Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 19, 2002, 339–374

LEFSCHETZ FIXED POINT THEOREM FOR ACYCLIC MAPS WITH MULTIPLICITY

Fritz von Haeseler — Heinz-Otto Peitgen — Gencho Skordev

Dedicated to the memory of Gilles Fournier

Presented by Lech Górniewicz

Abstract. The Lefschetz fixed point theorem for multivalued upper semicontinuous acyclic maps with multiplicity with respect to (w.r.t.) a given field $\mathbb{F}$ of $\mathbb{F}$-simplicial spaces is proved.

0. Introduction

There are many examples of single valued continuous maps $f\colon X \to Y$ with multiplicity in a given ring $\mathbb{K}$, so called m-maps w.r.t. $\mathbb{K}$, e.g. finite or ramified coverings; quotient maps of G-spaces, where G is a finite group; finite to one, open maps of compact closed manifolds. An important role in the investigation of these maps plays the transfer homomorphism $t(f)_*\colon H_*(Y,\mathbb{K}) \to H_*(X,\mathbb{K})$ (see [6], [13], [16], [42], [47]). Under certain assumptions the transfer homomorphism exists also for another important class of maps, namely fibrations (see [1], [12]).

There are several different ways to define a transfer homomorphism for m-maps. A straightforward definition in Čech homology lies within the framework of sheaf theory (cf. [16]). For our purposes the construction of transfer homomorphism using chain maps, which are chain approximations of the inverse map f^{-1}

2000 *Mathematics Subject Classification.* 55N20, 54H25.
Key words and phrases. Lefschetz fixed point theorem, acyclic maps, transfer homomorphisms.

of f, are more appropriate (cf. [8], [32]). A similar method applies in case of Vietoris maps $g\colon X \to Y$ (w.r.t. a field $\mathbb{F}$). In this situation the transfer $t(g)_*$ turns out to be the inverse of the homomorphism $g_*\colon H_*(X,\mathbb{F}) \to H_*(Y,\mathbb{F})$, ([46], [2]).

In Section 2 we construct, using chain maps, a transfer homomorphism in Čech homology for m-maps w.r.t. a given ring $\mathbb{K}$. It coincides with the transfer defined in [16], [47].

In Section 3 we consider the m-point (multivalued) maps – multivalued maps with multiplicity in a given ring $\mathbb{K}$. These maps, also called w-maps w.r.t. $\mathbb{K}$, are studied in [11], [24], [31], and the functor of singular homology is extended to the category of topological spaces with m-maps as morphisms. Since any m-point map $F\colon X \to Y$ is a composition $F = g \circ f^{-1}$ of the inverse of an m-map $f\colon Z \to X$ and a single valued continuous map $g\colon Z \to Y$, for compact spaces X, Y, Z, we may define an induced homomorphism in Čech homology by $F_* = g_* t(f)_*$. From the properties of the transfer $t(f)_*$ we obtain that the Čech homology functor can be extended to the category of all compact spaces with morphisms being m-point maps w.r.t. a given ring. This extension coincides with the extension of singular homology for compact ANR spaces (see [11], [24]).

In Section 4 we also extend the Čech homology (with coefficients in the field $\mathbb{F}$) functor to the category of compact spaces and m-acyclic (w.r.t. $\mathbb{F}$) maps.

In Section 5 we apply the definition of the induced homomorphism F_* and prove a Lefschetz fixed point theorem for m-acyclic maps w.r.t. $\mathbb{F}$ on $\mathbb{F}$-simplicial spaces introduced in [25]. Defining so called week n-approximative systems, for short $(n - wA)$-systems, we are able to prove a Lefschetz fixed point theorem for all upper semi continuous (u.s.c.) multivalued maps which have $(n - wA)$-systems, e.g. m-acyclic maps. $(n - wA)$-systems are a generalization of approximative systems, or A-systems, discussed in [36], [19, Chapter IV], [15] for u.s.c. acyclic, and for m-acyclic multivalued maps in [21].

Since the class of $\mathbb{F}$-simplicial spaces contains ANR-spaces, compact groups, etc. ([25], [26]), many known Lefschetz fixed point theorems (e.g. [3], [10], [17], [24], [26], [27], [29], [30], [33], [34], [37]–[40]) follow from Theorem 4 in Section 5.5.

1. Preliminaries

Here we shall define some of the notions we use and fix some notations.

As usual, we denote by $\mathbb{N}$ the natural numbers, by $\mathbb{Z}$ the integers, by $\mathbb{K}$ a commutative ring with 1, and by $\mathbb{F}$ a field.

All topological spaces X, Y, Z are assumed to be compact Hausdorff spaces.

For a given space X we denote by $\operatorname{Cov}(X)$ the set of all its finite open coverings. For λ, $\mu \in \operatorname{Cov}(X)$ we say that λ is a refinement of μ, denoted as $\lambda > \mu$, if for every element of $U \in \lambda$ there exists and element $V \in \mu$ such that $U \subset V$.

If A is a subset of X and $\lambda \in \mathrm{Cov}(X)$, we denote by $\mathrm{St}(A, \lambda)$ the union of all elements of the covering λ, which meet A. Moreover, $\mathrm{St}^{k+1}(A, \lambda)$ is defined as $\mathrm{St}(\mathrm{St}^k(A, \lambda), \lambda)$, where $k \in \mathbb{N}$ and $k \geq 1$.

The set $\mathrm{St}^k(\lambda)$ denotes the covering $\{\mathrm{St}^k(U, \lambda) \mid U \in \lambda\}$. For λ, $\mu \in \mathrm{Cov}(X)$ we say that λ is a star refinement of μ, denoted as $\lambda* > \mu$, if $\mathrm{St}(\lambda) > \mu$. We shall write $\lambda** > \mu$, if $\mathrm{St}^2(\lambda) > \mu$, and so forth. It is known that every open covering of a compact Hausdorff space has a star refinement, see [45, p. 47].

If $A \subset X$ and $\lambda \in \mathrm{Cov}(X)$, we shall write $A < \lambda$ if some element of λ contains A.

We shall use the standard definitions and notations for chain complexes, chain maps, homologies (see e.g. [18], [20]). For compact spaces we use the Čech homology ([18], [43], [2]) as defined by Vietoris (see [2], [43]).

An n-dimensional simplex σ^n of the space X is a set of $n+1$ points $\sigma^n = (x_0, \dots, x_n)$, where $x_0, \dots, x_n \in X$. For $\lambda \in \mathrm{Cov}(X)$ we write $\sigma^n < \lambda$ if $\{x_0, \dots, x_n\} < \lambda$. By $X(\lambda)$ we denote the simplicial complex of all simplices σ^n in X with $\sigma^n < \lambda$.

For λ, $\mu \in \mathrm{Cov}(X)$ such that $\lambda > \mu$ we denote by $i(\lambda, \mu)\colon X(\lambda) \to X(\mu)$ the simplicial map defined as the identical inclusion $X(\lambda) \subset X(\mu)$.

For a given continuous map $f\colon X \to Y$ and $\mu \in \mathrm{Cov}(Y)$ we denote by $f^{-1}(\mu)$ the covering $f^{-1}(\mu) = \{f^{-1}(U)\} \mid U \in \mu\}$. If $\lambda > f^{-1}(\mu)$, then f induces a simplicial map $f = f(\lambda, \mu)\colon X(\lambda) \to Y(\mu)$, which is defined by $f(\sigma) = (f(x_0), \dots, f(x_n))$ for a simplex $\sigma = (x_0, \dots, x_n)$ in $X(\lambda)$.

For $n \in \mathbb{N}$ we denote by $X(\lambda)^{(n)}$ the n-dimensional skeleton of the simplicial complex $X(\lambda)$, i.e. the simplicial complex generated by all i-dimensional simplexes of $X(\lambda)$ for $0 \leq i \leq n$.

For a given $\lambda \in \mathrm{Cov}(X)$ we denote by $N(\lambda)$ the nerve of the covering λ. The vertices of this (abstract) simplicial complex are the elements U of the covering λ. Furthermore, the $k+1$ vertices $U_0, \dots, U_k$ are vertices of a k-simplex in $N(\lambda)$ if and only if $U_0 \cap \ldots \cap U_k$ is nonempty.

For a given simplicial complex K and a ring of coefficients $\mathbb{K}$ we denote by $C_*(K, \mathbb{K})$ or $C_*(K)$ the chain complex of K with coefficients in $\mathbb{K}$. For a given simplicial map $g\colon K \to L$ we denote by $g\colon C_*(K) \to C_*(L)$ the induced chain map (we use the same notation for the induced chain map for simplicity and it will be clear from the context, whether we work with the simplicial map or with its induced chain map).

The support $\mathrm{supp}(c)$ of a given chain $c \in C_*(X(\lambda))$ is the smallest subset A of X such that c is a chain of the simplicial complex $A(\lambda|_A)$, where $\lambda|_A$ is the open covering $\{U \cap A \mid U \in \lambda\}$ of A.

For a chain $c \in C_*(N(\lambda))$ the support of c is the intersection of those subfamilies μ of λ such that $c \in C_*(N(\mu))$.

For a compact space X we consider the Čech homology groups $H_*(X) = (H_i(X,\mathbb{K}))_i$ and the reduced homology groups $\widetilde{H}_*(X)$, where our definition is based on Vietoris cycles with coefficients in $\mathbb{K}$ (see e.g. [2], [43]). A collection of cycles $\zeta = \{\zeta(\lambda) \mid \lambda \in \mathrm{Cov}(X)\}$, where $\zeta(\lambda)$ is a cycle in $C_i(X(\lambda))$, is called i-dimensional Vietoris cycle if $\zeta(\lambda)$ is homologous to $\zeta(\mu)$ on $X(\mu)$ for $\lambda > \mu$. The i-dimensional cycle ζ is a boundary if $\zeta(\lambda)$ is a boundary in $C_*(\lambda)$ for every $\lambda \in \mathrm{Cov}(X)$. The i-dimensional Čech homology group $H_*(X)$ is the factor group of the group of i-dimensional cycles by the subgroup of i-dimensional boundaries.

For a given $\lambda \in \mathrm{Cov}(X)$ we denote by $\pi(\lambda)_i\colon H_i(X) \to H_i(X(\lambda))$ the natural projection.

For a continuous map $f\colon X \to Y$ we denote by $f_* = (f_i)_i$ the homomorphism in the homology, induced by f.

The map f is called Vietoris map w.r.t. $\mathbb{K}$ if it is surjective and the reduced homology $\widetilde{H}_*(f^{-1}(x),\mathbb{K}) = 0$ for every $x \in X$. In other words, all sets $f^{-1}(x)$ are acyclic w.r.t. $\mathbb{K}$. For the Vietoris maps w.r.t. a field $\mathbb{F}$ the induced homomorphism $f_*\colon H_*(X,\mathbb{F}) \to H_*(Y,\mathbb{F})$ is an isomorphism (see [2]).

We shall use generalized traces of a linear map and generalized Lefschetz numbers of the linear maps f_* in the homologies with coefficients in a given field in the sense of J. Leray (see [28], [25]).

A multivalued map $F\colon X \to Y$ is a map which assigns to every point $x \in X$ a nonempty compact set $F(x)$. The graph $\mathcal{G}(F)$ of the map F is the subset of the space $X \times Y$ and is defined as $\mathcal{G}(F) = \{(x,y) \mid y \in F(x)\}$. The map F is called upper semi-continuous (u.s.c.) if the graph $\mathcal{G}(F)$ is a closed subset in $X \times Y$. For u.s.c. maps and their general properties see [4], [5], [19].

An u.s.c. map $F\colon X \to Y$ is called acyclic (w.r.t. $\mathbb{F}$) if the images $F(x)$ of all points $x \in X$ are acyclic sets (w.r.t. $\mathbb{F}$), i.e. $\widetilde{H}_*(F(x),\mathbb{F}) = 0$. For an acyclic (w.r.t. $\mathbb{F}$) map $F\colon X \to Y$ the induced homomorphism $F_* = (F_i)_i\colon H_*(X,\mathbb{F}) \to H_*(Y,\mathbb{F})$ is defined as follows. Let $p\colon \mathcal{G}(F) \to X$ and $q\colon \mathcal{G}(F) \to Y$ be the natural projections defined as $p(x,y) = x$ and $q(x,y) = y$, respectively. Then $F(x) = q(p^{-1}(x))$ and the map p is Vietoris map w.r.t. $\mathbb{F}$. This yields $F_* = q_*(p_*)^{-1}$.

2. Single valued maps with multiplicity (m-maps)

In this section we consider single valued continuous maps $f\colon X \to Y$, where X and Y are compact Hausdorff spaces, and $\mathbb{K}$ is a fixed commutative ring with 1. The chain complexes and the homologies are also with coefficients in $\mathbb{K}$.

Definition 1. Let $f\colon X \to Y$ be a continuous map. The map f is called *m-map with respect to $\mathbb{K}$ and multiplicity function $m\colon X \to \mathbb{K}$* if

(1) $f^{-1}(y) = \bigcup_{i=1}^{s(y)}\{x_i(y)\}$ consists of finitely many points, i.e. f is a finite to one map.

(2) Let $y_0 \in Y$ and let $U_i = U(x_i(y_0))$, $i = 1, \dots, s(y_0)$ be disjoint neighbourhoods of the points $x_i(y_0) \in f^{-1}(y_0)$, $i = 0, \dots, s(y_0)$. Then, for any neighbourhood V of y_0 such that $f^{-1}(V) \subset \bigcup_i U_i$, the map m satisfies

$$m(x_i(y_0)) = \sum_x \{m(x) \mid x \in U_i,\ f(x) = y\} \quad \text{for all } y \in V.$$

Remark 1. If Y is connected space, then for any $y \in Y$

$$m(f) = \sum_x \{m(x) \mid f(x) = y\}$$

does not depend on the point y. The constant $m(f) \in \mathbb{K}$ is called the multiplicity of f. In the sequel we consider m-maps $f\colon X \to Y$ with connected Y and $m(f) \neq 0$. Note that in this particular case f is surjective.

2.1. B-systems and w-carriers. In this section we define the technical notion B-systems, or Begle systems, for m-maps. They are modifications of systems of coverings used by E. Begle in [2] for Vietoris maps, and Begle's construction is based on an idea of L. Vietoris, see [46]. Using such systems of coverings E. Begle constructed chain maps, which are chain approximations of the inverse of a given Vietoris map.

In the construction of Begle appropriate acyclic carriers play an important role. However, in case of m-maps we need a more general notion of acyclic cariers, the so called w-acyclic carriers, or weak acyclic carriers, which have similar properties than acyclic carriers.

Starting from a B-system for an m-map $f\colon X \to Y$ we first construct a w-acyclic carrier Γ. The existence of a carrier Γ implies the existence of a chain approximation of the inverse map f^{-1} carried by Γ. Finally, the chain approximations of the map f^{-1} carried by different carriers Γ_1, Γ_2 are homotopic on sufficiently fine coverings of Y.

Definition 2. Let $f\colon X \to Y$ be an m-map with multiplicity function $m\colon X \to \mathbb{K}$ and let $\lambda \in \operatorname{Cov}(X)$ and $\mu \in \operatorname{Cov}(Y)$ be finite open coverings of X and Y, respectively, and $n \in \mathbb{N}$. A B-system $B(f, \lambda, \mu, n)$ consists of finite sequences of coverings $\mu_i \in \operatorname{Cov}(Y)$ and λ_i, $\lambda_i(U)$, $\overline{\lambda}_i(U) \in \operatorname{Cov}(X)$ and points $y(U) \in U$ with $U \in \mu_i$ and $i = 0, \dots, n+1$ such that

(1) $\lambda_0 > \lambda_1 > \dots > \lambda_n > \lambda_{n+1} = \lambda$ and $\mu_0{*} > \mu_1{*} > \dots > \mu_n{*} > \mu_{n+1} = \mu$,
(2) For $U \in \mu_i$ and $i = 0, \dots, n$ we have $\lambda_i > \overline{\lambda}_{i+1}(U){**} > \lambda_{i+1}(U) > \lambda_{i+1}$.
(3) For $U \in \mu_i$ the map f satisfies the following conditions:
(a) $f^{-1}(U) \subset \operatorname{St}(f^{-1}(y(U)), \overline{\lambda}_{i+1}(U))$,
(b) $\{\operatorname{St}(x, \lambda_{i+1}(U)) \mid x \in f^{-1}(y(U))\}$ is a disjoint family of sets,

(c) for $x_0 \in f^{-1}(y(U))$ and all $z \in U$ the multiplicity function m satisfies

$$m(x_0) = \sum_x \{m(x) \mid x \in f^{-1}(z) \cap \mathrm{St}(x, \lambda_{i+1}(U))\}.$$

The next lemma establishes the existence of B-systems.

