TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 347, Number 8, August 1995

LEFT ANNIHILATORS CHARACTERIZED BY GPIS

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ABSTRACT. Let R be a semiprime ring with extended centroid C, U the right Utumi quotient ring of R, S a subring of U containing R and ρ_1, ρ_2 two right ideals of R. In the paper we show that $l_S(\rho_1) = l_S(\rho_2)$ if and only if ρ_1 and ρ_2 satisfy the same generalized polynomial identities (GPIs) with coefficients in SC, where $l_S(\rho_i)$ denotes the left annihilator of ρ_i in S. As a consequence of the result, if ρ is a right ideal of R such that $l_R(\rho) = 0$, then ρ and U satisfy the same GPIs with coefficients in the two-sided Utumi quotient ring of R.

This paper is motivated by Chuang's paper [3] and Beidar's paper [2]. Recall that a ring R is said to be a left faithful ring if, for $a \in R$, $aR = 0$ implies $a = 0$. For a left faithful ring R, the right Utumi quotient ring of R can be characterized as a ring U satisfying the following axioms:

(1) R is a subring of U.

(2) For each $a \in U$, there exists a dense right ideal ρ of R such that $a\rho \subseteq R$.

(3) If $a \in U$ and $a\rho = 0$ for some dense right ideal ρ of R, then $a = 0$.

(4) For any dense right ideal ρ of R and for any right R-module homomorphism $\phi : \rho_R \to R_R$, there exists $a \in U$ such that $\phi(x) = ax$ for all $x \in \rho$.

Let R be a left faithful ring and ρ be a dense right ideal of R. We note that ρ itself is a left faithful ring. Furthermore, ρ and R have the same right Utumi quotient ring. More precisely, denote by $U(R)$ ($U(\rho)$ resp.) the right Utumi quotient ring of R (ρ resp.). Then there exists a ring isomorphism h from $U(\rho)$ onto $U(R)$ such that $h(x) = x$ for all $x \in \rho$. In [3] Chuang proved the theorem: Let R be a prime ring, U its right Utumi quotient ring and N_R a dense R-submodule of U_R. Then N and U satisfy the same generalized polynomial identities (GPIs) with coefficients in U. In this theorem we note that $N \cap R$ is always a dense right ideal of R. Since $N \cap R$ and R have the same right Utumi quotient ring, Chuang's theorem just says that R and U satisfy the same GPIs with coefficients in U. Also, in an earlier paper [2] Beidar proved that the same result remains true for semiprime rings. For a semiprime ring R we observe that $N \cap R$ is a dense right ideal of R for any dense R-submodule N_R of U_R. Also, $l_U(N \cap R)$, the left annihilator of $N \cap R$

Received by the editors April 11, 1994 and, in revised form, July 8, 1994; originally communicated to the *Proceedings of the AMS* by Ken Goodearl.

1991 *Mathematics Subject Classification.* Primary 16R50.

Key words and phrases. Semiprime ring, prime ring, Utumi quotient ring, GPI, orthogonal completion.

in U, is zero. In this paper we shall compare two left annihilators of two right ideals ρ_1 and ρ_2 of R in U by considering the GPIs satisfied by the two right ideals ρ_1 and ρ_2. From this we are able to generalize Chuang's and Beidar's results. For instance, if R is a semiprime ring and ρ is a right ideal of R such that $l_R(\rho) = 0$, we shall prove that ρ and U satisfy the same GPIs with coefficients in Q, the two-sided Utumi quotient ring of R. More explicitly, we prove in this paper the following

Main Theorem. *Let R be a semiprime ring with extended centroid C, U its right Utumi quotient ring, S a subring of U containing R and ρ_1, ρ_2 two right ideals of R. Then $l_S(\rho_1) = l_S(\rho_2)$ if and only if ρ_1 and ρ_2 satisfy the same GPIs with coefficients in SC.*

Throughout the paper, rings are always associative but not necessarily with unity. We shall fix some notation. For a semiprime ring R we denote by U its right Utumi quotient ring, by Q its two-sided Utumi quotient ring and by C the extended centroid of R. $U *_C C\{X_1, X_1, \dots\}$ stands for the free product of the C-algebra U and $C\{X_1, X_2, \dots\}$, the free C-algebra with indeterminates $X_1, X_2, \dots$. For these definitions and their basic properties we refer to [3], [4] and [6]. To prove the Main Theorem we need several lemmas. We begin the proof with the following easy observations.

