{i} \in\{ \pm 1\}$. This implies that $\chi_{\Gamma}(f(\gamma)) \equiv \chi_{\Gamma}(\gamma)(\bmod 2)$. Therefore, if $G_{\Gamma, g}=\mathcal{M}_{g}$ then, for any nonseparating simple closed curves $\gamma$ on $\Sigma_{g}$, we must have $\chi_{\Gamma}(\gamma)=1$-which is clearly impossible. Hence $G_{\Gamma, g}$ is a nontrivial proper subgroup of $\mathcal{M}_{g}$.

Let $\beta$ be a nonseparating simple closed curve with $\chi_{\Gamma}(\beta)=0$. Then, for a simple closed curve $\gamma$ on $\Sigma_{g}$ with $i_{2}(\beta, \gamma)=1$, we have $\chi_{\Gamma}\left(t_{\beta}(\gamma)\right) \not \equiv \chi_{\Gamma}(\gamma)(\bmod 2)$. Therefore, $t_{\beta} \notin G_{\Gamma, g}$.

Remark 4.3. By Lemma 4.2, we know that:

- if $\chi_{\Gamma}(c)=1$ then, for any $\gamma$,

$$
\chi_{\Gamma}\left(t_{c}(\gamma)\right)=\chi_{\Gamma}(\gamma)
$$

- if $\chi_{\Gamma}(c)=0$ then, for any $\gamma$,

$$
\chi_{\Gamma}\left(t_{c}(\gamma)\right)=\chi_{\Gamma}(\gamma)+i_{2}(c, \gamma)
$$

Lemma 4.4. For each pair of integers $(p, q)$ there is a basis $\mathcal{B}_{i}$ for $H_{1}\left(\Sigma_{5} ; \mathbb{Z}_{2}\right)$ (depending only on $(p, q)$ modulo 2 ) with the property that

$$
G_{F}\left(\xi_{p, q}\right) \leq G_{\Gamma_{i}, 5}
$$

but with $\chi_{\Gamma_{i}}\left(c_{2}\right)=\chi_{\Gamma_{i}}(d)=0$, where $\Gamma_{i}$ is the corresponding graph to a basis $\mathcal{B}_{i}$.
Proof. We will prove this in four cases.
Case 1: $p$ and $q$ are even integers. Let us consider a basis

$$
\mathcal{B}_{1}=\left\{c_{1}, a_{1}, a_{2}, b_{2}, a_{3}, b_{3}, a_{4}, a_{5}, B_{2}, B_{4}\right\}
$$

of $H_{1}\left(\Sigma_{5} ; \mathbb{Z}_{2}\right)$, where $\left\{a_{i}, b_{i}, c_{i}, d_{i}, B_{i}\right\}$ are simple closed curves on $\Sigma_{5}$ as in Figure 1 and Figure 3. Then the graph of $\mathcal{B}_{1}$,

$$
\Gamma_{1}=\Gamma\left(\left\{c_{1}, a_{1}, a_{2}, b_{2}, a_{3}, b_{3}, a_{4}, a_{5}, B_{2}, B_{4}\right\}\right)
$$

is as given in Figure 5.


Figure 5 Graph $\Gamma_{1}$

We can easily obtain the following relations in $H_{1}\left(\Sigma_{5} ; \mathbb{Z}_{2}\right)$ :

$$
\begin{aligned}
B_{0} & =a_{1}+a_{2}+a_{3}+a_{4}+a_{5}, \\
B_{1} & =B_{2}+a_{1}+a_{5}, \\
B_{3} & =B_{4}+a_{2}+a_{4}, \\
B_{5} & =a_{3}=\Phi_{K_{0,0}}\left(B_{5}\right) ; \\
\Phi_{K_{0,0}}\left(B_{4}\right) & =B_{4}+a_{2}, \\
\Phi_{K_{0,0}}\left(B_{3}\right) & =B_{4}+a_{2}+a_{4}+b_{2}, \\
\Phi_{K_{0,0}}\left(B_{2}\right) & =B_{2}+a_{1}+b_{2}+a_{2}, \\
\Phi_{K_{0,0}}\left(B_{1}\right) & =B_{2}+a_{1}+a_{2}+a_{5}+c_{1}+b_{2}, \\
\Phi_{K_{0,0}}\left(B_{0}\right) & =a_{3}+a_{4}+a_{5}+c_{1}+b_{2} .
\end{aligned}
$$

