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Left invariant metrics and curvatures on simply connected three-dimensional Lie groups

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For each simply connected three-dimensional Lie group we determine the automorphism group, classify the left invariant Riemannian metrics up to automorphism, and study the extent to which curvature can be altered by a change of metric. Thereby we obtain the principal Ricci curvatures, the scalar curvature and the sectional curvatures as functions of left invariant metrics on the three-dimensional Lie groups. Our results improve a bit of Milnor's results of [7] in the three-dimensional case, and Kowalski and Nikv́cević's results [6, Theorems 3.1 and 4.1].

1 Introduction

This paper concerns itself with three main problems:

1. to determine the automorphisms of all 3-dimensional Lie algebras $\mathfrak{g}$,
2. for each simply connected 3-dimensional Lie group G to classify all the left invariant Riemannian metrics on G up to automorphism of G, and
3. to study the extent to which curvature can be altered by a change of metric.

A non-zero Borel measure on G which is invariant under left multiplication is called a *left Haar measure* on G. If G is a Lie group, then a left Haar measure always exists and any two left Haar measures on G are propositional. The Lie group G is *unimodular* if every left Haar measure is a right Haar measure and vice versa. It is known that G is unimodular if and only if $|\det \mathrm{Ad}(t)| = 1$ for all $t \in G$ if and only if the trace of $\mathrm{ad}(X)$ is zero for all X in its Lie algebra $\mathfrak{g}$ if and only if $\mathfrak{g}$ is unimodular. The abelian, compact, semisimple, reductive, nilpotent Lie groups are well-known examples of unimodular Lie groups.

There are six simply connected three-dimensional unimodular Lie groups: the abelian Lie group $\mathbb{R}^3$, the nilpotent Lie group Nil, the special unitary group $\mathrm{SU}(2)$, the universal covering group $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ of the special linear group, the solvable Lie group Sol and the universal covering group $\widetilde{E}_0(2)$ of the connected component of the Euclidean group. Thus the Lie groups $\mathbb{R}^3$, Nil, $\mathrm{SU}(2)$, $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ are unimodular. We note that the two solvable Lie groups Sol and $\widetilde{E}_0(2)$ are also unimodular. There are uncountably many nonisomorphic non-unimodular three-dimensional Lie groups. These are all solvable and of the form $\mathbb{R}^2 \rtimes_\varphi \mathbb{R}$ where $\mathbb{R}$ acts on $\mathbb{R}^2$ via a linear map φ.

The investigations described here are motivated by the paper [1] in which all left invariant metrics on the real Heisenberg group are classified up to automorphisms. We establish the classification up to automorphism of the left invariant metrics on all simply connected three-dimensional Lie groups. Since the Riemannian connection is determined uniquely, due to the famous theorem of Levi–Civita, by the bracket product operation and the

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given metric, our classification of the left invariant metrics up to automorphism leads to the study of the left invariant metrics which leave all the curvature properties invariant. Utilizing Milnor's idea in [7] together with our complete list of left invariant metrics on simply connected three-dimensional Lie groups up to automorphism, we are able to understand completely the change of the signature of the Ricci transformation and the change of the sign of the scalar curvature by a change of metric. Our results of Section 4 extend completely Milnor's results of [7] in the three-dimensional case, and Kowalski and Nikv́cević's results [6, Theorems 3.1 and 4.1]. See Tables 1 and 2 for a summary.

All the calculations were done using the program Mathematica [8].

2 Automorphisms of three-dimensional Lie algebras

2.1 The three-dimensional Lie algebras

We list all the three dimensional Lie algebras. If $\{X, Y, Z\}$ is a basis for a Lie algebra then the following multiplication tables describe all the non-isomorphic Lie algebras of dimension 3 (for details see [5, I.4] and [7]).

(2.1.1) Abelian: $[X, Y] = [Y, Z] = [Z, X] = 0$.

(2.1.2) Nilpotent: $[X, Y] = Z,\ [Z, X] = [Z, Y] = 0$.

(2.1.3) Unimodular Solvable: There are only two non-isomorphic unimodular solvable Lie algebras and a basis may be chosen such that

(a) $[X, Y] = 0,\ [Z, X] = X,\ [Z, Y] = -Y$, or

(b) $[X, Y] = 0,\ [Z, X] = -Y,\ [Z, Y] = X$.

(2.1.4) Simple: There are only two non-isomorphic simple Lie algebras and a basis may be chosen such that

(a) $[X, Y] = 2Z,\ [Z, X] = 2Y,\ [Z, Y] = 2X$, or

(b) $[X, Y] = Z,\ [Z, X] = Y,\ [Z, Y] = -X$.

(2.1.5) Non-unimodular Solvable: There are uncountably many nonisomorphic non-unimodular solvable Lie algebras and a basis may be chosen such that

(a) $[X, Y] = 0,\ [Z, X] = X,\ [Z, Y] = Y$, or

(b) $[X, Y] = 0,\ [Z, X] = Y,\ [Z, Y] = -cX + 2Y$,

where $c \in \mathbb{R}$. Note that $\mathrm{ad}(Z) = \left[\begin{smallmatrix} 0 & -c \\ 1 & 2 \end{smallmatrix}\right]$ has trace 2 and determinant c.

2.2 The automorphisms of a Lie algebra

In this subsection we find all the automorphisms of each three dimensional Lie algebra. This results in finding all the automorphisms of the corresponding simply connected three-dimensional Lie group. For this purpose, we will use the ordered bases $\{X, Y, Z\}$ given in Section 2.1 for the three-dimensional Lie algebras.

2.2.1 The abelian Lie algebra $\mathbb{R}^3$

In the abelian case the Lie algebra is isomorphic to $\mathbb{R}^3$.

Proposition 2.1 *The Lie group* $\mathrm{Aut}\big(\mathbb{R}^3\big)$ *is isomorphic to* $\mathrm{GL}(3, \mathbb{R})$.

2.2.2 The Heisenberg Lie algebra $\mathfrak{n}$

In the nilpotent case the Lie algebra is isomorphic to the Heisenberg Lie algebra $\mathfrak{n}$ of all 3×3 strictly upper triangular real matrices. Choose the canonical basis $\{X_1, X_2, X_3\}$ in $\mathfrak{n}$ where

$$X_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $[X_1, X_2] = X_3$ and $[X_3, X_1] = [X_3, X_2] = 0$.

Proposition 2.2 *The Lie group* $\mathrm{Aut}(\mathfrak{n})$ *is isomorphic to*

$$\left\{ \begin{bmatrix} a & c & 0 \\ b & d & 0 \\ * & * & ad-bc \end{bmatrix} \;\middle|\; a,b,c,d,* \in \mathbb{R},\ ad-bc \neq 0 \right\}.$$

Proof. An automorphism on $\mathfrak{n}$ must map the center $\mathcal{Z}(\mathfrak{n}) = \langle X_3 \rangle \cong \mathbb{R}^1$ onto itself. Hence it maps the basis $\{X_1, X_2, X_3\}$ of $\mathfrak{n}$ as follows:

$$\begin{aligned} X_1 &\longmapsto aX_1 + bX_2 + kX_3, \\ X_2 &\longmapsto cX_1 + dX_2 + \ell X_3, \\ X_3 &\longmapsto rX_3. \end{aligned}$$

Thus we obtain that $\varphi[X_i, X_j] = [\varphi X_i, \varphi X_j]$ if and only if $r = ad - bc \neq 0$. □

2.2.3 The unimodular solvable Lie algebra $\mathbb{R}^2 \rtimes_\sigma \mathbb{R}$

In the unimodular solvable case the Lie algebra of (2.1.3) (a) is isomorphic to the semidirect product $\mathbb{R}^2 \rtimes_\sigma \mathbb{R}$, where $\sigma(t) = \left[\begin{smallmatrix} t & 0 \\ 0 & -t \end{smallmatrix}\right]$. We can choose a basis $\{X_1, X_2, X_3\}$ of $\mathbb{R}^2 \rtimes_\sigma \mathbb{R}$ where

$$X_1 = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, 0 \right), \quad X_2 = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, 0 \right), \quad X_3 = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 1 \right).$$

Then $[X_1, X_2] = 0$, $[X_3, X_1] = X_1$, $[X_3, X_2] = -X_2$.

Proposition 2.3 *The Lie group* $\mathrm{Aut}\big(\mathbb{R}^2 \rtimes_\sigma \mathbb{R}\big)$ *is isomorphic to*

$$S_1 \cup S_1 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \textit{where} \quad S_1 = \left\{ \begin{bmatrix} \alpha & 0 & * \\ 0 & \beta & * \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; \alpha, \beta, * \in \mathbb{R},\ \alpha\beta \neq 0 \right\}.$$

Proof. Let $\varphi \in \mathrm{Aut}\big(\mathbb{R}^2 \rtimes_\sigma \mathbb{R}\big)$. Then $\varphi[X_i, X_j] = [\varphi X_i, \varphi X_j]$ if and only if with respect to the basis $\{X_1, X_2, X_3\}$, $[\varphi]$ is of the form

$$[\varphi] = \begin{bmatrix} \alpha & 0 & \gamma \\ 0 & \beta & \delta \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \alpha & 0 & \gamma \\ 0 & \beta & \delta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

where $\alpha, \beta, \gamma, \delta$ are real numbers with $\alpha\beta \neq 0$. □

2.2.4 The unimodular solvable Lie algebra $\mathbb{R}^2 \rtimes \mathfrak{so}(2)$

In the unimodular solvable case the Lie algebra of (2.1.3) (b) is isomorphic to the Lie algebra $\mathbb{R}^2 \rtimes \mathfrak{so}(2)$. We choose a basis $\{X_1, X_2, X_3\}$ of $\mathbb{R}^2 \rtimes \mathfrak{so}(2)$, where

$$X_1 = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right), \quad X_2 = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right), \quad X_3 = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right).$$

Then $[X_1, X_2] = 0$, $[X_3, X_1] = -X_2$, $[X_3, X_2] = X_1$.

Proposition 2.4 *The Lie group* $\mathrm{Aut}\big(\mathbb{R}^2 \rtimes \mathfrak{so}(2)\big)$ *is isomorphic to*

$$S_2 \cup S_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \textit{where} \quad S_2 = \left\{ \begin{bmatrix} \mathbb{C}^* & & \gamma \\ & & \delta \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; \gamma, \delta \in \mathbb{R} \right\}.$$

Proof. Let $\varphi \in \operatorname{Aut}\big(\mathbb{R}^2 \rtimes \mathfrak{so}(2)\big)$. Then $\varphi[X_i, X_j] = [\varphi X_i, \varphi X_j]$ if and only if with respect to the basis $\{X_1, X_2, X_3\}$, $[\varphi]$ is of the form

$$[\varphi] = \begin{bmatrix} \alpha & \beta & \gamma \\ -\beta & \alpha & \delta \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \alpha & \beta & \gamma \\ -\beta & \alpha & \delta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

where $\alpha, \beta, \gamma, \delta$ are real numbers with $(\alpha, \beta) \neq (0,0)$. □

2.2.5 The simple Lie algebra $\mathfrak{sl}(2, \mathbb{R})$

In the simple case the Lie algebra of (2.1.4) (a) is isomorphic to the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of all 2×2 matrices of trace 0. We choose a basis $\{X_1, X_2, X_3\}$, where

$$X_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then $[X_1, X_2] = 2X_3,\ [X_3, X_1] = 2X_2,\ [X_3, X_2] = 2X_1$.

Proposition 2.5 *The Lie group* $\operatorname{Aut}(\mathfrak{sl}(2,\mathbb{R}))$ *is isomorphic to* $\mathrm{SO}(1,2)$.

Proof. Let $\varphi \in \operatorname{Aut}(\mathfrak{sl}(2,\mathbb{R}))$. With respect to the basis $\{X_1, X_2, X_3\}$,

$$\varphi X_j = a_{1j} X_1 + a_{2j} X_2 + a_{3j} X_3$$

for some a_{ij}. Observe that $\varphi[X_i, X_j] = [\varphi X_i, \varphi X_j]$ if and only if the classical adjoint $\operatorname{adj}[\varphi]$ of $[\varphi]$ is

$$\operatorname{adj}[\varphi] = \begin{bmatrix} a_{11} & -a_{21} & -a_{31} \\ -a_{12} & a_{22} & a_{32} \\ -a_{13} & a_{23} & a_{33} \end{bmatrix} = I_{1,2}\,[\varphi]^t\, I_{1,2}, \quad \text{where} \quad I_{1,2} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $(\det[\varphi]) I_3 = [\varphi](\operatorname{adj}[\varphi]) = [\varphi]\, I_{1,2} [\varphi]^t\, I_{1,2}$, we have $\det[\varphi] = 1$ and $[\varphi]^t\, I_{1,2}\,[\varphi] = I_{1,2}$. Hence $\varphi \in \operatorname{Aut}(\mathfrak{sl}(2,\mathbb{R}))$ if and only if $[\varphi] \in \mathrm{SO}(1,2)$. □

Remark 2.6 Note that the identity component $\mathrm{SO}_0(1,2)$ is isomorphic to $\mathrm{PSL}(2,\mathbb{R})$. The adjoint representation of $\mathrm{SL}(2,\mathbb{R})$ preserves the Cartan–Killing form which is a quadratic form of signature $(-,-,+)$. See also 3.2.5.