Lemma 1. *Let R be a simple Artinian ring and ρ_1, ρ_2 be two right ideals of R. Then $l_R(\rho_1) = l_R(\rho_2)$ if and only if $\rho_1 = \rho_2$.*

Proof. Since every right ideal of a simple Artinian ring is generated by one idempotent, there are two idempotents e and f in R such that $\rho_1 = eR$ and $\rho_2 = fR$. Assume that $l_R(\rho_1) = l_R(\rho_2)$. Then $1 - e \in l_R(\rho_1)$ and hence $(1-e)fR = 0$. This implies that $(1-e)f = 0$. That is, $f = ef$. Now, $\rho_2 = fR = efR \subseteq \rho_1$. Similarly, $\rho_1 \subseteq \rho_2$. Therefore $\rho_1 = \rho_2$. Of course, the converse is trivial. The proof is now complete.

Lemma 2. *Let R be a semiprime ring and ρ be a right ideal of R. Then ρ and ρU satisfy the same GPIs with coefficients in U.*

Proof. Let $f(X_1, \dots, X_t) \in U *_C C\{X_1, X_2, \dots\}$ be a GPI satisfied by ρ. Fix $y_1, \dots, y_t \in \rho U$. Write $y_i = \sum_{j=1}^{n(i)} a_{ij}u_{ij}$, where $a_{ij} \in \rho$ and $u_{ij} \in U$, $1 \le i \le t$. Since $\rho R \subseteq R$, $f(\sum_{j=1}^{n(1)} a_{1j}Y_{1j}, \dots, \sum_{j=1}^{n(t)} a_{tj}Y_{tj})$ is a GPI for R, where the Y_{ij} are distinct indeterminates. By [2], $f(\sum_{j=1}^{n(1)} a_{1j}Y_{1j}, \dots, \sum_{j=1}^{n(t)} a_{tj}Y_{tj})$ is also a GPI for U. In particular, set $Y_{ij} = u_{ij}$ for all i, j. Then $f(y_1, \dots, y_t) = 0$ as desired. This proves the lemma.

Lemma 3. *Let R be a prime ring and ρ be a nonzero right ideal of R. Suppose that $a_1, a_2, \dots, a_t \in U$ satisfy the following condition: if $\alpha_1, \alpha_2, \dots, \alpha_t \in C$ satisfy $(\alpha_1 a_1 + \cdots + \alpha_t \alpha_t)\rho = 0$, then $\alpha_i = 0$ for all i. Then there exists an element $u \in \rho$ such that $a_1 u, \dots, a_t u$ are C-independent unless R is a PI-ring.*

Proof. Since R is a prime ring, C itself is a field. Define $T_i \in \mathrm{Hom}_C(\rho C, U)$ by $T_i(y) = a_i y$ for all $y \in \rho C$, $1 \le i \le t$. Then $T_1, \dots, T_t$ are C-independent. Indeed, let $\beta_1, \dots, \beta_t \in C$ be such that $\beta_1 T_1 + \cdots + \beta_t T_t = 0$. That is, $(\beta_1 a_1 + \cdots + \beta_1 a_t)\rho C = 0$. By our assumption, $\beta_1 = \cdots = \beta_t = 0$.

