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**LEFT CENTRALIZERS AND ISOMORPHISMS OF GROUP  
ALGEBRAS**

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# LEFT CENTRALIZERS AND ISOMORPHISMS OF GROUP ALGEBRAS

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1. **Introduction.** Let  $G$  be a locally compact group with right invariant Haar measure  $m$  and real or complex group algebra  $L(G)$ <sup>1</sup>. A bounded linear transformation  $A$  of  $L(G)$  to itself which commutes with all of the operations of left multiplication is called a *left centralizer*. Examples of transformations  $A$  which satisfy the defining equation  $A(xy) = xAy$ , all  $x, y \in L(G)$ , include

(i) right multiplications:  $Ax = xz$  for some fixed  $z \in L(G)$ ;

(ii) right translations:  $A = R_{g_0}$  = the operation of translation on the right by some fixed  $g_0 \in G$ ; and

(iii) convolutions with measures:  $Ax = x * \mu$ , where  $\mu$  is a countably additive set function of bounded variation defined on the Borel sets of  $G$ , and  $x * \mu$  is defined by

$$(x * \mu)(g) = \int x(gh^{-1}) \mu(dh), \quad g \in G.$$

It is clear that (i) and (ii) are special cases of (iii); thus, given  $z \in L(G)$  we may define the appropriate  $\mu$  by  $\mu(E) = \int_E z(g) m(dg)$ , while for an assigned  $g_0 \in G$  the corresponding measure is defined by  $\mu(E) = 0$  or  $1$  according as  $E$  does not or does contain  $g_0^{-1}$ .

The principal result (Theorem 1) of Part I states that, conversely, *every left centralizer is a convolution with a regular measure*. Important auxiliary theorems (3 and 4) furnish a characterization of the right translations (up to scalar factors of unit modulus), and show that in the strong operator topology any left centralizer may be approximated by a finite linear combination of right translations.

In Part II these results are applied to obtain a generalization of a theorem proved in [5]. We showed there that if  $T$  is an isometric isomorphism of (real,

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<sup>1</sup>We are using the notation and terminology of [5]. In particular, for elements  $x$  and  $y$  of  $L(G)$  the symbol  $xy$  denotes the usual convolution-product.

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complex)  $L(G)$  onto the (real, complex) algebra  $L(G')$  belonging to a second locally compact group  $G'$  with right invariant Haar measure  $m'$ , then there exists a topological isomorphism  $\tau$  of  $G$  upon  $G'$  and a continuous character  $\chi$  such that  $T$  is given by

$$(*) \quad x'(g') \equiv (Tx) (\tau g) = c \chi(g) x(g),$$

for each  $x \in L(G)$  and almost all  $g \in G$ ,  $c$  being a measure-adjusting constant.<sup>2</sup> In the present paper we obtain the existence of  $\tau$  and the validity of (\*) by supposing only that  $T$  is an isomorphism which does not increase norm. (Of course, once (\*) is established it follows that  $T$  must be an isometry after all.) This generalization was suggested by, and extends, some results of Helson [2], which the author had the privilege of reading in manuscript. Helson obtained the theorem to be proved here for the special case of abelian  $G$ ,  $G'$  and complex algebras; his methods, unfortunately, seem to be essentially "abelian" in nature, in that he makes strong use of duality theory and the Bochner representation theorem for positive definite functions.

#### PART I. LEFT CENTRALIZERS

2. **The principal theorem.** We shall establish the following result:

**THEOREM 1.** *Let  $A$  be a left centralizer acting on (real, complex)  $L(G)$ . There exists a unique regular (real, complex) measure  $\mu$  of bounded variation such that  $A$  is given by  $Ax = x * \mu$ ; furthermore,  $\|A\| = \text{var } \mu$ .<sup>3</sup>*

*Proof.* Let  $\{V_\alpha\}$  be a basis for the neighborhoods of the identity element of  $G$ , with  $V_\alpha$  compact; we write  $\alpha \leq \beta$  in case  $V_\alpha \supseteq V_\beta$ , so that  $\{\alpha\}$  is a directed set. Let  $e_\alpha$  denote the characteristic function of  $V_\alpha$ , normalized through division by  $m(V_\alpha)$  so that  $\|e_\alpha\| = 1$ . It is well known that  $\{e_\alpha\}$  constitutes an approximate identity, in the sense that  $\lim_\alpha x e_\alpha = x$  for each  $x \in L(G)$ . Applying the transformation  $A$ , we obtain

$$\tilde{x} \equiv Ax = A \lim_\alpha x e_\alpha = \lim_\alpha A(x e_\alpha) = \lim_\alpha x A e_\alpha = \lim_\alpha x f_\alpha,$$

where  $f_\alpha \equiv A e_\alpha$  and  $\|f_\alpha\| \leq \|A\| \|e_\alpha\| = \|A\|$ .