2.2.6 The split simple Lie algebra $\mathfrak{so}(3)$

The simple Lie algebra of (2.1.4) (b) is isomorphic to the Lie algebra $\mathfrak{so}(3)$ of all 3×3 skew symmetric matrices. We choose the following basis $\{X_1, X_2, X_3\}$, where

$$X_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Then $[X_1, X_2] = X_3,\ [X_3, X_1] = X_2,\ [X_3, X_2] = -X_1$.

Proposition 2.7 *The Lie group* $\operatorname{Aut}(\mathfrak{so}(3))$ *is isomorphic to* $\mathrm{SO}(3)$.

Proof. Let $\varphi \in \operatorname{Aut}(\mathfrak{so}(3))$. With respect to the basis $\{X_1, X_2, X_3\}$

$$\varphi X_j = a_{1j} X_1 + a_{2j} X_2 + a_{3j} X_3$$

for some a_{ij}. Observe that $\varphi[X_i, X_j] = [\varphi X_i, \varphi X_j]$ if and only if $a_{ij} = (-1)^{i+j} \det[\varphi(i|j)]$, the (i,j) cofactor of $[\varphi]$ for all $i, j = 1, 2, 3$ if and only if $[\varphi]^t = \operatorname{adj}[\varphi]$, the classical adjoint of $[\varphi]$.

Since $[\varphi][\varphi]^t = [\varphi](\operatorname{adj}[\varphi]) = (\det[\varphi]) I_3$, we have $\det[\varphi] = 1$ and $[\varphi][\varphi]^t = I$. Hence $\varphi \in \operatorname{Aut}(\mathfrak{so}(3))$ if and only if $[\varphi] \in \mathrm{SO}(3)$. □

2.2.7 The non-unimodular solvable Lie algebras

All the three-dimensional non-unimodular Lie algebras are solvable. By (2.1.5), such a Lie algebras is isomorphic to either $\mathfrak{g}_I$ or $\mathfrak{g}_c$ for some $c \in \mathbb{R}$ where $\mathfrak{g}_I$ is the Lie algebra of (2.1.5) (a) and $\mathfrak{g}_c$ is the Lie algebra of (2.1.5) (b). In fact,

$$\mathfrak{g}_I \cong \mathbb{R}^2 \rtimes_{\sigma_I} \mathbb{R}, \quad \text{where} \quad \sigma_I(t) = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix};$$

$$\mathfrak{g}_c \cong \mathbb{R}^2 \rtimes_{\sigma_c} \mathbb{R}, \quad \text{where} \quad \sigma_c(t) = \begin{bmatrix} 0 & -ct \\ t & 2t \end{bmatrix}$$

with a basis

$$X = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, 0\right), \quad Y = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, 0\right), \quad Z = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 1\right)$$

satisfying

(a) $[X,Y] = 0,\ [Z,X] = X,\ [Z,Y] = Y$, or

(b) $[X,Y] = 0,\ [Z,X] = Y,\ [Z,Y] = -cX + 2Y$.

Proposition 2.8 (1) *The Lie group* $\mathrm{Aut}(\mathfrak{g}_I)$ *is isomorphic to*

$$\left\{ \begin{bmatrix} \mathrm{GL}(2,\mathbb{R}) & * \\ 0 & 1 \end{bmatrix} \,\middle|\, * \in \mathbb{R}^2 \right\}.$$

(2) *For each* $c \in \mathbb{R}$*, the Lie group* $\mathrm{Aut}(\mathfrak{g}_c)$ *is isomorphic to*

$$\left\{ \begin{bmatrix} \beta - \alpha & -c\alpha & * \\ \alpha & \beta + \alpha & * \\ 0 & 0 & 1 \end{bmatrix} \,\middle|\, \begin{array}{l} \alpha, \beta, * \in \mathbb{R}, \\ \beta^2 + (c-1)\alpha^2 \neq 0 \end{array} \right\}.$$

Proof. (1) It is easy to see that $\varphi \in \mathrm{Aut}(\mathfrak{g})$ if and only if with respect to the basis $\{X, Y, Z\}$, $[\varphi]$ is of the form $\left[\begin{smallmatrix} \mathrm{GL}(2,\mathbb{R}) & * \\ 0 & 1 \end{smallmatrix}\right]$ where $* \in \mathbb{R}^2$.

(2) Let $\varphi \in \mathrm{Aut}(\mathfrak{g}_c)$. Suppose that, with respect to the basis $\{X, Y, Z\}$, $[\varphi] = \left[\begin{smallmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{smallmatrix}\right]$. Note that $\varphi \in \mathrm{Aut}(\mathfrak{g}_c)$ if and only if

$$\begin{array}{ll} a_{32} = 0, & (a_{12} + 2a_{22})a_{31} = 0, \\ ca_{31} = 0, & (a_{12} + 2a_{22})a_{33} = 2a_{22} - ca_{21}, \\ c\,(a_{11} - a_{22}a_{33}) = 2a_{12}, & (a_{11} + 2a_{21})a_{33} - (a_{13} + 2a_{23})a_{31} = a_{22}, \\ c\,(a_{23}a_{31} - a_{21}a_{33}) = a_{12}. & \end{array}$$

If $a_{33} = 1$ or $c = 0$, then the above equations yield that $a_{31} = a_{32} = 0, a_{33} = 1$, $a_{12} = -ca_{21}$ and $a_{22} = a_{11} + 2a_{21}$.

Suppose that $a_{33} \neq 1$ and $c \neq 0$. Then the above equations yield that

$$\begin{array}{ll} a_{12} = -ca_{21}a_{33}, & a_{33}a_{11} - a_{22} = -2a_{21}a_{33}, \\ a_{22} = \dfrac{1}{2}ca_{21}(a_{33} + 1), & a_{11} - a_{33}a_{22} = -2a_{21}a_{33}. \end{array}$$

Thus $a_{33} \neq -1$, $-a_{11} = \frac{2a_{21}a_{33}}{a_{33}+1} = a_{22} = \frac{ca_{21}(a_{33}+1)}{2}$ and $c = \frac{4a_{33}}{(a_{33}+1)^2}$. Now consider the automorphism φ^2 and let $\left[\varphi^2\right] = \left[b_{ij}\right]$ with respect to the basis $\{X, Y, Z\}$. Then since $b_{33} = a_{33}^2$ and $b_{33} \neq 1$, we obtain that $c = \frac{4b_{33}}{(b_{33}+1)^2}$. Thus $\frac{4a_{33}}{(a_{33}+1)^2} = c = \frac{4a_{33}^2}{(a_{33}^2+1)^2}$ and so $a_{33} = 0$ or $a_{33} = 1$, a contradiction. □

3 Left invariant metrics on three-dimensional Lie groups

For each finite dimensional Lie algebra $\mathfrak{g}$ there is, up to isomorphism, a unique simply connected Lie group whose Lie algebra is isomorphic to $\mathfrak{g}$. In this section, we list all the three-dimensional simply connected Lie groups, and for each such G we classify all the left invariant Riemannian metrics on G up to automorphism of G and in the next section, we will study the extent to which curvature can be altered by a change of metric.

When we refer to left invariant metrics in this paper we mean *left invariant Riemannian metrics*.

3.1 Three-dimensional Lie groups

The simply connected unimodular three-dimensional Lie groups are well-known. We list them below together with the simply connected non-unimodular three-dimensional Lie groups for convenience.

3.1.1 The abelian Lie group $\mathbb{R}^3$

In the abelian case the corresponding Lie group is the abelian Lie group $\mathbb{R}^3$.

3.1.2 The Heisenberg group Nil

In the nilpotent case the corresponding Lie group is the Heisenberg group Nil of all 3×3 real matrices of the form

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

3.1.3 The solvable group Sol

The Lie group of the solvable Lie algebra $\mathbb{R}^2 \rtimes_\sigma \mathbb{R}$ is the 2-step solvable Lie group Sol, which is the semidirect product $\mathbb{R}^2 \rtimes_\varphi \mathbb{R}$ where $t \in \mathbb{R}$ acts on $\mathbb{R}^2$ by $\varphi(t) = \left[\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right]$.

3.1.4 The solvable group $\widetilde{E}_0(2)$

The solvable Lie algebra $\mathbb{R}^2 \rtimes \mathfrak{so}(2)$ is the Lie algebra of the Lie group $E_0(2) = \mathbb{R}^2 \rtimes \mathrm{SO}(2)$. The group $E_0(2)$ is not simply connected. The unique simply connected Lie group corresponding to the Lie algebra $\mathbb{R}^2 \rtimes \mathfrak{so}(2)$ is the universal covering group $\widetilde{E}_0(2)$ of $E_0(2)$. The group $\widetilde{E}_0(2) = \mathbb{C} \rtimes \mathbb{R}$, $(c, r)(d, s) = \big(c + e^{2\pi i r} d, r + s\big)$ has a faithful matrix representation in $\mathrm{GL}(3, \mathbb{C})$

$$(c, r) \longmapsto \begin{bmatrix} e^{2\pi i r} & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^r \end{bmatrix}, \quad r \in \mathbb{R}, \quad c \in \mathbb{C}.$$

3.1.5 The simple Lie groups $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$ and $\mathrm{SU}(2)$

There are two distinct simple Lie algebras of dimension 3: $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{so}(3)$. The Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ is the Lie algebra of the Lie group $\mathrm{SL}(2, \mathbb{R})$. The unique simply connected Lie group corresponding to the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ is the universal covering group $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$ of $\mathrm{SL}(2, \mathbb{R})$. The two fold covering group $\mathrm{SU}(2)$ of $\mathrm{SO}(3)$ is the unique simply connected Lie group corresponding to the Lie algebra $\mathfrak{so}(3)$.

3.1.6 The non-unimodular solvable Lie groups

The three-dimensional non-unimodular Lie algebra $\mathfrak{g}_I$ or $\mathfrak{g}_c$ is the Lie algebra of the three-dimensional simply connected Lie group

$$G_I \cong \mathbb{R}^2 \rtimes_{\varphi_I} \mathbb{R}, \quad \text{where} \quad \varphi_I(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix}, \quad \text{or}$$

$$G_c \cong \mathbb{R}^2 \rtimes_{\varphi_c} \mathbb{R}, \quad \text{where}$$

$$\varphi_c(t) = \begin{cases} e^t \frac{e^{zt}+e^{-zt}}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + e^t \frac{e^{zt}-e^{-zt}}{2z} \begin{bmatrix} -1 & -c \\ 1 & 1 \end{bmatrix} & \text{if} \quad z = \sqrt{1-c} \neq 0, \\ e^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + e^t t \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} & \text{if} \quad c = 1. \end{cases}$$

Remark 3.1 The group $\mathrm{SU}(2)$ is isomorphic to the group S^3 of unit quaternions and therefore lives on the 3-sphere. Hence the groups discussed in this paper are topologically either $\mathbb{R}^3$ or S^3.

3.2 Left invariant metrics on three-dimensional Lie groups

Let g be a Riemannian metric on a connected Lie group G, and let $\theta : G \to G$ be a diffeomorphism on G. Then θ induces a metric on G by $g_\theta(X,Y) = g\big(\theta_*^{-1}X, \theta_*^{-1}Y\big)$. If g is a left invariant metric on G, the induced metric g_θ is not necessarily left invariant.

Now to describe all the left invariant metrics on an n-dimensional connected Lie group, we fix a basis $\{X_1, X_2, \ldots, X_n\}$ for the Lie algebra of G and let $\{\omega_1, \omega_2, \ldots, \omega_n\}$ be its dual basis. Then every left invariant metric on G is of the form

$$g = \sum g_{ij}\, \omega_i \otimes \omega_j .$$

This yields a symmetric, positive definite matrix $[g] = [g_{ij}]$ and satisfies

$$g(X,Y) = [X]^t[g][Y]$$

where $[X]$ is the column vector $[a_1\, a_2 \ldots a_n]^t$ if and only if $X = a_1X_1 + a_2X_2 + \cdots + a_nX_n$.