By [3, Lemma 2], either there exists $u \in \rho$ such that $a_1u, \ldots, a_tu$ are C-independent, or there exists $\sum_{i=1}^{t} \delta_i T_i \neq 0$, where $\delta_i \in C$, which is of finite rank. If the first case occurs, then we are done. Therefore we assume the second situation. This implies that $\dim_C v\rho C < \infty$ and $v\rho C \neq 0$, where $v = \delta_1 a_1 + \cdots + \delta_t a_t$. By [3, Lemma 1], $\dim_C RC < \infty$ and hence R is a PI-ring. This completes the proof.

Let R be a prime ring and S be a subring of U containing R. It is well known that SC is a closed prime algebra over C [5]. Recall that $f \in SC *_C C\{X_1, X_2, \ldots\}$ is called *nontrivial* if f is nonzero. By a result of Martindale [8], if R satisfies a nontrivial GPI with coefficients in RC (in fact, in U), then RC is a strongly primitive ring. However, a nontrivial GPI for a right ideal ρ of R may only give a *trivial* identity for ρ. For instance, if there exist two C-independent elements $a, b \in U$ and $\beta \neq 0$ in C such that $(b + \beta a)\rho = 0$, then $aX_1bX_2 + \beta aX_1aX_2$ is a nontrivial GPI for ρ but gives a trivial identity for ρ. Therefore to handle this situation we must give a suitable adaptation for the idea of GPIs satisfied by one-sided ideals. We now follow a notion given by Chuang [3, p. 725]. Let B be a set of C-independent elements of S. By a B-monomial, we mean a monomial of the form $u_0Y_1u_1Y_2\cdots Y_nu_n$, where $\{u_0, u_1, \ldots, u_n\} \subseteq B$ and where $\{Y_1, \ldots, Y_n\} \subseteq \{X_1, X_2, \ldots\}$. Thus for each nonzero $f \in SC *_C C\{X_1, X_2, \ldots\}$ there exists a B such that f is a C-linear combination of B-monomials. A generalized polynomial $0 \neq f \in SC *_C C\{X_1, X_2, \ldots\}$ is called a *proper GPI* for a right ideal ρ of the prime ring R if f is of the form $g(X_1, \ldots, X_t)X_{t+1}$, where $g \in SC *_C C\{X_1, X_2, \ldots\}$, such that g lies in the C-span of B-monomials for some B, a finite set of C-independent elements of S, and furthermore if B satisfies the following condition:

If $\alpha_1, \ldots, \alpha_l \in C$ satisfy $(\alpha_1 b_1 + \cdots + \alpha_l b_l)\rho = 0$, where $B = \{b_1, \ldots, b_l\}$, then $\alpha_1 = \cdots = \alpha_l = 0$.

We remark that two right ideals ρ_1 and ρ_2 of the prime ring R satisfy the same GPIs with coefficients in SC if and only if ρ_1 and ρ_2 satisfy the same GPIs of the form $g(X_1, \ldots, X_t)X_{t+1}$, where $g \in SC *_C C\{X_1, X_2, \ldots\}$. Indeed, we need only give the proof of the "if" part. Let $f \in SC *_C C\{X_1, X_2, \ldots\}$ be a GPI for ρ_1. Let $X_1, \ldots, X_t$ be all indeterminates occurring in f. Fix any element $a \in S$. Then $f(X_1, \ldots, X_t)aX_{t+1}$ is a GPI for ρ_1. By our assumption, $f(X_1, \ldots, X_t)aX_{t+1}$ is also a GPI for ρ_2. Thus $f(x_1, \ldots, x_t)SCx_{t+1} = 0$ for all $x_1, \ldots, x_{t+1} \in \rho_2$. By the primeness of SC, $f(x_1, \ldots, x_t) = 0$ for all $x_1, \ldots, x_t \in \rho_2$. That is, f is a GPI for ρ_2. This proves our remark.