Lemma 1 (Existence of B-systems). *Let $f: X \to Y$ be an m-map with multiplicity function $m: X \to \mathbb{K}$. For given $\lambda \in \mathrm{Cov}(X)$ and $\mu \in \mathrm{Cov}(Y)$, and $n \in \mathbb{N}$ there exists a B-system $B(f, \lambda, \mu, n)$.*

Proof. (See also E. Begle [2, Lemmas 1, 2].) We define $\lambda_{n+1} = \lambda$ and $\mu_{n+1} = \mu$. For $y \in Y$ and $f^{-1}(y) = \{x(y)\}$ we consider a covering $\lambda_{n+1}(y) \in \mathrm{Cov}(Y)$ such that $\lambda_{n+1}(y) > \lambda$ and

$$\{\mathrm{St}(x(y), \lambda_{n+1}(y)) \mid x(y) \in f^{-1}(y)\}$$

is a disjoint family of sets. For $\lambda_{n+1}(y)$ we choose a covering $\overline{\lambda}_{n+1}(y) \in \mathrm{Cov}(X)$ such that $\overline{\lambda}_{n+1}(y) ** > \lambda_{n+1}(y)$. Let μ^*_{n+1} such that $\mu^*_{n+1}* > \mu_{n+1}$, then t here exists a neighbourhood $U(y)$ of y such that

$$U(y) \subset Y \setminus f(X \setminus \mathrm{St}(f^{-1}(y), \overline{\lambda}_{n+1}(y)))$$

and $U(y) < \mu^*_{n+1}$. Moreover, since f is an m-map we have, for all $z \in U(y)$,

$$m(x(y)) = \sum_x \{m(x) \mid x \in f^{-1}(z) \cap \mathrm{St}(f^{-1}(y).\overline{\lambda}_{n+1}(y))\}.$$

Now define $\mu_n \in \mathrm{Cov}(Y)$ to be a finite subcovering of $\{U(y) \mid y \in Y\}$, i.e. $\mu_n = \{U(y_1), \dots, U(y_L)\}$. For $U = U(y_i) \in \mu_n$ we denote $\lambda_{n+1}(U) = \lambda_{n+1}(y_i)$, and $\overline{\lambda}_{n+1}(U) = \overline{\lambda}_{n+1}(y_i)$, and, finally, $y(U) = y_i$. Now we define λ_n to be the covering of X such that $\lambda_n > \overline{\lambda}_{n+1}(U)$ for all $U \in \mu_n$.

Repeating the above construction for λ_n and μ_n gives λ_{n-1}, μ_{n-1}, $\overline{\lambda}_n(U)$, and $\lambda_n(U)$, $y(U)$ and after $n+1$ steps we obtain the desired B-system. □

To simplify the notation we write $B(\lambda, \mu, n)$ instead of $B(f, \lambda, \mu, n)$ if the map f is clear from the context.

Remark 2. Consider a B-system $B = B(f, \lambda, \mu, n)$ with defining sequences λ_i, μ_i, $\lambda_i(U)$, $\overline{\lambda}_i(U)$, for $i = 0, \dots, n+1$.

For any $m \le n$ the sequences λ_i, μ_i, $\lambda_i(U)$, $\overline{\lambda}_i(U)$ with $i = 0, \dots, m+1$ define a B-system $B(f, \lambda, \mu, m)$, the m-restriction of B.

Definition 3. Let $B_j = B(\lambda^j, \mu^j, n)$, $j = 0, 1$ be two B-systems for the m-map f. B_1 is finer than B_0, denoted as $B_1 > B_0$, if $\lambda^1_i > \lambda^0_i$, $\mu^1_i > \mu^0_i$, and $\lambda^1_i(V) > \lambda^0_i(U)$, $\overline{\lambda}^1_i(V) > \overline{\lambda^0_i(U)}$ for all $V \in \mu^1_{i-1}$ and all $U \in \mu^0_{i-1}$, $i = 1, \dots, n+1$.

REMARK 3. Let $B_j = B(\lambda^j, \mu^j, n), j = 0, 1$ be two arbitrary B-systems for the m-map f. The proof of Lemma 1 also shows the existence of a B-system $B(\lambda, \mu, n)$ which is finer than both B-systems B_j, $j = 0, 1$.

For the construction of a chain approximation T of the map f^{-1} we need the concept of w-acyclic carriers, which is a generalization of acyclic carriers, for the latter see e.g. [20, p. 200].

From now on $\overline{C}_* = (\overline{C}_k, \overline{\partial}_k, \overline{\varepsilon})$ and $C_* = (C_k, \partial_k, \varepsilon)$ denote chain complexes of $\mathbb{K}$-modules with augmentations $\overline{\varepsilon}: \overline{C}_* \to \mathbb{K}$, and $\varepsilon: C_* \to \mathbb{K}$, respectively, see [18], [20, Chapter 10]. Moreover, we suppose that $\overline{C}_k$ are chain complexes of free $\mathbb{K}$-modules with basis $\{\sigma^k\}$ and we call the chain complex $\overline{C}_*$ free.

A carrier $\Gamma: \overline{C}_* \to C_*$ is a function which assigns to each element σ^k of the basis of $\overline{C}_k$ an augmented subcomplex $\Gamma(\sigma^k) \subset C_k$ such that for $\partial_k \sigma^k = \sum k_i \sigma_i^k$ each $\Gamma(\sigma_i^k)$ is subcomplex of $\Gamma(\sigma^k)$.

DEFINITION 4. ([20, p. 54]). A carrier $\Gamma: \overline{C}_* \to C_*$ is called *w-acyclic carrier* if $H_j(\Gamma(\sigma^k), \mathbb{K}) = 0$ for all $j \geq 1$ and all $k \geq 1$. Furthermore, for every $\sigma^1 \in \overline{C}_1$, there exists a decomposition of $\Gamma(\sigma^1)$

$$\Gamma(\sigma^1) = \bigoplus_{r=1}^{s(\sigma^1)} \Gamma(\sigma^1)_r$$

such that each $\Gamma(\sigma^1)_r$ is an augmented subcomplex of $\Gamma(\sigma^1)$ and the reduced homology $\widetilde{H}_*(\Gamma(\sigma^1)_r, \mathbb{K}) = 0$ for all $r = 1, \ldots, s$, i.e. the chain complexes $\Gamma(\sigma^1)_r$ are acyclic.

DEFINITION 5. A chain map $\varphi: \overline{C}_* \to C_*$ is *carried by a w-acyclic carrier* Γ if

(1) $\varphi(\sigma^k) \subset \Gamma(\sigma^k)$ for all basis elements $\sigma^k \in \overline{C}_*$,
(2) $\varphi(\partial\sigma^1) = \sum_{r=1}^{s} \varphi(\partial\sigma^1)_r$, where $s = s(\sigma^1)$ and $\varphi(\partial\sigma^1)_r \in \Gamma(\sigma^1)_r$ and $\varepsilon(\varphi(\partial\sigma^1)_r) = 0$ for all $r = 1, \ldots, s$ and $\sigma^1 \in \overline{C}_1$.

The following extension lemma will be used several times throughout the paper.

LEMMA 2 (Extension lemma). *Let $\Gamma: \overline{C}_* \to C_*$ be a w-acyclic carrier and let $\varphi_0: \overline{C}_0 \to C_0$ be a $\mathbb{K}$-homomorphism which is carried by Γ such that for all $\sigma^1 \in \overline{C}_1$ one has $\varphi(\partial\sigma^1) = \sum_r \varphi(\partial\sigma^1)_r$ and $\varepsilon(\varphi_0(\partial\sigma^1)_r) = 0$. Then*

(1) *There exists a chain map $\varphi: \overline{C}_* \to C_*$ such that φ is carried by Γ and $\varphi(\sigma^0) = \varphi_0(\sigma^0)$ for all $\sigma^0 \in \overline{C}_0$.*
(2) *Every two chain maps φ and ψ which are extensions of φ_0, i.e. they both satisfy* (1)*, are homotopic with a homotopy $D: \overline{C}_* \to C_*$ such that $D(\sigma^0) = 0$ and $D(\sigma^k) \subset \Gamma(\sigma^k)$.*

Proof. We only prove the extension of φ_0 to dimension 1. The further extension is a well known procedure for acyclic carriers, [20, p. 200].

Let $\sigma^1 \in \overline{C}_1$ be a basis element and $\partial_1\sigma^1 = \sum k_j \sigma_j^1$ be its boundary. Since the homomorphism φ_0 is carried by the carrier Γ, i.e. $\varphi_0(\sigma_j^1) \in \Gamma(\sigma_j^1) \subset \Gamma(\sigma^1)$, we conclude that

$$\varphi_0(\partial_1\sigma^1) \in \Gamma(\sigma^1) = \bigoplus_{r=1}^{s} \Gamma(\sigma^1)_r.$$

This yields the representation $\varphi_0(\partial_1\sigma^1) = \sum_{r=1}^{s} \varphi_0(\partial_1\sigma^1)_r$ with $\varphi_0(\partial_1\sigma^1)_r \in \Gamma(\sigma^1)_r$ and $\varepsilon(\varphi_0(\partial\sigma^1)_r) = 0$. Since the reduced homology $\widetilde{H}_*(\Gamma\sigma^1)_r, \mathbb{K}) = 0$, there is a chain $\varphi(\sigma^1)_r \in \Gamma(\sigma^1)_r$ such that $\partial_1\varphi(\sigma^1)_r = \varphi_0(\partial_1\sigma^1)_r$. Now define $\varphi(\sigma^1)$ as

$$\varphi(\sigma^1) = \sum_{r=1} \varphi(\sigma^1)_r \in \Gamma(\sigma^1).$$

This proves the first assertion. The second one follows from standard arguments in the theory of acyclic carriers as stated in [20, p. 200]. □

Let σ^i be an i-dimensional simplex in $Y(\mu_0)^{(n+1)}$ and let $U \in \mu_i$ be such that $\sigma^i \subset U$. We denote such an U by $U(\sigma^i)$ and the corresponding point $y(U(\sigma^i))$ is denoted by $y(\sigma^i)$. The covering $\lambda_{i+1}(U(\sigma^i))$ is simply denoted by $\lambda_{i+1}(\sigma^i)$, and the same notation is used for the coverings $\{\overline{\lambda}_i\}$.

Let $f^{-1}(y(\sigma^i)) = \{x_r(\sigma'') \mid r = 1, \dots, s(\sigma'')\}$. The stars $\mathrm{St}(x_r, \lambda_{i+1}(\sigma^i))$, with $x_r = x_r(\sigma^i) \in f^{-1}(y)(\sigma^i)$ are disjoint, and each

$$\lambda_{i+1}(\sigma^i)_r = \{W \in \lambda_{i+1}(\sigma^i) \mid x_r \in W\}$$

is a covering of $\mathrm{St}(x_r, \lambda_{i+1}(\sigma^i))$. Furthermore, the chain complex

$$C_*(\mathrm{St}(x_r, \lambda_{i+1}(\sigma^i))(\lambda_{i+1}(\sigma^i)_r))$$

is acyclic.

Lemma 3. *Let $B(f, \lambda, \mu, n)$ be a B-system. Then there exists a w-acyclic carrier $\Gamma\colon C_*(Y(\mu_0)^{(n+1)}) \to C_*(X(\lambda))$ such that*

$$\Gamma(\sigma^0) = \bigoplus_{r=1}^{s(\sigma^0)} C_*(\mathrm{St}(x_r, \lambda_1(\sigma^0))(\lambda_1(\sigma^0)_r)), \tag{1}$$

where $\sigma^0 = \{a\}$ is a zero dimensional simplex in $Y(\mu_0)^{(n+1)}$.

Proof. The chain complex $C_*(Y(\mu_0))$ is free, and all simplexes σ^k with $\sigma^k < \mu_0$ constitute a basis. It suffices to define the carrier Γ on these simplexes σ^k. We shall do this by induction on the dimension, using the Lemmas 1 and 2.

For the zero dimensional simplex σ^0 we define $\Gamma(\sigma^0)$ using extension (1).

Suppose $\Gamma(\sigma^i)$ is defined for all i-dimensional simplexes $\sigma \in Y(\mu_0)^{(n+1)}$, where $i \leq k \leq n$, such that Γ is a w-acyclic carrier up to dimension k.

Fix $U(\sigma^i) \in \mu_i$, $y(\sigma^i) \in U(\sigma^i)$, see Definition 2, due to our assumption we have

$$\Gamma(\sigma^i) = \bigoplus C_*(\mathrm{St}(x_r(\sigma^i), \lambda_{i+1}(\sigma^i))(\lambda_{i+1}(\sigma^i)_r)),$$

where $f^{-1}(y(\sigma^i)) = \{x_r(\sigma^i) \mid r = 1, \dots, s(\sigma^i)\}$, and $U(\sigma^i_t) \subset U(\sigma^i)$ for the boundary simplex σ^i_t of σ^i.

Now let σ^{k+1} be a $(k+1)$-dimensional simplex in $Y(\mu_0)^{(n+1)}$ with $\partial_{k+1}\sigma^{k+1} = \sum \sigma^{k+1}_t$ and there exists $V \in \mu_0$ such that $\sigma^{k+1} \subset V$.

From the induction hypothesis we have $U(\sigma^{k+1}_t) \in \mu_k$ and the corresponding points $y(\sigma^{k+1}_t)$ are elements of $U(\sigma^{k+1}_t)$. Since $\mu_0 > \mu_k$ there exists $\widetilde{V} \in \mu_k$ such that $V \subset \widetilde{V}$ and $\mathrm{St}(\widetilde{V}, \mu_k) \subset U(\sigma^{k+1})$. With $y(\sigma^{k+1})$ we denote the corresponding point in $U(\sigma^{k+1})$.

Now we define $\Gamma(\sigma^{k+1})$ as

$$\Gamma(\sigma^{k+1}) = \bigoplus \{C_*(\mathrm{St}(x_r, \lambda_{k+2}(\sigma^{k+1}))(\lambda_{k+2}(\sigma^{k+1})_r)) \mid r = 1, \dots, s(\sigma^{k+1})\}.$$

Since

$$y(\sigma^{k+1}_t) \in U(\sigma^{k+1}_t) \subset U(\sigma^{k+1}),$$

and

$$f^{-1}(y(\sigma^{k+1}_t)) \subset f^{-1}(U(\sigma^{k+1}_t)) \subset \mathrm{St}(f^{-1}(y(\sigma^{k+1})), \overline{\lambda}_{k+2}(\sigma^{k+1})),$$

we obtain

$$\begin{aligned} \mathrm{St}(f^{-1}(y(\sigma^{k+1}_t)), \lambda_k(\sigma^{k+1}_t)) &\subset \mathrm{St}^2(f^{-1}(y(\sigma^{k+1})), \overline{\lambda}_{k+2}(\sigma^{k+1})) \\ &\subset \mathrm{St}(f^{-1}(y(\sigma^{k+1})), \lambda_{k+2}(\sigma^{k+1})). \end{aligned}$$

This yields $\Gamma(\sigma^{k+1}_t) \subset \Gamma(\sigma^{k+1})$. □

REMARK 4. There are several different ways to define $\Gamma(\sigma_0)$, e.g., $\Gamma(\sigma_0) = C_*(f^{-1}(a)(\lambda_0))$ for $\sigma_0 = \{a\}$, or $\Gamma(\sigma_0) = C_*(f^{-1}(y(U(\sigma^0)))(\lambda_0))$. We shall use these definitions later on. We say that the carrier Γ, constructed in the previous lemma, corresponds to the B-system B.

The next lemma guarantees the existence of refinements of w-cyclic carriers.

LEMMA 4. *If Γ^j, $j = 0, 1$, are two w-acyclic carriers corresponding to the B-system $B = B(f, \lambda, \mu, n)$, then there exists a w-acyclic carrier Γ corresponding to B such that*

$$\Gamma^j(\sigma^k) \subset \Gamma(\sigma^k)$$

for $j = 0, 1$ and for all k with $0 \leq k \leq n + 1$.