Hence the graph yields

$$
\chi_{\Gamma_{1}}\left(a_{i}\right)=\chi_{\Gamma_{1}}\left(B_{i}\right)=\chi_{\Gamma_{1}}\left(\Phi_{K_{0,0}}\left(B_{i}\right)\right)=1 \quad \text { for } i=0,1,2,3,4,5
$$

and $\chi_{\Gamma_{1}}\left(c_{1}\right)=\chi_{\Gamma_{1}}\left(c_{6}\right)=1$. So we have

$$
\left\{t_{B_{i}}, \Phi_{K_{0,0}}\left(t_{B_{i}}\right), t_{a_{j}}, t_{b_{3}}, t_{b_{3}^{\prime}}, t_{c_{1}}, t_{c_{6}} \mid i=0,1,2,3,4,5, j=1,2,3,4,5\right\}
$$

and each generator of the group $G_{F}\left(\Phi_{K_{0,0}}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2}\right)$ is an element of $G_{\Gamma_{1}, 5}$. This implies that $G_{F}\left(\Phi_{K_{0,0}}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2}\right) \leq G_{\Gamma_{1}, 5}$.

But we have

$$
\chi_{\Gamma_{1}}\left(c_{j}\right)=\chi_{\Gamma_{1}}(d)=0
$$

for $j=2,3,4,5$ and therefore

$$
t_{c_{2}}, t_{c_{3}}, t_{c_{4}}, t_{c_{5}}, t_{d} \notin G_{\Gamma_{1}, 5}
$$

This implies that $t_{c_{2}}, t_{d} \notin G_{F}\left(\Phi_{K_{0,0}}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2}\right)$.
Since the $\mathbb{Z}_{2}$-homology class of $\Phi_{K_{2 p, 2 q}}\left(B_{i}\right)$ and $\Phi_{K_{0,0}}\left(B_{i}\right)$ are the same for any $p, q \in \mathbb{Z}$, we get

$$
\chi_{\Gamma_{1}}\left(\Phi_{K_{2 p, 2 q}}\left(B_{i}\right)\right)=\chi_{\Gamma_{1}}\left(\Phi_{K_{0,0}}\left(B_{i}\right)\right)
$$

for $i=0,1,2,3,4,5$. This implies that $G_{F}\left(\Phi_{K_{2 p, 2 q}}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2}\right) \leq G_{\Gamma_{1}, 5}$, so we have $t_{c_{2}}, t_{d} \notin G_{F}\left(\Phi_{K_{2 p, 2 q}}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2}\right)$.

Case 2: $p$ is an odd integer and $q$ is an even integer. Let us consider a basis $\mathcal{B}_{2}=\left\{a_{3}, b_{3}, B_{1}, B_{2}, B_{3}, B_{4}, d_{1}, d_{2}, d_{3}, d_{4}\right\}$ of $\mathbb{Z}_{2}$-vector space $H_{1}\left(\Sigma_{5} ; \mathbb{Z}_{2}\right)$ and its graph

$$
\Gamma_{2}=\Gamma\left(\left\{a_{3}, b_{3}, B_{1}, B_{2}, B_{3}, B_{4}, d_{1}, d_{2}, d_{3}, d_{4}\right\}\right)
$$

here $\left\{a_{i}, b_{i}, c_{i}, d_{i}, B_{i}\right\}$ are simple closed curves on $\Sigma_{5}$ as in Figure 1, Figure 3, and Figure 6. Then the graph $\Gamma_{2}$ is as in Figure 7.


Figure 6 Simple closed curves $d_{i}$


Figure 7 Graph $\Gamma_{2}$

Since $\Phi_{K_{1,0}}=t_{c_{2}} \circ t_{a_{2}} \circ t_{b_{2}}^{-1} \circ t_{a_{1}}^{-1} \circ t_{b_{1}}$, we get the following relations in $H_{1}\left(\Sigma_{5} ; \mathbb{Z}_{2}\right)$ :