Lemma 4. *Let R be a prime ring, S a subring of U containing R and ρ_1, ρ_2 two right ideals of R such that $l_S(\rho_1) = l_S(\rho_2)$. Suppose that ρ_i has no proper GPI in $SC *_C C\{X_1, X_2, \ldots\}$ for $i = 1, 2$. Then ρ_1 and ρ_2 satisfy the same GPIs with coefficients in SC.*

Proof. We note first that $l_{SC}(\rho_1) = l_{SC}(\rho_2)$. Indeed, if $y\rho_1 = 0$ where $y \in SC$, then there exists a nonzero ideal I of R such that $Iy \subseteq S$, since $R \subseteq S \subseteq U$, and hence $(Iy)\rho_1 = 0$, which implies $Iy\rho_2 = 0$. Thus $y\rho_2 = 0$ follows. This proves $l_{SC}(\rho_1) \subseteq l_{SC}(\rho_2)$. Thus $l_{SC}(\rho_1) = l_{SC}(\rho_2)$.

Let $0 \neq f \in SC *_C C\{X_1, X_2, \ldots\}$ be a GPI for ρ_1. Assume for the moment that f is of the form $h(X_1, \ldots, X_l)X_{l+1}$, where $h \in SC *_C C\{X_1, X_2, \ldots\}$. Then there is a finite set B of C-independent elements of S such that h lies in the C-span of B-monomials. Say that $B = \{b_1, \dots, b_t\}$. We proceed by induction on t, the number of elements in B. Since ρ_1 has no proper GPI in $SC *_C C\{X_1, X_2, \dots\}$, we may assume that $(\alpha_1 b_1 + \cdots + \alpha_{t-1} b_{t-1} + \alpha_t b_t)\rho_1 = 0$ for some $\alpha_1, \dots, \alpha_t \in C$, not all zero. Without loss of generality we can assume $\alpha_t = 1$. Then we have $(\alpha_1 b_1 + \cdots + \alpha_{t-1} b_{t-1} + b_t)\rho_2 = 0$ since $l_{SC}(\rho_1) = l_{SC}(\rho_2)$. Also, let g be the GP obtained from h by replacing the coefficient b_t in f by $-(\alpha_1 b_1 + \cdots + \alpha_{t-1} b_{t-1})$. Set $B_0 = \{b_1, \dots, b_{t-1}\}$. Then gX_{l+1} is also a GPI for ρ_1 and furthermore g lies in the C-span of B_0-monomials, where $|B_0| = t-1$. Applying the induction hypothesis yields that gX_{l+1} is a GPI for ρ_2. Now the fact that $(\alpha_1 b_1 + \cdots + \alpha_{t-1} b_{t-1} + b_t)\rho_2 = 0$ implies that f is a GPI for ρ_2. Similarly, by the assumption that ρ_2 has no proper GPI in $SC *_C C\{X_1, X_2, \dots\}$ we deduce that every GPI of the form $h(X_1, \dots, X_l)X_{l+1}$ in $SC *_C C\{X_1, X_2, \cdots\}$ for ρ_2 is satisfied by ρ_1. Therefore, ρ_1 and ρ_2 satisfy the same GPIs with coefficients in SC by the remark given before this lemma. The proof of Lemma 4 is complete.

With Lemma 4 in hand we are now able to prove the Main Theorem when R is a prime ring.

Lemma 5. *The Main Theorem holds when R is a prime ring.*

Proof. We may assume that $\rho_1 \neq 0$ and $\rho_2 \neq 0$. By Lemma 4, we may assume that ρ_1 has a proper GPI $f \in SC *_C C\{X_1, X_2, \dots\}$. Write $f = g(X_1, \dots, X_l)X_{l+1}$. Thus there exists a finite set $B = \{b_1, \dots, b_t\}$ of C-independent elements of S such that g lies in the C-span of B-monomials. Also, we have that if $(\alpha_1 b_1 + \cdots + \alpha_t b_t)\rho_1 = 0$ where $\alpha_i \in C$, then $\alpha_1 = \cdots = \alpha_t = 0$. We claim that R satisfies a nontrivial GPI with coefficients in U. If R is a PI-ring, the claim holds trivially. Suppose that R is not a PI-ring. Then by Lemma 3 there exists an element $u \in \rho_1$ such that $b_1 u, \dots, b_t u$ are C-independent. Then $g(uX_1, \dots, uX_l)uX_{l+1}$ is a nontrivial GPI for R since $uR \subseteq \rho_1$. This proves the claim. By Chuang's theorem [3], R and S satisfy the same GPIs with coefficients in U. By Martindale's theorem [8], SC is a strongly primitive ring. In particular, $\mathrm{Soc}(SC)$, the socle of SC, is nonzero. Set $\sigma = \mathrm{Soc}(SC) \neq 0$. Then σ is a simple ring with minimal right ideals.