PROOF. It suffices to consider dimension 0. Due to the properties of Γ^j, $j = 0, 1$, there exists for every 0-dimensional simplex $\sigma^0 \in Y(\mu_0)^{(n+1)}$ with

$\sigma^0 = \{a\}$ open sets $U^j(\sigma^0) \in \mu_0$ and points $y^j(\sigma^0) \in U^j(\sigma^0)$ such that

$$\Gamma^j(\sigma^0) = \bigoplus_{x_r^j} C_*(\mathrm{St}(x_r^j, \lambda_1^j(\sigma^0))(\lambda_1^j(\sigma^0)_r)),$$

where the sum is over $x_r^j \in f^{-1}(y^j(\sigma^0))$ for $j = 0, 1$. Furthermore $\lambda_1^j(\sigma^0) = \lambda_1(U^j(\sigma^0))$. Since $a \in U^j(\sigma^0)$ and $\mu_0 * > \mu_1$, there is an $U(\sigma^0) \in \mu_1$ and a corresponding point $y(\sigma^0) \in U(\sigma^0)$ such that $U^j(\sigma^0) \subset U(\sigma^0)$ and

$$f^{-1}(U(\sigma^0)) \subset \mathrm{St}(f^{-1}(y(\sigma_0)), \overline{\lambda}_2(\sigma_0)).$$

Now define $\Gamma(\sigma^0)$ by setting

$$\Gamma(\sigma^0) = \bigoplus_{x_u} uC_*(\mathrm{St}(x_u, \lambda_2(\sigma^0))(\lambda_2(\sigma^0)_u)),$$

where $x_u \in f^{-1}(y(\sigma_0))$. From the construction we obtain that $\Gamma^j(\sigma^0) \subset \Gamma(\sigma^0)$, $j = 0, 1$. $\square$

Lemma 5. *For two B-systems $B^j = B(\lambda^j, \mu^j, n)$ and corresponding w-acyclic carriers Γ^j, $j = 0, 1$, there exists a w-acyclic carrier Γ corresponding to a common refinement $B(\lambda, \mu, n)$ of the B-systems B^0 and B^1 such that*

$$\Gamma^j(\sigma^k) \subset \Gamma(\sigma^k)$$

for $j = 0, 1$ and for $k = 0, \dots, n + 1$.

Proof. Due to Remark 3, there exists a common refinement $B(\lambda, \mu, n)$ of B^0 and B^1. Using the construction in the proof of Lemma 4 we obtain a w-acyclic carrier Γ with the desired properties. $\square$

2.2. Chain approximations of m-maps. In this section we show how to construct a chain approximation of f^{-1} for an m-map $f\colon X \to Y$ with multiplicity function $m\colon X \to \mathbb{K}$.

Lemma 6. *Let $f\colon X \to Y$ be an m-map and let $B = B(f, \lambda, \mu, n)$ be a B-system with corresponding w-acyclic carrier Γ. If the $\mathbb{K}$-homomorphism*

$$T\colon C_0(Y(\mu_0)^{(0)}) \to C_0(X(\lambda))$$

is defined by

$$T(\sigma^0) = \sum \{m(x)x \mid x \in f^{-1}(a)\},$$

where $\sigma^0 = \{a\}$, then T has an extension $T\colon C_*(Y(\mu_0)^{(n)}) \to C_*(X(\lambda))$ which is a chain map carried by Γ.

Proof. We apply the Extension Lemma 2. By definition of T, it is obvious that $T(\sigma^0) \in \Gamma(\sigma^0)$. Now let σ^1 be an one-dimensional simplex of the simplicial complex $Y(\mu_0)^{(n)}$, then

$$\Gamma(\sigma^1) = \bigoplus_r C_*(\mathrm{St}(x_r, \lambda_2(\sigma^1))(\lambda_2(\sigma^1)_r)).$$

For $z \in U(\sigma^1)$ and the 0-dimensional simplex $\sigma^0 = \{z\}$ we have that $T(\sigma^0) \subset \Gamma(\sigma^1)$, therefore $T(\sigma^0) = \sum_r T(\sigma^0)_r$, where $T(\sigma^0)_r \in C_0(\mathrm{St}(x_r, \lambda_2(\sigma^1))(\lambda_2(\sigma^1)_r))$. This leads to

$$T(\sigma^0)_r = \sum \{m(x)x \mid x \in f^{-1}(z) \cap \mathrm{St}(x_r, \lambda_2(\sigma^1))\},$$

with $x_r = x_r(\sigma^1) \in f^{-1}(y(\sigma^1))$. For the augmentation ε we obtain

$$\varepsilon(T(\sigma^0)_r) = \varepsilon\Big(\sum \{m(x)x \mid x \in f^{-1}(z) \cap \mathrm{St}(x_r, \lambda_2(\sigma^1))\}\Big) = m(x_r(\sigma^1)),$$

due to Definition 1. This yields $\varepsilon(T(\partial\sigma^0)_r) = 0$ and the assertion follows from the Extension Lemma. □

REMARK 5. (1) It is also possible to define $T(\sigma^k) = \sum T(\sigma^k)_r$ in such a way that

$$T(\sigma^k)_r = x_r(\sigma^k) * T(\partial\sigma^k)_r,$$

where $T(\partial\sigma^k) = \sum T(\partial\sigma^k)_r$ and $T(\partial\sigma^k) \subset \Gamma(\sigma^k)_r$ and $x_r(\sigma^k) * T(\partial\sigma^k)_r$ denotes the cone (joint) over $T(\partial\sigma^k)$ with vertex $x_r(\sigma^k)$. Then T is an extension of $T\colon C_0(Y(\mu_0)^{(0)}) \to C_0(X(\lambda))$ which is carried by Γ.

(2) The map $f\colon (X, A) \to (Y, B)$ is called m-map w.r.t. $\mathbb{K}$ if the maps $f\colon X \to Y$ and $f_A = f \circ i_A\colon A \to B$, i_A is the identity inclusion, are m-maps with multiplicity functions $m\colon X \to \mathbb{K}$ and $m \circ i_A\colon X \to \mathbb{K}$, respectively.

Let $f\colon (X, A) \to (Y, B)$ be an m-map w.r.t $\mathbb{K}$ and let $B(f, \lambda, \mu, n)$ be a B-system belonging to f. Then there exists a B-system $B(f_A, \alpha, \beta, n)$ for f_A such that

$$\alpha_i > \lambda_i \cap A, \quad \beta_i > \mu_i \cap B, \quad \overline{\alpha}_i(U) > \overline{\lambda}_i(V) \cap A, \quad \beta_i(U) > \mu_i(V)$$

for all $U \in \beta_{i-1}$ and $V \in \mu_{i-1}$. The proof is similar to the proof of Lemma 1. We call the B-system $B(f_A, \alpha, \beta, n)$ a restriction of $B(f, \lambda, \mu, n)$.

(3) Let $f\colon (X, A) \to (Y, B)$ be an m-map w.r.t. $\mathbb{K}$ and let $B(f_A, \alpha, \beta, n)$ be a restriction of $B(f, \lambda, \mu, n)$. There exists a chain map

$$T\colon C_*(Y(\mu_0)^{(n+1)}, B(\beta_0)^{(n+1)}) \to C_*(X(\lambda), A(\alpha))$$

such that $T\colon C_*(Y(\mu_0)^{(n+1)}) \to C_*(X(\lambda))$ and $T\colon C_*(B(\beta_0)^{(n+1)}) \to C_*(A(\alpha))$ are chain maps corresponding to the B-systems $B(f, \lambda, \mu, n)$ and $B(f_A, \alpha, \beta, n)$, respectively. The proof follows the same lines as the proof in Lemma 3.

So far the construction of chain approximations depends on the given w-acyclic carrier as well as on the given B-system. The following corollaries show that this dependence is not essential.

Corollary 1. *Let f be an m-map. If T^j, $j = 0, 1$, with w-acyclic carriers Γ^j and B-systems $B^j = B(f, \lambda^j, \mu^j, n)$ are chain approximations of f, then there exists a covering $\mu > \mu_0^j$, $j = 0, 1$ such that the chain maps T^0 and T^1 are chain homotopic on $Y(\mu)^{(n)}$.*

Proof. Lemma 4 guarantees the existence of a w-acyclic carrier Γ corresponding to a common refinement $B(\lambda, \mu, n)$ of B^0 and B^1 such that $\Gamma^j(\sigma^k) \subset \Gamma(\sigma^k)$ for $j = 0, 1$ and $0 \leq k \leq n - 1$, i.e. Γ carries T^0 and T^1 on $Y(\mu)$, and the Extension Lemma 2 implies the existence of a homotopy between T^0 and T^1 on $Y(\mu)^{(n)}$. □

The next corollary deals with m-restrictions of B-systems.

Corollary 2. *Let $1 \leq m \leq n$. Let T be a chain map corresponding to the B-system $B(f, \lambda, \mu, n)$ and let T_m be the chain map corresponding to the m-restriction of $B(f, \lambda, \mu, n)$. Then*

$$T_m, T \colon C_*(Y(\mu_0)^{(m)}) \to C_*(X(\lambda)^m)$$

are homotopic.

Proof. For $\mu_0 \in B(f, \lambda, \mu, n)$ there is a w-acyclic carrier $\Gamma \colon C_*(Y(\mu_0)^{(m)}) \to C_*(X(\lambda)^m)$ which carries T_m and T. □

From Corollaries 1 and 2 we obtain:

Corollary 3. *Let T and T' be the corresponding chain maps of the B-systems $B(f, \lambda, \mu, n)$ and $B(f, \lambda', \mu', m)$, respectively, such that $n \geq m$ and $\lambda' > \lambda$. Then there exists a covering $\beta_0 \in \operatorname{Cov}(Y)$ with $\beta_0 > \mu_0$, μ_0', where $\mu_0 \in B(f, \lambda, \mu, n), \mu_0' \in B(f, \lambda', \mu', n)$, such that*

$$T, T' \colon C_*(Y(\beta_0)^{(m)}) \to C_*(X(\lambda)^m)$$

are homotopic.

2.3. Transfer homomorphism for m-maps. We consider m-maps $f \colon X \to Y$ with multiplicity function $m \colon X \to \mathbb{K}$ and assume that the value of m is independent of $x \in X$ and not equal to 0. Under this assumption we call $m(f) = m(x)$ the multiplicity of f.

Theorem 1. *If $f \colon X \to Y$ is an m-map w.r.t. $\mathbb{K}$ and multiplicity $m(f) \neq 0$, then there exists a homomorphism*

$$t(f)_* \colon H_*(Y, \mathbb{K}) \to H_*(X, \mathbb{K})$$

such that $f_* t(f)_* = m(f) \mathrm{id}_*$

Proof. We first establish the existence of the homomorphism $t(f)_*$.

Let $[\zeta] \in H_i(Y, \mathbb{K})$, and let $\zeta = \{\zeta(\mu) \mid \mu \in \mathrm{Cov}(Y)\}$ be an i-dimensional Vietoris cycle corresponding to the homology class $[\zeta]$. Furthermore, choose $n > i+1$. For $\lambda \in \mathrm{Cov}(X)$ and $\mu \in \mathrm{Cov}(Y)$ and n we consider the chain map $T\colon C_*(Y(\mu_0)^{n+1}) \to C_*(X(\lambda))$ corresponding to a B-system $B(f, \lambda, \mu, n)$ and we define

$$t(\zeta)(\lambda) = T(\zeta(\mu_0)) \in C_i(X(\lambda))$$

and

$$t(f)_i(\zeta) = \{t(\zeta)(\lambda) \mid \lambda \in \mathrm{Cov}(X)\}.$$

It remains to show that $t(\zeta)(\lambda')$ and $t(\zeta)(\lambda'')$ are homologous in $X(\lambda')$ if $\lambda'' > \lambda'$. Associated with the coverings λ', λ'' are chain maps T' and T'', respectively, corresponding to the B-systems $B(f, \lambda', \mu, n)$ and $B(f, \lambda'', \mu, n)$, respectively. Corollary 3 ensures the existence of a covering $\beta_0 > \mu_0', \mu_0''$ such that

$$T', T''\colon C_*(Y(\beta_0)^{(n)}) \to C_*(X(\lambda')^n)$$

are homotopic. Since $T'(\zeta(\mu_0'))$ and $T''(\zeta(\mu_0''))$ are homologous to $T'(\zeta(\beta_0))$ and $T'(\zeta(\beta_0))$ in $X(\lambda')$, respectively, it follows that $t(\zeta)(\lambda')$ and $t(\zeta)(\lambda'')$ are homologous in $X(\lambda')$. Therefore the homomorphism $t(f)_i$ is well defined for $n > i+1$. Corollary 3 implies that $t(f)_i$ is independent on the choice of the number n.

We conclude the proof by showing that $f_* t(f)_* = m(f)\mathrm{id}_*$. To this end we consider

$$T\colon C_*(Y(\mu_0)^{(n+1)}) \to C_*(X(\lambda)^{(n+1)})$$

defined as follows

$$T(\sigma^k) = \sum_r x_r(\sigma^k) * T(\partial\sigma^k)_r,$$

(see (1) of Remark 5). As a first step, we demonstrate

$$f(t(\sigma^k)) = m(f)\mathbf{b}(\sigma^k), \tag{2}$$

where $\mathbf{b}(\sigma^k)$ is a barycentric subdivision of the simplex σ^k and additionally $\mathbf{b}(\sigma^k) \in C_k(Y(\mu_k)^{(n)})$. Since (2) holds for $k = 0$, the proof follows from simple induction argument.

Now, let $[\zeta] \in H_i(Y, \mathbb{K})$ and $\zeta = \{\zeta(\mu) \mid \mu \in \mathrm{Cov}(Y)\}$. For $\mu \in \mathrm{Cov}(Y)$ and $\lambda = f^{-1}(\mu)$ and $n > i+1$ there exists a B-system $B(f, \lambda, \mu, n)$ and a corresponding chain map T. Therefore $t(f)(\zeta(\lambda)) = T(\zeta(\mu_0))$ and we obtain

$$f(t(f)(\zeta)(\lambda)) = f(T(\zeta(\mu_0))) = m(f)\mathbf{b}(\zeta(\mu_0)),$$

where $\mathbf{b}(\zeta(\mu_0))$ is a first barycentric subdivision of the chain $\zeta(\mu_0) \in C_*(Y(\mu_i))$ such that $\zeta(\mu_0)$ and $b(\zeta(\mu_0))$ are homotopic in $Y(\mu_i)$. Therefore they are homotopic in $Y(\mu)$. This yields $f_* t(f)_* = m(f)\mathrm{id}_*$. $\square$

The homomorphism $t(f)_* = (t(f)_i)_i$ is called the *transfer homomorphism* of the map f.

Remark 6. For every closed, open, perfect, zero-dimensional map $g\colon X \to Y$ and Y locally connected space, Zarelua, [47], defined a summing homomorphism $\sigma\colon g_* g^* \mathcal{A} \to \mathcal{A}$ over the sheaf of $\mathbb{K}$-modules $\mathcal{A}$, see [6]. The maps g_* and g^* are the g_*-direct image and g^*-inverse image of sheaves w.r.t. g.

If $\underline{\mathbb{K}} = Y \times \mathbb{K}$ denotes the constant sheaf, then Zarelua's construction could be applied to any m-map $f\colon X \to Y$ w.r.t. $\mathbb{K}$ and we obtain a summing homomorphism between sheaves

$$\sigma\colon f_* f^* \underline{\mathbb{K}} = f_* \underline{\mathbb{K}} \to \underline{\mathbb{K}}.$$

This homomorphism induces a homomorphism

$$\sigma^*\colon H^*(Y, f_*\underline{\mathbb{K}}) \to H^*(Y, \underline{\mathbb{K}}).$$

There is an isomorphism $i^*\colon H^*(X, \underline{\mathbb{K}}) \to H^*(Y, f^*\underline{\mathbb{K}})$. Define

$$t^*(f) = \sigma^* i^*\colon H^*(X, \underline{\mathbb{K}}) \to H^*(X, \underline{\mathbb{K}}).$$

We defined the transfer homomorphism

$$t_*(f)\colon H_*(Y, \underline{\mathbb{K}}) \to H_*(X, \underline{\mathbb{K}})$$

using chain maps T, corresponding to B-system for f. These chain maps induce the homomorphism $t^*(f)$.

We now consider composition of m-maps. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be m-maps with multiplicity functions $m\colon X \to \mathbb{K}$ and $\overline{m}\colon Y \to \mathbb{K}$, respectively. The composition $gf\colon X \to Z$ is an m-map with multiplicity function $\widehat{m}\colon X \to \mathbb{K}$ with $\widehat{m}(x_{ij}) = m(x_{ij})\overline{m}(y(z))$, where $z \in Z$, $g^{-1}(z) = \{y_i(z) \mid i\}$, $f^{-1}(y_i(z)) = \{x_{ij} \mid j\}$.

With $t(f)_*$, $t(g)_*$, and $t(gf)_*$ we denote the respective transfer homomorphisms of f, g, and gf.