$$
\begin{aligned}
B_{0} & =B_{1}+B_{2}+B_{3}+B_{4}+a_{3} \\
\Phi_{K_{1,0}}\left(B_{0}\right) & =B_{1}+B_{2}+B_{4}+b_{3}+d_{1}+d_{2}+d_{4} \\
\Phi_{K_{1,0}}\left(B_{1}\right) & =B_{1}+B_{3}+B_{4}+a_{3}+d_{2} \\
\Phi_{K_{1,0}}\left(B_{2}\right) & =B_{2}+B_{3}+B_{4}+b_{3}+d_{1}+d_{2}+d_{3} \\
\Phi_{K_{1,0}}\left(B_{3}\right) & =B_{3}+b_{3}+d_{3} \\
\Phi_{K_{1,0}}\left(B_{4}\right) & =B_{3}+B_{4}+b_{3}+d_{2}+d_{4} \\
\Phi_{K_{1,0}}\left(B_{5}\right) & =B_{5}=a_{3} \\
c_{2} & =a_{3}+b_{3}+d_{4}+B_{4} \\
d & =B_{3}+B_{4}+d_{1}+d_{2} .
\end{aligned}
$$

A computation of $\chi_{\Gamma_{2}}$ shows that

$$
\begin{equation*}
\chi_{\Gamma_{2}}\left(B_{i}\right)=\chi_{\Gamma_{2}}\left(\Phi_{K_{1,0}}\left(B_{i}\right)\right)=\chi_{\Gamma_{2}}\left(b_{3}\right)=\chi_{\Gamma_{2}}\left(b_{3}^{\prime}\right)=\chi_{\Gamma_{2}}\left(a_{3}\right)=1 \tag{4.1}
\end{equation*}
$$

for each $i=0,1,2,3,4,5$ and that

$$
\begin{equation*}
\chi_{\Gamma_{2}}\left(c_{1}\right)=\chi_{\Gamma_{2}}\left(c_{2}\right)=\chi_{\Gamma_{2}}\left(a_{1}\right)=\chi_{\Gamma_{2}}\left(a_{2}\right)=\chi_{\Gamma_{2}}\left(b_{2}\right)=\chi_{\Gamma_{2}}(d)=0 \tag{4.2}
\end{equation*}
$$

Hence $G_{F}\left(\Phi_{K_{1,0}}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2}\right) \leq G_{\Gamma_{2}, 5}$ and, since $t_{c_{2}}, t_{d} \notin G_{\Gamma_{2}, 5}$, we get

$$
t_{c_{2}}, t_{d} \notin G_{F}\left(\Phi_{K_{1,0}}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2}\right)
$$

Furthermore, since $\Phi_{K_{2 p+1,2 q}}\left(B_{i}\right)$ and $\Phi_{K_{1,0}}\left(B_{i}\right)$ represent the same element in $H_{1}\left(\Sigma_{2} ; \mathbb{Z}_{2}\right)$, we get $\chi_{\Gamma_{2}}\left(\Phi_{K_{2 p+1,2 q}}\left(B_{i}\right)\right)=\chi_{\Gamma_{2}}\left(\Phi_{K_{1,0}}\left(B_{i}\right)\right)=1$; this implies that

$$
t_{c_{2}}, t_{d} \notin G_{F}\left(\Phi_{K_{2 p+1,2 q}}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2}\right)
$$

for any $p, q \in \mathbb{Z}$ because $G_{F}\left(\Phi_{K_{2 p+1,2 q}}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2}\right) \leq G_{\Gamma_{2}, 5}$.
Case 3: $p$ is an even integer and $q$ is an odd integer. We want to find a graph

$$
\Gamma_{3}=\Gamma\left(\left\{\gamma_{1}, \gamma_{2}, \ldots, \gamma_{10}\right\}\right)
$$

satisfying

$$
\begin{equation*}
\chi_{\Gamma_{3}}\left(B_{i}\right)=\chi_{\Gamma_{3}}\left(\Phi_{K_{0,1}}\left(B_{i}\right)\right)=\chi_{\Gamma_{3}}\left(b_{3}\right)=\chi_{\Gamma_{3}}\left(b_{3}^{\prime}\right)=\chi_{\Gamma_{3}}\left(a_{3}\right)=1 \tag{4.3}
\end{equation*}
$$

for $i=0,1,2,3,4,5$ and

$$
\begin{equation*}
\chi_{\Gamma_{3}}\left(c_{2}\right)=\chi_{\Gamma_{3}}(d)=0 \tag{4.4}
\end{equation*}
$$

