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LEFT ABSOLUTELY FLAT GENERALIZED INVERSE SEMIGROUPS

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ABSTRACT. A semigroup S is called (left, right) absolutely flat if all of its (left, right) S-sets are flat. S is a (left, right) generalized inverse semigroup if S is regular and its set of idempotents $E(S)$ is a (left, right) normal band (i.e. a strong semilattice of (left zero, right zero) rectangular bands). In this paper it is proved that a generalized inverse semigroup S is left absolutely flat if and only if S is a right generalized inverse semigroup and the (nonidentity) structure maps of $E(S)$ are constant. In particular all inverse semigroups are left (and right) absolutely flat (see [**1**]). Other consequences are derived.

1. Introduction. Let S be a semigroup. S-Ens (respectively, Ens-S) will denote the class of all left (right) S-sets. For $A \in$ Ens-S and $B \in S$-Ens, let τ denote the smallest equivalence relation on $A \times B$ containing all pairs $((as, b), (a, sb))$ for $a \in A$, $b \in B$, and $s \in S$. The tensor product $A \otimes B$ (or, more precisely, $A \otimes_S B$) is defined to be the set $(A \times B)/\tau$, and possesses the customary universal mapping property with respect to balanced maps from $A \times B$ to an arbitrary set. For $a \in A$ and $b \in B$, $a \otimes b$ represents the τ-class of (a, b). B is called flat (in S-Ens) if and only if, for all embeddings $A \to C$ in Ens-S, the induced map $A \otimes B \to C \otimes B$ is an embedding. S is called left absolutely flat if all of its left S-sets are flat. Right absolute flatness is defined similarly, and S is called absolutely flat if it is both left and right absolutely flat.

In [**4**], M. Kil'p proves that every left absolutely flat semigroup is regular, and in [**5**] that every inverse union of groups is absolutely flat. The present authors show that, in fact, every inverse semigroup is absolutely flat [**1**] and in the same paper characterize those Rees matrix semigroups (with or without zero) which are left absolutely flat. Furthermore, it is known that every left absolutely flat union of groups must be a semilattice of right groups (see, for example [**6**]). In this paper we shall characterize the generalized inverse semigroups (and in particular the strong semilattices of right groups) which are left absolutely flat.

In his study of amalgamation of semigroups, T. E. Hall developed the Free Representation Extension Property [**3**]. A semigroup is left absolutely flat if and only if it has this property *in the class of all semigroups* (see [**2**, Proposition 1.1]).

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Amalgamation bases (in the class of all semigroups) often have this property and its dual, i.e. they are often absolutely flat. The authors intend to employ the material developed in this paper in subsequent studies on amalgamation.

2. Left absolutely flat generalized inverse semigroups. If S is a semigroup, S^1 denotes the monoid obtained by adjoining a new identity element 1 to S. Let $A \in$ Ens-S, a, $\hat{a} \in A$, $B \in S$-Ens, and b, $\hat{b} \in B$. Following Lemma 1.2 of [**1**] it is easy to show that $a \otimes b = \hat{a} \otimes \hat{b}$ in $A \otimes_S B$ if and only if there exist $a_1, \ldots, a_n \in A$, $b_2, \ldots, b_n \in B$, $s_1, \ldots, s_n t_1, \ldots, t_n \in S^1$ such that

$$\begin{aligned} a &= a_1 s_1 & & \\ a_1 t_1 &= a_2 s_2 & s_1 b &= t_1 b_2 \\ a_2 t_2 &= a_3 s_3 & s_2 b_2 &= t_2 b_3 \\ &\vdots & &\vdots \\ a_n t_n &= \hat{a} & s_n b_n &= t_n \hat{b} \end{aligned}$$

(where it is assumed that S^1 acts unitally on A and B). The system of equalities above is called a *scheme over A and B of length n joining* (a, b) *to* $(\hat{a}, \hat{b})$. From this description we see that a left S-set B is flat if and only if, for every right S-set A, and every a, $\hat{a} \in A$, b, $\hat{b} \in B$ such that there exists a scheme over A and B joining (a, b) to $(\hat{a}, \hat{b})$, there exists a scheme (of possibly different length) over $aS^1 \cup \hat{a}S^1$ and B joining (a, b) to $(\hat{a}, \hat{b})$. (See Lemma 2.2 of [**1**].)