Note that $\rho_1\sigma$ and ρ_1 satisfy the same GPIs with coefficients in U. Indeed, by Lemma 2, a GPI for ρ_1 is satisfied by $\rho_1\sigma$. Conversely, let $h(X_1, \dots, X_k)$ be a GPI for $\rho_1\sigma$ with coefficients in U. Fix k elements $y_1, \dots, y_k \in \rho_1$. Then $h(y_1X_1, \dots, y_kX_1)$ is a GPI for σ. Since σ_R is a dense submodule of U_R, by [3] U satisfies $h(y_1X_1, \dots, y_kX_1)$. In particular, set $X_1 = 1$. Then $h(y_1, \dots, y_k) = 0$. Therefore $h(X_1, \dots, X_k)$ is a GPI for ρ_1. This proves that $\rho_1\sigma$ and ρ_1 satisfy the same GPIs with coefficients in U. Of course, $\rho_2\sigma$ and ρ_2 also satisfy the same GPIs with coefficients in U.

Assume first that $l_S(\rho_1) = l_S(\rho_2)$, and let $f \in SC *_C C\{X_1, X_2, \dots\}$ be a GPI for ρ_1. Write $f = f(X_1, \dots, X_l)$. Let $t \in \sigma$. Then $tf(X_1t, \dots, X_lt) \in \sigma *_C C\{X_1, X_2, \dots\}$ is a GPI for $\rho_1\sigma$. Let $d_1, \dots, d_m$ be the coefficients occurring in $tf(X_1t, \dots, X_lt)$. Note that $d_i \in \sigma$ for each i. By Litoff's theorem [7, Theorem 3, p. 90], there exists an idempotent $e \in \sigma$ such that $d_i \in e\sigma e$ for $i = 1, 2, \dots, m$. Thus $e\rho_1\sigma e$ satisfies the GPI $tf(X_1t, \dots, X_lt)$. It follows from the fact $l_S(\rho_1) = l_S(\rho_2)$ that $l_\sigma(\rho_1\sigma) = l_\sigma(\rho_2\sigma)$ and hence $l_{e\sigma e}(e\rho_1\sigma e) = l_{e\sigma e}(e\rho_2\sigma e)$. Note that $e\sigma e$ is now a simple Artinian ring and that $e\rho_1\sigma e$ and $e\rho_2\sigma e$ are two right ideals of $e\sigma e$. Applying Lemma 1, we have $e\rho_1\sigma e = e\rho_2\sigma e$. Now $e\rho_2\sigma e$ satisfies the GPI $tf(X_1t, \dots, X_lt)$ and hence $\rho_2\sigma$ satisfies $tf(X_1t, \dots, X_lt)$.

So if we fix l elements $x_1, \dots, x_l \in \rho_2\sigma$, then σ satisfies the GPI $X_1f(x_1X_1, \dots, x_lX_1)$. Since σ_R is a dense submodule of U_R, by [3] U satisfy $X_1f(x_1X_1, \dots, x_lX_1)$. In particular, set $X_1 = 1$. Then $f(x_1, \dots, x_l) = 0$. Therefore $\rho_2\sigma$ and hence ρ_2 satisfy $f(X_1, \dots, X_l)$. Up to now we have proved that every GPI in $SC *_C C\{X_1, X_2, \dots\}$ for ρ_1 is also a GPI for ρ_2. Thus ρ_1 and ρ_2 satisfy the same GPIs with coefficients in SC.