Note that we observe the following relations in $H_{1}\left(\Sigma_{5} ; \mathbb{Z}_{2}\right)$.

|  | $\Phi_{K_{0,0}}\left(B_{i}\right)$ | $\Phi_{K_{0,1}}\left(B_{i}\right)$ |
| :---: | :---: | :---: |
| $B_{0}$ | $B_{0}+a_{1}+b_{1}+a_{2}+b_{2}$ | $B_{0}+a_{1}+b_{1}+a_{2}+b_{2}$ |
| $B_{1}$ | $B_{1}+b_{1}+b_{2}+a_{2}$ | $B_{1}+b_{1}+a_{2}+b_{2}+d$ |
| $B_{2}$ | $B_{2}+a_{1}+b_{2}+a_{2}$ | $B_{2}+a_{1}+b_{2}+a_{2}$ |
| $B_{3}$ | $B_{3}+b_{2}$ | $B_{3}+b_{2}$ |
| $B_{4}$ | $B_{4}+a_{2}$ | $B_{4}+a_{2}+d$ |
| $B_{5}$ | $B_{5}$ | $B_{5}$ |

From equation (4.3), we may assume that $B_{i}(i=1,2,3,4), b_{3}$, and $a_{3}$ are in the generating set, which we will extend to a basis of $H_{1}\left(\Sigma_{5} ; \mathbb{Z}_{2}\right)$. For each $i=0,1,2,3,4,5, B_{i}$ and $\Phi_{K_{0,1}}\left(B_{i}\right)$ are elements of $G_{\Gamma_{3}, 5}$ at the same time. Since $i_{2}\left(\Phi_{K_{0,0}}\left(B_{0}\right), d\right)=0$, we get $\chi_{\Gamma_{3}}\left(\Phi_{K_{0,1}}\left(B_{0}\right)\right)=\chi_{\Gamma_{3}}\left(\Phi_{K_{0,0}}\left(B_{0}\right)\right)$. We also know that

$$
\begin{aligned}
i_{2}\left(B_{0}, b_{1}\right) & =i_{2}\left(t_{b_{1}}\left(B_{0}\right), a_{1}\right)=i_{2}\left(t_{a_{1}}^{-1}\left(t_{b_{1}}\left(B_{0}\right)\right), b_{2}\right) \\
& =i_{2}\left(t_{b_{2}}^{-1}\left(t_{a_{1}}^{-1}\left(t_{b_{1}}\left(B_{0}\right)\right)\right), a_{2}\right)=1
\end{aligned}
$$

So by Lemma 4.2 and Remark 4.3 it follows that

$$
\chi_{\Gamma_{3}}\left(\Phi_{K_{0,0}}\left(B_{0}\right)\right)=\chi_{\Gamma_{3}}\left(B_{0}\right)+\left|\left\{a_{1}, b_{1}, a_{2}, b_{2}\right\}-G_{\Gamma_{3}, 5}\right|=\chi_{\Gamma_{3}}\left(B_{0}\right) .
$$

Therefore, if $B_{0}$ and $\Phi_{K_{0,1}}\left(B_{0}\right)$ are elements of $G_{\Gamma_{3}, 5}$ at the same time, then an even number of elements in $\left\{a_{1}, b_{1}, a_{2}, b_{2}\right\}$ must have $\chi_{\Gamma_{3}}=0$. By the same method, we derive the following statements:

- an even number of elements in $\left\{b_{1}, b_{2}, a_{2}, d\right\}$ must have $\chi_{\Gamma_{3}}=0$ because $\chi_{\Gamma_{3}}\left(\Phi_{K_{0,1}}\left(B_{1}\right)\right)=\chi_{\Gamma_{3}}\left(B_{1}\right) ;$
- an even number of elements in $\left\{a_{1}, b_{2}, a_{2}\right\}$ must have $\chi_{\Gamma_{3}}=0$ because $\chi_{\Gamma_{3}}\left(\Phi_{K_{0,1}}\left(B_{2}\right)\right)=\chi_{\Gamma_{3}}\left(B_{2}\right) ;$
- an even number of elements in $\left\{b_{2}\right\}$ must have $\chi_{\Gamma_{3}}=0$ because $\chi_{\Gamma_{3}}\left(\Phi_{K_{0,1}}\left(B_{3}\right)\right)=\chi_{\Gamma_{3}}\left(B_{3}\right) ;$
- an even number of elements in $\left\{a_{2}, d\right\}$ must have $\chi_{\Gamma_{3}}=0$ because $\chi_{\Gamma_{3}}\left(\Phi_{K_{0,1}}\left(B_{4}\right)\right)=\chi_{\Gamma_{3}}\left(B_{4}\right)$.