Let $\varphi \in \mathrm{Aut}(\mathfrak{g})$ and let g be a left invariant metric on G. Define g_φ by $g_\varphi(X,Y) = g\big(\varphi^{-1}X, \varphi^{-1}Y\big)$ for all $X, Y \in \mathfrak{g}$. Then g_φ is a left invariant metric on G, because

$$g_\varphi(X,Y) = g\big(\varphi^{-1}X, \varphi^{-1}Y\big) = g\big(\varphi^{-1}\big(\ell_p^{-1}\big)_*X, \varphi^{-1}\big(\ell_p^{-1}\big)_*Y\big) = \big(\ell_p\big)^* g_\varphi(X,Y).$$

Further $[g_\varphi] = \big[\varphi^{-1}\big]^t[g]\big[\varphi^{-1}\big]$. Therefore $\mathrm{Aut}(\mathfrak{g})$ acts on the set of all left invariant metrics on G by the rule $(\varphi, g) \mapsto g_\varphi$ or equivalently $([\varphi],[g]) \mapsto [g_\varphi]$.

A left invariant metric g' on G is *equivalent up to automorphism* to a left invariant metric g, written $g' \sim g$, if there exists $\varphi \in \mathrm{Aut}(\mathfrak{g})$ such that $[g'] = [\varphi]^t[g][\varphi]$. In this case we often say that $[g']$ is equivalent up to automorphism to $[g]$ or $[g']$ is *cogredient* to $[g]$ by $[\varphi]$.

Let $[g] = [g_{ij}]$ be a positive definite matrix. Then for any $(x_1,x_2,x_3) \neq (0,0,0)$, $g_{11}x_1^2 + g_{22}x_2^2 + g_{33}x_3^2 + 2(g_{12}x_1x_2 + g_{13}x_1x_3 + g_{23}x_2x_3) > 0$. This implies that

$$g_{11}, g_{22}, g_{33} > 0, \quad g_{ii}g_{jj} > g_{ij}^2 .$$

In particular, $g_{ii} + g_{jj} > 2|g_{ij}|$.

In this section we will classify all the left invariant metrics up to automorphism on the three-dimensional simply connected Lie groups. For this purpose, we will use the bases $\{X, Y, Z\}$ given in Section 2.1 for the three-dimensional Lie algebras. Let $\{\omega_1, \omega_2, \omega_3\}$ be its dual basis. Then every left invariant metric g is of the form $g = \sum g_{ij}\omega_i \otimes \omega_j$. If $\{X_1', X_2', X_3'\}$ is another basis with its dual basis $\{\omega_1', \omega_2', \omega_3'\}$, then the left invariant metric g will be of the form $g = \sum g_{ij}'\, \omega_i' \otimes \omega_j'$ so that $[g_{ij}'] = [p_{ij}]^t[g_{ij}][p_{ij}]$ where $X_i' = p_{i1}X_1 + p_{i2}X_2 + p_{i3}X_3$. We must note that it is not necessarily true that $[p_{ij}] \in \mathrm{Aut}(\mathfrak{g})$.

3.2.1 The abelian Lie group $\mathbb{R}^3$

Recalling that $\mathrm{Aut}(\mathbb{R}^3) \cong \mathrm{GL}(3,\mathbb{R})$ we obtain

Theorem 3.2 *Any left invariant metric on $\mathbb{R}^3$ is equivalent up to automorphism to the metric whose associated matrix is the identity matrix.*

Proof. Let $[g] = [g_{ij}]$ be a symmetric and positive definite 3×3 real matrix. Since $[g]$ is symmetric, there is $T \in \mathrm{O}(3)$ such that $T^t[g]T = D$, a diagonal matrix. Since $[g]$ is positive definite, the diagonal entries are all positive and hence there is $S \in \mathrm{GL}(3, \mathbb{R})$ such that $S^t[g]S = I$. $\square$

3.2.2 The Heisenberg group Nil

Recalling Proposition 2.2, we obtain

Theorem 3.3 *Any left invariant metric on* Nil *is equivalent up to automorphism to a metric whose associated matrix is of the form*

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $\lambda > 0$.

Proof. Let $[g] = [g_{ij}]$ be a symmetric and positive definite real 3×3 matrix. Let $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{g_{13}}{g_{33}} & -\frac{g_{23}}{g_{33}} & 1 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{n})$. Then $B^t[g]B = \begin{bmatrix} [g_1] & \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \\ 0\ 0 & 1 \end{bmatrix}$ where $[g_1]$ is a symmetric and positive definite matrix. Since $[g_1]$ is symmetric and positive definite and $\begin{bmatrix} \mathrm{SO}(2) & \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \\ 0\ 0 & 1 \end{bmatrix} \subset \mathrm{Aut}(\mathfrak{n})$, we may assume that $B^t[g]B = \mathrm{diag}\{d_1, d_2, 1\}$ where $d_1, d_2 > 0$. Let $D = \begin{bmatrix} \sqrt[4]{\frac{d_2}{d_1}} & 0 & 0 \\ 0 & \sqrt[4]{\frac{d_1}{d_2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{n})$. Then $D^tB^t[g]BD = \mathrm{diag}\big\{\sqrt{d_1d_2}, \sqrt{d_1d_2}, 1\big\}$.

Therefore $[g]$ is equivalent up to automorphism to a diagonal matrix $\mathrm{diag}\{\lambda, \lambda, 1\}$ with $\lambda > 0$. Finally it is easy to see that any two such distinct diagonal matrices are not equivalent. $\square$

3.2.3 The solvable Lie group Sol

Recalling Proposition 2.3, we obtain

Theorem 3.4 *Any left invariant metric on* Sol *is equivalent up to automorphism to a metric whose associated matrix is of the form*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$$

where $\mu > 1$ *and* $\nu > 0$.

Proof. Let $[g] = [g_{ij}]$ be a symmetric and positive definite 3×3 real matrix. Suppose $g_{12} = 0$ and let $B = \begin{bmatrix} \frac{1}{\sqrt{g_{11}}} & 0 & -\frac{g_{13}}{g_{11}} \\ 0 & \frac{1}{\sqrt{g_{22}}} & -\frac{g_{23}}{g_{22}} \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aut}\big(\mathbb{R}^2 \rtimes_\sigma \mathbb{R}\big)$. Then $B^t[g]B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix}$. Next suppose $g_{12} \neq 0$. Since $g_{11}g_{22} > g_{12}^2$, $B = \begin{bmatrix} \frac{1}{\sqrt{g_{11}}} & 0 & \frac{g_{13}g_{22} - g_{12}g_{23}}{g_{12}^2 - g_{11}g_{22}} \\ 0 & \frac{\sqrt{g_{11}}}{g_{12}} & \frac{g_{11}g_{23} - g_{12}g_{13}}{g_{12}^2 - g_{11}g_{22}} \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aut}\big(\mathbb{R}^2 \rtimes_\sigma \mathbb{R}\big)$ and then $B^t[g]B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$ with $\mu > 1$. Finally, inspection shows that any two matrices of the form $\begin{bmatrix} 1 & 1 & 0 \\ 1 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix}$ are equivalent if and only if they are equal. $\square$

3.2.4 The solvable Lie group $\widetilde{E}_0(2)$

Recalling Proposition 2.4, we have

Theorem 3.5 *Any left invariant metric on $\widetilde{E}_0(2)$ is equivalent up to automorphism to a metric whose associated matrix is of the form*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$$

where $0 < \mu \leq 1$ and $\nu > 0$.

Proof. Let $[g] = [g_{ij}]$ be a symmetric and positive definite 3×3 real matrix. Since $\begin{bmatrix} \mathrm{SO}(2) & 0 \\ 0 & 1 \end{bmatrix} \subset \mathrm{Aut}\big(\mathbb{R}^2 \rtimes \mathfrak{so}(2)\big)$, we may assume that $g_{12} = 0$. Let $B = \begin{bmatrix} \frac{1}{\sqrt{g_{11}}} & 0 & -\frac{g_{13}}{g_{11}} \\ 0 & \frac{1}{\sqrt{g_{11}}} & -\frac{g_{23}}{g_{22}} \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aut}\big(\mathbb{R}^2 \rtimes \mathfrak{so}(2)\big)$. Then $B^t[g]B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$ where $\mu, \nu > 0$. If $\mu > 1$, then $C = \begin{bmatrix} 0 & \frac{1}{\sqrt{\mu}} & 0 \\ -\frac{1}{\sqrt{\mu}} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aut}\big(\mathbb{R}^2 \rtimes \mathfrak{so}(2)\big)$ and $C^t B^t [g] B C^t = \mathrm{diag}\{1, \frac{1}{\mu}, \nu\}$. Thus we may assume that $\mu \leq 1$. Therefore $[g]$ is equivalent up to automorphism to a matrix $\mathrm{diag}\{1, \mu, \nu\}$ where $0 < \mu \leq 1$ and $\nu > 0$.

Inspection shows that any two matrices of the form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$ where $0 < \mu \leq 1$ and $\nu > 0$ are equivalent if and only if they are equal. □

3.2.5 The simple Lie group $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$

Since $\mathrm{SL}(2,\mathbb{R})$ is a simple Lie group with center $\{\pm I\}$, by [2, Corollaries II.5.2 and II.6.5], $\mathrm{SL}(2,\mathbb{R})/\{\pm I\} \cong \mathrm{Int}(\mathfrak{sl}(2,\mathbb{R})) = \mathrm{Aut}_0(\mathfrak{sl}(2,\mathbb{R})) = \mathrm{SO}_0(1,2)$. Here $\mathrm{Int}(\mathfrak{sl}(2,\mathbb{R}))$ is the *adjoint group* of $\mathfrak{sl}(2,\mathbb{R})$. In fact, the map $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{SO}_0(1,2)$ is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} \frac{1}{2}\big(a^2+b^2+c^2+d^2\big) & \frac{1}{2}\big(a^2-b^2+c^2-d^2\big) & -ab-cd \\ \frac{1}{2}\big(a^2+b^2-c^2-d^2\big) & \frac{1}{2}\big(a^2-b^2-c^2+d^2\big) & -ab+cd \\ -ac-bd & -ac+bd & ad+bc \end{bmatrix}.$$

The kernel of this homomorphism is $\{\pm I\}$ and its image is $\mathrm{SO}_0(1,2)$. It thus implements the isomorphism $\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{SO}_0(1,2)$ mentioned in Remark 2.6. Note that the group $\mathrm{SO}(1,2)$ has two components

$$\mathrm{SO}_0(1,2) = \left\{ \begin{bmatrix} \frac{1}{2}\big(a^2+b^2+c^2+d^2\big) & \frac{1}{2}\big(a^2-b^2+c^2-d^2\big) & -ab-cd \\ \frac{1}{2}\big(a^2+b^2-c^2-d^2\big) & \frac{1}{2}\big(a^2-b^2-c^2+d^2\big) & -ab+cd \\ -ac-bd & -ac+bd & ad+bc \end{bmatrix} \right\},$$

$$\mathrm{SO}_0(1,2) \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $ad - bc = 1$. Thus in particular each of these matrices can be connected by a (continuous) arc to that one of the two matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

in its component (but there is no arc connecting these two matrices). It is known that the group $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ has no faithful finite dimensional linear representation. It is the only simply connected three-dimensional real Lie group with this property. (Nil has a not simply connected quotient with this property). A description of parametrization of $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ and its $\mathbb{R}^3$ geometry is given for instance in the book [3], notably pp. 414–434, and in [4], notably p. 18.

Recalling from Proposition 2.5 that $\mathrm{Aut}(\mathfrak{sl}(2,\mathbb{R})) \cong \mathrm{SO}(1,2)$ we obtain

Theorem 3.6 *Any left invariant metric on $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ is equivalent up to automorphism to a metric whose associated matrix is of the form*

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$$

where $\lambda > 0$, $\mu \geq \nu > 0$.