Proposition 1. *If $f\colon X \to Y$ and $g\colon Y \to Z$ are m-maps w.r.t. $\mathbb{K}$ then*

$$t(gf)_* = t(f)_* t(g)_*.$$

Proof. For given $\lambda \in \mathrm{Cov}(X)$, $\mu \in \mathrm{Cov}(Y)$, $\nu \in \mathrm{Cov}(Z)$ and $n \in \mathbb{N}$ we shall construct B-systems and corresponding chain maps

$$\begin{aligned} T(f)\colon C_*(Y(\mu_{n+1})^{(n+1)}) &\to C_*(X(\lambda)^{(n+1)}), \\ T(g)\colon C_*(Z(\nu_0)^{(n+1)}) &\to C_*(Y(\mu)^{(n+1)}), \end{aligned}$$

such that the composition $T(f)T(g)$ is a chain map corresponding to a B-system of the map gf for λ, ν, and $n \in \mathbb{N}$.

Firstly, we shall construct sequences of coverings λ_i, μ_j, ν_j, for $0 \le i \le n+1$ and $0 \le j \le 2n+1$. Denote $\lambda_{n+1} = \lambda$, $\mu = \mu_{2n+2}$, $\nu = \nu_{2n+2}$ and consider the points $z \in Z$ and $g^{-1}(z) = \{y_i(z) \mid i\}$, and $f^{-1}(y_i(z)) = \{x_{ij} \mid j\}$. For the chosen point z there are coverings $\overline{\lambda}_{n+1}(z)$, $\lambda_{n+1}(z) \in \mathrm{Cov}(X)$ such that

$$\overline{\lambda}(z)** > \lambda_{n+1}(z) > \lambda_{n+1},$$

and such that the set $\{\mathrm{St}(x_{ij}, \lambda_{n+1}) \mid i,j\}$ is a disjoint family of sets. Let $U_i(z)$ be a neighbourhood of $y_i(z)$ such that

$$U_i(z) < \mu_{2n+2},$$
$$U_i(z) \subset Y \setminus f(X \setminus \mathrm{St}(f^{-1}(y_i(z)), \overline{\lambda}_{n+1}(z))).$$

Choose a covering $\mu_{2n+2}(z) \in \mathrm{Cov}(Y)$ such that $\mathrm{St}^3(y_i(z), \mu_{2n+2}(z)) \subset U_i(z)$ and consider a neighbourhood $U(z)$ of $z \in Z$ such that

$$U(z) \subset Z \setminus g(Y \setminus \mathrm{St}(g^{-1}(z), \mu_{2n+2}(z)))$$

and $U(z) < \nu^*_{2n+2}$, where $\nu^*_{2n+2} > \nu_{2n+2}$. Now let ν_{2n+1} be a finite subcovering of the covering $\{U(z) \mid z \in Z\}$. For $U = U(z) \in \nu_{2n+1}$ we denote $z = z(U)$, $\mu_{2n+2} = \mu_{2n+2}(z(U))$, $\overline{\lambda}_{n+1}(U) = \overline{\lambda}_{n+1}(z(U))$, $\lambda_{n+1}(U) = \lambda_{n+1}(z(U))$. The coverings ν_{2n+1}, $\mu_{2n+2(U)}$, and $\overline{\lambda}_{n+1}(U)$ satisfy the following conditions:

(1) $\nu_{2n+1}* > \nu_{2n+2}$,
(2) $g^{-1}(U) \subset \mathrm{St}(g^{-1}(z(U)), \mu_{2n+2})$ for $U \in \nu_{2n+1}$,
(3) $\{\mathrm{St}^3(y(U), \mu_{2n+2}(U)) \mid y(U) \in g^{-1}(z(U))\}$ is a disjoint family of sets,
(4) $f^{-1}(\mathrm{St}^3(y(U), \mu_{2n+2}(U))) \subset \mathrm{St}(f^{-1}(y(U)), \overline{\lambda}_{n+1}(U))$,
(5) $\{\mathrm{St}(x(U), \lambda_{n+1}(U)) \mid x(U) \in (gf)^{-1}(z(U))\}$ is a disjoint family of sets,
(6) $\overline{\lambda}_{n+1}(U)** > \lambda_{n+1}$,
(7) $\mu_{n+2}(U) > \mu_{2n+2}$.

Choose $\lambda_n \in \mathrm{Cov}(X)$, $\mu_{2n+1} \in \mathrm{Cov}(Y)$ such that $\lambda_n > \overline{\lambda}_{n+1}(U)$ and $\mu_{2n+1} > \mu_{2n+2}(U)$ for all $U \in \nu_{2n+1}$. Now starting with λ_n, μ_{2n+1}, ν_{2n+1} we construct λ_{n-1}, μ_{2n}, ν_{2n}, $\overline{\lambda}(V)$, $\lambda(V)$, $\mu_{2n+1}(V)$ for $V \in \nu_{2n}$ having the above seven properties. Continuing this construction we obtain systems of coverings λ_k, $\overline{\lambda}_{k+1}(V)$, $\lambda_{k+1}(V)$, $\mu_{n+k+1}(V)$, μ_{n+k+1}, ν_{n+k+1} for $k = 0, \dots, n+1$, $V \in \nu_{n+k+2}$ satisfying the above seven requirements. The sequences $\{\lambda_i\}$, $\{\nu_{n+i+1}\}$, $\{\overline{\lambda}_i(U)\}$, $\{\lambda_i(U)\}$ with $i = 0, \dots, n+1\}$ form a B-system $B(gf, \lambda, \nu, n)$. Furthermore, we have the B-systems $B(f, \lambda, \mu, n)$ and $B(g, \mu_{n+1}, \nu_{n+1}, n)$.

Let $T(g)\colon C_*(Z(\nu_0)^{(n+1)}) \to C_*(Y(\mu_{(n+1)})^{n+1})$ be a chain map corresponding to the B-system $B(g, \mu_{n+1}, \nu_{n+1}, n)$ with w-acyclic carrier $\Gamma(g)$ and let $T(f)\colon C_*(Y(\mu_{n+1})^{(n+1)}) \to C_*(X(\lambda)^{(n+1)})$ be a chain map corresponding to the B-system $B(f, \lambda, \mu, n)$ with w-acyclic carrier $\Gamma(f)$.

In order to prove the assertion we construct a w-acyclic carrier $\widetilde{\Gamma}$ corresponding to $B(gf, \lambda, \mu, n)$ such that $\widetilde{\Gamma}$ carries the chain map $T(f)T(g)\colon C_*(Z(\nu_0)^{(n+1)})$

$\to C_*(X(\lambda)^{(n+1)})$. We start with the construction of $\widetilde{\Gamma}$ on $Z(\nu_0)^{(0)}$. Let $\sigma^o \in Z(\nu_0)^{(0)}, \sigma^0 = \{a\}$, then there is $U(\sigma^0) \in \nu_0$ and a corresponding point $z(\sigma^0)$ such that

$$\Gamma(g)(\sigma^0) = \bigoplus_r C_*(\mathrm{St}(y_r, \mu_1(\sigma^0))(\mu_1(\sigma^0)_r)),$$

where $y_r \in g^{-1}(z(\sigma^0))$. Since $\nu_0 > \mu_{n+1}$ there is a $\widetilde{U}(\sigma^0) \in \nu_{n+1}$ such that $U(\sigma^0) \subset \widetilde{U}(\sigma^0)$ and there exists a corresponding point $\widetilde{z}(\sigma^0)$. Now define

$$\widetilde{\Gamma}(\sigma^0) = \bigoplus_s C_*(\mathrm{St}(x_s, \lambda_1(\sigma^0))(\lambda_1(\sigma^0)_s)),$$

where $x_s \in (gf)^{-1}(\widetilde{z}(\sigma^0))$. Due to the construction $T(f)T(g)(\sigma^0) \in \widetilde{\Gamma}(\sigma^0)$ holds.

Let us assume that $\widetilde{\Gamma}$ carries $T(g)T(f)$ up to the dimension $k-1$ such that:

(a) For for the l-dimensional simplex $\sigma^l \in Z(\nu_0)^{(l)}$, $l = 0, \dots, k$, there exists $\widetilde{U}(\sigma^l) \in \nu_{n+l+1}$ and a corresponding point $\widetilde{z}(\sigma^l)$ such that

$$\widetilde{\Gamma}(\sigma^l) = \bigoplus_s \mathrm{St}(x_s, \lambda_{l+1}(\sigma^l))(\lambda_{l+1}(\sigma^l)_s).$$

(b) There is $\widetilde{U}(\sigma^l) \in \nu_l$ and a corresponding point $z(\sigma^l)$ such that $z(\sigma^l) \in \widetilde{U}(\sigma^l)$.

Let $\sigma^k \in \nu_0$ and $\sigma^k_j \in \partial\sigma^k$. The construction of $\Gamma(g)$ implies the existence of $U(\sigma^k_j) \in \nu_{k-1}$, $U(\sigma^k) \in \nu_k$ with corresponding points $z(\sigma^k_j)$, $z(\sigma^k)$, respectively. Due to the induction hypothesis there is $\widetilde{U}(\sigma^k_j) \in \nu_{n+k}$ with corresponding point $\widetilde{z}(\sigma^k_j)$. Since $\nu_{k-1}* > \nu_k* > \nu_{n+k} > \nu_{n+k+1}$ there exists $\widetilde{U}(\sigma^k) \in \nu_{n+k+1}$ and a point $\widetilde{z}(\sigma^k)$.

We define

$$\widetilde{\Gamma}(\sigma^k) = \bigoplus_r C_*(\mathrm{St}(\widetilde{x}_r, \lambda_{k+1}(\sigma^k))(\lambda_{k+1}(\sigma^k)_r)),$$

where $\widetilde{x}_r \in (gf)^{-1}(\widetilde{z}(\sigma^k))$.

The above seven properties imply that $\widetilde{\Gamma}$ is a w-acyclic carrier of $T(f)T(g)$ up to dimension k with the desired properties. Therefore there is an extension of $\widetilde{\Gamma}$ on $C_*(Z(\nu_0)^{(n)})$ which is a w-acyclic carrier of $T(f)T(g)$. An application of the construction performed in the proof of Lemma 3 for $B(gf, \lambda, \mu, n)$ extends $\widetilde{\Gamma}$ to a w-acyclic carrier on $Z(\nu_{n+1})^{(n)}$ corresponding to $B(gf, \lambda, \mu, n)$. The definition of the transfer yields the desired assertion. □

Proposition 2. *Let $f\colon (X, A) \to (Y, B)$ be an m-map w.r.t. $\mathbb{K}$. Then the following diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(Y,B,\mathbb{K}) \arrow[r] \arrow[d, "t(f)_n"'] \arrow[dr, phantom, "\mathrm{(I)}"] & H_{n-1}(B,\mathbb{K}) \arrow[r] \arrow[d, "t(f_A)_{n-1}"'] \arrow[dr, phantom, "\mathrm{(II)}"] & H_{n-1}(Y,\mathbb{K}) \arrow[r] \arrow[d, "t(f)_{n-1}"] & \cdots \\
\cdots \arrow[r] & H_n(X,A,\mathbb{K}) \arrow[r] & H_{n-1}(A,\mathbb{K}) \arrow[r] & H_{n-1}(X,\mathbb{K}) \arrow[r] & \cdots
\end{tikzcd}$$

with horizontal homology sequences of pairs (X,A), (Y,B), *and the vertical maps being the respective transfer maps, commutes.*

Proof. Due to Proposition 1, diagram (II) commutes. To prove the commutativity of diagram (I) we consider a B-system $B = B(f,\lambda,\mu,n)$ and its restriction $B_A = B(f_A,\alpha,\beta,n)$. Point (2) of Remark 5 guarantees the existence of a chain map

$$T\colon C_*((Y(\mu_0)^{(n+1)}, B(\beta_0)^{(n+1)}) \to C_*((X(\lambda)^{(n+1)}, A(\alpha)^{(n+1)})$$

such that the chain maps

$$T\colon C_*(Y(\mu_0)^{(n+1)}) \to C_*(X(\lambda)^{(n+1)}) \text{ and } T\colon C_*(B(\beta_0)^{(n+1)}) \to C_*(A(\alpha)^{(n+1)})$$

are corresponding to the B-systems B and B_A, respectively. The definition of the transfer maps of m-maps f and f_A and $f\colon (X,A) \to (Y,B)$ using the chain approximations T yields the assertion. □

3. Extension of the Čech homology functor for m-point multivalued maps

3.1. m-diagrams and induced homomorphisms. We consider m-maps $f\colon X \to Y$, $g\colon W \to Z$ with multiplicity functions $m\colon X \to \mathbb{K}$ and $\overline{m}\colon W \to Z$, respectively. Furthermore, we assume that there exist single value maps p, q such that the pull back diagram

$$\begin{array}{ccc} W & \xrightarrow{\ p\ } & X \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Z & \xrightarrow[\ q\ ]{} & Y \end{array} \tag{3}$$

commutes. For $z_0 \in Z$ we set $y_0 = q(z_0)$, $f^{-1}(y_0) = \{x_i(y_0) \mid i\}$, $p^{-1}(x_i(y_0)) \cap g^{-1}(z_0) = \{\omega_{ij} \mid j\}$.

Definition 6. A commutative diagram of the form (3) is called *m-diagram* if, for all i,

$$m(x_i(y_0)) = \sum_j \overline{m}(\omega_{ij}(y_0)).$$

Examples 1. Let $f\colon X \to Y$ be an m-map with multiplicity function $m\colon X \to \mathbb{K}$ and $q\colon Z \to Y$ any single valued continuous map. Define

$$W = \{(z,x) \in Z \times X \mid q(z) = f(x)\}$$

and $g(z,x) = z$ and $p(z,x) = x$. Then $g\colon W \to Z$ is a m-map with multiplicity function $\overline{m}(z,x) = m(x)$ and the pull back diagram is an m-diagram.

Lemma 7. *For every m-diagram $t(f)_* q_* = p_* t(g)_*$, where $t(f)_*$, $t(g)_*$ are the transfer homomorphisms.*

Proof. Let $\lambda \in \operatorname{Cov}(X)$, $\mu \in \operatorname{Cov}(Y)$, $\alpha \in \operatorname{Cov}(W)$ and $\beta \in \operatorname{Cov}(Z)$, and $n \in \mathbb{N}$, such that $\alpha > p^{-1}(\lambda)$, $\beta > q^{-1}(\mu)$. Consider $B(f,\lambda,\mu,n)$ and $B(g,\alpha,\beta,n)$ such that

$$\beta_i > q^{-1}(\mu_i), \quad \alpha_i > p^{-1}(\lambda_i), \quad \overline{\alpha}(V) > p^{-1}(\overline{\lambda}_i(U)), \quad \alpha_i(V) > p^{-1}(\lambda_i(U))$$

for every $U \in \mu_{i+1}$ and $V \in \beta_{i+1}$. With $T(f)$ and $T(g)$ we denote chain approximations corresponding to the B-systems $B(f,\lambda,\mu,n)$ and $B(g,\alpha,\beta,n)$, respectively. Since (3) is an m-diagram it follows

$$pT(g)(\sigma^0) = T(f)q(\sigma^0)$$

for all zero-dimensional simplices $\sigma^0 \in Z(\beta_0)$. Any w-acyclic carrier

$$\Gamma(f)\colon C_*(Y(\mu_0)^{(n+1)}) \to C_*(X(\lambda)^{(n+1)}),$$

corresponding to $B(f,\lambda,\mu,n)$ induces a w-acyclic carrier

$$\widetilde{\Gamma}\colon C_*(Z(\beta_0)^{(n+1)}) \to C_*(X(\lambda)^{(n+1)})$$

of $T(f)q$ with $\widetilde{\Gamma}(\sigma^k) = \Gamma(f)(q(\sigma^k))$.

For a w-acyclic carrier $\Gamma(f)$ we construct a w-acyclic carrier $\widetilde{\Gamma}$ in such a way that $\widetilde{\Gamma}$ carries $pT(g)$. The Extension Lemma implies the homotopy of the chain maps $T(f)q$ and $pT(g)$, which yields the assertion.