For the converse, let $x\rho_1 = 0$ where $x \in S$. Then ρ_1 satisfies $xX_1 \in SC *_C C\{X_1, X_2, \dots\}$. By the assumption, $x\rho_2 = 0$. Therefore, $l_S(\rho_1) \subseteq l_S(\rho_2)$. Similarly, $l_S(\rho_2) \subseteq l_S(\rho_1)$, and so $l_S(\rho_1) = l_S(\rho_2)$. This completes the proof.

To prove the Main Theorem we must generalize Lemma 5 to the case of semiprime rings. To arrive at this aim we need some results about orthogonal completions for semiprime rings given in [1]. Let R be a semiprime ring. Recall that a subset $T \subseteq U$ is called orthogonally complete if $0 \in T$ and given any set of orthogonal idempotents $\{e_\omega\} \subseteq C$ and any subset $\{x_\omega\} \subseteq T$, $\omega \in \Omega$, there exists $x \in T$ such that $e_\omega x = e_\omega x_\omega$ for all $\omega \in \Omega$. For any subset $K \subseteq U$, denote by $\widehat{K}$ the orthogonal completion of K in U, which is defined as the intersection of all orthogonally complete subsets of U containing K. Note that $\widehat{K}$ itself is an orthogonally complete subset of U. Now we prove

Lemma 6. *Let R be a semiprime ring, S a subring of U containing R and ρ a right ideal of R. Then the following statements hold.*

(i) *ρ and $\hat{\rho}$ satisfy the same GPIs with coefficients in U.*

(ii) *For any two right ideals ρ_1, ρ_2 of R, $l_S(\rho_1) = l_S(\rho_2)$ if and only if $l_{\widehat{S}}(\hat{\rho}_1) = l_{\widehat{S}}(\hat{\rho}_2)$.*

Proof. For (i), let $f(X_1, \dots, X_l) \in U *_C C\{X_1, X_2, \dots\}$ be a GPI for ρ. To prove that f is a GPI for $\hat{\rho}$ it suffices to assume that f only involves one indeterminant, say $f = f(X)$. For $x \in \hat{\rho}$, by the definition of $\hat{\rho}$ we have $x = \sum_\omega^\perp e_\omega x_\omega$, where $\{e_\omega\}_{\omega\in\Omega}$ is a set of orthogonal idempotents of C such that $\sum_\omega Ce_\omega$ is an essential ideal of C and where $x_\omega \in \rho$ for all $\omega \in \Omega$ [1]. Note that f contains no constant term. Thus we have

$$e_\omega f(x) = f(e_\omega x) = f(e_\omega x_\omega) = e_\omega f(x_\omega) = 0$$

for all $\omega \in \Omega$, since $e_\omega x = e_\omega x_\omega$ and $f(x_\omega) = 0$. This implies $f(x)(\sum_\omega Ce_\omega) = 0$. By [1, Lemma 1] U_C is a nonsingular C-module, which implies $f(x) = 0$. This proves (i).

For (ii), assume first that $l_S(\rho_1) = l_S(\rho_2)$. Let $x = \sum_\omega^\perp e_\omega x_\omega \in \widehat{S}$ satisfy $x\hat{\rho}_1 = 0$, where $\sum_\omega Ce_\omega$ is an essential ideal of C and $x_\omega \in S$ for all ω. Then $e_\omega x\rho_1 = 0$, that is, $e_\omega x_\omega \rho_1 = 0$. Note that $l_{SC}(\rho_1) = l_{SC}(\rho_2)$, since $l_S(\rho_1) = l_S(\rho_2)$. We have $e_\omega x_\omega \rho_2 = 0$. But $r_U(e_\omega x_\omega)$ is an orthogonally complete subset of U, which implies $e_\omega x_\omega \hat{\rho}_2 = 0$ and hence $x\hat{\rho}_2 = 0$. In other words, $x \in l_{\widehat{S}}(\hat{\rho}_2)$. Therefore, $l_{\widehat{S}}(\hat{\rho}_1) \subseteq l_{\widehat{S}}(\hat{\rho}_2)$. Similarly, $l_{\widehat{S}}(\hat{\rho}_2) \subseteq l_{\widehat{S}}(\hat{\rho}_1)$ and hence $l_{\widehat{S}}(\hat{\rho}_1) = l_{\widehat{S}}(\hat{\rho}_2)$.