The elements  $f_\alpha$  may be thought of as linear functionals on the space  $C_0(G)$  of continuous functions vanishing at  $\infty$ ; as such their norms are equal to their  $L$  norms, and thus are bounded. Since the unit sphere in  $C_0(G)^*$  is weak\*-compact

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<sup>2</sup>The notations  $G'$ ,  $m'$  used here replace  $\Gamma$ ,  $\mu$  of [5].

<sup>3</sup>Suggestions supplied by the referee have made possible considerable simplification in the proof of this result, as well as in that of Theorem 4.

the functionals  $f_\alpha$  have a weak\*limit point which, by the Kakutani-Riesz theorem is uniquely representable as a regular measure  $\mu$ , of bounded variation; in fact,  $\text{var } \mu \leq \|A\|$ .

Let  $K$  denote the subspace of  $C_0(G)$  consisting of functions having compact support. If  $x$  and  $y$  belong to  $K$ , then so does the function  $z$  defined by

$$z(g) = \int y(h) x(hg^{-1}) m(dh).$$

Let  $\epsilon > 0$  and  $\alpha_0$  be arbitrary; since  $\mu$  is a weak\*limit point of the  $f_\alpha$ , there is an  $\alpha_1 > \alpha_0$  such that

$$|\int z(g) \mu(dg) - \int z(g) f_{\alpha_1}(g) m(dg)| < \epsilon.$$

Replacing  $z(g)$  by its definition and rearranging the iterated integrals, we get

$$|\int y(h) m(dh) \int x(hg^{-1}) \mu(dg) - \int y(h) m(dh) \int x(hg^{-1}) f_{\alpha_1}(g) m(dg)| < \epsilon.$$

Since  $xf_\alpha \rightarrow \tilde{x}$  in  $L(G)$  and therefore as functionals, we have, on passing to the limit through a suitable cofinal subset  $\{\alpha_1\}$  of  $\{\alpha\}$ ,

$$|\int y(h) m(dh) \int x(hg^{-1}) \mu(dg) - \int y(h) \tilde{x}(h) m(dh)| \leq \epsilon.$$

Therefore

$$\int y(h) m(dh) \int x(hg^{-1}) \mu(dg) = \int y(h) \tilde{x}(h) m(dh);$$

this shows that  $x * \mu$  and  $\tilde{x}$  are equal as functionals on  $K$ , hence on  $C_0(G)$ , and so finally as elements of  $L(G)$ .

Thus we have shown that  $Ax = x * \mu$ , at least for  $x \in K$ ; a density argument shows that the equation actually holds for all  $x \in L(G)$ . Since  $\|A\| \leq \text{var } \mu$  it follows from the reverse inequality above that  $\|A\| = \text{var } \mu$ . If  $x * \mu_1 = x * \mu_2$  for all  $x \in K$ , then it is easy to see that  $\mu_1 = \mu_2$ , from which it follows that  $\mu$  is unique, and the proof is completed.

A result equivalent to Theorem 1 is the following:

**THEOREM 2.** *In the strong operator topology, the set of convolution operators on  $L(G)$  is a closed subset of the ring of all bounded operators.*

*Proof that Theorem 1 implies Theorem 2.* Let  $\{\mu_\alpha\}$  be a directed sequence of convolution operators which converges to an operator  $A$  in the strong topology. Clearly  $A$  is a left centralizer, and the result follows.

*Proof that Theorem 2 implies Theorem 1.* As in the proof of Theorem 1, we

have  $Ax = \lim_{\alpha} x f_{\alpha}$ ; but right multiplication by  $f_{\alpha}$  may be regarded as a convolution operator  $\mu_{\alpha}$ , and  $\mu_{\alpha}$  tends strongly to  $A$ . Hence  $A$  is a convolution operator.