In order to construct $\widetilde{\Gamma}$ it is sufficient to construct $\widetilde{\Gamma}$ for simplices of the form $q(\sigma^k)$, where σ^k is a k-dimensional simplex in $Z(\beta_0)^{(n)}$. For $q(\sigma^0)$ we define $\widetilde{\Gamma}(\sigma^0) = \Gamma(f)(q(\sigma^0))$. Suppose that the w-carrier $\widetilde{\Gamma}$ is defined on $Z(\nu_0)^{(k-1)}$ and satisfies

(i) $\widetilde{\Gamma}$ is a w-carrier on $Z(\nu_0)^{(k-1)}$ which carries $pT(g)$,

(ii) For every simplex $\sigma^l \in Z(\nu_0)^{(k-1)}$ there exists $U(q(\sigma^l)) \in \mu_{l-1}$ with corresponding point $y(q(\sigma^l))$, and $q(\sigma^l) \in U(q(\sigma^l))$ such that

$$U(q(\sigma^l_j)) \subset U(q(\sigma^l)) \quad \text{for } \sigma^l_j \in \partial_l \sigma^l.$$

(iii) $\widetilde{\Gamma}(\sigma^l)$ is defined as in the proof of Lemma 3 using the point $y(q(\sigma^l))$ and the neighbourhood $U(q(\sigma^l))$.

Let $\sigma^k \in Z(\beta_0)^{(n)}$ be k-dimensional simplex. Then there exists an element $V(\sigma^k) \in \beta_k$ with corresponding point $z(\sigma^k)$ such that

$$T(g)(\sigma^k) \in \Gamma(g)(\sigma^k),$$

where $T(g)(\sigma^k)$ is defined with $V(\sigma^k)$ and $z(\sigma^k)$. For $\sigma^k_t \in \partial\sigma^k$ there is $U(q(\sigma^k_t))$ $\in \mu_{k-1}$. There are also $U'(q(\sigma^k)) \in \mu_k$ and $U''(q(\sigma^k)) \in \mu_k$ such that $U(q(\sigma^k_t)) \subset$

$U(q(\sigma^k))$ and $V(\sigma^k) \subset p^{-1}(U''(q(\sigma^k)))$. Therefore there is $U(q(\sigma^k)) \in \mu_{k+1}$, which contains $U'(q(\sigma^k))$ and $U''(q(\sigma^k))$.

Define $\Gamma(q(\sigma^k))$ with $U(q(\sigma^k))$ and the corresponding point $z(q(\sigma^k))$. $\square$

3.2. m-point maps. In this section we define the induced homomorphism $F_*: H_*(X, \mathbb{K}) \to H_*(Y, \mathbb{K})$ for a class of multivalued upper semi continuous (u.s.c.) maps, the so called m-point maps $F: X \to Y$ with a multiplicity function $m: \mathcal{G}(F) \to \mathbb{K}$, where $\mathcal{G}(F) = \{(x, y) \mid y \in F(x)\}$ is the graph of the map F.

Definition 7. Let $F: X \to Y$ be multivalued map and $m: \mathcal{G}(F) \to \mathbb{K}$ a multiplicity function of F. The map F is called *m-point multivalued map* (or m-point map) with multiplicity function m if

(1) $F(x)$ consists of finitely many points $\{y_i(x) \mid i = 1, \dots, s\}, s = s(x)$,

(2) for all $x_0 \in X$ with $F(x_0) = \{y_i(x_0) \mid i = 1, \dots, s\}$, $s = s(x_0)$ and disjoint neighbourhoods U_i of $y_i(x_0)$ exists a neighbourhood U of x_0 such that
  - (a) $F(U) \subset \bigcup_{i=1}^{s} U_i$,
  - (b) $m(x_0, y_i(x_0)) = \sum\{m(x, y(x)) \mid y(x) \in U\}$ for all $x \in U$ and $i = 1, \dots, s$.

Remark 7. In [11] these maps are called weighted maps and in [24] they were named m-functions.

Examples 2. (1) Let $R: \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ be a rational map on the compactified complex line. Then the inverse map R^{-1}defined as $R^{-1}(z) = \{w \in \widehat{\mathbf{C}} \mid R(w) = z\}$ is an m-point map w.r.t. the integers $\mathbb{Z}$ and with multiplicity function $m(z, w) =$ local degree of R at w. For polynomial maps see [22]. Moreover, the multiplicity $m(R^{-1})$ of the map R^{-1} is equal to the degree of the map R.

(2) Let K, L be finite polyhedra and let $p: K \to L$ a ramified covering with multiplicity map $\mu: K \to \mathbb{N}$ (see [42]). The map $p^{-1}: L \to K$ is an m-point map (w.r.t. $\mathbb{Z}$) with multiplicity function $m: \Gamma(p^{-1}) \to \mathbb{Z}$, defined as $m(x, y) = \mu(y)$.

(3) Let G be a finite group which acts (from the left) on X, and let G_x denote the stabilizer of $x \in X$, i.e., the subgroup $G_x = \{g \in G \mid gx = x\}$. Then the inverse map p^{-1} of the projection $p: X \to X/G$ is an m-map w.r.t. $\mathbb{Z}$. If $p^{-1}y = \{x_1, \dots, x_s\}$, $s = s(y)$ the multiplicity function is defined by $m(y, x_i) = \operatorname{card}(G_{x_i})$, the cardinality of G_x.

(4) Let $SP^n(Y)$ be the n-th symmetric product of the space Y (see [30]), and let $f: X \to SP^n(Y)$ be a single valued continuous map. Then f induces an m-point map $F: X \to Y$ w.r.t. $\mathbb{Z}$, defined with $F(x) = \pi(f(x))$, where

$$\pi: SP^n(Y) \to Y, \quad \pi(y_1^{k_1} \dots y_s^{k_s}) = \{y_1, \dots, y_s\}, \quad k_1 + \dots + k_s = n.$$

The multiplicity function is given by $m(x, y_i) = k_i$, and F has multiplicity n.

On the other hand, if $F: X \to Y$ is an m-point map w.r.t. $\mathbb{Z}$ with multiplicity $m(F) = n > 0$ and $m(x, y_i) > 0$ for all $x \in X$, then the map F induces a single valued continuous map $f: X \to SP^n(Y)$, defined by $f(x) = y_1^{k_1} \dots y_s^{k_s}$ and $F(x) = \{y_1, \dots , y_s\}$, $k_i = m(x, y_i)$.

For other examples see [9] and [35].

Definition 8. Let $F: X \to Y$ be an m-point map with multiplicity function $m: \mathcal{G}(F) \to \mathbb{K}$. A triple (p, q, Z) is called a *representation of* F, if Z is a compact space and $p: Z \to X$, $q: Z \to Y$ are continuous single valued maps such that

(1) p is an m-map with multiplicity function $\overline{m}: Z \to \mathbb{K}$,
(2) $F(x) = q(p^{-1}(x))$ for all $x \in X$,
(3) $m(x, y(x)) = \sum \overline{m}(z)$, where the sum is over all z with $p(z) = x$ and $q(z) = y(x)$.

Remark 8. (1) Every m-point map $F: X \to Y$ (w.r.t. $\mathbb{K}$) has the representation $(p_0, q_0, \mathcal{G}(F))$, where $p_0: \mathcal{G}(F) \to X$, $q_0: \mathcal{G}(F) \to X$ are the natural projections $p_0(x, y) = x$, $q_0(x, y) = y$. We call this representation a minimal representation.

(2) If (p, q, Z) is a representation of $F: X \to Y$, then the map $f: Z \to \mathcal{G}(F)$ defined as $f(z) = (p(z), q(z))$ is such that the following diagram commutes

$$\begin{tikzcd}
X & \mathcal{G}(F) \arrow[l, "p_0"'] \arrow[r, "q_0"] & Y \\
& Z \arrow[ul, "p"] \arrow[u, "f"'] \arrow[ur, "q"'] &
\end{tikzcd}$$

Lemma 8. *Let (p, q, Z) be a representation of the m-point map $F: X \to Y$ w.r.t. $\mathbb{K}$ and let $(p_0, q_0, \mathcal{G}(F))$ be the minimal representation. Then*

$$q_* t(p)_* = (q_0)_* t(p_0)_*,$$

where $t(p)_*$ and $t(p_0)_*$ are the transfer homomorphisms of p and p_0, respectively.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "p"'] & \mathcal{G}(F) \arrow[d, "p_0"] \\
X \arrow[r, equal, "\mathrm{id}"] & X
\end{tikzcd}$$

Since f is an m-map, Lemma 7 yields that $f_* t(p)_* = t(p_0)_*$, and the equality $q_0 f = q$ implies the assertion. $\square$

Remark 9. The homomorphism $q_* t(p)_*$ does not depend on the representation (p, q, Z).

DEFINITION 9. Let $F: X \to Y$ be an m-point map w.r.t. $\mathbb{K}$ and (p, q, Z) be a representation of F. The homomorphism

$$F_* = q_* t(p)_* : H_*(X, \mathbb{K}) \to H_*(Y, \mathbb{K})$$

is called the *induced homomorphism of* F in the Čech homology.

REMARK 10. If $F: X \to Y$ is single valued and continuous, then F is an m-point map w.r.t. any ring $\mathbb{K}$, where $m: \mathcal{G}(F) \to \mathbb{K}$ is defined as $m(x, F(x)) = 1$. The induced homomorphism of F in the previous definition coincides with the standard induced homomorphism in the homology.

Now consider the composition of m-point maps $F_1: X \to Y$ and $F_2: Y \to Z$. Then $F = F_2F_1: X \to Z$ is also an m-point map (Theorem 4 in [11], Proposition 2.6 in [24]).

With m-$\underline{\mathrm{Top}}(\mathbb{K})$ we denote the category of all compact Hausdorff spaces and m-point maps w.r.t. $\mathbb{K}$. The category $\underline{\mathrm{Top}}$ of all compact Hausdorff spaces and single valued continuous maps is a subcategory of m-$\underline{\mathrm{Top}}(\mathbb{K})$.

LEMMA 9. *If $F_1: X \to Y$, $F_2: Y \to Z$ are m-point maps w.r.t. $\mathbb{K}$, then*

$$(F_2F_1)_* = (F_2)_*(F_1)_*.$$

PROOF. Consider representations (p_i, q_i, Z_i) of F_i, $i = 1, 2$. Let

$$\begin{tikzcd} W \arrow[r, "\psi"] \arrow[d, "g"'] & Z_2 \arrow[d, "p_2"] \\ Z_1 \arrow[r, "q_1"'] & Y \end{tikzcd}$$

be the pullback diagram. Due to Lemma 7, we have the identity

$$\psi_* t(g)_* = t(p_2)_*(q_1)_*.$$

Since $(p_1g, q_2\psi, W)$ is a representation of F_2F_1, we have $(F_1F_2)_* = (q_2\psi)_* t(p_1g)_*$. Now Proposition 1 together with the above identity proves the assertion. □

As a next step, we define a notion of homotopy for m-point maps.

DEFINITION 10. The m-point maps $F_0, F_1: X \to Y$ with multiplicity functions $m_k: \mathcal{G}(F_k) \to \mathbb{K}$, $k = 0, 1$ are *m-point homotopic* if there exists an m-point map $H: X \times I \to Y$, with $I = [0, 1]$, with multiplicity function $m: \mathcal{G}(H) \to \mathbb{K}$ such that

(1) $F_k(x) = H(x, k)$ for $k = 0, 1$,
(2) $m((x, k), y) = m_k(x, y)$ for $(x, y) \in \mathcal{G}(F_k)$ and $k = 0, 1$.

LEMMA 10. *If F_0 and F_1 are m-point homotopic maps, then $(F_0)_* = (F_1)_*$*

PROOF. With $(p_k, q_k, \mathcal{G}(F_k))$ we denote the minimal representation of F_k, $k = 0, 1$. Furthermore, let $H: X \times I \to Y$ be a homotopy of F_0 and F_1 and $(p, q, \mathcal{G}(H))$ its minimal representation. For $k = 0, 1$ we consider the diagram

$$\begin{tikzcd}
 & X \times I \arrow[dr, "H"] & \\
X \arrow[ur, "i_k"] & \mathcal{G}(H) \arrow[u, "p"] \arrow[r, "q"] & Y \\
 & \mathcal{G}(F_k) \arrow[ul, "p_k"] \arrow[u, "j_k"] \arrow[ur, "q_k"'] &
\end{tikzcd}$$

where $i_k(x) = (x, k)$, $j_k(x, y) = (x, y, k)$ are the natural inclusions. Since p_k is the pullback of the map p w.r.t. the inclusion i_k, Proposition 1 implies

$$t(p)_*(i_k)* = (j_k)_* t(p_k)_*.$$

Therefore $H_*(i_k)_* = (F_k)_*$. Since $(i_0)_* = (i_1)_*$, we obtain $(F_0)_* = (F_1)_*$. □

THEOREM 2. *There exists an extension of Čech-homology functor H_* with coefficients in $\mathbb{K}$ on the category m-point-$\underline{\mathrm{Top}}(\mathbb{K})$, satisfying the axioms of Eilenberg–Steenrod.*

PROOF. As we have already seen, every m-point map $F: X \to Y$, i.e. a morphism in the category m-point-$\underline{\mathrm{Top}}(\mathbb{K})$) induces a homomorphism

$$F_*: H_*(X, \mathbb{K}) \to H_*(Y, \mathbb{K}).$$

Due to Lemma 10, we know that m-point-homotopic m-point maps induce the same homomorphism in the homology.

It remains to show that the homology sequence of a pair (X, A), where of A is a closed subspace of X, is functorial w.r.t. m-point maps. For a closed subspace A of X we denote by ι_A the identity inclusion from A in X. If $F: X \to Y$ is an m-point map with multiplicity function $m: \mathcal{G}(F) \to \mathbb{K}$ and if there are closed subspaces $A \subset X$, $B \subset Y$ such that $F(A) \subseteq B$, then $F: (X, A) \to (Y, B)$ is an m-point map w.r.t. $\mathbb{K}$. This follows from the fact that $F_A = F\iota_A: A \to B$ is an m-point map with multiplicity function $m\iota_{\mathcal{G}(F_A)}: \mathcal{G}(F_A) \to \mathbb{K}$, $\iota_{\mathcal{G}(F_A)}$ is the identity inclusion of $\mathcal{G}(F_A)$ in $\mathcal{G}(F)$

For the minimal representations $(p, q, \mathcal{G}(F))$, $(p_A, q_A, \mathcal{G}(F_A))$ the following diagram

$$\begin{tikzcd}
X & \mathcal{G}(F) \arrow[l, "p"'] \arrow[r, "q"] & Y \\
A \arrow[u, "\iota_A"] & \mathcal{G}(F_A) \arrow[l, "p_A"] \arrow[r, "q_A"'] \arrow[u, "\iota_{\mathcal{G}(F_A)}"] & B \arrow[u, "\iota_B"]
\end{tikzcd}$$

is commutative. Now $p\colon (\mathcal{G}(F),\mathcal{G}(F_A)) \to (X,A)$ is an m-map w.r.t. $\mathbb{K}$ and $p_A = p|_A$. Proposition 2 applied to the map p and the commutativity of the above diagram prove the assertion. □

4. Extension of the Čech homology functor for m-acyclic maps

From now on the ring $\mathbb{K}$ will be replaced by the field $\mathbb{F}$.

4.1. m-acyclic multivalued maps. In this section we introduce the notion of m-acyclic multivalued maps, also called w-carriers in [9], [35]. m-acyclic maps are generalizations of m-point multivalued maps. Loosely speaking, in Definition 7 we replace each point $y_i(x) \in F(x)$ by a connected component $C_i(x)$, and assume that these components are acyclic w.r.t. $\mathbb{F}$.

To be more precise, we consider an u.s.c. multivalued map $F\colon X \to Y$ such that for every $x \in X$ the set $F(x)$ is the finite union of nonempty, compact, connected sets. Two points (x_1, y_1) and (x_2, y_2) of the graph $\mathcal{G}(F)$ are called equivalent, denoted as $(x_1, y_1) \sim (x_2, y_2)$, if and only if $x_1 = x_2$ and y_1, y_2 belong to the same connected component of $F(x)$. Clearly, this defines an equivalence relation. The quotient is denoted by $\widetilde{\mathcal{G}}(F) = \mathcal{G}(F)/\sim$, its elements are denoted by $(x, C(x))$, where $C(x)$ is the connected component of $F(x)$ containing x. Moreover, we equip the quotient with the topology induced by the equivalence relation.

Definition 11 ([9], [10]). Let $F\colon X \to Y$ be an u.s.c. multivalued map and let $m\colon \widetilde{\mathcal{G}}(F) \to \mathbb{K}$ be a map. F is called *m-multivalued map* with multiplicity function m if the following two conditions are satisfied:

(1) $F(x)$ consists of finitely many connected components $C_i(x)$, $i = 1, \ldots,$ $s = s(x)$ for each $x \in X$,
(2) For all $x_0 \in X$ with $F(x_0) = C_1(x_0) \cup \ldots \cup C_s(x_0)$, $s = s(x_0)$, and disjoint neighbourhoods U_i of $C_i(x_0)$ there exists a neighbourhood U of x_0 such that
 (a) $F(U) \subset \bigcup_{i=1}^{s} U_i$,
 (b) $m(x_0, C_i(x_0)) = \sum m(x, C(x))$, for all $x \in U$, where the sum is over all $C(x) \subset U_i$ and $i = 1, \ldots, s$.