P r o o f. Let $[g] = [g_{ij}]$ be a symmetric and positive definite 3×3 real matrix. Since $\begin{bmatrix} 1 & 0 \\ 0 & \mathrm{SO}(2) \end{bmatrix} \subset \mathrm{SO}(1,2)$, we may assume that $g_{23} = 0$. Suppose that $g_{22} = g_{33}$ or $g_{13} = 0$. Let t_1 be a solution of the equation $g_{13}\cos t + g_{12}\sin t = 0$. Then $\sin t_1 = \mp\frac{g_{13}}{\sqrt{g_{12}^2+g_{13}^2}}$ and $\cos t_1 = \pm\frac{g_{12}}{\sqrt{g_{12}^2+g_{13}^2}}$ and hence $g_{12}\cos t - g_{13}\sin t = \pm\sqrt{g_{12}^2+g_{13}^2}$. Taking $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t_1 & \sin t_1 \\ 0 & -\sin t_1 & \cos t_1 \end{bmatrix} \in \mathrm{SO}(1,2)$, we get $B^t[g]B = [g']$ with $g'_{12} = \pm\sqrt{g_{12}^2+g_{13}^2}$ and $g'_{13} = g'_{23} = 0$. Hence we may assume that the matrix $[g]$ has $g_{23} = 0$ and $g_{13} = 0$ (if $g_{22} = g_{33}$). Next we take $C = \begin{bmatrix} \cosh t_2 & \sinh t_2 & 0 \\ \sinh t_2 & \cosh t_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{SO}(1,2)$ where t_2 is a solution of the equation $(g_{11}+g_{22})\sinh(2t) + 2g_{12}\cosh(2t) = 0$. Then $C^t[g]C = \mathrm{diag}\{\lambda,\mu,\nu\}$ for some $\lambda,\mu,\nu > 0$. Clearly we may assume further that $\mu \geq \nu$. If $g_{12} = 0$ then the matrix $[g]$ is cogredient to a matrix $[g']$ with $g'_{13} = g'_{23} = 0$ and then, by the above argument, $[g']$ and hence $[g]$ is cogredient to the matrix of the form $\mathrm{diag}\{\lambda,\mu,\nu\}$ with $\lambda > 0, \mu \geq \nu > 0$.

Suppose that $g_{22} \neq g_{33}$, $g_{12} \neq 0$ and $g_{13} \neq 0$. It suffices to show that $[g]$ is cogredient to $[g']$ where $g'_{23} = 0$ and $g'_{13} = 0$. This follows by taking

$$\begin{aligned}
a &= 1,\\
b &= \Big\{11g_{11}^2g_{12}^6 - 44g_{11}g_{12}^7 + 44g_{12}^8 + 22g_{11}^2g_{12}^4g_{13}^2 - 110g_{11}g_{12}^5g_{13}^2 + 132g_{12}^6g_{13}^2\\
&\quad + 11g_{11}^2g_{12}^2g_{13}^4 - 88g_{11}g_{12}^3g_{13}^4 + 143g_{12}^4g_{13}^4 - 22g_{11}g_{12}g_{13}^6 + 66g_{12}^2g_{13}^6\\
&\quad + 11g_{13}^8 - 8g_{11}^2g_{12}^4(g_{12}^2+g_{13}^2) + 32g_{11}g_{12}^5(g_{12}^2+g_{13}^2) - 32g_{12}^6(g_{12}^2+g_{13}^2)\\
&\quad - 8g_{11}^2g_{12}^2g_{13}^2(g_{12}^2+g_{13}^2) + 48g_{11}g_{12}^3g_{13}^2(g_{12}^2+g_{13}^2) - 64g_{12}^4g_{13}^2(g_{12}^2+g_{13}^2)\\
&\quad + 2g_{11}^2g_{13}^4(g_{12}^2+g_{13}^2) + 16g_{11}g_{12}g_{13}^4(g_{12}^2+g_{13}^2) - 48g_{12}^2g_{13}^4(g_{12}^2+g_{13}^2)\\
&\quad - 16g_{13}^6(g_{12}^2+g_{13}^2) - 3g_{11}^2g_{12}^2(g_{12}^2+g_{13}^2)^2 + 12g_{11}g_{12}^3(g_{12}^2+g_{13}^2)^2\\
&\quad - 12g_{12}^4(g_{12}^2+g_{13}^2)^2 + 6g_{11}g_{12}g_{13}^2(g_{12}^2+g_{13}^2)^2 - 12g_{12}^2g_{13}^2(g_{12}^2+g_{13}^2)^2\\
&\quad - 3g_{13}^4(g_{12}^2+g_{13}^2)^2 + 22g_{11}g_{12}^6g_{22} - 44g_{12}^7g_{22} + 44g_{11}g_{12}^4g_{13}^2g_{22}\\
&\quad - 110g_{12}^5g_{13}^2g_{22} + 22g_{11}g_{12}^2g_{13}^4g_{22} - 88g_{12}^3g_{13}^4g_{22} - 22g_{12}g_{13}^6g_{22}\\
&\quad - 16g_{11}g_{12}^4(g_{12}^2+g_{13}^2)g_{22} + 32g_{12}^5(g_{12}^2+g_{13}^2)g_{22} - 16g_{11}g_{12}^2g_{13}^2(g_{12}^2+g_{13}^2)g_{22}\\
&\quad + 48g_{12}^3g_{13}^2(g_{12}^2+g_{13}^2)g_{22} + 4g_{11}g_{13}^4(g_{12}^2+g_{13}^2)g_{22} + 16g_{12}g_{13}^4(g_{12}^2+g_{13}^2)g_{22}\\
&\quad - 6g_{11}g_{12}^2(g_{12}^2+g_{13}^2)^2g_{22} + 12g_{12}^3(g_{12}^2+g_{13}^2)^2g_{22} + 6g_{12}g_{13}^2(g_{12}^2+g_{13}^2)^2g_{22}\\
&\quad + 11g_{12}^6g_{22}^2 + 22g_{12}^4g_{13}^2g_{22}^2 + 11g_{12}^2g_{13}^4g_{22}^2 - 8g_{12}^4(g_{12}^2+g_{13}^2)g_{22}^2\\
&\quad - 8g_{12}^2g_{13}^2(g_{12}^2+g_{13}^2)g_{22}^2 + 2g_{13}^4(g_{12}^2+g_{13}^2)g_{22}^2 - 3g_{12}^2(g_{12}^2+g_{13}^2)^2g_{22}^2\Big\}\\
&\quad \Big/ \Big\{\big(4g_{13}^3(g_{12}^2+g_{13}^2)^{3/2}(g_{11}^2 - 4g_{12}^2 - 4g_{13}^2 + 2g_{11}g_{22} + g_{22}^2)\big)\Big\},
\end{aligned}$$

$$c = \frac{g_{12} - \sqrt{g_{12}^2 + g_{13}^2}}{g_{13}},$$
$$d = \text{arbitrary.}$$

(The calculations here were done using the program Mathematica [8].) Here we observe the following. Since $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \in \mathrm{SO}_0(1,2)$, we may assume that $g_{12}g_{13} > 0$. Since $[g] = [g_{ij}]$ is positive definite, $g_{11} + g_{22} > 2|g_{12} + g_{13}|$ and so $(g_{11} + g_{22})^2 > 4\big(g_{12} + g_{13}\big)^2 = 4\big(g_{12}^2 + g_{13}^2\big) + 8g_{12}g_{13} > 4\big(g_{12}^2 + g_{13}^2\big)$. This shows that the denominator of our choice b is non-zero. Hence all such real numbers a, b, c, d exist.

Inspection shows that any two matrices of the form $\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$ where $\lambda > 0, \mu \geq \nu > 0$ are equivalent if and only if they are equal. □

3.2.6 The simple Lie group $\mathrm{SU}(2)$

Recalling from Proposition 2.7 that $\mathrm{Aut}(\mathfrak{su}(2)) = \mathrm{Aut}(\mathfrak{so}(3)) \cong \mathrm{SO}(3)$ we obtain

Theorem 3.7 *Any left invariant metric on* $\mathrm{SU}(2)$ *is equivalent up to automorphism to a metric whose associated matrix is of the form*

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$$

where $\lambda \geq \mu \geq \nu > 0.$

Proof. Let $[g] = [g_{ij}]$ be a symmetric and positive definite 3×3 real matrix. Since $[g]$ is symmetric, it is orthogonally diagonalizable; that is, there exists an orthogonal matrix P such that $P^{-1}[g]P = P^t[g]P$ is a diagonal matrix. If $P \notin \mathrm{O}(3)$, we can write $P = Q\sigma$ with $Q \in \mathrm{SO}(3)$ and $\sigma = \mathrm{diag}\{-1, 1, 1\}$. Then $Q^t[g]Q = P^t[g]P$ and $[g]$ is equivalent to a diagonal matrix with positive entries as $[g]$ is positive definite.

On the other hand, since $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \in \mathrm{SO}(3)$, we can switch the diagonal entries of a diagonal matrix. Therefore $[g]$ is equivalent up to automorphism to a diagonal matrix $\mathrm{diag}\{\lambda, \mu, \nu\}$ with $\lambda \geq \mu \geq \nu > 0$. Inspection shows that any two matrices of the form $\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$ where $\lambda \geq \mu \geq \nu > 0$ are equivalent if and only if they are equal. □

3.2.7 The non-unimodular Lie groups

Recalling Proposition 2.8, we obtain the following:

Theorem 3.8 *Any left invariant metric on* G_I *is equivalent up to automorphism to a metric whose associated matrix is of the form*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix}$$

where $\nu > 0.$

Proof. Let $[g] = [g_{ij}]$ be a symmetric and positive definite real 3×3 matrix. Since $g_{11}g_{22} > g_{12}^2$, $B = \begin{bmatrix} 1 & 0 & \frac{g_{13}g_{22} - g_{12}g_{23}}{g_{12}^2 - g_{11}g_{22}} \\ 0 & 1 & \frac{g_{11}g_{23} - g_{12}g_{13}}{g_{12}^2 - g_{11}g_{22}} \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{g}_I)$ and then $B^t[g]B = \begin{bmatrix} [g_1] & & 0 \\ & & 0 \\ 0 & 0 & \lambda \end{bmatrix}$ where $[g_1]$ is a symmetric, positive definite matrix and $\lambda > 0$. Since $[g_1]$ is symmetric and positive definite and $\begin{bmatrix} \mathrm{SO}(2) & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{bmatrix} \subset \mathrm{Aut}(\mathfrak{g})$, we may assume that there is $B \in \mathrm{Aut}(\mathfrak{g}_I)$ such that $B^t[g_1]B = \mathrm{diag}\{d_1, d_2, 1\}$ where $d_1, d_2 > 0$. Let $D = \begin{bmatrix} \frac{1}{\sqrt{d_1}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{d_2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{g}_I)$. Then $D^tB^t[g]BD = \mathrm{diag}\{1, 1, \nu\}$. Therefore $[g]$ is equivalent up to automorphism to a diagonal matrix $\mathrm{diag}\{1, 1, \nu\}$ with $\nu > 0$. Finally it is easy to see that any two such distinct diagonal matrices are not equivalent. □

Lemma 3.9 *Let $[g] = [g_{ij}]$ be a left invariant metric on G_c. Then*

(1) *$[g]$ is cogredient to a matrix $[g']$ with $g'_{13} = g'_{23} = 0$.*

(2) *If $g_{13} = g_{23} = 0$ and if $g_{12} \leq 0$ or $g_{12} \geq 2g_{11}$, then $[g]$ is cogredient to a matrix of the form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$ where $\nu > 0$, and if $c = 0$ then $\mu > 0$; if $c \neq 0$ then $0 < \mu \leq |c|$.*

Proof. (1) Since $g_{11}g_{22} > g_{12}^2$, we have $B = \begin{bmatrix} 1 & 0 & \frac{g_{13}g_{22} - g_{12}g_{23}}{g_{12}^2 - g_{11}g_{22}} \\ 0 & 1 & \frac{g_{11}g_{23} - g_{12}g_{13}}{g_{12}^2 - g_{11}g_{22}} \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{g}_c)$ and $B^t[g]B = [g']$ where $g'_{13} = g'_{23} = 0$.