A regular semigroup S is called a (left, right) generalized inverse semigroup (see [**7**]) provided its set of idempotents $E(S)$ forms a (left, right) normal band, or, equivalently, provided S is regular and

(1) $$(xef = xfe,\ efx = fex) \quad xefy = xfey$$

for all $x, y \in S$, $e, f \in E(S)$.

Proposition 2.1. *Every left absolutely flat generalized inverse semigroup is a right generalized inverse semigroup.*

Proof. Let S be a left absolutely flat generalized inverse semigroup. We need only show that if $e, f \in E(S)$, $efe = e$, and $fef = f$, then $ef = f$. Let $\theta_L(e, f)$ denote the smallest left congruence on S^1 which identifies e and f. Clearly $e \otimes \bar{1} = f \otimes \bar{1}$ in $S^1 \otimes S^1/\theta_L(e, f)$. Hence $e \otimes \bar{1} = f \otimes \bar{1}$ in $(eS \cup fS) \otimes S^1/\theta_L(e, f)$ and so either $e = f$ (in which case $ef = f$) or there exist $a_1, \ldots, a_n \in eS \cup fS$ and $s_1, \ldots, s_n t_1, \ldots, t_n \in S^1$ such that $\{s_i, t_i\} = \{e, f\}$ for $i = 1, \ldots, n$ and

$$\begin{aligned} e &= a_1 s_1 \\ a_1 t_1 &= a_2 s_2 \\ &\vdots \\ a_n t_n &= f \end{aligned}$$

(see [**1**, Lemma 1.1]). By induction we now establish $e = a_i fe$ for $i = 1, \ldots, n$. If $i = 1$, $a_1 fe = a_1 fs_1 fe = a_1 s_1 fe = efe = e$. Now suppose $a_{k-1} fe = e$ for some k, $1 < k < n$. Then $a_k fe = a_k fs_k fe = a_k s_k fe = a_{k-1} t_{k-1} fe = a_{k-1} ft_{k-1} fe = a_{k-1} fe = e$, and the induction is complete. In particular $e = a_n fe$ and so $ef = a_n fef = a_n ft_n f = a_n t_n f = f$ as required. □

We shall need the following lemma to establish a second condition which is necessary for a (right) generalized inverse semigroup to be absolutely flat.

LEMMA 2.2. *Let S be a right generalized inverse semigroup, and let e_1, e_2, f_1, $f_2 \in E(S)$ be such that $e_1 \mathscr{R} e_2$, $f_1 \mathscr{R} f_2$, $f_1 e_1 = f_1$ and $f_2 e_2 = f_2$. Then $(f_1, f_2) \in \theta_L(e_1, f_1) \vee \theta_L(e_2, f_2)$ implies $f_1 = f_2$.*

PROOF. It will be convenient to introduce the abbreviations $\theta_i = \theta_L(e_i, f_i)$, $i = 1, 2$, and $\Phi = \theta_1 \circ \theta_2$. Since $\theta_1 \vee \theta_2 = \bigcup_{n=1}^{\infty} \Phi^n$, it suffices to prove by induction that, for every $n \in N$, $(f_1, f_2) \in \Phi^n$ implies $f_1 = f_2$. In the process we shall use the (easily verified) fact that, for $x, y \in S^1$, $(x, y) \in \theta_i$ if and only if $(x = y)$ or $(xe_i = x$, $ye_i = y$, and $xf_i = yf_i)$, $i = 1, 2$.