**3. Auxiliary theorems.** The right translations are characterized as follows:

**THEOREM 3.** *Let  $A$  be a bounded linear mapping of  $L(G)$  to itself.  $A$  has the form  $A = \lambda R_{g_0}$  for some scalar  $\lambda$  of unit modulus and some  $g_0 \in G$  if and only if*

- 3A)  $A$  is a left centralizer; and
- 3B)  $A$  preserves norm.<sup>4</sup>

*Proof.* Since the mappings  $\lambda R_{g_0}$  clearly have the indicated properties we have only to prove the reverse implication.

Let  $\mu$  be the measure determined by  $A$  in accordance with Theorem 1. Then the assumption 3B) means that

$$(1) \quad \int m(dg) \left| \int x(gh^{-1}) \mu(dh) \right| = \|x\|, \quad x \in L(G).$$

Let  $|\mu| (E) =$  the total variation of  $\mu$  on  $E$ ;  $|\mu|$  is a regular measure and

$$(2) \quad \left| \int x(gh^{-1}) \mu(dh) \right| \leq \int |x(gh^{-1})| |\mu| (dh).$$

We assert that equality holds in (2) for almost all  $g$ . Supposing the contrary, let strict inequality hold on a set of positive measure, and integrate (2) over  $g$  with respect to  $m$ . From (1) we obtain, by means of Fubini's theorem,

$$\begin{aligned} \|x\| &= \int m(dg) \left| \int x(gh^{-1}) \mu(dh) \right| < \int m(dg) \int |x(gh^{-1})| |\mu| (dh) \\ &= \int |\mu| (dh) \int |x(gh^{-1})| m(dg) = \int |\mu| (dh) \int |x(g)| m(dg) \\ &= (\text{var } \mu) \|x\| = \|A\| \|x\| = \|x\|. \end{aligned}$$

This contradiction proves the assertion.

Let  $x \in K$ . Then both members of (2) are continuous functions of  $g$ , so that equality holds everywhere; set  $g = i =$  identity of  $G$  and replace the function  $x(h)$  by  $x(h^{-1})$ , which is again an element of  $K$ . We obtain

$$(3) \quad \left| \int x(h) \mu(dh) \right| = \int |x(h)| |\mu| (dh) \equiv J(x).$$

<sup>4</sup> A similar result has been given by Kawada [4], for real  $L(G)$ . In Kawada's theorem the condition  $\|Ax\| = \|x\|$  is replaced by  $Ax \geq 0$  if and only if  $x \geq 0$ , where of course the inequalities are meant in the almost everywhere sense. Theorem 3 can be deduced from this, and conversely.

For  $J(x) \neq 0$  and  $x$  a nonnegative function, there is a unique constant  $\lambda_x$  of unit modulus such that

$$\int x(h) \mu(dh) = \lambda_x \int x(h) |\mu| (dh).$$

If  $y$  is a nonnegative function in  $K$  for which  $J(y) \neq 0$ , then similarly we may write

$$\int y(h) \mu(dh) = \lambda_y \int y(h) |\mu| (dh),$$

and

$$\int \{x(h) + y(h)\} \mu(dh) = \lambda_{x+y} \int \{x(h) + y(h)\} |\mu| (dh).$$

But the left member of the last equation is also equal to

$$\lambda_x \int x(h) |\mu| (dh) + \lambda_y \int y(h) |\mu| (dh).$$

Therefore  $\lambda_x = \lambda_y = \lambda_{x+y}$ . In other words, there is a unique constant  $\lambda$  of unit modulus such that for all nonnegative  $x \in K$  for which  $J(x) \neq 0$  we have

$$\int x(h) \mu(dh) = \lambda \int x(h) |\mu| (dh).$$

Hence for all Borel sets  $E$  we have  $\mu(E) = \lambda |\mu| (E)$ , and we may as well suppose that  $\mu$  is nonnegative. Equation (3) then becomes

$$(4) \quad \left| \int x(h) \mu(dh) \right| = \int |x(h)| \mu(dh), \quad x \in K.$$

By the regularity of  $\mu$ , this equation actually holds for all real continuous  $x$  having limits at  $\infty$ ; that is, for all  $x$  in the space  $C(G)$ , where  $G$  is made compact, if necessary, by adjoining an ideal point.