Assume next that $l_{\widehat{S}}(\hat{\rho}_1) = l_{\widehat{S}}(\hat{\rho}_2)$. Since the proof that $l_S(\rho_1) = l_S(\rho_2)$ is trivial, we omit it.

We are now ready to prove the Main Theorem.

Proof of the Main Theorem. Note that the "if" part is trivial. Therefore it suffices to prove the "only if" part. Suppose that $l_S(\rho_1) = l_S(\rho_2)$. By Lemma 6, $l_{\widehat{S}}(\hat{\rho}_1) = l_{\widehat{S}}(\hat{\rho}_2)$. Note that $\widehat{R}$ is also a semiprime ring and that $\widehat{S}$ is a subring of U containing $\widehat{R}$. Moreover, $\hat{\rho}_1$ and $\hat{\rho}_2$ are two right ideals of $\widehat{R}$. Denote by B the complete Boolean algebra of idempotents of C [1]. Fix a maximal ideal Δ of B. Let ϕ be the canonical homomorphism from U onto $U/\Delta U$. By [1, Theorem 1], $\phi(\widehat{R})$ is a prime ring with right ideals $\phi(\hat{\rho}_1)$ and $\phi(\hat{\rho}_2)$. Moreover, $\phi(U) = U/\Delta U$ is a right quotient ring of $\phi(\widehat{R})$ and $\phi(\widehat{R}) \subseteq \phi(\widehat{S}) \subseteq \phi(U)$. We claim that $l_{\phi(\widehat{S})}(\phi(\hat{\rho}_1)) = l_{\phi(\widehat{S})}(\phi(\hat{\rho}_2))$. Let $\phi(x) \in l_{\phi(\widehat{S})}(\phi(\hat{\rho}_1))$, where $x \in \widehat{S}$. Then $x\hat{\rho}_1 \subseteq \Delta U$. Now $x\hat{\rho}_1$ is an orthogonally complete subset of U since $\hat{\rho}_1$ is. By [1, Lemma 2(3)], there is $e \in B - \Delta$ such that $ex\hat{\rho}_1 = 0$. But $ex \in \widehat{S}$ since $B\widehat{S} \subseteq \widehat{S}$. By the fact that $l_{\widehat{S}}(\hat{\rho}_1) = l_{\widehat{S}}(\hat{\rho}_2)$, we have $ex\hat{\rho}_2 = 0$, and hence $\phi(x) \in l_{\phi(\widehat{S})}(\phi(\hat{\rho}_2))$ by [1, Lemma 2(3)] again. This proves our claim.

Let $f \in SC *_C C\{X_1, X_2, \dots\}$ be a GPI for ρ_1. By Lemma 6, f is also a GPI for $\hat{\rho}_1$. Denote by f_ϕ the GP obtained from f via replacing each coefficient occurring in f by its image under ϕ. Then f_ϕ has coefficients in $\phi(\widehat{S}C)$ and f_ϕ is a GPI for $\phi(\hat{\rho}_1)$. Since $\phi(\widehat{R})$ is a prime ring and $l_{\phi(\widehat{S})}(\phi(\hat{\rho}_1)) = l_{\phi(\widehat{S})}(\phi(\hat{\rho}_2))$, by Lemma 5 f_ϕ is also a GPI for $\phi(\hat{\rho}_2)$. Write $f = f(X_1, \dots, X_l)$. Then we have $f(x_1, \dots, x_l) \in \Delta U$ for all $x_i \in \hat{\rho}_2$. But $\bigcap\{\Delta U \mid \Delta$ is a maximal ideal of $B\} = 0$; we obtain $f(x_1, \dots, x_l) = 0$ for all $x_i \in \hat{\rho}_2$. That is, f is a GPI for $\hat{\rho}_2$ and hence for ρ_2. This completes the proof of the Main Theorem.