First, then, suppose $(f_1, f_2) \in \Phi^1$, so that $f_1 \theta_1 z \theta_2 f_2$ for some $z \in S^1$. If $z = f_1$ then $(f_1, f_2) \in \theta_2$ results, which in particular entails $f_1 e_2 = f_1$. But also $f_1 e_2 = f_2 e_2 = f_2$, so $f_1 = f_2$ as claimed. A similar argument succeeds if $z = f_2$. If $f_1 \neq z \neq f_2$ then we have $f_1 e_1 = f_1$, $ze_1 = z$, $f_1 = zf_1$ (since $(f_1, z) \in \theta_1$) and $ze_2 = z$, $f_2 e_2 = f_2$, $zf_2 = f_2$ (since $(z, f_2) \in \theta_2$). Note that $z \neq 1$. If z' denotes any inverse of z in S, and we recall that zz', $z'z \in E(S)$, then we may calculate $f_1 = zf_1 = zf_1 e_1 = zf_1 z' z e_1 = zf_1 z' z = zf_1 z' z e_2 = zf_1 e_2 = zf_2 = f_2$, as desired.

Now assume that $(f_1, f_2) \in \Phi^n$ implies $f_1 = f_2$ for some $n \geqslant 1$, and let $z_1, z_2 \in S^1$ be such that $f_1 \Phi^n z_1 \theta_1 z_2 \theta_2 f_2$. If $z_1 = z_2$ then $(f_1, f_2) \in \Phi^n$ immediately results (since $\Phi^n \circ \theta_2 = \Phi^n$) and the inductive hypothesis finishes the argument. Otherwise, if $z_1 \neq z_2$ (and without loss of generality we assume $z_i \neq f_i$, $i = 1, 2$) we see that $f_1 = f_2 f_1 = z_2 f_2 f_1 = z_2 f_1$, which together with $f_1 e_1 = f_1$ and $z_2 e_1 = z_2$ implies $(f_1, z_2) \in \theta_1$. Hence, $f_1 \Phi f_2$ which by the $n = 1$ case yields $f_1 = f_2$ and the proof is complete. □

Every right normal band E is a strong semilattice of right zero bands, i.e. $E = \mathscr{S}(\Gamma; R_\gamma; \phi_{\alpha,\beta})$ where Γ is a semilattice, each R_γ $(\gamma \in \Gamma)$ is a right zero band, $E = \bigcup_{\gamma \in \Gamma} R_\gamma$, and the maps $\phi_{\alpha,\beta}$: $R_\alpha \to R_\beta$ $(\alpha \geqslant \beta)$ are the structure maps. We shall say *E has constant structure maps* if $\phi_{\alpha,\beta}$ is a constant function whenever $\alpha > \beta$ $(\alpha, \beta \in \Gamma)$. It is not difficult to prove that E has constant structure maps if and only if

$$(2) \qquad (\forall e, f, g \in E)(efg = fg \text{ or } efg = egf).$$

PROPOSITION 2.3. *If S is a left absolutely flat right generalized inverse semigroup then $E(S)$ has constant structure maps.*

PROOF. Refer to (2) and assume $e, f, g \in E(S)$ and $efg \neq fg$. Let $e_1 = fg$, $e_2 = gf$, $f_1 = efg$ and $f_2 = egf$ and notice that e_1, e_2, f_1, f_2 satisfy the hypotheses of Lemma 2.2. As before, suppose $\theta_i = \theta_L(e_i, f_i)$, $i = 1, 2$, and let $\theta_R(e_1, e_2)$ denote the smallest right congruence on S which identifies e_1 and e_2. Note that the only nonsingleton class of $\theta_R(e_1, e_2)$ is $\{e_1, e_2\}$. Because of this and the fact that $efg \neq fg$, μ: $f_1 S \cup f_2 S \to S/\theta_R(e_1, e_2)$ defined by $\mu(s) = \bar{s}$ for all $s \in f_1 S \cup f_2 S$ is a monomorphism. Hence,

$$\mu \otimes \mathrm{id}\colon (f_1 S \cup f_2 S) \otimes S^1/\theta_1 \vee \theta_2 \to S/\theta_R(e_1, e_2) \otimes S^1/\theta_1 \vee \theta_2$$