We now appeal to a theorem due to Kakutani [3], stating that, if  $f$  is a functional of norm 1 on  $C(\Omega)$  such that  $x \geq 0$  implies  $f(x) \geq 0$  and such that  $\min(x, y) = 0$  implies  $\min(f(x), f(y)) = 0$ , then  $f$  is a point functional:  $f(x) = x(\omega_0)$  for some fixed  $\omega_0 \in \Omega$ . We apply the theorem to the functional  $f(x) = \int x(h) \mu(dh)$ . Clearly  $\|f\| = 1$ , since  $\text{var } \mu = 1$ . The functional is certainly order-preserving. Finally, if  $\min(x, y) = 0$  then  $x(g) + y(g) = |x(g) - y(g)|$ , and therefore

$$\int \{x(g) + y(g)\} \mu(dg) = \int |x(g) - y(g)| \mu(dg) = \int \{x(g) - y(g)\} \mu(dg)$$

by (4). Consequently  $f(x) + f(y) = |f(x) - f(y)|$ , so that  $\min(f(x), f(y)) = 0$ .

The functional  $\int x(g) \mu(dg)$  is thus seen to satisfy the hypotheses of Kakutani's theorem, and therefore is a point functional, for some point other than the point at infinity, since the functional does not vanish identically on  $K$ . Therefore  $\mu$  is concentrated at one point, and the operator  $A$  has the desired form.

Our next result states that the right translations span the space of all left centralizers. Precisely, we prove:

**THEOREM 4.** *Every left centralizer  $A$  is a strong limit point of the set of finite linear combinations of right translations.*

*Proof.* By the Hahn-Banach theorem, it suffices to show that, if  $F$  is any strongly continuous linear functional on the operators on  $L(G)$ , which vanishes on the right translations, then  $F(A) = 0$ . A strongly continuous linear functional  $F$  on the operators  $\{T\}$  on a Banach space  $X$  is given by an expression of the form

$$F(T) = \sum_{i=1}^n x_i^*(Tx_i),$$

where  $x_i \in X$ ,  $x_i^* \in X^*$ ,  $i = 1, 2, \dots, n$  (see, for example, the proof of Theorem 2 in [1]). When  $X = L(G)$ , we have

$$F(T) = \sum_{i=1}^n \int v_i(g) (Tx_i)(g) m(dg),$$

where the  $v_i$  are bounded measurable functions.

Suppose now that  $F$  vanishes on right translations. This means that

$$\sum_{i=1}^n \int v_i(g) x_i(gh^{-1}) m(dg) = 0 \quad (h \in G).$$

Computing  $F(A)$ , we have

$$\begin{aligned} F(A) &= \sum_{i=1}^n \int v_i(g) (Ax_i)(g) m(dg) = \sum_{i=1}^n \int v_i(g) m(dg) \int x_i(gh^{-1}) \mu(dh) \\ &= \int \mu(dh) \sum_{i=1}^n \int v_i(g) x_i(gh^{-1}) m(dg) = 0, \end{aligned}$$

as we wished to show.

## PART II. ISOMORPHISM OF GROUP ALGEBRAS

**4. The isomorphism theorem.** In this and succeeding §§,  $G'$  is a second locally compact group having right invariant Haar measure  $m'$ , group algebra  $L(G')$ , and right translation operators  $R'_g$ . The chief result to be established

is the following:

**THEOREM 5.** *Let  $T$  be an algebra isomorphism of  $L(G)$  onto  $L(G')$ , both algebras real or complex, which does not increase norms. There exists a bicontinuous isomorphism  $\tau$  of  $G$  upon  $G'$  and a continuous character  $\chi$  (real or complex with  $L(G)$ ) on  $G$  such that*

$$5A) \quad TR_g T^{-1} = \chi(g^{-1}) R'_{\tau g}, \quad g \in G, \tau g \in G';$$

$$5B) \quad (Tx)(\tau g) = c \chi(g) x(g);$$

hence  $T$  is actually an isometry. The number  $c$  is the constant value of the ratio  $m(E)/m'(\tau E)$ .