(2) Suppose that $g_{12}(g_{12} - 2g_{11}) \geq 0$. Let $B = \begin{bmatrix} \beta - \alpha & -c\alpha & 0 \\ \alpha & \beta + \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ where

$$\alpha = 2g_{12},$$
$$\beta = (g_{11}c - g_{22}) + \sqrt{g_{11}^2 c^2 - 2(2g_{11}g_{12} - 2g_{12}^2 + g_{11}g_{22})c + (2g_{12} - g_{22})^2}.$$

Since $g_{11}g_{22} - g_{12}^2$, we have $4g_{12}(2g_{11} - g_{12})\big(g_{11}g_{22} - g_{12}^2\big) \leq 0$, which implies that the value $g_{11}^2c^2 - 2\big(2g_{11}g_{12} - 2g_{12}^2 + g_{11}g_{22}\big)c + \big(2g_{12} - g_{22}\big)^2$ is nonnegative for any c. Thus β is well-defined real number. Moreover, we can show that $\beta^2 + (c-1)\alpha^2 \neq 0$ and so $B \in \mathrm{Aut}(\mathfrak{g})$ and $B^t[g]B = \mathrm{diag}\{d_1, d_2, d_3\}$ where $d_i > 0$. Let $C = \begin{bmatrix} \frac{1}{\sqrt{d_1}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{d_1}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{g}_c)$. Then $C^t \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & \nu \end{bmatrix} C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$. Let $C = \begin{bmatrix} \beta - \alpha & -c\alpha & 0 \\ \alpha & \beta + \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ where $\alpha = \frac{c - \mu}{\sqrt{\mu((c-\mu)^2 + 4\mu)}}$ and $\beta = \frac{c + \mu}{\sqrt{\mu((c-\mu)^2 + 4\mu)}}$. Since $c \neq 0$, $C \in \mathrm{Aut}(\mathfrak{g}_c)$ and $C^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix} C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{c^2}{\mu} & 0 \\ 0 & 0 & \nu \end{bmatrix}$. Thus we have the result. □

Theorem 3.10 (Case: $c < 0$) *Any left invariant metric on G_c, where $c < 0$, is equivalent up to automorphism to a metric whose associated matrix is of the form*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$$

where $0 < \mu \leq |c|$ and $\nu > 0$.

Proof. Let $[g] = [g_{ij}]$ be a symmetric and positive definite real 3×3 matrix. By Lemma 3.9, we may assume that $g_{13} = g_{23} = 0$ and $0 < g_{12} < 2g_{11}$. Then $B = \begin{bmatrix} 2 & c & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{g}_c)$ and $B^t[g]B = [g'_{ij}]$ where $g'_{13} = g'_{23} = 0$ and $g'_{12} = (2g_{11} - g_{12})c$. Since $c < 0$ and $g_{12} < 2g_{11}$, we have $g'_{12} < 0$. Thus by Lemma 3.9 (2), $[g]$ is equivalent up to automorphism to a diagonal matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$ where $1 < \mu \leq |c|$ and $\nu > 0$. Finally it is easy to see that any two such distinct matrices are not equivalent. □

Theorem 3.11 (Case: $c = 0$) *Any left invariant metric on G_0 is equivalent up to automorphism to a metric whose associated matrix is of the form*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix} \quad or \quad \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix}$$

where $\mu, \nu > 0$.

Proof. Let $[g] = [g_{ij}]$ be a symmetric and positive definite real 3×3 matrix. By Lemma 3.9 (1), we may assume that $g_{13} = g_{23} = 0$. If $g_{12} \neq 0$ and $g_{22} \neq 2g_{12}$, then $B = \begin{bmatrix} -\frac{g_{22}}{g_{12}} & 0 & 0 \\ 1 & 2 - \frac{g_{22}}{g_{12}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{g}_0)$ and $B^t[g]B = \mathrm{diag}\{d_1, d_2, d_3\}$. Thus by Lemma 3.9 (2), $[g]$ is equivalent up to automorphism to a diagonal matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$ where $\mu, \nu > 0$. Suppose that $g_{22} = 2g_{12}$. Let $\alpha = 0$ and β be a solution of the equation $(g_{11} - 2g_{12})b^2 - 2(g_{11} + g_{12})b + (g_{11} - 2g_{12}) = 0$. Then $B = \begin{bmatrix} \beta - \alpha & 0 & 0 \\ \alpha & \beta + \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{g}_0)$ and $B^t[g]B = \begin{bmatrix} 2\mu & \mu & 0 \\ \mu & 2\mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$ for some $\mu > 0$. Let $C = \begin{bmatrix} \frac{1}{\sqrt{2\mu}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2\mu}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then $C \in \mathrm{Aut}(\mathfrak{g}_0)$ and $C^t \begin{bmatrix} 2\mu & \mu & 0 \\ \mu & 2\mu & 0 \\ 0 & 0 & \nu \end{bmatrix} C = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix}$. Finally it is easy to see that any two such distinct matrices are not equivalent. □

Theorem 3.12 (Case: $c = 1$) *Any left invariant metric on G_1 is equivalent up to automorphism to a metric whose associated matrix is of the form*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix} \quad or \quad \begin{bmatrix} 1 & \lambda & 0 \\ \lambda & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix}$$

where $0 < \lambda < 1, 0 < \mu \leq 1$ and $\nu > 0$.

Proof. Let $[g] = [g_{ij}]$ be a symmetric and positive definite real 3×3 matrix. By Lemma 3.9 (1), we may assume that $g_{13} = g_{23} = 0$. Let

$$m = \big((g_{11} + g_{22} - 2g_{12})\big((g_{11} + g_{22} - 2g_{12})^2 + 4g_{11}g_{22}\big)\big)^{-\frac{1}{2}},$$
$$\alpha = m(g_{11} - g_{22}), \quad \beta = 2m(g_{11} + g_{22} - 2g_{12}).$$

Then $B = \begin{bmatrix} \beta - \alpha & -\alpha & 0 \\ \alpha & \beta + \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{g}_1)$ and $B^t[g]B = \begin{bmatrix} 1 & \lambda & 0 \\ \lambda & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix}$ where $|\lambda| < 1$ and $\nu > 0$. If $-1 < \lambda \leq 0$, by Lemma 3.9 (2), $\begin{bmatrix} 1 & \lambda & 0 \\ \lambda & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix}$ is equivalent up to automorphisms to a matrix of the form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$ where $0 < \mu \leq 1$ and $\nu > 0$. Finally it is easy to see that any two such distinct matrices are not equivalent. □

Theorem 3.13 (Case: $c > 1$) *Any left invariant metric on G_c with $c > 1$ is equivalent up to automorphism to a metric whose associated matrix is of the form*

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$$

where $1 < \mu \leq c$ and $\nu > 0$.

P r o o f. Let $[g] = [g_{ij}]$ be a symmetric and positive definite real 3×3 matrix. By Lemma 3.9, we may assume that $g_{13} = g_{23} = 0$ and $0 < g_{12} < 2g_{11}$. Suppose $g_{11} \neq g_{12}$. Note that for any $c > 1$,

$$D(c) := g_{11}^2 c^2 - 2g_{11}g_{22}c - 4g_{11}g_{12}c + 4g_{12}^2 c + 4g_{11}g_{22} - 4g_{12}g_{22} + g_{22}^2 \geq 0.$$

Let $B_1 = \begin{bmatrix} \beta - 1 & -c & 0 \\ 1 & \beta + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ with $\beta = \frac{2g_{11} + g_{22} - 2g_{12} - cg_{11} + \sqrt{D(c)}}{2(g_{11} - g_{12})}$. Then $B_1 \in \mathrm{Aut}(\mathfrak{g}_c)$, $B_1^t[g]B_1 = \begin{bmatrix} \lambda & \lambda & 0 \\ \lambda & k & 0 \\ 0 & 0 & \nu \end{bmatrix}$ for some $\lambda, k > 0$. Let $B_2 = \begin{bmatrix} \frac{1}{\sqrt{\lambda}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{g}_c)$. Thus $B_2^t B_1^t [g] B_1 B_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$ where $\mu > 1$ and $\nu > 0$. Observe also that for any $[g] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$, we have $C = \begin{bmatrix} \frac{1}{\sqrt{\mu - 1}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{\mu - 1}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{g}_c)$ and $C^t[g]C = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 + \frac{(c-1)^2}{\mu - 1} & 0 \\ 0 & 0 & \nu \end{bmatrix}$. Therefore $[g]$ is equivalent up to automorphism to a diagonal matrix $\begin{bmatrix} 1 & 1 & 0 \\ 1 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$ where $1 < \mu \leq c$. Finally it is easy to see that any two such distinct matrices are not equivalent. □

Remark 3.14 Consider the Lie algebra $\mathfrak{g}_c$ where $0 < c < 1$. Recall that $\mathfrak{g}_c$ has a basis $\{X, Y, Z\}$ so that

$$[X, Y] = 0, \quad [Z, X] = Y, \quad [Z, Y] = -cX + 2Y.$$

Putting $X_1 = -cX + (1 - z)Y,\ X_2 = -cX + (1 + z)Y,\ X_3 = Z$ where $z = \sqrt{1 - c}$, we obtain a new basis $\{X_1, X_2, X_3\}$ for $\mathfrak{g}_c$ satisfying

$$[X_1, X_2] = 0, \quad [X_3, X_1] = (1 - z)X_1, \quad [X_3, X_2] = (1 + z)X_2.$$

With respect to this new basis, the Lie group $\mathrm{Aut}(\mathfrak{g}_c)$ is isomorphic to

$$\left\{ \begin{bmatrix} \gamma & 0 & * \\ 0 & \delta & * \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; \gamma, \delta, * \in \mathbb{R},\ \gamma\delta \neq 0 \right\}.$$

In fact, given γ, δ,

$$\begin{bmatrix} \dfrac{1+z}{-2cz} & \dfrac{1}{-2z} & 0 \\ \dfrac{1-z}{2cz} & \dfrac{1}{2z} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \gamma & 0 & * \\ 0 & \delta & * \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{1+z}{-2cz} & \dfrac{1}{-2z} & 0 \\ \dfrac{1-z}{2cz} & \frac{1}{2z} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \beta - \alpha & -c\alpha & * \\ \alpha & \beta + \alpha & * \\ 0 & 0 & 1 \end{bmatrix}$$

where $z = \sqrt{1 - c}$, $\alpha = \frac{\delta - \gamma}{2z}$ and $\beta = \frac{\delta + \gamma}{2}$.

Theorem 3.15 (Case: $0 < c < 1$) *Any left invariant metric on G_c with $0 < c < 1$ is equivalent up to automorphism to a metric whose associated matrix is of the form*

$$\begin{bmatrix} \dfrac{1+z}{-2cz} & \dfrac{1}{-2z} & 0 \\ \dfrac{1-z}{2cz} & \dfrac{1}{2z} & 0 \\ 0 & 0 & 1 \end{bmatrix}^t \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix} \begin{bmatrix} \dfrac{1+z}{-2cz} & \dfrac{1}{-2z} & 0 \\ \dfrac{1-z}{2cz} & \dfrac{1}{2z} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $z = \sqrt{1 - c}$, $0 \leq \mu < 1$ and $\nu > 0$.

Proof. Let $\{X_1, X_2, X_3\}$ be the basis for $\mathfrak{g}_c$ given in Remark 3.14, i.e.,

$$X_1 = -cX + (1-z)Y, \quad X_2 = -cX + (1+z)Y, \quad X_3 = Z$$

where $z = \sqrt{1-c}$. Then

$$[X_1, X_2] = 0, \quad [X_3, X_1] = (1-z)X_1, \quad [X_3, X_2] = (1+z)X_2$$

and the Lie group $\mathrm{Aut}(\mathfrak{g}_c)$ is isomorphic to $\left\{ \begin{bmatrix} \gamma & 0 & * \\ 0 & \delta & * \\ 0 & 0 & 1 \end{bmatrix} \,\middle|\, \gamma, \delta, * \in \mathbb{R},\ \gamma\delta \neq 0 \right\}$.