When combined with these constraints, equation (4.4) yields

$$
\begin{aligned}
& \chi_{\Gamma_{3}}\left(a_{1}\right)=\chi_{\Gamma_{3}}\left(a_{2}\right)=0, \\
& \chi_{\Gamma_{3}}\left(b_{1}\right)=\chi_{\Gamma_{3}}\left(b_{2}\right)=1 .
\end{aligned}
$$

Hence $\left\{B_{1}, B_{2}, B_{3}, B_{4}, b_{1}, b_{2}, b_{3}, a_{3}\right\}$ might be a subset of $G_{\Gamma_{3}, 5}$, and we will extend it to a basis of $H_{1}\left(\Sigma_{5} ; \mathbb{Z}_{2}\right)$ by adding two simple closed curves $d_{1}, d_{2}$ as in Figure 6. Let

$$
\Gamma_{3}=\Gamma\left(\left\{B_{1}, B_{2}, B_{3}, B_{4}, b_{1}, b_{2}, b_{3}, a_{3}, d_{1}, d_{2}\right\}\right)
$$

then $\Gamma_{3}$ is the graph in Figure 8 and satisfies equations (4.3) and (4.4).


Figure 8 Graph $\Gamma_{3}$

Therefore, $G_{F}\left(\Phi_{K_{0,1}}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2}\right) \leq G_{\Gamma_{3}, 5}$ and, since $t_{c_{2}}, t_{d} \notin G_{\Gamma_{3}, 5}$, we get

$$
t_{c_{2}}, t_{d} \notin G_{F}\left(\Phi_{K_{0,1}}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2}\right)
$$

and

$$
t_{c_{2}}, t_{d} \notin G_{F}\left(\Phi_{K_{2 p, 2 q+1}}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2}\right)
$$

for any $p, q \in \mathbb{Z}$.
Case 4: $p$ and $q$ are odd integers. We want to find a graph

$$
\Gamma_{4}=\Gamma\left(\left\{\gamma_{1}, \gamma_{2}, \ldots, \gamma_{10}\right\}\right)
$$

satisfying

$$
\begin{equation*}
\chi_{\Gamma_{4}}\left(B_{i}\right)=\chi_{\Gamma_{4}}\left(\Phi_{K_{1,1}}\left(B_{i}\right)\right)=\chi_{\Gamma_{4}}\left(b_{3}\right)=\chi_{\Gamma_{4}}\left(b_{3}^{\prime}\right)=\chi_{\Gamma_{4}}\left(a_{3}\right)=1 \tag{4.6}
\end{equation*}
$$

for $i=0,1,2,3,4,5$ and

$$
\begin{equation*}
\chi_{\Gamma_{4}}\left(c_{2}\right)=\chi_{\Gamma_{4}}(d)=0 \tag{4.7}
\end{equation*}
$$

We may assume that each element of $\left\{B_{1}, B_{2}, B_{3}, B_{4}, a_{3}, b_{3}\right\}$ is in the generating set, and we will extend it to a basis of $H_{1}\left(\Sigma_{5} ; \mathbb{Z}_{2}\right)$.

Note that we observe the following relations in $H_{1}\left(\Sigma_{5} ; \mathbb{Z}_{2}\right)$.

|  | $\Phi_{K_{0,0}}\left(B_{i}\right)$ | $\Phi_{K_{1,1}}\left(B_{i}\right)$ |
| :---: | :---: | :---: |
| $B_{0}$ | $B_{0}+a_{1}+b_{1}+a_{2}+b_{2}$ | $B_{0}+a_{1}+b_{1}+a_{2}+b_{2}$ |
| $B_{1}$ | $B_{1}+b_{1}+b_{2}+a_{2}$ | $B_{1}+b_{1}+a_{2}+b_{2}+c_{2}+d$ |
| $B_{2}$ | $B_{2}+a_{1}+b_{2}+a_{2}$ | $B_{2}+a_{1}+b_{2}+a_{2}+c_{2}$ |
| $B_{3}$ | $B_{3}+b_{2}$ | $B_{3}+b_{2}+c_{2}$ |
| $B_{4}$ | $B_{4}+a_{2}$ | $B_{4}+a_{2}+c_{2}+d$ |
| $B_{5}$ | $B_{5}$ | $B_{5}$ |