The proof of Theorem 5, which will be given in §6 after some necessary lemmas have been developed, is based on the following idea, due in part to Kawada [4]. First of all, it is clear that  $T$  induces a 1-1 mapping of the left centralizers of  $L(G)$  onto those of  $L(G')$  by means of the formula

$$A' = TAT^{-1},$$

the boundedness of  $T^{-1}$  being guaranteed by that of  $T$  together with the 1-1-ness. In particular, then, translations on  $L(G)$  are carried into left centralizers. It turns out, moreover, that the image of an arbitrary translation  $R_g$  is an isometric left centralizer; this is proved in Lemma 1. Therefore, by Theorem 3, it follows that the operator  $TR_g T^{-1}$  is a scalar multiple of a translation on  $L(G')$ ; we write

$$TR_g T^{-1} = \lambda_g R'_{\tau g},$$

where  $|\lambda_g| = 1$ . We then show (Lemmas 2, 3) that  $\tau$  is 1-1, onto, and bicontinuous, and that  $\lambda_g$  (or rather, its inverse) is a continuous character, thereby establishing 5A). The formula 5B) follows quickly with the aid of Theorem 4.

**5. The mapping of translations.** We shall first prove:

**LEMMA 1.** *Let  $R_g$  be a right translation on  $L(G)$  and set  $Z'_g = TR_g T^{-1}$ . Then for some  $\lambda_g$  of modulus unity, and for some  $g' \equiv \tau g \in G'$ , we have*

$$Z'_g = TR_g T^{-1} = \lambda_g R'_{g'}.$$

*Proof.* In view of Theorem 3, we have only to show that  $Z'_g$  is isometric, since it certainly commutes with all left multiplications. As used in the proof of Theorem 1, let  $\{e_\alpha\}$  be an approximate identity of  $L(G)$ , and put  $Te_\alpha = e'_\alpha$ . Choose  $x' \in L(G')$ ; clearly  $x'e'_\alpha \rightarrow x'$ , since  $xe_\alpha \rightarrow x$  for all  $x \in L(G)$ . Then



$$Z'_g x' = \lim_{\alpha} x' Z'_g e'_\alpha = \lim_{\alpha} x' TR_g T^{-1} T e_\alpha = \lim_{\alpha} x' TR_g e_\alpha,$$

and therefore

$$\begin{aligned} \|Z'_g x'\| &\leq \overline{\lim}_{\alpha} \|x' TR_g e_\alpha\| \leq \|x'\| \overline{\lim}_{\alpha} \|TR_g e_\alpha\| \leq \|x'\| \overline{\lim}_{\alpha} \|R_g e_\alpha\| \\ &= \|x'\| \overline{\lim}_{\alpha} \|e_\alpha\| = \|x'\|, \end{aligned}$$

which shows that  $Z'_g$  is a contraction. But  $Z'_g{}^{-1} = Z'_{g^{-1}}$  is also a contraction, by the same argument. Therefore  $Z'_g$  is an isometry, as we had to show.

LEMMA 2. *The mappings  $g \rightarrow \lambda g$  and  $g \rightarrow \tau g$  defined above are continuous homomorphisms of  $G$  to, respectively, the multiplicative group of scalars of unit modulus and the group  $G'$ ;  $\tau$  is  $1 - 1$ .*

*Proof.* The fact that the mappings are homomorphisms follows from the equations

$$\lambda_{gh} R'_{\tau(gh)} = TR_{gh} T^{-1} = TR_g T^{-1} TR_h T^{-1} = \lambda_g R'_{\tau g} \lambda_h R'_{\tau h} = \lambda_g \lambda_h R'_{\tau g \tau h}.$$

The function  $\tau$  is  $1 - 1$ ; for if  $\tau g = i'$ , the identity of  $G'$ , then

$$\lambda_g I' = TR_g T^{-1}, \text{ and } R_g = \lambda_g I, \text{ so that } g = i, \lambda_g = 1.$$

In order to prove that  $\tau$  is continuous, we observe that it is the product of the mappings  $M_1, M_2, M_3$  defined by:

$$\begin{aligned} M_1 : g &\rightarrow R_g, & g \in G; \\ M_2 : R_g &\rightarrow TR_g T^{-1} = \lambda_g R'_{\tau g}, & g \in G; \\ M_3 : \lambda R'_{g'} &\rightarrow g', & \lambda \text{ a scalar of unit modulus, } g' \in G'. \end{aligned}$$

It is well known that, in the strong operator topology,  $R_g$  is a continuous function of  $g$ , so that  $M_1$  is continuous.  $M_2$  is continuous, since  $T$  and its inverse are bounded. The mapping  $M_3$  is a homomorphism of the group of operators  $\{\lambda R'_{g'}\}$  onto  $G'$ ; hence in order to prove its continuity everywhere it is sufficient to consider merely neighborhoods of the identity  $I'$ .