Let g be a left invariant metric on $\mathfrak{g}_c$. Then with respect to the basis $\{X_1, X_2, X_3\}$, $[g] = [g_{ij}]$ is a symmetric and positive definite real 3×3 matrix. Since $B = \begin{bmatrix} \frac{1}{\sqrt{g_{11}}} & 0 & \frac{g_{13}g_{22} - g_{12}g_{23}}{g_{12}^2 - g_{11}g_{22}} \\ 0 & \frac{\pm 1}{\sqrt{g_{22}}} & \frac{g_{11}g_{23} - g_{12}g_{13}}{g_{12}^2 - g_{11}g_{22}} \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{g}_c)$ and $B^t[g]B = \begin{bmatrix} 1 & \pm\mu & 0 \\ \pm\mu & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix}$, $[g]$ is equivalent up to automorphism to a matrix of the form $\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix}$ where $0 \leq \mu < 1$ and $\nu > 0$. Moreover it is easy to see that any two such distinct matrices are not equivalent. Since

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \dfrac{1+z}{-2cz} & \dfrac{1}{-2z} & 0 \\ \dfrac{1-z}{2cz} & \dfrac{1}{2z} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix},$$

Remark 3.14 yields the result. □

4 Curvatures on three-dimensional unimodular Lie Groups

In this and the next sections, we study the extent to which curvature can be altered by a change of left invariant metric. Let g be a Riemannian metric on a connected Lie group G. Suppose that $\varphi \in \mathrm{Aut}(\mathfrak{g})$ with $[g'] = [\varphi]^t[g][\varphi]$. Let ∇ and ∇' be the Levi–Civita connections determined by the left invariant metrics g and g', respectively, on G. Then by [7, (5.3)]

$$\begin{aligned} g\big(\varphi\nabla'_X Y, \varphi Z\big) &= g'(\nabla'_X Y, Z) \\ &= \frac{1}{2}\big(g'([X,Y],Z) - g'([Y,Z],X) + g'([Z,X],Y)\big) \\ &= \frac{1}{2}\big(g([\varphi X, \varphi Y], \varphi Z) - g([\varphi Y, \varphi Z], \varphi X) + g([\varphi Z, \varphi X], \varphi Y)\big) \\ &= g(\nabla_{\varphi X}\varphi Y, \varphi Z), \end{aligned}$$

for all $X, Y, Z \in \mathfrak{g}$. This reduces to $\varphi\nabla'_X Y = \nabla_{\varphi X}\varphi Y$. In other words, we have the following commuting diagram:

$$\begin{tikzcd} \mathfrak{g} \times \mathfrak{g} \arrow[r, "\nabla'"] \arrow[d, "\varphi\times\varphi"'] & \mathfrak{g} \arrow[d, "\varphi"] \\ \mathfrak{g} \times \mathfrak{g} \arrow[r, "\nabla"] & \mathfrak{g} \end{tikzcd}$$

Therefore, the classification of the left invariant metrics up to automorphism is equivalent to the study of the left invariant metrics which leave all the curvature properties invariant.

Let g be a left invariant metric on a connected Lie group G and let ∇ be the Levi–Civita connection determined by g. Then the Riemann curvature tensor R associates to each smooth vector fields X, Y the linear transformation

$$R_{XY} = \nabla_{[X,Y]} - \nabla_X \nabla_Y + \nabla_Y \nabla_X$$

from smooth vector fields to smooth vector fields. If U and V are orthonormal, the number

$$K = \kappa(U, V) = \langle R_{UV}(U), V\rangle$$

is called the *sectional curvature* associated with U and V. If $Y_1, Y_2, \ldots, Y_n$ is any orthonormal basis for $\mathfrak{g}$, then for a unit vector field U the number

$$r(U) = \sum_i \kappa(U, Y_i) = \sum_i \langle R_{UY_i}(U), Y_i\rangle$$

is called *Ricci curvature*, and the number

$$\rho = r(Y_1) + r(Y_2) + \cdots + r(Y_n) = 2\sum_{i<j} \kappa(Y_i, Y_j)$$

is called the *scalar curvature*.

In particular, we review curvatures of left invariant metrics on the 3-dimensional simply connected unimodular Lie groups from [7]. Let $\mathfrak{g}$ be a 3-dimensional unimodular Lie algebra with a positive definite metric and with a preferred orientation. One can choose an orthonormal basis $\{Y_1, Y_2, Y_3\}$ which is positively oriented such that

$$[Y_2, Y_3] = \xi_1 Y_1, \quad [Y_3, Y_1] = \xi_2 Y_2, \quad [Y_1, Y_2] = \xi_3 Y_3.$$

Define numbers $\zeta_1, \zeta_2, \zeta_3$ by the formula

$$\zeta_i = \frac{1}{2}(\xi_1 + \xi_2 + \xi_3) - \xi_i.$$

Then:

1. the orthonormal basis Y_1, Y_2, Y_3 diagonalizes the Ricci transformation $\hat{r}$, the principal Ricci curvatures being given by

$$r(Y_1) = 2\zeta_2\zeta_3, \quad r(Y_2) = 2\zeta_3\zeta_1, \quad r(Y_3) = 2\zeta_1\zeta_2,$$

2. the scalar curvature is given by the formula

$$\rho = 2(\zeta_2\zeta_3 + \zeta_3\zeta_1 + \zeta_1\zeta_2),$$

3. for any pair of orthonormal vectors U and V, the sectional curvature $\kappa(U, V)$ associated with U and V is given by the formula

$$\kappa(U, V) = \|U \times V\|^2 \frac{\rho}{2} - r(U \times V).$$

In particular, for each $1 \leq i < j \leq 3$,

$$\kappa(Y_i, Y_j) = \zeta_1\zeta_2 + \zeta_2\zeta_3 + \zeta_3\zeta_1 - 2\zeta_i\zeta_j.$$

If $U = \sum u_i Y_i$ and $V = \sum v_i Y_i$ are linearly independent unit vectors, then

$$\kappa(U, V) = (u_2v_3 - u_3v_2)^2\kappa(Y_2, Y_3) + (u_3v_1 - u_1v_3)^2\kappa(Y_3, Y_1) + (u_1v_2 - u_2v_1)^2\kappa(Y_1, Y_2),$$

where

$$\kappa(Y_i, Y_j) = \zeta_1\zeta_2 + \zeta_2\zeta_3 + \zeta_3\zeta_1 - 2\zeta_i\zeta_j.$$

Lie algebra ($\mathfrak{g}$)	Associated simply connected Lie group	Automorphisms of a Lie algebra	Left invariant metrics	Signature of Ricci transformation / Sign of scalar curvature
Abelian ($\mathbb{R}^3$)	$\mathbb{R}^3$	$\mathrm{GL}(3,\mathbb{R})$	$\begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix}$	$(0,0,0)$ $\rho=0$
Nilpotent ($\mathfrak{n}$)	$\mathrm{Nil}=\left\{ \begin{bmatrix} 1&x&z\\0&1&y\\0&0&1 \end{bmatrix} \middle\vert\, x,y,z\in\mathbb{R} \right\}$ $=$ Heisenberg group	$\left\{ \begin{bmatrix} a&c&0\\b&d&0\\ *&*&ad-bc \end{bmatrix} \middle\vert\, \begin{matrix} a,b,c,d,*\in\mathbb{R},\\ ad-bc\neq 0 \end{matrix} \right\}$	$\begin{bmatrix} \lambda&0&0\\0&\lambda&0\\0&0&1 \end{bmatrix}$ $\lambda>0$	$(+,-,-)$ $\rho<0$ in all cases
Unimodular solvable (a) $\mathbb{R}^2\rtimes_\sigma\mathbb{R}$	$\mathrm{Sol} = \mathbb{R}^2\rtimes_\varphi\mathbb{R}$ where $\varphi(t)=\begin{bmatrix} e^t&0\\0&e^{-t} \end{bmatrix}$	$S_1\cup S_1\begin{bmatrix} 0&1&0\\1&0&0\\0&0&-1 \end{bmatrix}$, where $S_1 = \left\{ \begin{bmatrix} \alpha&0&*\\0&\beta&*\\0&0&1 \end{bmatrix} \middle\vert\, \begin{matrix} \alpha,\beta,*\in\mathbb{R},\\ \alpha\beta\neq 0 \end{matrix} \right\}$	$\begin{bmatrix} 1&1&0\\1&\mu&0\\0&0&\nu \end{bmatrix}$ $\begin{matrix} \mu>1\\ \nu>0 \end{matrix}$ $\begin{bmatrix} 1&0&0\\0&1&0\\0&0&\nu \end{bmatrix}$ $\nu>0$	$(+,-,-)$ $\rho<0$ in all cases $(0,0,-)$ $\rho<0$ in all cases
(b) $\mathbb{R}^2\rtimes\mathfrak{so}(2)$	$\widetilde{E}_0(2)$: the universal covering of $\mathbb{R}^2\rtimes\mathrm{SO}(2)$	$S_2\cup S_2\begin{bmatrix} 1&0&0\\0&-1&0\\0&0&-1 \end{bmatrix}$, where $S_2 = \left\{ \begin{bmatrix} \mathbb{C}^* & & \gamma\\ & & \delta\\ 0&0&1 \end{bmatrix} \middle\vert\, \gamma,\delta\in\mathbb{R} \right\}$	$\begin{bmatrix} 1&0&0\\0&\mu&0\\0&0&\nu \end{bmatrix}$ $\begin{matrix} 0<\mu\leq 1\\ \nu>0 \end{matrix}$	$(+,-,-)$ if $\mu<1$ $(0,0,0)$ if $\mu=1$ $\rho<0$ if $\mu<1$ $\rho=0$ if $\mu=1$
Simple (a) $\mathfrak{sl}(2,\mathbb{R})$	$\widetilde{\mathrm{PSL}}(2,\mathbb{R})$	$\mathrm{SO}(1,2)$	$\begin{bmatrix} \lambda&0&0\\0&\mu&0\\0&0&\nu \end{bmatrix}$ $\begin{matrix} \lambda>0\\ \mu\geq\nu>0 \end{matrix}$	$(+,-,-)$ if $\lambda<\mu+\nu$ $(0,0,-)$ if $\lambda=\mu+\nu$ $(-,+,-)$ if $\lambda>\mu+\nu$ $\rho<0$ in all cases
(b) $\mathfrak{so}(3)$	$\mathrm{SU}(2)$	$\mathrm{SO}(3)$	$\begin{bmatrix} \lambda&0&0\\0&\mu&0\\0&0&\nu \end{bmatrix}$ $\lambda\geq\mu\geq\nu>0$	$(+,+,+)$ if $\lambda<\mu+\nu$ $(+,0,0)$ if $\lambda=\mu+\nu$ $(+,-,-)$ if $\lambda>\mu+\nu$ $\rho>0$ if $\sqrt{\lambda}<\sqrt{\mu}+\sqrt{\nu}$ $\rho=0$ if $\sqrt{\lambda}=\sqrt{\mu}+\sqrt{\nu}$ $\rho<0$ if $\sqrt{\lambda}>\sqrt{\mu}+\sqrt{\nu}$

Table 1 Unimodular Case

In [7], Milnor studied the extent to which curvature is altered by a change of metric on each three-dimensional Lie group. Since we have the left invariant metrics in hand, we can take an orthonormal basis $\{Y_1, Y_2, Y_3\}$. We compute directly the principal Ricci curvatures $r(Y_1), r(Y_2), r(Y_3)$, the scalar curvature $\rho = r(Y_1) + r(Y_2) + r(Y_3)$ and the sectional curvatures κ. Then these are expressed explicitly as functions of left invariant metrics. Therefore we can understand completely the change of the signature of the Ricci transformation and the change of the sign of the scalar curvature by a change of metric. In particular, given a simply connected three-dimensional Lie group G with any left invariant metric on it, it is possible to determine the necessary and sufficient conditions for three real numbers r_1, r_2, r_3 in order that they be the principal Ricci curvatures of the Lie group G. Actually this kind of problem was treated in [6] by means of the formulas given in [7], which relate the Ricci curvatures to the structure constants of G. See Table 1 for a summary.

Recall that the left invariant metrics are the metrics $[g_{ij}]$ obtained after fixing the ordered bases $\{X, Y, Z\}$ which are given in Section 2.1 for the three-dimensional Lie algebras.

Theorem 4.1 *For any left invariant metric on* $\mathbb{R}^3$*, the Ricci transformation has signature* $(0, 0, 0)$ *and the scalar curvature* ρ *is zero.*

Theorem 4.2 *For any left invariant metric on* Nil*, the Ricci transformation has signature* $(+, -, -)$ *and the scalar curvature* ρ *is strictly negative. Furthermore, the Ricci transformation is diagonalized as* $\mathrm{diag}\{-\rho, \rho, \rho\}$.