Hence, by Lemma 4.2 and (4.6)-(4.8), we have the following statements:

- an even number of elements in $\left\{a_{1}, b_{1}, a_{2}, b_{2}\right\}$ must have $\chi_{\Gamma_{4}}=0$ because $\chi_{\Gamma_{3}}\left(\Phi_{K_{1,1}}\left(B_{0}\right)\right)=\chi_{\Gamma_{3}}\left(B_{0}\right) ;$
- an even number of elements in $\left\{a_{2}, b_{1}, b_{2}, c_{2}, d\right\}$ must have $\chi_{\Gamma_{4}}=0$ because $\chi_{\Gamma_{3}}\left(\Phi_{K_{1,1}}\left(B_{1}\right)\right)=\chi_{\Gamma_{3}}\left(B_{1}\right)$;
- an even number of elements in $\left\{a_{1}, a_{2}, b_{2}, c_{2}\right\}$ must have $\chi_{\Gamma_{4}}=0$ because $\chi_{\Gamma_{3}}\left(\Phi_{K_{1,1}}\left(B_{2}\right)\right)=\chi_{\Gamma_{3}}\left(B_{2}\right) ;$
- an even number of elements in $\left\{b_{2}, c_{2}\right\}$ must have $\chi_{\Gamma_{4}}=0$ because $\chi_{\Gamma_{3}}\left(\Phi_{K_{1,1}}\left(B_{3}\right)\right)=\chi_{\Gamma_{3}}\left(B_{3}\right) ;$
- an even number of elements in $\left\{a_{2}, c_{2}, d\right\}$ must have $\chi_{\Gamma_{4}}=0$ because $\chi_{\Gamma_{3}}\left(\Phi_{K_{1,1}}\left(B_{4}\right)\right)=\chi_{\Gamma_{3}}\left(B_{4}\right)$.
This implies that

$$
\begin{aligned}
& \chi_{\Gamma_{3}}\left(a_{1}\right)=\chi_{\Gamma_{3}}\left(a_{2}\right)=1, \\
& \chi_{\Gamma_{3}}\left(b_{1}\right)=\chi_{\Gamma_{3}}\left(b_{2}\right)=0,
\end{aligned}
$$

so $\left\{B_{1}, B_{2}, B_{3}, B_{4}, a_{1}, a_{2}, b_{3}, a_{3}\right\}$ might be a subset of $G_{\Gamma_{4}, 5}$. We will extend this subset to a basis of $H_{1}\left(\Sigma_{5} ; \mathbb{Z}_{2}\right)$ by adding two simple closed curves $d_{3}, d_{4}$ as in Figure 6. Let

$$
\Gamma_{4}=\Gamma\left(\left\{B_{1}, B_{2}, B_{3}, B_{4}, a_{1}, a_{2}, a_{3}, b_{3}, d_{3}, d_{4}\right\}\right)
$$

then $\Gamma_{4}$ is graphed as in Figure 9 and satisfies equations (4.6) and (4.7).


Figure 9 Graph $\Gamma_{4}$

Therefore, $G_{F}\left(\Phi_{K_{1,1}}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2}\right) \leq G_{\Gamma_{4,5}}$ and, since $t_{c_{2}}, t_{d} \notin G_{\Gamma_{4}, 5}$, we get

$$
t_{c_{2}}, t_{d} \notin G_{F}\left(\Phi_{K_{1,1}}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2}\right)
$$

and

$$
t_{c_{2}}, t_{d} \notin G_{F}\left(\Phi_{K_{2 p+1,2 q+1}}\left(\eta_{1,2}^{2}\right) \cdot \eta_{1,2}^{2}\right)
$$

for any $p, q \in \mathbb{Z}$.
Remark 4.5. We can double-check the preceding statements by using the representation of a mapping class group in a symplectic group (this approach was suggested by S. Humphries [12]). There is a natural map