Let  $V'$  be an arbitrary neighborhood of  $i' \in G'$ ; we shall construct a strong neighborhood of  $I'$  whose image under  $M_3$  is contained in  $V'$ . Let  $W'$  be a neighborhood of  $i'$  having finite measure  $w$  and satisfying  $W'W'^{-1} \subseteq V'$ . Let  $x' \in L(G')$  be the characteristic function of  $W'$ . We shall show that if  $\|\lambda R'_{g'} x' - x'\| < 2w$  then  $g' \in V'$ . In fact, suppose that  $g' \notin V'$ ; then  $W'$  and  $W'g'$  are disjoint, so that

$$\begin{aligned} ||\lambda R'_{g'}x' - x'|| &= \int |\lambda x'(h'g') - x'(h')| m'(dh') \\ &= \int \{|\lambda x'(h'g')| + |x'(h')|\} m'(dh') \\ &= ||\lambda R'_{g'}x'|| + ||x'|| = 2||x'|| = 2w. \end{aligned}$$

Hence  $M_3$  and *a fortiori*  $\tau = M_3 M_2 M_1$  are continuous in  $g$ . Finally,  $\lambda_g$  is continuous since  $\lambda_g I$  is the product of the uniformly bounded and continuous functions  $TR_g T^{-1}$  and  $R'_{\tau g^{-1}}$ .

LEMMA 3. *The mapping  $\tau$  exhausts  $G'$  and is a homeomorphism of  $G$  onto  $G'$ .*

*Proof.* We first show that  $\tau G$  is closed in  $G'$ . Suppose that  $\{\tau g_\alpha\}$  is a directed sequence of elements which converges to an element  $h'$  of  $G'$ . Then the corresponding translations  $R'_{\tau g_\alpha}$  tend to  $R'_{h'}$ . Mapping back to the algebra of operators on  $L(G)$ , we see that  $T^{-1}R'_{\tau g_\alpha}T$  tends to  $T^{-1}R'_{h'}T$ . But  $T^{-1}R'_{\tau g}T = \lambda_g^{-1}R'_g$  by definition of  $\tau$  and  $\lambda$ . Therefore the operators  $\lambda_g^{-1}R'_g$  converge in the strong topology to an operator which is clearly on isometric left centralizer, and which therefore has the form  $\lambda R'_h$  for some scalar  $\lambda$  and  $h \in G$ . Returning to  $L(G')$ , we readily see that  $\lambda TR'_h T^{-1} = R'_{h'}$ , so that  $h' = \tau h$ ,  $\lambda = \lambda_h^{-1}$ .

Next we note that the continuity of  $\tau^{-1}$  on  $\tau G$  follows from the equation

$$T^{-1}R'_{g'}T = \lambda_{\tau^{-1}g'}^{-1}R'_{\tau^{-1}g'} \quad (g' = \tau g)$$

just as that of  $\tau$  was obtained from its defining equation.

Finally we establish the fact that  $\tau G = G'$ . Suppose if possible that  $h' \in G'$  has no counterimage in  $G$ ;  $T^{-1}R'_{h'}T \equiv A$  is nevertheless a left centralizer on  $L(G)$ , and therefore, by Theorem 4, may be expressed as the strong limit of a directed sequence  $\{A_\alpha\}$  of finite linear combinations of translations. Then  $TAT^{-1} = R'_{h'}$  is the strong limit of operators  $A_\alpha$  each of which is a finite linear combination of translations  $R'_{g'}$ , with  $g' \in \tau G$ . Let  $W'$  be a neighborhood of  $i'$  so small that  $m'(W') < \infty$  and  $h'W'^{-1}W' \cap \tau G$  is empty; the existence of such a  $W'$  is assured by the fact that  $\tau G$  is closed. Let  $x'$  be the characteristic function of  $W'$ , and set  $x'_\alpha = A_\alpha x'$ . Let  $k'$  be any element of  $W'h'^{-1}$ . Then  $k'g' \notin W'$ , for each  $g' \in \tau G$ . Hence  $x'(k'h') = 1$  implies that  $x'_\alpha(k') = 0$ . Therefore

$$\begin{aligned} ||x'_\alpha - R'_{h'}x'|| &\geq \int_{W'h'^{-1}} |x'_\alpha(k') - x'(k'h')| m'(dk') \\ &= \int_{W'h'^{-1}} 1 m'(dk') = m'(W'). \end{aligned}$$

But this contradicts the assertion that  $A'_\alpha$  tends strongly to  $R'_{h'}$ . Hence no such  $h'$  can exist; this completes the proof of the lemma.