is injective. Since $\bar{f}_1 \otimes \bar{1} = \bar{f}_2 \otimes \bar{1}$ in $S/\theta_R(e_1, e_2) \otimes S^1/\theta_1 \vee \theta_2$, it follows that $f_1 \otimes \bar{1} = f_2 \otimes \bar{1}$ in $(f_1S \cup f_2S) \otimes S^1/\theta_1 \vee \theta_2$. We therefore obtain a scheme

$$\begin{aligned} f_1 &= a_1 s_1 & & \\ a_1 t_1 &= a_2 s_2 & s_1\bar{1} &= t_1\bar{b}_2 \\ &\vdots & &\vdots \\ a_n t_n &= f_2 & s_n\bar{b}_n &= t_n\bar{1} \end{aligned}$$

where for $i = 1,\ldots,n$, $a_i \in f_1S \cup f_2S$, $s_i, t_i \in S^1$ and $b_2,\ldots,b_n \in S^1$. From this it follows that $(f_1, f_2) \in \theta_1 \vee \theta_2$ which by Lemma 2.2 yields $f_1 = f_2$, i.e. $efg = egf$, as required. □

Our main result is the following theorem in which we demonstrate that the necessary conditions developed in Propositions 2.1 and 2.3 are also sufficient.

THEOREM 2.4. *Let S be a right generalized inverse semigroup in which $E(S)$ has constant structure maps. Then S is left absolutely flat.*

PROOF. Suppose $A \in$ Ens-S and $B \in S$-Ens. We will prove by induction on n that any scheme

$$\begin{aligned} a &= a_1 s_1 & & \\ a_1 t_1 &= a_2 s_2 & s_1 b &= t_1 b_2 \\ a_2 t_2 &= a_3 s_3 & s_2 b_2 &= t_2 b_3 \\ &\vdots & &\vdots \\ a_n t_n &= \hat{a} & s_n b_n &= t_n \hat{b} \end{aligned}$$

over A and B joining (a, b) to $(\hat{a}, \hat{b})$ may be replaced by one over $aS^1 \cup \hat{a}S^1$ and B. Throughout this proof s_i' (resp. t_i'), $i = 1,\ldots,n$, will denote a fixed inverse of $s_i(t_i)$ in S.

If $n = 1$ (Σ) is

$$\begin{aligned} a &= a_1 s_1 & & \\ a_1 t_1 &= \hat{a} & s_1 b &= t_1 \hat{b}. \end{aligned}$$

If $s_1 = 1$ or $t_1 = 1$ the scheme itself serves as the replacement. Otherwise the following scheme serves:

$$\begin{aligned} a &= (as_1')s_1 & & \\ (as_1')t_1 &= (as_1't_1t_1')t_1 & s_1 b &= t_1\hat{b} \\ (as_1't_1t_1')s_1 &= (\hat{a}t_1')s_1 & t_1\hat{b} &= s_1 b \\ (\hat{a}t_1')t_1 &= \hat{a} & s_1 b &= t_1\hat{b} \end{aligned}$$

Only the third line on the left may need explanation:

$$\begin{aligned} as_1't_1t_1's_1 &= a_1s_1s_1't_1t_1's_1 \\ &= a_1t_1t_1's_1s_1's_1 \quad \text{(by (1))} \\ &= \hat{a}t_1's_1. \end{aligned}$$

Now assume that for some $n > 1$ all schemes of length less than n can be "replaced" and consider (Σ) as above. Before proceeding we first note that whenever $a_i t_i = a_{i+1}s_{i+1} \in aS^1$ for some $i = 1, \ldots, n-1$, then (Σ) can be resolved into the two shorter schemes

$$\begin{aligned} a &= a_1 s_1 & & \\ a_1 t_1 &= a_2 s_2 & s_1 b &= t_1 b_2 \\ &\vdots & &\vdots \\ a_i t_i &= a_{i+1} s_{i+1} & s_i b_i &= t_i b_{i+1} \end{aligned}$$

and

$$\begin{aligned} a_i t_i &= a_{i+1} s_{i+1} & & \\ a_{i+1} t_{i+1} &= a_{i+2} s_{i+2} & s_{i+1} b_{i+1} &= t_{i+1} b_{i+2} \\ &\vdots & &\vdots \\ a_n t_n &= \hat{a} & s_n b_n &= t_n \hat{b}. \end{aligned}$$