If X is connected and x_1, $x_2 \in X$, then

$$\sum m(x_1, C(x_1)) = \sum m(x_2, C(x_2)),$$

where the sums are over all connected components of $F(x_1)$, $F(x_2)$, respectively, see Lemma 2.3 in [24]. Therefore, for X connected it makes sense to speak of the *multiplicity* of the map F by setting $m(F) = \sum m(x, C(x))$ summing over all connected components of $F(x)$.

From now on X is assumed to be connected.

Remark 11. If F is an m-multivalued map with multiplicity $m(F) \neq 0$ w.r.t. to the field $\mathbb{F}$, then we may consider F as an m-multivalued map with multiplicity 1 and multiplicity function $m_1(x, C(x)) = m(x, C(x))m(F)^{-1}$.

Definition 12. Let $F: X \to Y$ be an m-multivalued map with multiplicity $m(F) \neq 0$. The map F is called *m-acyclic map* (w.r.t. $\mathbb{F}$) if for each $x \in X$ the connected components $C(x)$ of $F(x)$ are acyclic compact spaces with respect to Čech homology with coefficients in $\mathbb{F}$, i.e. the reduced homology $\widetilde{H}_*(X, \mathbb{F}) = 0$.

Example 3. (1) Single valued continuous maps, u.s.c. acyclic (w.r.t. $\mathbb{F}$) maps and m-point maps provide examples for m-acyclic maps (w.r.t. a field $\mathbb{F}$).

(2) $(1, n)$-maps F, with $n \equiv 1 \bmod 2$, are m-acyclic maps w.r.t. the field $\mathbb{F}_2$ and multiplicity function $m(x, C(x)) = 1$ for all $(x, C(x)) \in \widetilde{\mathcal{G}}(F)$ (see [15] and [33]).

4.2. Induced homomorphism of the Čech homology for m-acyclic maps. Let $F: X \to Y$ be an m-acyclic map w.r.t. $\mathbb{F}$ and $\mathcal{G}(F)$ the graph of F. The natural projection $\pi_1: \mathcal{G}(F) \to \widetilde{\mathcal{G}}(F)$ from the space $\mathcal{G}(F)$ onto the quotient space $\widetilde{\mathcal{G}}(F)$ is a Vietoris map w.r.t. $\mathbb{F}$, since the components $C(x)$ of $F(x)$ are acyclic w.r.t. $\mathbb{F}$, i.e. $\widetilde{H}_*(\pi_1^{-1}(x, C(x))) = 0$ for all $(x, C(x)) \in \widetilde{\mathcal{G}}(F)$.

With $\pi_2: \widetilde{\mathcal{G}}(F) \to X$ we denote the natural projection $\pi_2(x, C(x)) = x$. It is an m-map w.r.t. $\mathbb{F}$ with multiplicity function induced by the multiplicity function of the m-acyclic map F. The following diagram

$$\begin{tikzcd} X & \mathcal{G}(H) \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "\pi_1"] & Y \\ & \widetilde{\mathcal{G}}(F) \arrow[ul, "\pi_2"] & \end{tikzcd}$$

where $p(x, y) = x$, $q(x, y) = y$ is commutative. Since $\mathbb{F}$ is a field, Vietoris' Theorem (see [2], [46]) implies that the homomorphism

$$(\pi_1)_*: H_*(\mathcal{G}(F)) \to H_*(\widetilde{\mathcal{G}}(F))$$

is an isomorphism. Therefore the following definition is meaningful.

Definition 13. Let $F: X \to Y$ be an m-acyclic map w.r.t. the field $\mathbb{F}$. The *induced homomorphism* $F_*: H_*(X) \to H_*(Y)$ is the homomorphism

$$F_* = q_*((\pi_1)_*)^{-1} t(\pi_2)_*.$$

Remark 12. In case of a commutative ring the same definition of the induced homomorphism $F_*: H_*(X, \mathbb{K}) \to H_*(Y, \mathbb{K})$ is possible, if one supposes that the map $\pi_1: \mathcal{G}(F) \to \widetilde{\mathcal{G}}(F)$ is a Vietoris map of order n for every $n \in \mathbb{N}$ (see [2]).

Consider the category m-acyclic-$\underline{\mathrm{Top}}(\mathbb{F})$ with objects compact Hausdorff spaces and morphisms m-acyclic maps w.r.t. $\mathbb{F}$. Obviously the category m-point-$\underline{\mathrm{Top}}(\mathbb{F})$ is a subcategory of m-acyclic-$\underline{\mathrm{Top}}(\mathbb{K})$.

Corollary 4. *There exists an extension of Čech homology functor with coefficient in the field $\mathbb{F}$ from m-point-$\underline{\mathrm{Top}}(\mathbb{F})$ to m-acyclic-$\underline{\mathrm{Top}}(\mathbb{F})$, satisfying the Eilenberg–Steenrod axioms.*

Proof. The assertion follows from Theorem 2 and the relative form of Vietoris' Theorem, [19, pp. 39–40]. □

Remark 13. In [11], [24], [9], the authors introduce and study induced homomorphisms $F_*^s: H_*^s(X,\mathbb{F}) \to H_*^s(Y,\mathbb{F})$ for m-acyclic maps $F: X \to Y$ in singular homology theory for arbitrary spaces X, Y. In case X, Y are compact we have defined the homomorphism $F_*: H_*(X,\mathbb{F}) \to H_*(Y,\mathbb{F})$ in Čech homology, see Definition 13. If X and Y are compact ANR spaces, then the homomorphisms F_* and F_*^s are equal.

5. Lefschetz fixed point theorem

In this section we prove a general Lefschetz fixed point theorem for m-acyclic maps w.r.t. a field $\mathbb{F}$.

Let X, Y be a compact Hausdorff spaces and let $\mathbb{F}$ be a field. In this section we shall consider chains and homologies with coefficients in $\mathbb{F}$.

5.1. $(n-M)$-systems.

Definition 14. Let $n \in \mathbb{N}$ and let X, Y be compact Hausdorff spaces. An $(n-M)$-*system* is a system of chain maps such that for $\lambda \in \mathrm{Cov}(X)$, $\mu \in \mathrm{Cov}(Y)$ there exists a covering $\widehat{\lambda} = \widehat{\lambda}(\mu) \in \mathrm{Cov}(X)$, $\widehat{\lambda} > \lambda$ with the following property: for every $\nu > \widehat{\lambda}$ there exists a chain map

$$T(\nu,\mu,n): C_*(X(\nu)^{(n+1)}) \to C_*(Y(\mu)^{(n+1)})$$

and, moreover, for $\mu', \mu'' \in \mathrm{Cov}(Y)$, $\mu'' > \mu'$ there exists $\overline{\nu} = \overline{\nu}(\mu', \mu'')$ such that for $\nu' > \widehat{\lambda}(\mu')$ and $\nu'' > \widehat{\lambda}(\mu'')$ the diagram

$$\begin{tikzcd}
 & H_i(X(\nu')^{(n+1)}) \arrow[r, "T(\nu',\mu',n)_i"] & H_i(Y(\mu')^{(n+1)}) \arrow[dd, "i(\mu',\mu'')"] \\
H_i(X(\nu)^{(n+1)}) \arrow[ur, "i(\nu,\nu')"] \arrow[dr, "i(\nu,\nu'')"'] & & \\
 & H_i(X(\nu'')^{(n+1)}) \arrow[r, "T(\nu',\mu'',n)_i"'] & H_i(Y(\mu'')^{(n+1)})
\end{tikzcd}$$

is commutative for $\nu > \overline{\nu}$, where

$$i(\nu,\nu')\colon X(\nu)^{(n+1)} \to X(\nu')^{(n+1)},$$
$$i(\nu,\mu'')\colon X(\nu)^{(m+1)} \to X(\nu'')^{(n+1)},$$
$$i(\mu',\mu'')\colon Y(\mu'') \to Y(\mu')$$

are the identity inclusions. We denote an $(n-M)$-system by

$$\mathcal{T}_{X,Y} = \{T(\nu,\mu,n) \mid \lambda \in \operatorname{Cov}(X),\ \mu \in \operatorname{Cov}(Y),\ \nu > \widehat{\lambda}(\mu)\}.$$

In case $X = Y$ an $(n-M)$-system is denoted by $\mathcal{T}_X$.

Lemma 11. *Every $(n-M)$-system $\mathcal{T}_{X,Y}$ induces a homomorphisms*

$$(\mathcal{T}_{X,Y})_i\colon H_i(X) \to H_i(Y), \quad 0 \le i \le n,$$

such that the diagram

$$\begin{tikzcd} H_i(X) \arrow[r, "(\mathcal{T}_{X,Y})_i"] \arrow[d, "\pi(\nu)_i"'] & H_i(Y) \arrow[d, "\pi(\mu)_i"'] \\ H_i(X(\nu)^{(n+1)}) \arrow[r, "T(\nu,\mu,n)_i"'] & H_i(Y(\mu)^{(n+1)}) \end{tikzcd}$$

commutes for $\nu > \widehat{\lambda}(\mu)$, $0 \le i \le n$, where the vertical maps $\pi(\nu)_i$ and $\pi(\mu)_i$ are the natural projections, and $T(\nu,\mu,n)_i$ is induced by the chain map $T(\nu,\mu,n) \in \mathcal{T}_{X,Y}$ with $\nu > \widehat{\lambda}(\mu)$ and the homology is with coefficients in $\mathbb{F}$.

Proof. Let $\mu \in \operatorname{Cov}(Y)$, $n \in \mathbb{N}$ and

$$T(\nu,\mu,n)\colon C_*(X(\nu)^{(n+1)}) \to C_*(Y(\mu)^{(n+1)})$$

in $\mathcal{T}_{X,Y}$. For $i \in \mathbb{N}$ with $i \le n$ we consider the homology class

$$[\zeta] \in H_i(X,\mathbb{F}), \quad [\zeta] = \{\zeta(\lambda) \mid \lambda \in \operatorname{Cov}(X)\}$$

and define

$$(T_{X,Y})_i([\zeta]) = \{T(\widehat{\lambda}(\mu),\mu,n)(\zeta(\widehat{\lambda}(\mu))) \mid \mu \in \operatorname{Cov}(Y)\}.$$

This definition is correct. Indeed, for μ', $\mu'' \in \operatorname{Cov}(Y)$ such that $\mu'' > \mu'$ there exists $\overline{\nu} = \overline{\nu}(\mu',\mu'') \in \operatorname{Cov}(X)$ such that the diagram in Definition 14 is commutative in the homology with coefficients $\mathbb{F}$ up to the dimension n.

Therefore the cycles $T(\overline{\nu},\mu',n)(\zeta(\overline{\nu}))$ and $T(\overline{\nu},\mu'',n)(\zeta(\overline{\nu}))$ are homologous in $Y(\mu')^{(n+1)}$. Since the cycle $\zeta(\overline{\nu})$ is homologous to $\zeta(\widehat{\lambda}(\mu'))$ in $X(\lambda(\mu'))^{(n+1)}$ and since the cycles $\zeta(\overline{\nu})$ and $\zeta(\widehat{\lambda}(\mu''))$ are homologous in $X(\lambda(\mu'')^{(n+1)})$, it follows that the cycles $T(\overline{\nu},\mu',n)(\zeta(\overline{\nu}))$ and $T(\overline{\nu},\mu',n)(\zeta(\widehat{\lambda}(\mu')))$ are homologous in $Y(\mu')^{(n+1)}$ and that $T(\overline{\nu},\mu'',n)(\zeta(\overline{\nu}))$ and $T(\overline{\nu},\mu'',n)(\zeta(\widehat{\lambda}(\mu'')))$ are homologous in $Y(\mu'')^{(n+1)}$.

Therefore $T(\overline{\nu},\mu',n)(\zeta(\widehat{\lambda}(\mu')))$ is homologous to $T(\overline{\nu},\mu'',n)(\zeta(\widehat{\lambda}(\mu'')))$ in $Y(\mu')^{(n+1)}$.

Definition 14 yields that the homomorphism $(T_{X,Y})_i$ does not depend on the choice of $n \in \mathbb{N}$ and the commutativity of the diagram in Lemma 11 is a consequence of the definition of the homomorphism $(T_{X,Y})_i$. □

5.2. $\mathbb{F}$-simplicial spaces. Here we remind the definition of $\mathbb{F}$-simplicial spaces, see [25] for the case of compact Hausdorff spaces.

Definition 15. A compact Hausdorff space X is called $\mathbb{F}$-*simplicial space* if for every covering $\lambda \in \operatorname{Cov}(X)$ there exists a covering $\lambda^\natural \in \operatorname{Cov}(X)$ with $\lambda^\natural > \lambda$ such that for every $\gamma \in \operatorname{Cov}(X)$ there exists a chain map

$$\omega(\lambda^\natural,\gamma)\colon C_*(X(\lambda^\natural)) \to C_*(X(\gamma))$$

with the following properties

(1) $\omega(\lambda^\natural,\gamma)$ is augmentation preserving,
(2) $\omega(\lambda^\natural,\gamma)$ is subordinate to the covering λ, i.e.

$$\operatorname{supp}(\omega(\lambda^\natural,\gamma)(c)) \subset \operatorname{St}(\operatorname{supp}(c),\lambda)$$

for every chain $c \in C_*(X(\lambda^\natural))$

Example 4 ([25], [26]). Compact absolute neighbourhood retracts (ANR) are $\mathbb{F}$-simplicial spaces. Quasicomplexes ([27]), and semi-complexes ([7], [44]), are $\mathbb{F}$-simplicial spaces. Compact topological groups are $\mathbb{F}$-simplicial spaces if $\mathbb{F}$ is a field of characteristic zero ([26]).

We need the following

Lemma 12 ([25, Lemma 2.2]). *Let X be a compact space. Then, for every covering $\lambda \in \operatorname{Cov}(X)$, there exists a covering $\lambda_0 \in \operatorname{Cov}(X)$, $\lambda_0 > \lambda$, such that any augmentation preserving chain map $\varphi\colon C_*(X(\mu)) \to C_*(X(\lambda))$, which is subordinated to λ_0 and $\mu \in \operatorname{Cov}(X)$ is refinement of λ_0, is chain homotopic to the chain map induced by the identity inclusion $i(\mu,\lambda)\colon X(\mu) \to X(\lambda)$.*

5.3. Traces of maps in homology, induced by an $(n-M)$-system of $\mathbb{F}$-simplicial space. Consider an $(n-M)$-system $\mathcal{T}_X$ (w.r.t. $\mathbb{F}$) and its induced homomorphisms $(\mathcal{T}_X)_i\colon H_i(X) \to H_i(X)$, $0 \le i \le n$. The $(n-M)$-system $\mathcal{T}_X$ is of a finite type if the dimensions of the images of $(\mathcal{T}_X)_i$ as a vector space over $\mathbb{F}$ are finite, i.e., $\dim_{\mathbb{F}}(\operatorname{Im}(\mathcal{T}_X)_i) < \infty$, for all $0 \le i \le n$.

Lemma 13. *Let X be a $\mathbb{F}$-simplicial space and let $\mathcal{T}_X$ be an $(n-M)$-system of finite type w.r.t. $\mathbb{F}$. For every sufficiently fine covering $\lambda \in \operatorname{Cov}(X)$ there exist coverings ν, $\mu \in \operatorname{Cov}(X)$ with*

(1) *ν, $\mu > \lambda$,*

(2) *the linear map* $T(\nu,\mu,n)_i\omega(\mu,\nu)_i \colon H_i(X(\nu)^{(n+1)}) \to H_i(X(\mu)^{(n+1)})$ *is defined and have a* (*generalized*) *trace* $\mathrm{Tr}(T(\nu,\mu,n)_i\omega(\mu,\nu)_i)$,

(3) $\mathrm{Tr}((\mathcal{T}_X)_i) = \mathrm{Tr}(T(\nu,\mu,n)_i\omega(\mu,\nu)_i)$.

Proof. We follow the proof of Theorem 2.7 in [25]. Let A_i be the image $\mathrm{Im}(\mathcal{T}_X)_i$ of the homomorphism $(\mathcal{T}_X)_i$. Since $\dim_{\mathbb{F}}(A_i) < \infty$ there exists a covering $\lambda \in \mathrm{Cov}(X)$ such that the projections

$$\pi(\lambda)_i|_{A_i} \colon A_i \to H_i(X(\lambda)^{(n+1)})$$

are monomorphisms for all $0 \le i \le n$.