P r o o f. We may assume that the metric $\langle , \rangle$ is associated with the matrix $\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}$ with $\lambda > 0$. Recalling that the Lie algebra of Nil has a basis $\{X, Y, Z\}$ so that $[X, Y] = Z$, $[Z, X] = [Z, Y] = 0$, we see that

$$\langle X, X\rangle = \langle Y, Y\rangle = \lambda, \quad \langle Z, Z\rangle = 1, \quad \langle X, Y\rangle = \langle X, Z\rangle = \langle Y, Z\rangle = 0.$$

Taking $Y_1 = \frac{1}{\sqrt{\lambda}} X$, $Y_2 = \frac{1}{\sqrt{\lambda}} Y, Y_3 = Z$, they form an orthonormal basis and satisfy

$$[Y_2, Y_3] = 0, \quad [Y_3, Y_1] = 0, \quad [Y_1, Y_2] = \frac{1}{\lambda} Y_3,$$

with $\xi_1 = 0$, $\xi_2 = 0$, $\xi_3 = \frac{1}{\lambda}$. The principal Ricci curvatures are

$$r(Y_1) = -\frac{1}{2\lambda^2}, \quad r(Y_2) = -\frac{1}{2\lambda^2}, \quad r(Y_3) = \frac{1}{2\lambda^2},$$

and the scalar curvature is strictly negative

$$\rho = r(Y_1) + r(Y_2) + r(Y_3) = -\frac{1}{2\lambda^2}.$$

The Ricci transformation $\hat{r}$ is diagonalized as $\mathrm{diag}\{\rho, \rho, -\rho\}$ and has signature $(-, -, +)$. Taking the orthonormal basis $\{Y_3, Y_1, Y_2\}$, we see that the Ricci transformation $\hat{r}$ is diagonalized as $\mathrm{diag}\{-\rho, \rho, \rho\}$ and has signature $(+, -, -)$. □

Corollary 4.3 *The sectional curvatures for the metric* $\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *with* $\lambda > 0$ *on* Nil *are*

$$\kappa(X, Y) = -\frac{3}{4\lambda^2}, \quad \kappa(Y, Z) = \kappa(Z, X) = \frac{1}{4\lambda^2}.$$

Theorem 4.4 *The Ricci transformation for* Sol *has signature either* $(0, 0, -)$ *or* $(+, -, -)$ *depending on the choice of left invariant metric which is equivalent up to automorphism to a metric whose associated matrix is of the form*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix} \quad or \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$$

where $\mu > 1, \nu > 0$*; and the scalar curvature is always strictly negative.*

P r o o f. Suppose that the metric $\langle , \rangle$ is associated with the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix}$ with $\nu > 0$. Recalling that the Lie algebra of Sol has a basis $\{X, Y, Z\}$ so that $[X, Y] = 0$, $[Z, X] = X$, $[Z, Y] = -Y$, we see that

$$\langle X, X\rangle = \langle Y, Y\rangle = 1, \quad \langle Z, Z\rangle = \nu, \quad \langle X, Y\rangle = \langle X, Z\rangle = \langle Y, Z\rangle = 0.$$

Taking $Y_1 = \frac{1}{\sqrt{2}}(X + Y)$, $Y_2 = -\frac{1}{\sqrt{2}}(X - Y)$, $Y_3 = \frac{1}{\sqrt{\nu}}Z$, they form an orthonormal basis and satisfy $[Y_2, Y_3] = \frac{1}{\sqrt{\nu}}Y_1$, $[Y_3, Y_1] = -\frac{1}{\sqrt{\nu}}Y_2$, $[Y_1, Y_2] = 0$. The principal Ricci curvatures are

$$r(Y_1) = 0 = r(Y_2), \quad r(Y_3) = -\frac{2}{\nu},$$

and the Ricci transformation $\hat{r}$ is diagonalized as $\mathrm{diag}\{r(Y_1), r(Y_2), r(Y_3)\}$. Thus the signature of the Ricci transformation $\hat{r}$ is $(0, 0, -)$.

Suppose that the metric $\langle , \rangle$ is associated with the matrix $\begin{bmatrix} 1 & 1 & 0 \\ 1 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$ with $\mu > 1, \nu > 0$. With respect to the basis $\{X, Y, Z\}$ for the Lie algebra of Sol subject to $[X, Y] = 0$, $[Z, X] = X$, $[Z, Y] = -Y$, we see that

$$\langle X, X\rangle = \langle X, Y\rangle = 1, \quad \langle Y, Y\rangle = \mu, \quad \langle Z, Z\rangle = \nu, \quad \langle X, Z\rangle = \langle Y, Z\rangle = 0.$$

Taking $Y_1 = \frac{1}{\sqrt{2}\sqrt{\mu+\sqrt{\mu}}}\big(\sqrt{\mu}X + Y\big)$, $Y_2 = \frac{1}{\sqrt{2}\sqrt{\mu-\sqrt{\mu}}}\big(-\sqrt{\mu}X + Y\big)$, $Y_3 = \frac{1}{\sqrt{\nu}}Z$, they form an orthonormal basis and satisfy

$$[Y_2, Y_3] = \frac{\sqrt{\mu + \sqrt{\mu}}}{\sqrt{\nu}\sqrt{\mu - \sqrt{\mu}}}Y_1, \quad [Y_3, Y_1] = \frac{-\sqrt{\mu - \sqrt{\mu}}}{\sqrt{\nu}\sqrt{\mu + \sqrt{\mu}}}Y_2, \quad [Y_1, Y_2] = 0.$$

The principal Ricci curvatures are

$$r(Y_1) = \frac{2\sqrt{\mu}}{\nu(\mu - 1)}, \quad r(Y_2) = -\frac{2\sqrt{\mu}}{\nu(\mu - 1)}, \quad r(Y_3) = -\frac{2\mu}{\nu(\mu - 1)},$$

and the Ricci transformation $\hat{r}$ is diagonalized as $\mathrm{diag}\{r(Y_1), r(Y_2), r(Y_3)\}$. Thus the signature of the Ricci transformation $\hat{r}$ is $(+, -, -)$. In particular,

$$\rho = r(Y_1) + r(Y_2) + r(Y_3) = \begin{cases} -\dfrac{2}{\nu} \\ -\dfrac{2\mu}{\nu(\mu - 1)}. \end{cases}$$

Hence the scalar curvature ρ is strictly negative. □

Corollary 4.5 *The sectional curvatures for the metric* $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix}$ *with* $\nu > 0$ *or* $\left[\begin{smallmatrix} 1 & 1 & 0 \\ 1 & \mu & 0 \\ 0 & 0 & \nu \end{smallmatrix}\right]$ *with* $\mu > 1, \nu > 0$ *on* Sol *are*

$$\kappa(X, Y) = \begin{cases} \dfrac{1}{\nu} \\ \dfrac{\mu}{\nu\sqrt{\mu^2 - 1}}, \end{cases} \qquad \kappa(Y, Z) = \begin{cases} \dfrac{-1}{\nu} \\ \dfrac{(2 - \mu)}{\nu(\mu - 1)}, \end{cases} \qquad \kappa(Z, X) = \begin{cases} \dfrac{-1}{\nu} \\ \dfrac{2 - \mu}{\nu(\mu - 1)}. \end{cases}$$

Theorem 4.6 *The Ricci transformation for $\widetilde{E}_0(2)$ has signature either $(+,-,-)$ or $(0,0,0)$ depending on the choice of left invariant metric which is equivalent up to automorphism to a metric whose associated matrix is of the form*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$$

where $0 < \mu \leq 1$ and $\nu > 0$. Moreover, a left invariant metric on $\widetilde{E}_0(2)$ is flat if and only if it is equivalent up to automorphism to a metric whose associated matrix is of the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix}$$

where $\nu > 0$ if and only if the scalar curvature ρ is zero.

Proof. We may assume that the metric $\langle , \rangle$ is associated with the matrix $\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{smallmatrix}\right]$ with $0 < \mu \leq 1$ and $\nu > 0$. Recalling that the Lie algebra of $\widetilde{E}_0(2)$ has a basis $\{X, Y, Z\}$ so that $[X, Y] = 0$, $[Z, X] = -Y$, $[Z, Y] = X$, we see that

$$\langle X, X\rangle = 1, \quad \langle Y, Y\rangle = \mu, \quad \langle Z, Z\rangle = \nu, \quad \langle X, Y\rangle = \langle X, Z\rangle = \langle Y, Z\rangle = 0.$$

Taking $Y_1 = X$, $Y_2 = \frac{1}{\sqrt{\mu}} Y$, $Y_3 = \frac{1}{\sqrt{\nu}} Z$, they form an orthonormal basis and satisfy

$$[Y_1, Y_2] = 0, \quad [Y_3, Y_1] = -\frac{\sqrt{\mu}}{\sqrt{\nu}} Y_2, \quad [Y_3, Y_2] = \frac{1}{\sqrt{\mu\nu}} Y_1.$$

The principal Ricci curvatures are

$$r(Y_1) = \frac{(1+\mu)(1-\mu)}{2\mu\nu}, \quad r(Y_2) = -\frac{(1+\mu)(1-\mu)}{2\mu\nu}, \quad r(Y_3) = -\frac{(1-\mu)^2}{2\mu\nu},$$

and the Ricci transformation $\hat{r}$ is diagonalized as $\operatorname{diag}\{r(Y_1), r(Y_2), r(Y_3)\}$. Therefore $r(Y_1) \geq 0$, $r(Y_2) \leq 0$, $r(Y_3) \leq 0$, and $r(Y_1) = 0$ if and only if $r(Y_2) = 0 = r(Y_3)$. Therefore the signature of the Ricci transformation $\hat{r}$ is $(+,-,-)$ or $(0,0,0)$ according as $\mu < 1$ or $\mu = 1$. Moreover, the metric is flat if and only if $r(Y_i) = 0$ if and only if $\mu = 1$. In particular, the scalar curvature is non-positive

$$\rho = r(Y_1) + r(Y_2) + r(Y_3) = -\frac{(1-\mu)^2}{2\mu\nu}.$$

Hence the last assertion holds. □

Corollary 4.7 *The sectional curvatures for the metric* $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$ *with $0 < \mu \leq 1$ and $\nu > 0$ on $\widetilde{E}_0(2)$ are*

$$\kappa(X, Y) = \frac{(1-\mu)^2}{4\mu\nu}, \quad \kappa(Y, Z) = -\frac{(1-\mu)(3+\mu)}{4\mu\nu}, \quad \kappa(Z, X) = \frac{(1-\mu)(1+3\mu)}{4\mu\nu}.$$

Theorem 4.8 *The Ricci transformation for $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$ has signature either $(+,-,-)$ or $(0,0,-)$ depending on the choice of left invariant metric which is equivalent up to automorphism to a metric whose associated matrix is of the form*

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$$

according as $\lambda \neq \mu + \nu$, or $\lambda = \mu + \nu$; and the scalar curvature is always strictly negative.

Proof. We may assume that the metric $\langle\,,\rangle$ is associated with the matrix $\left[\begin{smallmatrix}\lambda&0&0\\0&\mu&0\\0&0&\nu\end{smallmatrix}\right]$ with $\lambda > 0$ and $\mu \geq \nu > 0$. Recalling that the Lie algebra $\mathfrak{so}(1,2)$ of $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ has a basis $\{X,Y,Z\}$ so that $[X,Y] = 2Z$, $[Z,X] = 2Y$, $[Z,Y] = 2X$, we see that

$$\langle X,X\rangle = \lambda, \quad \langle Y,Y\rangle = \mu, \quad \langle Z,Z\rangle = \nu, \quad \langle X,Y\rangle = \langle X,Z\rangle = \langle Y,Z\rangle = 0.$$

Taking $Y_1 = \frac{1}{\sqrt{\lambda}}X_1$, $Y_2 = \frac{1}{\sqrt{\mu}}X_2$, $Y_3 = \frac{1}{\sqrt{\nu}}X_3$, they form an orthonormal basis and satisfy

$$[Y_2,Y_3] = \frac{-2\lambda}{\sqrt{\lambda\mu\nu}}Y_1, \quad [Y_3,Y_1] = \frac{2\mu}{\sqrt{\lambda\mu\nu}}Y_2, \quad [Y_1,Y_2] = \frac{2\nu}{\sqrt{\lambda\mu\nu}}Y_3.$$

The principal Ricci curvatures are

$$r(Y_1) = \frac{2\big(\lambda^2 - (\mu-\nu)^2\big)}{\lambda\mu\nu}, \quad r(Y_2) = \frac{2\big(\mu^2 - (\nu+\lambda)^2\big)}{\lambda\mu\nu}, \quad r(Y_3) = \frac{2\big(\nu^2 - (\lambda+\mu)^2\big)}{\lambda\mu\nu},$$

and the Ricci transformation $\hat{r}$ is diagonalized as $\mathrm{diag}\{r(Y_1), r(Y_2), r(Y_3)\}$. Thus $r(Y_3) < 0$, and $r(Y_1) > 0$, $= 0$, < 0 if and only if $r(Y_2) < 0, = 0, > 0$, respectively. Therefore the signature of the Ricci transformation $\hat{r}$ is $(+,-,-)$, $(0,0,-)$ or $(-,+,-)$ according as $\lambda+\nu > \mu$, $\lambda+\nu = \mu$ or $\lambda+\nu < \mu$. In particular, the scalar curvature

$$\begin{aligned}\rho &= r(Y_1) + r(Y_2) + r(Y_3)\\ &= \frac{2\big(\lambda^2+\mu^2+\nu^2-(\lambda+\mu)^2-(\mu-\nu)^2-(\nu+\lambda)^2\big)}{\lambda\mu\nu}\\ &= \frac{-2\big((\lambda+\mu-\nu)^2+4\lambda\nu\big)}{\lambda\mu\nu}\\ &< 0\end{aligned}$$

is strictly negative. □

Corollary 4.9 *The sectional curvatures for the metric* $\left[\begin{smallmatrix}\lambda&0&0\\0&\mu&0\\0&0&\nu\end{smallmatrix}\right]$ *with* $\lambda > 0$, $\mu \geq \nu > 0$ *on* $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ *are*

$$\begin{aligned}\kappa(X,Y) &= \frac{(\lambda+\mu-\nu)^2 + 4\nu(\mu-\nu)}{\lambda\mu\nu},\\ \kappa(Y,Z) &= -\frac{(\lambda+\mu+\nu)^2 + 2\big(\lambda^2-\mu^2-\nu^2\big)}{\lambda\mu\nu},\\ \kappa(Z,X) &= \frac{(\lambda-\mu+\nu)^2 - 4\mu(\mu-\nu)}{\lambda\mu\nu}.\end{aligned}$$