By the inductive hypothesis, the first scheme may be replaced by one over aS^1 and B joining (a, b) to $(a_{i+1}s_{i+1}, b_{i+1})$ and the second may be replaced by one over $aS^1 \cup \hat{a}S^1$ and B joining $(a_i t_i, b_{i+1})$ to $(\hat{a}, \hat{b})$. The two new schemes may then be spliced together to join (a, b) and $(\hat{a}, \hat{b})$ over $aS^1 \cup \hat{a}S^1$ and B as required.

Now we show that without loss of generality we may assume $s_i \neq 1$ and $t_i \neq 1$ for all i. In fact, if $s_1 = 1$ then $a_1 t_1 \in aS^1$ and, hence, the scheme may be replaced using the preceding argument. If $s_i = 1$ for some $i > 1$ it is an easy matter to see that (Σ) may be replaced by a scheme of length $n - 1$ allowing the use of the inductive hypothesis. If $t_i = 1$ for some i, similar considerations apply.

We henceforth assume that $s_i, t_i \in S$ for $i = 1, \ldots, n$. It will be useful to establish the following notation:

$$z_1 = s_1', \qquad z_{i+1} = z_i t_i s_{i+1}', \tag{3}$$

$$z_1' = s_1, \qquad z_{i+1}' = s_{i+1} t_i' z_i', \tag{3'}$$

$$w_n = t_n', \qquad w_i = w_{i+1} s_{i+1} t_i', \tag{4}$$

$$w_n' = t_n, \qquad w_i' = t_i s_{i+1}' w_{i+1}' \tag{4'}$$

for $1 \leqslant i \leqslant n-1$. The following equalities can be verified for $1 \leqslant i \leqslant n$:

$$w_1 s_1 = w_i z_i', \tag{5}$$

$$z_n t_n = z_i w_i', \tag{6}$$

$$a z_i = a_i z_i' z_i, \tag{7}$$

$$\hat{a} w_i = a_i w_i' w_i. \tag{8}$$

For example, to establish (7), we use induction on i. If $i = 1$ the result is clear.

Otherwise,

$$\begin{aligned}
az_{i+1} &= az_i t_i s'_{i+1} && \text{(by (3))}\\
&= a_i z'_i z_i t_i s'_{i+1} && \text{(inductive hypothesis)}\\
&= a_i z'_i z_i t_i t'_i t_i s'_{i+1} &&\\
&= a_i t_i t'_i z'_i z_i t_i s'_{i+1} && \text{(by (1))}\\
&= a_{i+1} s_{i+1} t'_i z'_i z_i t_i s'_{i+1} &&\\
&= a_{i+1} z'_{i+1} z_{i+1} && \text{(by (3) and (3′)).}
\end{aligned}$$

Now let

(9) $$e_i = t'_i z'_i z_i t_i$$

and

(10) $$f_i = s'_i w'_i w_i s_i$$

for $i = 1, \ldots, n$. If $e_i t'_i t_i s'_{i+1} s_{i+1} = t'_i t_i s'_{i+1} s_{i+1}$ for some i $(1 \leqslant i \leqslant n-1)$, then $t_i e_i s'_{i+1} s_{i+1} = t_i s'_{i+1} s_{i+1}$ and so

$$\begin{aligned}
a_i t_i &= a_i t_i s'_{i+1} s_{i+1}\\
&= a_i t_i e_i s'_{i+1} s_{i+1}\\
&= a_i t_i t'_i z'_i z_i t_i s'_{i+1} s_{i+1} && \text{(by (9))}\\
&= a_i z'_i z_i t_i s'_{i+1} s_{i+1} && \text{(by (1))}\\
&= az_i t_i s'_{i+1} s_{i+1} && \text{(by (7))}
\end{aligned}$$

which shows $a_i t_i \in aS^1$ and a previous argument applies. If $f_{i+1} s'_{i+1} s_{i+1} t'_i t_i = s'_{i+1} s_{i+1} t'_i t_i$ for some i $(1 \leqslant i \leqslant n-1)$ the proof is similar.