For the covering λ we consider the covering $\widehat{\lambda} = \widehat{\lambda}(\lambda)$ with $\widehat{\lambda} > \lambda$ and the chain map $T(\widehat{\lambda},\lambda,n) \in \mathcal{T}_X$ and also the covering $(\widehat{\lambda})_0$ whose relation to $\widehat{\lambda}$ is explained in Lemma 12. Then for every $\mu > (\widehat{\lambda})_0$ and every augmentation preserving chain map $\varphi \colon C_*(X(\mu)^{(n+1)}) \to C_*(X(\widehat{\lambda})^{(n+1)})$, which is subordinated to $(\widehat{\lambda})_0$, the chain map φ is chain homotopic to the inclusion $i(\mu,\widehat{\lambda})$.

Now we choose a special covering μ. Since X is a $\mathbb{F}$-simplicial space, Definition 15 guarantees the existence of a covering $((\widehat{\lambda})_0)^\natural$ related to $(\widehat{\lambda})_0$. Choose $\mu > ((\widehat{\lambda})_0)^\natural$ and for every covering $\nu > ((\widehat{\lambda})_0)^\natural$ there exists a chain map

$$\omega(\mu,\nu) \colon C_*(X(\mu)) \to C_*(X(\nu)),$$

which is subordinate to the covering $((\widehat{\lambda})_0)^\natural$. Since $((\widehat{\lambda})_0)^\natural > (\widehat{\lambda})_0$, the chain mapping $\omega(\mu,\nu)$ is subordinate to the covering $(\widehat{\lambda})_0$. Then Lemma 12 implies that the chain maps $i(\nu,((\widehat{\lambda})_0)^\natural)\omega(\mu,\nu)$ and $i(\mu,((\widehat{\lambda})_0)^\natural)$ are chain homotopic, which implies that the diagram

$$\begin{tikzcd}
H_i(X(\nu)^{(n+1)}) \arrow[r, "\omega(\nu,\mu)_i"] \arrow[dr, "i(\mu,((\widehat{\lambda})_0)^\sharp)_i"'] & H_i(X(\nu)^{(n+1)}) \arrow[d, "\pi(((\widehat{\lambda})_0)^\natural)_i"] \arrow[dl, phantom, "\mathrm{(I)}"] \\
& H_i(X(((\widehat{\lambda}_0)^\natural)^{(n+1)}))
\end{tikzcd}$$

is commutative, the vertical maps are the natural inclusion and projection, respectively.

Now we shall choose a special covering ν. To the covering μ there exists a covering $\widehat{\mu}$ and the chain map $T(\widehat{\mu},\mu,n) \in \mathcal{T}_X$. Choose $\nu > \widehat{\mu}$. Then we have the chain map $T(\nu,\mu,n) \in \mathcal{T}_X$. Lemma 11 implies that the diagram

$$\begin{tikzcd}
H_i(X) \arrow[r, "(\mathcal{T}_X)_i"] \arrow[d, "\pi(\nu)_i"'] \arrow[dr, phantom, "\mathrm{(II)}"] & H_i(X) \arrow[d, "\pi(\mu)_i"] \\
H_i(X(\nu)^{(n+1)}) \arrow[r, "T(\nu,\mu,n)_i"'] & H_i(X(\mu)^{(n+1)})
\end{tikzcd}$$

is commutative for $i \le i \le n$. Furthermore, the diagram

$$\begin{tikzcd}
H_i(X(\nu)^{(n+1)}) \arrow[r, "T(\nu,\mu,n)_i"] \arrow[d, "Vi(\nu,((\lambda)_0)^\sharp)_i"'] \arrow[dr, phantom, "\mathrm{(III)}"] & H_i(X(\mu)^{(n+1)}) \arrow[d, "i(\mu,\nu)_i"] \\
H_i(X(((\widehat{\lambda})_0)^{(n+1)}) \arrow[r, "T(((\lambda)_0)^\sharp,\lambda,n)_i"'] & H_i(X(\lambda)^{(n+1)})
\end{tikzcd}$$

is commutative, where the vertical homomorphisms are induced by the natural inclusions.

From the continuity of the Čech homologies and Lemma 12 and for ν sufficiently fine it follows that

$$\mathrm{Im}(T(\nu,\mu,n)_i) = \mathrm{Im}(\pi(\mu)_i(\mathcal{T})_i). \tag{4}$$

To achieve the assertion we have to impose an additional condition for the covering ν. To this end, we consider the diagram

$$\begin{tikzcd}
H_i(X) \arrow[r, equal, "\mathrm{id}"] \arrow[d, "\pi(\mu)_i"'] & H_i(X) \arrow[r, "(\mathcal{T}_X)_i"] \arrow[d] \arrow[dr, phantom, "\mathrm{(II)}"] & H_i(X) \arrow[d, "\pi(\mu)_i"] \\
H_i(X(\mu)^{(n+1)}) \arrow[r, "\omega(\nu,\mu)_i"] \arrow[dr, "i(\mu,((\widehat{\lambda})_0)^\sharp)_i"'] & H_i(X(\mu)^{(n+1)}) \arrow[r, "T(\nu,\mu,n)_i"] \arrow[d] \arrow[l, phantom, "\mathrm{(I)}" near end, shift right=4] \arrow[dr, phantom, "\mathrm{(III)}"] & H_i(X(\mu)^{(n+1)}) \arrow[d, "i(\mu,\nu)_i"] \\
 & H_i(X(((\widehat{\lambda})_0)^{(n+1)}) \arrow[r, "T(((\lambda)_0)^\sharp,\lambda,n)_i"'] & H_i(X(\lambda)^{(n+1)})
\end{tikzcd}$$

Since the subdiagrams (I)–(III) are commutative and since

$$\begin{aligned}
\pi(\lambda)_i(\mathcal{T}_X)_i &= T(((\widehat{\lambda})_0)^\natural,\lambda,n)_i i(\mu,((\widehat{\lambda})_0)^\natural)_i\pi(\mu)_i \\
&= T(((\widehat{\lambda})_0)^\natural,\lambda,n)_i i(\nu,\mu)_i\omega(\nu,\mu)_i\pi(\mu)_i \\
&= i(\mu,\lambda)_i T(\nu,\mu,n)_i\omega(\nu,\mu)_i\pi(\mu)_i,
\end{aligned} \tag{5}$$

the diagram is commutative. Now let $r(\lambda)_i\colon H_i(X(\lambda)^{(n+1)}) \to H_i(X)$ be the homomorphism satisfying

$$\mathrm{Im}(r(\lambda)_i) \subset \mathrm{Im}(\mathcal{T}_X)_i, \quad r(\lambda)_i\pi(\lambda)_i|_{A_i} = \mathrm{id}_{A_i}. \tag{6}$$

From (5) and (6) it follows that $(\mathcal{T}_X)_i = r(\lambda)_i i(\mu,\lambda)_i T(\nu,\mu,n)_i\omega(\nu,\mu)_i\pi(\mu)_i$, which gives $\mathrm{Tr}((\mathcal{T}_X)_i) = \mathrm{Tr}((r(\lambda)_i i(\mu,\lambda)_i T(\nu,\mu,n)_i\omega(\nu,\mu)_i\pi(\mu)_i)$ and, by the standard properties of the trace, yields

$$\begin{aligned}
&\mathrm{Tr}(r(\lambda)_i i(\mu,\lambda)_i T(\nu,\mu,n)_i\omega(\nu,\mu)_i\pi(\mu)_i \\
&\qquad = \mathrm{Tr}(\pi(\mu)_i r(\lambda)_i i(\mu,\lambda)_i T(\nu,\mu,n)_i\omega(\nu,\mu)_i).
\end{aligned} \tag{7}$$

Equation (4) implies that the homomorphism $\pi(\mu)_i r(\lambda)_i i(\mu,\lambda)_i$ is the identity on the image of $T(\nu,\mu,n)_i$. Therefore (7) implies the assertion of the lemma. □

5.4. Weak approximation systems of upper semi-continuous maps.

Definition 16. Let $F: X \to Y$ be an upper semi-continuous multivalued map. An $(n-M)$-system $\mathcal{T}_{X,Y}$ (w.r.t. $\mathbb{F}$) is called *n-weak approximation system, $(n - wA)$-system*) of F if, for every $\lambda \in \operatorname{Cov}(X)$, $\mu \in \operatorname{Cov}(Y)$, there exist $\widehat{\lambda} = \widehat{\lambda}(\mu) \in \operatorname{Cov}(X)$ with $\widehat{\lambda} > \lambda$ and a chain map $T(\widehat{\lambda}, \mu, n) \in \mathcal{T}_{X,Y}$ with

$$T(\widehat{\lambda}, \mu, n): C_*(X(\widehat{\lambda})^{(n+1)}) \to C_*(Y(\mu)^{(n+1)}),$$

such that for every simplex $\sigma^k \in X(\widehat{\lambda})^{(n+1)}$ there exists an open set $U(\sigma^k)$ in X with

(1) $U(\sigma^k) < \lambda$,
(2) $\operatorname{supp}(T(\widehat{\lambda}, \mu, n)(\sigma^k)) \subset \operatorname{St}(F(U(\sigma^k)), \mu)$.

We denote a $(n - wA)$-system of the map F also by $(n - wA)(F)$.

Definition 17. The n-weak approximation system $(n-wA)(F)$ of the u.s.c. map $F: X \to Y$ is called n-approximation system ($(n - A)$-system) if the chain maps $T(\widehat{\lambda}, \mu, n)$ are augmentation preserving. In this case it is denoted by $A(F)$.

An $(n-A)$-system of an u.s.c. map F is also denoted by $(n-A)(F)$ or $A(F)$.

Remark 14. Approximation systems for upper semi-continuous acyclic maps are used by L. Vietoris (see [46]), and later by S. Eilenberg and D. Montgomery (see [17]), and by E. Begle ([2], [3]), B. O'Neil ([33]). The explicit definition is given in [36], [38], [15], [19], [41].

Since every $(n - wA)$-system, $\mathcal{T}_{X,Y}$, is also an $(n - M)$-system, it induces homomorphisms $(\mathcal{T}_{X,Y})_i: H_i(X, \mathbb{F}) \to H_i(Y, \mathbb{F})$ for all $0 \leq i \leq n$. These are the homomorphisms induced by the $(n - wA)$-system of the map F, and they are denote as $(n - wA(F))_i$ or simply by $(n - A(F))_i$.

Lemma 14. *Every acyclic (w.r.t. $\mathbb{F}$) map $F: X \to Y$ has an $(n - A)$-system for every $n \in \mathbb{N}$. Furthermore, for $i \in \mathbb{N}$, $i \leq n$, we have $(n-A(F))_i = F_i$, where $F_i: H_i(X, \mathbb{F}) \to H_i(X, \mathbb{F})$ is the induced homomorphism of the acyclic map F.*

Proof. Let $\mathcal{G}(F) = \{(x, y) \mid y \in F(x)\}$ be the graph of the map F. Denote by $p: \Gamma(F) \to X$ and $q: \Gamma(F) \to Y$ the natural projections. The map p is a Vietoris map w.r.t. $\mathbb{F}$, i.e. the reduced Čech homology $\widetilde{H}_*(F(x), \mathbb{F}) = 0$ and we have $F(x) = q(p^{-1}(x))$ for all $x \in X$.

Let $\lambda \in \operatorname{Cov}(X)$, $\mu \in \operatorname{Cov}(Y)$ and $\nu \in \operatorname{Cov}(\mathcal{G}(F))$ with $\nu > p^{-1}(\lambda)$, $q^{-1}(\mu)$.

Consider the Vietoris map $p: \mathcal{G}(F) \to X$, the coverings ν, λ and a natural number $n \in \mathbb{N}$. Let

$$T(\widehat{\lambda}(\nu), \nu, n): C_* X(\lambda(\nu))^{(n+1)}) \to C_*(\Gamma(F)(\nu))$$

be the augmentation preserving chain map constructed in Lemma 2 in [2].

The map $q: \mathcal{G}(F) \to Y$ induces an augmentation preserving chain map

$$q: C_*(\Gamma(F)(\nu)) \to C_*(Y(\nu))$$

and Lemma 3 in [2] implies that the system of augmentation preserving chain maps $\{qT(\widehat{\lambda}(\nu), \nu, n) \mid \lambda \in \mathrm{Cov}(X),\ \mu \in \mathrm{Cov}(Y),\ \nu \in \mathrm{Cov}(\mathcal{G}(F)), \nu > \lambda, \nu\}$ is an $(n-A)$-system (w.r.t. $\mathbb{F}$) of the map F. We denote it by $n - A(F)$. The assertion $(n - A(F))_i = F_i$ follows from results in [2]. $\square$

Lemma 14 with X and Y finite polyhedra is discussed in [36].

Remark 15. In case of continuous single valued maps $f: X \to Y$ the field $\mathbb{F}$ can be replaced by a commutative ring $\mathbb{K}$. Every single valued continuous map $f: X \to Y$ has an A-system w.r.t. $\mathbb{K}$.

Let $\lambda \in \mathrm{Cov}(X)$, $\mu \in \mathrm{Cov}(Y)$ and $\widehat{\lambda} = \widehat{\lambda}(\mu)$ be a covering with $\widehat{\lambda} > \lambda$, $f^{-1}(\mu)$. The map f induces the simplicial map $f(\widehat{\lambda}, \mu): X(\widehat{\lambda}) \to Y(\mu)$, which is used for the definition of the induced homomorphism $f_*: H_*(X, \mathbb{K}) \to H_*(Y, \mathbb{K})$. For a simplex $\sigma = (x_0, \dots, x_k) \in X(\widehat{\lambda})$ we have $f(\widehat{\lambda}, \mu)(\sigma) = (f(x_0), \dots, f(x_k))$. The system of augmentation preserving chain maps, induced by the simplicial maps $f(\widehat{\lambda}, \mu)$, $\{f(\widehat{\lambda}, \mu) \mid \lambda \in \mathrm{Cov}(X), \mu \in \mathrm{Cov}(Y)\}$ is an $(n-A)$-system w.r.t. $\mathbb{K}$ for every $n \in \mathbb{N}$. The homomorphisms $(n - A(f))_i$ in the homology with coefficients in $\mathbb{K}$ are equal to the homomorphism $f_i: H_i(X, \mathbb{K}) \to H_i(Y, \mathbb{K})$

Lemma 15. *Let $f: X \to Y$ be a surjective m-map w.r.t. a commutative ring $\mathbb{K}$ with* 1. *The m-point map $f = f^{-1}: Y \to X$ (w.r.t. $\mathbb{K}$) has an $n - wA$-system $(n - wA)(f^{-1})$ w.r.t. $\mathbb{K}$ for every $n \in \mathbb{N}$. The homomorphisms $(n - wA(f^{-1}))_i: H_i(X, \mathbb{K}) \to H_i(Y, \mathbb{K})$ induced by $(n - wA)(f^{-1})$, coincide with the transfer homomorphisms $t(f)_i: H_i(Y, \mathbb{K}) \to H_i(X, \mathbb{K})$.*

Proof. Let $\lambda \in \mathrm{Cov}(X)$, $\mu \in \mathrm{Cov}(Y)$, and $n \in \mathbb{N}$. Denote by

$$T(\widehat{\mu}, \lambda, n): C_*(Y(\widehat{\mu})^{(n+1)}) \to C_*(X(\lambda)^{(n+1)}$$

the chain map constructed in Lemma 6, and let $\widehat{\mu} = \widehat{\mu}(\lambda)$. Corollaries 1–3 imply that the system of chain maps $\{T(\widehat{\mu}, \lambda, n) \mid \lambda \in \mathrm{Cov}(X),\ \mu \in \mathrm{Cov}(Y),\ n \in \mathbb{N}\}$ is an $(n - wA)$-system of f^{-1} w.r.t. $\mathbb{K}$. We denote it by $wA(f^{-1})$.

The assertion about the homomorphisms induced in homology follows direct from the definitions. $\square$

The case X, Y being finite polyhedra is discussed in [21].