$$
\psi_{n}: \mathcal{M}_{5} \xrightarrow{\psi} \operatorname{Sp}(10, \mathbb{Z}) \xrightarrow{q_{n}} \operatorname{Sp}(10, \mathbb{Z} / n \mathbb{Z})
$$

where, for each $t_{\gamma} \in \mathcal{M}_{5}$,

$$
\psi\left(t_{\gamma}\right): H_{1}\left(\Sigma_{5}, \mathbb{Z}\right) \rightarrow H_{1}\left(\Sigma_{5}, \mathbb{Z}\right)
$$

is an integral matrix representation of the mapping class group action on the integral first homology group. We then reduce the coefficient of the symplectic group to $\mathbb{Z} / n \mathbb{Z}$ by taking a quotient map $q_{n}$. It is easy to check that

$$
\psi_{2}\left(t_{c_{2}}^{2}\right)=\operatorname{Id}_{10 \times 10}=\psi_{2}\left(t_{d}^{2}\right)
$$

which implies that

$$
\psi_{2}\left(G_{F}\left(\xi_{p, q}\right)\right)=\psi_{2}\left(G_{F}\left(\xi_{r, s}\right)\right) \quad \text { if }(p, q) \equiv(r, s)(\bmod 2)
$$

An explicit group order computation (using a computer algebra system such as GAP [7] or Sagemath [20]) shows that

$$
\begin{aligned}
\operatorname{Order}\left(\psi_{2}\left(G_{F}\left(\xi_{p, q}\right)\right)\right) & =50030759116800, \\
\operatorname{Order}\left(\left\langle\psi_{2}\left(G_{F}\left(\xi_{p, q}\right) \cup\left\{t_{c_{2}}\right\}\right)\right\rangle\right) & =24815256521932800, \\
\operatorname{Order}\left(\left\langle\psi_{2}\left(G_{F}\left(\xi_{p, q}\right) \cup\left\{t_{d}\right\}\right)\right\rangle\right) & =24815256521932800, \\
\operatorname{Order}\left(\psi_{2}\left(\mathcal{M}_{5}\right)\right) & =24815256521932800,
\end{aligned}
$$

and this implies that

$$
t_{c_{2}}, t_{d} \notin G_{F}\left(\xi_{p, q}\right) \quad \text { for any } p, q \in \mathbb{Z} .
$$

THEOREM 4.6. $\quad \xi_{p, q}$ is not marked equivalent to $\xi_{r, s}$ if $(p, q) \not \equiv(r, s)(\bmod 2)$.
Proof. Let us consider the $\Gamma_{1}$ case, in which
$i_{2}\left(\Phi_{K_{0,0}}\left(B_{i}\right), c_{2}\right)=\left\{\begin{array}{ll}1, & i=1,2,3,4, \\ 0, & i=0 ;\end{array} \quad i_{2}\left(\Phi_{K_{0,0}}\left(B_{i}\right), d\right)= \begin{cases}1, & i=1,4, \\ 0, & i=0,2,3 .\end{cases}\right.$
Then, by Lemma 4.2, it follows that $\chi_{\Gamma_{1}}\left(\Phi_{K_{1,0}}\left(B_{i}\right)\right)=0$ for $i=1,2,3,4$, $\chi_{\Gamma_{1}}\left(\Phi_{K_{0,1}}\left(B_{i}\right)\right)=0$ for $i=1,4$, and $\chi_{\Gamma_{1}}\left(\Phi_{K_{1,1}}\left(B_{i}\right)\right)=0$ for $i=2,3$; this gives the result for $\Gamma_{1}$. Other rows are obtained by the same method.
$G_{\Gamma_{i}, 5}$ does not contain