We conclude this paper with two applications of the Main Theorem. Recall that we denote by Q the two-sided Utumi quotient ring of R, a semiprime ring.

Theorem 1. *Let R be a semiprime ring and ρ a right ideal of R such that $l_R(\rho) = 0$. Then ρ and U satisfy the same GPIs with coefficients in Q.*

Proof. We claim that $l_Q(\rho Q) = 0$. Indeed, let $x \in Q$ be such that $x\rho Q = 0$. Then by the semiprimeness of Q we have $x\rho = 0$. By the definition of Q, there exists a dense left ideal λ of R such that $\lambda x \subseteq R$. Thus $(\lambda x)\rho = 0$ and hence $\lambda x \subseteq l_R(\rho) = 0$. This implies $x = 0$. So $l_Q(\rho Q) = 0 = l_Q(Q)$. By the Main Theorem, ρQ and Q satisfy the same GPIs with coefficients in Q $(= QC)$. But Q_R is a dense R-submodule of U_R; applying [2, Theorem 2] and Lemma 2 yields that ρ and U satisfy the same GPIs with coefficients in Q. This completes the proof.

Theorem 2. *Let R be a semiprime ring and ρ a right ideal of R. Then, for each positive integer m, ρ^m and ρ satisfy the same GPIs with coefficients in U.*

Proof. By the Main Theorem, it suffices to prove that $l_U(\rho) = l_U(\rho^m)$. The fact that $l_U(\rho) \subseteq l_U(\rho^m)$ is clear. For the converse, let $x \in l_U(\rho^m)$. Then $x\rho^m = 0$. That is, ρ satisfies the GPI xX^m. By Lemma 2, $x(\rho U)^m = 0$. Now this implies $(x\rho U)^m = 0$, since $\rho U x \subseteq \rho U$. By the semiprimeness of U, $x\rho U = 0$ follows. Therefore $x\rho = 0$. This gives $l_U(\rho^m) = l_U(\rho)$. The proof is now complete.

Remark. In Theorem 1, we cannot conclude that ρ and U satisfy the same GPIs with coefficients in U even if R is a domain. Indeed, there exists a domain R but U is not a domain. Choose $a \in U - \{0\}$ such that $r_U(a) \neq 0$. Set $\rho = R \cap r_U(a)$. Then ρ is a nonzero right ideal of R such that $a\rho = 0$, but $aU \neq 0$.

References

1. K. I. Beidar, *Rings of quotients of semiprime rings*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **1978**, no. 5, 36–42; English transl., Moscow Univ. Math. Bull. **33** (1978), no. 5, 29–34.
2. ——, *Rings with generalized identities.* III, Vestnik Moskov. Univ. Ser. I Math. Mekh. **1978**, no. 4, 66–73; English transl., Moscow Univ. Math. Bull. **33** (1978), no. 4, 53–58 .
3. C. L. Chuang, *GPIs having coefficients in Utumi quotient rings*, Proc. Amer. Math. Soc. **103** (1988), 723–728.
4. ——, *Differential identities with automorphisms and antiautomorphisms.* I, J. Algebra **149** (1992), 371–404.
5. T. S. Erickson, W. S. Martindale, and J. Osbern, *Prime nonassociative algebras*, Pacific J. Math. **60** (1975), 49–63.
6. C. Faith, *Lectures on injective modules and quotient rings*, Lecture Notes in Math., vol. 49, Springer-Verlag, Berlin and New York, 1967.
7. N. Jacobson, *Structure of rings*, 2nd ed., Amer. Math. Soc., Providence, RI, 1964.
8. W. S. Martindale III, *Prime rings satisfying a generalized polynomial identity*, J. Algebra **12** (1969), 576–584.

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