Definition 18. Let $F_1: X \to Y$, $F_2: Y \to Z$ be upper semi-continuous maps with n-weak approximation systems $(n - wA)(F_1)$, $(n - wA)(F_2)$ w.r.t. $\mathbb{F}$, respectively. For given $\mu \in \mathrm{Cov}(Y)$, $\nu \in \mathrm{Cov}(Z)$, let

$$T_2(\widehat{\mu}(\nu), \nu, n): C_*(Y(\widehat{\mu}(\nu))^{(n+1)}) \to C_*(Z(\nu)^{(n+1)})$$

be a chain map belonging to $(n-wA)(F_2)$. For the coverings $\lambda \in \mathrm{Cov}(X)$ and $\widehat{\mu}(\nu)$ let

$$T_1(\widehat{\lambda}(\widehat{\mu}(\nu)), \nu, n)\colon C_*(X(\widehat{\lambda}(\widehat{\mu}(\nu), \nu)^{(n+1)}) \to C_*(Y(\widehat{\mu}(\nu))^{(n+1)})$$

be a chain map belonging to $(n-wA)(F_1)$. The system of chain maps

$$\{T_2(\widehat{\mu}(\nu), \nu, n)T_1(\widehat{\lambda}(\widehat{\mu}(\nu)), \widehat{\mu}(\nu)) \mid \lambda \in \mathrm{Cov}(X), \mu \in \mathrm{Cov}(Y), \nu \in \mathrm{Cov}(Z)\}$$

is called the composition of the $(n-wA)$-systems $(n-wA)(F_1)$ and $(n-wA)(F_2)$ and is denoted $(n-wA(F_2)) \circ (n-wA(F_1))$.

Lemma 16. *Let $F_1\colon X \to Y$, $F_2\colon Y \to Z$ be upper semi-continuous maps with n-weak approximation systems $(n-wA)(F_1)$, $(n-wA)(F_2)$ w.r.t. $\mathbb{F}$, respectively. The composition $(n-wA)(F_2) \circ (n-wA)(F_1)$ is an $(n-wA)$-system for the map $F_2F_1\colon X \to Z$ w.r.t. $\mathbb{F}$.*

The proof is a direct consequence of the definition of $(n-wA)$-systems.

Lemma 17. *Every m-point map (w.r.t. $\mathbb{K}$) $F\colon X \to Y$ has an $(n-wA)$-system $(n-wA)(F)$ w.r.t. $\mathbb{K}$ for every $n \in \mathbb{N}$. The homomorphisms in the Čech homology, induced by $(n-wA)(F)$, coincide with the homomorphisms $F_i\colon H(X,\mathbb{K}) \to H(Y,\mathbb{K})$ induced by the map F.*

Proof. Consider the minimal representation of F, (see (1) of Remark 8)

$$X \xleftarrow{p_0} \mathcal{G}(F) \xrightarrow{q_0} Y$$

The map $p_0\colon \Gamma(F) \to X$ is an m-map w.r.t. $\mathbb{K}$ and $q_0\colon \Gamma(F) \to Y$ is single valued and continuous. Then, see Remark 15 $(n-A)(q_0) \circ (n-wA(p_0^{-1})$ is an $(n-wA)$-system for F w.r.t. $\mathbb{K}$. The assertion about the induced homomorphisms in the homology follows direct from the definitions. $\square$

Lemma 18. *Every m-acyclic map $F\colon X \to Y$ w.r.t. $\mathbb{F}$ has an $(n-wA)$-system w.r.t. $\mathbb{F}$ for every $n \in \mathbb{N}$. The homomorphisms in the Čech homology, induced by $(n-wA)(F)$, coincide with the homomorphisms $F_i\colon H(X,\mathbb{K}) \to H(Y,\mathbb{K})$ induced by F*

Proof. Consider the diagram

$$\begin{tikzcd} X & \mathcal{G}(F) \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "\pi_1"] & Y \\ & \widetilde{\mathcal{G}}(F) \arrow[ul, "\pi_2"] & \end{tikzcd}$$

as in Section 4.2. The map $\pi_1\colon \mathcal{G}(F) \to \widetilde{\mathcal{G}}(F)$ is Vietoris map w.r.t. $\mathbb{F}$. The map $\pi_2\colon \widetilde{\mathcal{G}}(F) \to X$ is an m-map w.r.t. $\mathbb{F}$ such that $F(x) = q(\pi_1^{-1}(\pi_2^{-1}(x)))$. Then $(n-wA)(q) \circ (n-wA)(\pi_2^{-1}) \circ (n-wA)(\pi_1^{-1})$ is an $(n-wA)$-system for F w.r.t. $\mathbb{F}$.

The assertion about the induced homomorphisms in the homology follows direct from the definitions. □

5.5. Lefschetz fixed point theorem for upper semi-continuous maps with $n - wA$-systems. Before stating the main result of this section we state a theorem of C. H. Dowker.

THEOREM 3 ([14]). *For every covering $\lambda \in \mathrm{Cov}(X)$ there exist two augmentation preserving chain mappings*

$$k(\lambda): C_*(N(\lambda)) \to C_*(X(\lambda)) \quad \textit{and} \quad l(\lambda): C_*(X(\lambda)) \to C_*(N(\lambda))$$

such that

(1) *$k(\lambda)$ and $l(\lambda)$ are chain homotopy inverse,*
(2) $\mathrm{supp}(k(\lambda)(c_1)) \subset \mathrm{supp}(c_1)$ *for $c_1 \in C_*(N(\lambda))$,*
$\mathrm{supp}(l(\lambda)(c_2)) \subset \mathrm{St}(\mathrm{supp}(c_2), \lambda)$ *for $c_2 \in C_*(X(\lambda))$.*

LEMMA 19. *Let $F: X \to X$ be an upper semi-continuous map of the $\mathbb{F}$-simplicial space X such that $\dim_{\mathbb{F}} Im(F_*) < \infty$. Furthermore, for every n sufficiently large there exists an $(n - wA)$-system w.r.t. $\mathbb{F}$ for F. If the Lefschetz number $L(F) = \sum_i (-1)^i Tr(F_i)$ is different from zero, then F has a fixed point, i.e. there exists $x_0 \in X$ such that $x_0 \in F(x_0)$.*

PROOF. We use H. Hopf's proof of the Lefschetz fixed point theorem [23, pp. 5–13], see also the proof of the Theorem 2.7 in [25].

Suppose that the u.s.c. map $F: X \to X$ has no fixed point, i.e. $x \notin F(x)$ for all $x \in X$. Then there exists a covering $\lambda \in \mathrm{Cov}(X)$ such that

$$\mathrm{St}(U, \lambda) \cap \mathrm{St}^2(F(U), \lambda) = \emptyset \tag{8}$$

for every $U \in \lambda$. Choose $n \in \mathbb{N}$ such that $n > \max\{\dim_{\mathbb{F}} F_*, \dim_{\mathbb{F}} N(\lambda)\}$ and let $(n-wA)(F)$ be an $(n-wA)$-system for F. Applying Lemma 13 to the $(n-M)$-system $(n-wA)(F)$ and the covering λ we obtain coverings μ, $\nu \in \mathrm{Cov}(X)$ such that

$$\mathrm{Tr}(F_i) = \mathrm{Tr}(T(\nu, \mu, n)_i \omega(\mu, \nu)_i)$$

and, for every simplex $\sigma \in X(\mu)^{(n+1)}$, there exists $U(\sigma) < \nu$ with

$$\mathrm{supp}(T(\nu, \mu, n)(\sigma)) \subset St(F(U(\sigma)), \nu). \tag{9}$$

Additionally, we choose ν such that $\nu{*}{*}{*} > \lambda$ and consider the chain map

$$\psi = l(\mu) T(\nu, \mu, n) \omega(\mu, \nu) k(\mu): C_*(N(\mu)) \to C_*(N(\mu)),$$

and the homomorphism in the homology induced by it

$$\psi_i: H_i(N(\mu)) \to H_i(N(\mu)).$$

Then

$$\begin{aligned}\operatorname{Tr}(\psi_i) &= \operatorname{Tr}((l(\mu)T(\nu,\mu,n)\omega(\mu,\nu)k(\mu))_i) \\ &= \operatorname{Tr}((k(\mu)l(\mu)T(\nu,\mu,n)\omega(\mu,\nu))_i) = \operatorname{Tr}((T(\nu,\mu,n)\omega(\mu,\nu))_i) = \operatorname{Tr}(F_i).\end{aligned}$$

Then Hopf's Lemma ([23, pp. 5–13]), on the Lefschetz numbers implies that the Lefschetz number $L(F) = \sum_i (-1)^i \operatorname{Tr}(F_i)$ of F is equal to $\sum_i (-1)^i \operatorname{Tr}(\psi_i)$. Since $L(F) \neq 0$, there exists an i_0, $0 \leq i_0 \leq n$ such that the trace of the homomorphism $\psi_{i_0} \colon C_{i_0}(N(\mu)) \to C_{i_0}(N(\mu))$ is not zero. I.e. there exists an i_0-dimensional simplex $\sigma^{i_0} \in N(\mu)$ which is contained in the chain $\psi_{i_0}(\sigma^{i_0})$ having a nonzero coefficient. This implies $\sigma^{i_0} \subset \operatorname{supp}(\psi_{i_0}(\sigma^{i_0}))$. The last inclusion contradicts (8). $\square$

Finally, we can state a generalized Lefschetz fixed point theorem.

Theorem 4. *Let $\mathbb{F}$ be a field and let X be an $\mathbb{F}$-simplicial space. If $F \colon X \to X$ is an m-acyclic map w.r.t. $\mathbb{F}$ such that $\dim_{\mathbb{F}} \operatorname{Im}(F_*) < \infty$ and such that the Lefschetz number of F is different from zero, then F has a fixed point, i.e. there is a point $x_0 \in X$ such that $x_0 \in F(x_0)$.*

It follows from Lemmas 14 and 19.

References

[1] J. C. Becker and D. H. Gottlieb, *Transfer maps for fibrations and duality*, Compositio Math. **33** (1976), 107–133.

[2] E. Begle, *The Vietoris mapping theorem for bicompact spaces*, Ann. Math. **51** (1950), 534–543.

[3] ———, *A fixed point theorem*, Ann. Math. **51** (1950), 544–550.

[4] C. Berge, *Espaces Topologiques. Fonctions Multivoques*, Dunod, Paris, 1959.

[5] Y. Borisovich, E. Gelman, A. Myshkis and V. Obukhovskiĭ, *Topological methods in fixed point theory for multivalued mappings*, Russian Math. Surveys **35** (1980), 65–143.

[6] G. Bredon, *Sheaf Theory*, Mc Graw Hill, 1967.

[7] F. Browder, *On the fixed point index for continuous mappings of locally compact spaces*, Summa Brasil. Math. **4** (1960), 253–293.

[8] G. Brumfield and R. Kubelka, *A homology transfer for a class of simplicial maps*, Manuscr. Math. **38** (1982), 99–128.

[9] G. Conti and J. Pejsachovicz, *Fixed point theorem for multivalued weighted maps*, Ann. Mat. Pura Appl. (4) **126** (1980), 319–341.

[10] G. Darbo, *Grado topologico e teoremi di esistenza di punti uniti per trasformazioni plurivalenti di bizelle*, Rend. Sem. Mat. Univ. Padova **19** (1950), 371–395.

[11] ———, *Teoria dell'omologia in una categoria di mappe plurivalenti ponderate*, Rend. Sem. Mat. Univ. Padova **28** (1958), 188–220.

[12] A. Dold, *The fixed point transfer of fiber-preserving maps*, Math. Z. **148** (1976), 215–244.

[13] ______, *Ramified coverings, orbit projections and symmetric powers*, Math. Proc. Cambridge Philos. Soc. **99** (1986), 66–72.

[14] D. H. Dowker, *Homology groups of relations*, Ann. Math. **56** (1952), 84–95.

[15] Z. Dzedzej, *Fixed point index for a class of nonacyclic multivalued maps*, Disserttiones Math. **253** (1985), 1–58.

[16] A. Edmonds, *Branched coverings and orbits maps*, Michigan Math. J. **23** (1977), 289.

[17] S. Eilenberg and D. Montgomery, *Fixed points for multivalued transformations*, Amer. J. Math. **69** (1946), 214–222.

[18] S. Eilenberg and N. Steenrod, *Foundations of Algebraic Topology*, Princeton Univ. Press., 1952.

[19] L. Górniewicz, *Topological Fixed Point Theory of Multivalued Mappings*, Kluwer Acad. Publ., Dordrecht, 1999.

[20] M. Greenberg and J. Harper, *Algebraic Topology*, Addison-Wesley Publ. Co., Reedwood City, 1981.

[21] F. von Haeseler and G. Skordev, *Borsuk–Ulam theorem, fixed point index and chain approximation for maps with multiplicity*, Pacific J. Math. **153** (1992), 369–395.

[22] M. Henriksen and J. Isbell, *On the continuity of the real roots of algebraic equations*, Proc. Amer. Math. Soc. **4** (1953), 431–434.

[23] H. Hopf, *Selecta*, Springer–Verlag, Berlin, 1964.

[24] R. Jerrard, *Homology with multivalued functions applied to fixed points*, Trans. Amer, Math. Soc. **213** (1975), 407–427.

[25] R. Knill, *Q-simplicial spaces*, Illinois J. Math. **14** (1970), 40–51.

[26] ______, *The Lefschetz fixed point theorem for compact groups*, Proc. Amer. Math. Soc. **66** (1977), 148–152.

[27] S. Lefschetz, *Algebraic Topology*, Amer. Math. Soc., New York, 1942.

[28] J. Leray, *Theorie des points fixes: indices total et nombre de Lefschetz*, Bull. Soc. Math. France **87** (1959), 221–233.

[29] S. Masih, *Fixed points of symmetric product mappings of polyhedra and metric* ANR, Fund. Math. **80** (1973), 149–156.

[30] Ch. Maxwell, *Fixed points of symmetric product mappings*, Proc. Amer. Math. Soc. **8** (1957), 808–815.

[31] ______, *The degree of multiple-valued maps of spheres*, Lecture Notes in Math. **664** (1978), 123–141.

[32] H. Munkholm, *A chain level transfer homomorphism for PL fibration*, Math. Z. **166** (1979), 183–186.

[33] B. O'Neill, *Induced homology homomorphism for set-valued maps*, Pacific J. Math. **7** (1957), 1179–1184.

[34] J. Pejsachowicz, *A Lefschetz fixed point theorem for multivalued weighted maps*, Boll. Un. Math. Ital. A **14** (1977), 391–397.

[35] ______, *The homotopy theory of weighted maps*, Boll. Un. Math. Ital. B **14** (1977), 702–721.

[36] H.-W. Siegberg and G. Skordev, *Fixed point index and chain approximations*, Pacific J. Math. **102** (1982), 455–486.

[37] G. Skordev, *Coincidence of maps in Q-simplicial spaces*, Fund. Math. **113** (1981), 67–79.

[38] ______, *Homological properties and fixed points of continuous maps*, Diss. Univ. Sofia, 1982. (Bulgarian)

[39] ______, *Coincidence of maps in homogeneous spaces*, C. R. Acad. Bulgare **38** (1985), 1279–1281.

[40] ———, *The Lefschetz fixed point theorem*, Annuaire Univ. Sofia Fac. Math. Mech. **79** (1985), 201–213.

[41] ———, *The multiplicity property of the fixed point index for multivalued maps*, Serdica Math. J. **15** (1989), 160–170.

[42] L. Smith, *Transfer and ramified coverings*, Math. Proc. Cambridge Philos. Soc. **93** (1983), 485–493.

[43] E. H. Spanier, *Cohomology theory for general spaces*, Ann. Math. **49** (1948), 407–427.

[44] R. Thompson, *A unified approach to local and global fixed point indices*, Adv. Math. **3** (1969), 1–72.

[45] J. W. Tukey, *Convergence and Uniformity in Topology*, Ann. of Math. Stud., Nr. 2, Princeton Univ. Pres, 1940.

[46] L. Vietoris, *Über den höheren Zusammenhang kompakter Räume und eine Klasse von zusammenhangstreuen Abbildungen*, Math. Ann. **97** (1927), 454–472.

[47] A. V. Zarelua, *Limits of local systems of sheaves and zero dimensional mappings*, Proc. Steklov Inst. Math. **4** (1984), 105–120.

Manuscript received October 22, 2001

Fritz von Haeseler
Katholieke Universiteit Leuven
Department of Electrical Engineering
Kasteelpark Arenberg 10
B-3001, Leuven, BELGIUM

E-mail address: F.v.Haeseler@esat.kuleuven.ac.be

Heinz-Otto Peitgen and Gencho Skordev
Center for Complex Systems and Visualization
University of Bremen
Universitätsallee 29
28359 Bremen, GERMANY

E-mail address: peitgen@proxy.cevis.uni-bremen.de, skordev@proxy.cevis.uni-bremen.de