$$
\begin{aligned}
& \Gamma_{1} \quad t_{\Phi_{K_{1,0}, 0}\left(B_{j}\right)}(j=1,2,3,4), t_{\Phi_{K_{0,1}}\left(B_{1}\right)}, t_{\Phi_{K_{0,1}}\left(B_{4}\right)}, t_{\Phi_{K_{1,1}}\left(B_{2}\right)}, t_{\Phi_{K_{1,1}}\left(B_{3}\right)} \\
& \Gamma_{2} \quad t_{\Phi_{K_{0,0}, 0}\left(B_{j}\right)}(j=1,2,3,4), t_{\Phi_{K_{0,1}}\left(B_{2}\right)}, t_{\Phi_{K_{0,1}}\left(B_{3}\right)}, t_{\Phi_{K_{1,1}, 1}\left(B_{1}\right)}, t_{\Phi_{K_{1,1}}\left(B_{4}\right)} \\
& \Gamma_{3} \quad t_{\Phi_{K_{0,0}\left(B_{1}\right)}}, t_{\Phi_{K_{0,0}\left(B_{4}\right)},}, t_{\Phi_{K_{1,0}\left(B_{2}\right)}}, t_{\Phi_{K_{1,0}\left(B_{3}\right)}}, t_{\Phi_{K_{1,1}\left(B_{j}\right)}}(j=1,2,3,4) \\
& \Gamma_{4} \quad t_{\Phi_{K_{0,0}, 0}\left(B_{2}\right)}, t_{\Phi_{K_{0,0}}\left(B_{3}\right)}, t_{\Phi_{K_{1,0}\left(B_{1}\right)}}, t_{\Phi_{K_{1,0}, 0}\left(B_{4}\right)}, t_{\Phi_{\Pi_{0,1}( }\left(B_{j}\right)}(j=1,2,3,4)
\end{aligned}
$$

It is clear that $t_{\Phi_{K_{p, q}}\left(B_{j}\right)}$ is contained in $G_{\Gamma_{i}, 5}$ if and only if $t_{\Phi_{K_{s_{p}, \varepsilon_{q}}}\left(B_{j}\right)}$ is contained in $G_{\Gamma_{i}, 5}$, where $\varepsilon_{p}, \varepsilon_{q} \in\{0,1\}$ such that $p \equiv \varepsilon_{p}$ and $q \equiv \varepsilon_{q}$ modulo 2 . The reason is that $\chi_{\Gamma_{i}}\left(\Phi_{K_{p, q}}\left(B_{j}\right)\right)=\chi_{\Gamma_{i}}\left(\Phi_{K_{\varepsilon_{p}, \varepsilon_{q}}}\left(B_{j}\right)\right)$, which implies that

$$
\xi_{p, q} \not \neq \xi_{r, s} \quad \text { if }(p, q) \not \equiv(r, s)(\bmod 2) .
$$

For example, if $(p, q) \equiv(0,0)$ and $(r, s) \equiv(1,0)$ modulo 2 , then

$$
t_{\Phi_{K_{p, q}}\left(B_{j}\right)} \notin G_{\Gamma_{2}, 5} \quad(j=1,2,3,4)
$$

and $G_{F}\left(\xi_{r, s}\right) \leq G_{\Gamma_{2}, 5}$. Hence $t_{\Phi_{K_{p, q}\left(B_{j}\right)}} \in G_{F}\left(\xi_{p, q}\right)$, but $t_{\Phi_{K_{p, q}}\left(B_{j}\right)} \notin G_{F}\left(\xi_{r, s}\right)$ for $j=$ $1,2,3,4$. This implies that $G_{F}\left(\xi_{p, q}\right) \neq G_{F}\left(\xi_{r, s}\right)$ and $\xi_{p, q} \neq \xi_{r, s}$.

Corollary 4.7. If $p \not \equiv q$ modulo 2 , then the knot surgery 4-manifold $E(2)_{K_{p, q}}$ has at least two nonisomorphic genus 5 Lefschetz fibration structures.

Proof. This follows from Lemma 3.1. Since $K_{p, q}$ is equivalent to $K_{q, p}$, we get a diffeomorphism $E(n)_{K_{p, q}} \approx E(n)_{K_{q, p}}$. However, by Theorem 4.6 we know that $\xi_{p, q} \neq \xi_{q, p}$.

Remark 4.8. We are interested in the question of whether the knot surgery 4manifold $E(2)_{K}$ admits infinitely many nonisomorphic Lefschetz fibrations over $S^{2}$ with the same generic fiber. In Theorem 3.4 we constructed a family of simply connected genus 5 Lefschetz fibrations over $S^{2}$, all of whose underlying spaces are diffeomorphic, from a pair of inequivalent prime fibered knots. We expect that these knots are strong candidates for admitting infinitely many nonisomorphic Lefschetz fibrations. We leave this problem for a future research project.

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