Finally, by (2), we may assume

(11) $$e_i t'_i t_i s'_{i+1} s_{i+1} = e_i s'_{i+1} s_{i+1} t'_i t_i$$

and

(12) $$f_{i+1} t'_i t_i s'_{i+1} s_{i+1} = f_{i+1} s'_{i+1} s_{i+1} t'_i t_i$$

for all i $(1 \leqslant i \leqslant n-1)$. (11) and (12) imply

(13) $$s_{i+1} e_i = z'_{i+1} z_{i+1} s_{i+1}$$

and

(14) $$t_i f_{i+1} = w'_i w_i t_i.$$

For example,

$$\begin{aligned}
t_i f_{i+1} &= t_i t'_i t_i f_{i+1} s'_{i+1} s_{i+1}\\
&= t_i f_{i+1} t'_i t_i s'_{i+1} s_{i+1} && \text{(by (1))}\\
&= t_i f_{i+1} s'_{i+1} s_{i+1} t'_i t_i && \text{(by (12))}\\
&= t_i f_{i+1} t'_i t_i\\
&= t_i s'_{i+1} w'_{i+1} w_{i+1} s_{i+1} t'_i t_i && \text{(by (10))}\\
&= w'_i w_i t_i && \text{(by (4) and (4′)).}
\end{aligned}$$

Without loss of generality we may assume that n, the length of (Σ), is even. (Otherwise one could consider the scheme

$$\begin{array}{rlrl} a &= (as_1')s_1 & & \\ (as_1')s_1 &= a_1 s_1 & s_1 b &= s_1 b \\ a_1 t_1 &= a_2 s_2 & s_1 b &= t_1 b_2 \\ &\vdots & &\vdots \\ a_n t_n &= \hat{a} & s_n b_n &= t_n \hat{b} \end{array}$$

of length $n + 1$ joining (a, b) to $(\hat{a}, \hat{b})$, which still possesses properties (11) and (12).) We now show that (Σ) may be replaced by the scheme

$$\begin{array}{rlrl} a &= (az_1)s_1 & & \\ (az_1)t_1 &= (az_2)s_2 & s_1 b &= t_1 b_2 \\ &\vdots & &\vdots \\ (az_{n-1})t_{n\text{-}1} &= (az_n)s_n & s_{n-1}b_{n-1} &= t_{n-1}b_n \\ (az_n)t_n &= (az_n t_n w_n)t_n & s_n b_n &= t_n \hat{b} \\ (az_n t_n w_n)s_m &= (az_n t_n w_{n-1})t_{n-1} & t_n \hat{b} &= s_n b_n \\ &\vdots & &\vdots \\ (az_n t_n w_{n/2+2})s_{n/2+2} &= (az_n t_n w_{n/2+1})t_{n/2+1} & t_{n/2+2}b_{n/2+3} &= s_{n/2+2}b_{n/2+2} \\ (az_n t_n w_{n/2+1})s_{n/2+1} &= (\hat{a}w_1 s_1 z_{n/2})t_{n/2} & t_{n/2+1}b_{n/2+2} &= s_{n/2+1}b_{n/2+1} \\ (\hat{a}w_1 s_1 z_{n/2})s_{n/2} &= (\hat{a}w_1 s_1 z_{n/2-1})t_{n/2-1} & t_{n/2}b_{n/2+1} &= s_{n/2}b_{n/2} \\ &\vdots & &\vdots \\ (\hat{a}w_1 s_1 z_2)s_2 &= (\hat{a}w_1 s_1 z_1)t_1 & t_2 b_3 &= s_2 b_2 \\ (\hat{a}w_1 s_1 z_1)s_1 &= (\hat{a}w_1)s_1 & t_1 b_2 &= s_1 b \\ (\hat{a}w_1)t_1 &= (\hat{a}w_2)s_2 & s_1 b &= t_1 b_2 \\ &\vdots & &\vdots \\ (\hat{a}w_{n-1})t_{n-1} &= (\hat{a}w_n)s_n & s_{n-1}b_{n-1} &= t_{n-1}b_n \\ (\hat{a}w_n)t_n &= \hat{a} & s_n b_n &= t_n \hat{b}. \end{array}$$

We must check that these equalities hold. Because the equalities on the right appear in the original scheme we need only consider those on the left. The reader will notice that these have been presented in five groups.