6. **Proof of the isomorphism theorem.** If we define  $\chi(g) = \lambda_g^{-1}$ , it is clear that the lemmas of the preceding § establish the existence and properties of  $\tau$  and  $\chi$  and the validity of 5A). In order to prove 5B), let  $U$  be the isomorphism of  $L(G)$  onto  $L(G')$  which 5B) defines; that is,

$$(Ux)(\tau g) = c\chi(g)x(g), \quad x \in L(G).$$

Then

$$(U^{-1}x')(g) = c^{-1}\chi(g^{-1})x'(\tau g) \quad \text{for } x' \in L(G').$$

Hence

$$\begin{aligned} (UR_h U^{-1}x')(\tau g) &= c\chi(g)(R_h U^{-1}x')(g) = c\chi(g)(U^{-1}x')(gh) \\ &= c\chi(g)c^{-1}\chi(h^{-1}g^{-1})x'(\tau g\tau h) = \chi(h^{-1})x'(\tau g\tau h) \\ &= \chi(h^{-1})(R_{\tau h}x')(\tau g), \end{aligned}$$

showing that  $UR_h U^{-1} = \chi(h^{-1})R_{\tau h}$ ,  $h \in G$ . Therefore  $UR_g U^{-1} = TR_g T^{-1}$ , and consequently  $T^{-1}UR_g = R_g T^{-1}U$ . Let  $S = T^{-1}U$ ; we see that  $S$  is a bicontinuous automorphism of  $L(G)$  which commutes with all right translations. We shall show that  $S$  is the identity mapping, which will prove that  $U = T$ , as desired.

Let  $z \in L(G)$ , and let  $A$  be the left centralizer defined by right multiplication by  $z$ :  $Ax = xz$ , all  $x \in L(G)$ . Let  $\{A_\alpha\}$  be a directed sequence of combinations of translations which converges to  $A$  in the strong topology. We have

$$Ax = \lim_\alpha A_\alpha x.$$

Therefore

$$SAx = S \lim_\alpha A_\alpha x = \lim_\alpha SA_\alpha x = \lim_\alpha A_\alpha Sx = ASx.$$

In other words, using the fact that  $S$  is an automorphism, we have

$$Sx Sz = S(xz) = SAx = ASx = (Sx)z,$$

so that  $Sz = z$ . Since  $z \in L(G)$  is arbitrary,  $S = I$ , and the proof is completed.

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# Pacific Journal of Mathematics

Vol. 2, No. 2

February, 1952

L. Carlitz, <i>Some theorems on Bernoulli numbers of higher order</i> . . . . .	127
Watson Bryan Fulks, <i>On the boundary values of solutions of the heat equation</i> . . . . .	141
John W. Green, <i>On the level surfaces of potentials of masses with fixed center of gravity</i> . . . . .	147
Isidore Heller, <i>Contributions to the theory of divergent series</i> . . . . .	153
Melvin Henriksen, <i>On the ideal structure of the ring of entire functions</i> . . . .	179
James Richard Jackson, <i>Some theorems concerning absolute neighborhood retracts</i> . . . . .	185
Everett H. Languier, <i>Homology bases with applications to local connectedness</i> . . . . .	191
Janet McDonald, <i>Davis's canonical pencils of lines</i> . . . . .	209
J. D. Niblett, <i>Some hypergeometric identities</i> . . . . .	219
Elmer Edwin Osborne, <i>On matrices having the same characteristic equation</i> . . . . .	227
Robert Steinberg and Raymond Moos Redheffer, <i>Analytic proof of the Lindemann theorem</i> . . . . .	231
Edward Silverman, <i>Set functions associated with Lebesgue area</i> . . . . .	243
James G. Wendel, <i>Left centralizers and isomorphisms of group algebras</i> . . .	251
Kosaku Yosida, <i>On Brownian motion in a homogeneous Riemannian space</i> . . . . .	263