That the first equation in the first group holds is obvious. Moreover, for $1 \leqslant i \leqslant n - 1$,

$$\begin{aligned} az_i t_i &= a_i z_i' z_i t_i && \text{(by (7))} \\ &= a_i t_i t_i' z_i' z_i t_i && \text{(by (1))} \\ &= a_{i+1} s_{i+1} t_i' z_i' z_i t_i && \\ &= a_{i+1} s_{i+1} e_i && \text{(by (9))} \\ &= a_{i+1} z_{i+1}' z_{i+1} s_{i+1} && \text{(by (13))} \\ &= az_{i+1} s_{i+1} && \text{(by (7)).} \end{aligned}$$

The equations in the fifth group hold for reasons of symmetry.

That the first equation in the second group holds is clear. Furthermore, for $n/2+1 \leqslant i \leqslant n-1$,

$$\begin{aligned} az_n t_n w_{i+1} s_{i+1} &= az_{i+1} w'_{i+1} w_{i+1} s_{i+1} && \text{(by (6))} \\ &= az_i t_i s'_{i+1} w'_{i+1} w_{i+1} s_{i+1} && \text{(by (3))} \\ &= az_i t_i f_{i+1} && \text{(by (10))} \\ &= az_i w'_i w_i t_i && \text{(by (14))} \\ &= az_n t_n w_i t_i && \text{(by (6)).} \end{aligned}$$

The equations in the fourth group also hold, for analogous reasons.

Finally, the middle equation holds. In fact,

$$\begin{aligned} az_n t_n w_{n/2+1} s_{n/2+1} &= az_{n/2} w'_{n/2} w_{n/2+1} s_{n/2+1} && \text{(by (6))} \\ &= a_{n/2} z'_{n/2} z_{n/2} w'_{n/2} w_{n/2+1} s_{n/2+1} && \text{(by (7))} \\ &= a_{n/2} z'_{n/2} z_{n/2} t_{n/2} s'_{n/2+1} w'_{n/2+1} w_{n/2+1} s_{n/2+1} && \text{(by (4'))} \\ &= a_{n/2} z'_{n/2} z_{n/2} t_{n/2} f_{n/2+1} && \text{(by (10))} \\ &= a_{n/2} z'_{n/2} z_{n/2} w'_{n/2} w_{n/2} t_{n/2} && \text{(by (14))} \\ &= a_{n/2} w'_{n/2} w_{n/2} z'_{n/2} z_{n/2} t_{n/2} && \text{(by (1))} \\ &= \hat{a} w_{n/2} z'_{n/2} z_{n/2} t_{n/2} && \text{(by (8))} \\ &= \hat{a} w_1 s_1 z_{n/2} t_{n/2} && \text{(by (5)).} \end{aligned}$$

This completes the proof of Theorem 2.4. □

Among the consequences of Theorem 2.4 are

COROLLARY 2.5 [**1**, THEOREM 4.2]. *Inverse semigroups are absolutely flat.*

COROLLARY 2.6. *Let S be a strong semilattice of completely simple semigroups. Then S is left absolutely flat if and only if S is a strong semilattice of right groups and $E(S)$ has constant structure maps.*

PROOF. If S is left absolutely flat, then each of its completely simple components must be a right group [**6**, or **1**, proof of Theorem 4.3]. Hence $E(S)$ is right normal and has constant structure maps by Theorem 2.4. The converse clearly holds. □

COROLLARY 2.7. (1) *A normal band is left absolutely flat if and only if it is right normal and has constant structure maps.*

(2) *A right normal band is left absolutely flat if and only if it has constant structure maps.*

(3) *A left normal band is left absolutely flat if and only if it is a semilattice.*

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