Theorem 4.10 *The Ricci transformation for SU*(2) *has signature either* $(+,+,+)$, $(+,0,0)$ *or* $(+,-,-)$ *depending on the choice of left invariant metric which is equivalent up to automorphism to a metric whose associated matrix is of the form*

$$\begin{bmatrix}\lambda&0&0\\0&\mu&0\\0&0&\nu\end{bmatrix}$$

according as $\lambda < \mu+\nu$, $\lambda = \mu+\nu$ *or* $\lambda > \mu+\nu$*; and the scalar curvature is positive, zero or negative if and only if* $\sqrt{\lambda} < \sqrt{\mu}+\sqrt{\nu}$, $\sqrt{\lambda} = \sqrt{\mu}+\sqrt{\nu}$ *or* $\sqrt{\lambda} > \sqrt{\mu}+\sqrt{\nu}$*, respectively.*

Proof. We may assume that the metric $\langle\,,\rangle$ is associated with the matrix $\left[\begin{smallmatrix}\lambda&0&0\\0&\mu&0\\0&0&\nu\end{smallmatrix}\right]$ with $\lambda \geq \mu \geq \nu > 0$.

Recalling that the Lie algebra $\mathfrak{su}(3)$ of $\mathrm{SU}(2)$ has a basis X_1, X_2, X_3 so that $[X_1, X_2] = X_3$, $[X_3, X_1] = X_2$, $[X_3, X_2] = -X_1$, we see that

$$\langle X, X\rangle = \lambda, \quad \langle Y, Y\rangle = \mu, \quad \langle Z, Z\rangle = \nu, \quad \langle X, Y\rangle = \langle X, Z\rangle = \langle Y, Z\rangle = 0.$$

Taking $Y_1 = \frac{1}{\sqrt{\lambda}}X_1$, $Y_2 = \frac{1}{\sqrt{\mu}}X_2$, $Y_3 = \frac{1}{\sqrt{\nu}}X_3$, they form an orthonormal basis and satisfy

$$[Y_2, Y_3] = \frac{\lambda}{\sqrt{\lambda\mu\nu}}Y_1, \quad [Y_3, Y_1] = \frac{\mu}{\sqrt{\lambda\mu\nu}}Y_2, \quad [Y_1, Y_2] = \frac{\nu}{\sqrt{\lambda\mu\nu}}Y_3.$$

The principal Ricci curvatures are

$$r(Y_1) = \frac{\lambda^2 - (\mu-\nu)^2}{2\lambda\mu\nu} > 0, \quad r(Y_2) = \frac{\mu^2 - (\nu-\lambda)^2}{2\lambda\mu\nu}, \quad r(Y_3) = \frac{\nu^2 - (\lambda-\mu)^2}{2\lambda\mu\nu},$$

and the Ricci transformation $\hat{r}$ is diagonalized as $\mathrm{diag}\{r(Y_1), r(Y_2), r(Y_3)\}$. Thus $r(Y_1) > 0$, and $r(Y_2) > 0$, $= 0$, < 0 if and only if $r(Y_3) > 0, = 0, < 0$, respectively. Therefore the signature of the Ricci transformation $\hat{r}$ is $(+,+,+)$, $(+,0,0)$ or $(+,-,-)$ according as $\lambda < \mu+\nu$, $\lambda = \mu+\nu$ or $\lambda > \mu+\nu$. In particular, the scalar curvature is

$$\begin{aligned}\rho &= r(Y_1) + r(Y_2) + r(Y_3)\\ &= \frac{\lambda^2+\mu^2+\nu^2-(\lambda-\mu)^2-(\mu-\nu)^2-(\nu-\lambda)^2}{2\lambda\mu\nu}\\ &= -\frac{\big(\sqrt{\lambda}+\sqrt{\mu}-\sqrt{\nu}\big)\big(\sqrt{\lambda}-\sqrt{\mu}+\sqrt{\nu}\big)\big(\sqrt{\lambda}+\sqrt{\mu}+\sqrt{\nu}\big)\big(\sqrt{\lambda}-\sqrt{\mu}-\sqrt{\nu}\big)}{2\lambda\mu\nu}.\end{aligned}$$

Since $\lambda \geq \mu \geq \nu > 0$, we have that $\rho > 0$, $= 0$ or < 0 if and only if $\sqrt{\lambda} - \sqrt{\mu} - \sqrt{\nu} < 0$, $= 0$ or > 0, respectively. □

Corollary 4.11 *The sectional curvatures for the metric* $\left[\begin{smallmatrix}\lambda & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & \nu\end{smallmatrix}\right]$ *with* $\lambda \geq \mu \geq \nu > 0$ *on* $\mathrm{SU}(2)$ *are*

$$\begin{aligned}\kappa(X,Y) &= \frac{(\lambda-\mu+\nu)^2 + 4\nu(\mu-\nu)}{4\lambda\mu\nu},\\ \kappa(Y,Z) &= \frac{(\lambda+\mu-\nu)^2 - 4\lambda(\lambda-\nu)}{4\lambda\mu\nu},\\ \kappa(Z,X) &= \frac{(\lambda-\mu-\nu)^2 + 4\mu(\lambda-\mu)}{4\lambda\mu\nu}.\end{aligned}$$

5 Curvatures on three-dimensional non-unimodular Lie groups

Recall that G_I and G_c are simply connected three-dimensional non-unimodular Lie groups whose Lie algebras are $\mathfrak{g}_I$ and $\mathfrak{g}_c$, respectively. Now we study curvatures on simply connected three-dimensional non-unimodular Lie groups. Utilizing Milnor's idea again together with our complete list of left invariant metrics on simply connected three-dimensional non-unimodular Lie groups up to automorphism, we are able to understand completely the change of the signature of the Ricci transformation by a change of metric. We show that the scalar curvature of any left invariant metric on all three-dimensional simply connected non-unimodular Lie groups is always strictly negative. Our results extend Milnor's result [7, Theorem 4.11] in great detail. See Table 2 for a summary.

Recall again that the left invariant metrics are the metrics $[g_{ij}]$ obtained after fixing the bases $\{X, Y, Z\}$ which are given in Section 2.1 for the three-dimensional Lie algebras.

Theorem 5.1 *The Ricci transformation for* G_I *has signature* $(-,-,-)$ *depending on the choice of left invariant metric which is equivalent up to automorphism to a metric whose associated matrix is of the form*

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \nu\end{bmatrix}$$

where $\nu > 0$; and the scalar curvature is always strictly negative.

P r o o f. We may assume that the metric $\langle , \rangle$ is associated with the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix}$ with $\nu > 0$. We see that

$$\langle X, X\rangle = \langle Y, Y\rangle = 1, \quad \langle Z, Z\rangle = \nu, \quad \langle X, Y\rangle = \langle X, Z\rangle = \langle Y, Z\rangle = 0.$$

Taking $Y_1 = \frac{1}{\sqrt{\nu}} Z$, $Y_2 = X$, $Y_3 = Y$, they form an orthonormal basis and satisfy

$$[Y_1, Y_2] = \frac{1}{\sqrt{\nu}} Y_2, \quad [Y_1, Y_3] = \frac{1}{\sqrt{\nu}} Y_3, \quad [Y_2, Y_3] = 0.$$

With respect to the basis $\{Y_1, Y_2, Y_3\}$, the associated matrix of the Ricci transformation $\hat{r}$ is of the form

$$[\hat{r}] = \begin{bmatrix} -\frac{2}{\nu} & 0 & 0 \\ 0 & -\frac{2}{\nu} & 0 \\ 0 & 0 & -\frac{2}{\nu} \end{bmatrix}.$$

Thus the principal Ricci curvatures are

$$r_1 = r_2 = r_3 = -\frac{2}{\nu}.$$

Therefore the signature of the Ricci transformation $\hat{r}$ is $(-, -, -)$. Clearly the scalar curvature

$$\rho = r(Y_1) + r(Y_2) + r(Y_3) = -\frac{6}{\nu}$$

is strictly negative. □

Corollary 5.2 *The sectional curvatures for the metric* $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix}$ *with $\nu > 0$ on G_I are*

$$\kappa(X, Y) = \kappa(Y, Z) = \kappa(Z, X) = -\frac{1}{\nu}.$$

Lemma 5.3 *The Ricci transformation for G_c has signature $(+, -, -)$ depending on the choice of left invariant metric which is equivalent up to automorphism to a metric whose associated matrix is of the form*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$$

where μ, $\nu > 0$; and the scalar curvature is always strictly negative.

P r o o f. Suppose that the metric $\langle , \rangle$ is associated with the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$ with μ, $\nu > 0$. Note that

$$\langle X, X\rangle = 1, \quad \langle Y, Y\rangle = \mu, \quad \langle Z, Z\rangle = \nu, \quad \langle X, Y\rangle = \langle X, Z\rangle = \langle Y, Z\rangle = 0.$$

Taking $Y_1 = \frac{1}{\sqrt{\nu}} Z$, $Y_2 = X$, $Y_3 = \frac{1}{\sqrt{\mu}} Y$, they form an orthonormal basis and satisfy

$$[Y_1, Y_2] = \frac{\sqrt{\mu}}{\sqrt{\nu}} Y_3, \quad [Y_1, Y_3] = \frac{-c}{\sqrt{\mu\nu}} Y_2 + \frac{2}{\sqrt{\nu}} Y_3, \quad [Y_2, Y_3] = 0.$$

With respect to the basis $\{Y_1, Y_2, Y_3\}$, the associated matrix of the Ricci transformation $\hat{r}$ is of the form

$$[\hat{r}] = \begin{bmatrix} -\frac{(\mu-c)^2+8\mu}{2\mu\nu} & 0 & 0 \\ 0 & -\frac{\mu^2-c^2}{2\mu\nu} & -\frac{2\sqrt{\mu}}{\nu} \\ 0 & -\frac{2\sqrt{\mu}}{\nu} & \frac{\mu^2-8\mu-c^2}{2\mu\nu} \end{bmatrix}.$$

Thus the principal Ricci curvatures are

$$r_1 = -\frac{4\mu - \sqrt{(\mu^2-4\mu-c^2)^2+16\mu^3}}{2\mu\nu} = \frac{\left(\mu^2-c^2\right)^2+8\mu\left(\mu^2+c^2\right)}{2\mu\nu(4\mu+\sqrt{(\mu^2-4\mu-c^2)^2+16\mu^3})},$$

$$r_2 = -\frac{4\mu + \sqrt{(\mu^2-4\mu-c^2)^2+16\mu^3}}{2\mu\nu},$$

$$r_3 = -\frac{(\mu-c)^2+8\mu}{2\mu\nu}.$$

Note that $r_1 > 0, r_2 < 0$ and $r_3 < 0$. Therefore the signature of the Ricci transformation $\hat{r}$ is $(+,-,-)$. Clearly the scalar curvature

$$\rho = r(Y_1) + r(Y_2) + r(Y_3) = -\frac{(\mu-c)^2+16\mu}{2\mu\nu}.$$

is strictly negative. □

Corollary 5.4 *The sectional curvatures for the metric* $\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{smallmatrix}\right]$ *on* G_c *are*

$$\kappa(X,Y) = \frac{(\mu-c)^2}{4\mu\nu}, \quad \kappa(Y,Z) = \frac{\mu^2+2(c-8)\mu-3c^2}{4\mu\nu}, \quad \kappa(Z,X) = -\frac{(\mu-c)(c+3\mu)}{4\mu\nu}.$$

Theorem 5.5 (Case: $c < 0$) *The Ricci transformation for* G_c *with* $c < 0$ *has signature* $(+,-,-)$ *depending on the choice of left invariant metric which is equivalent up to automorphism to a metric whose associated matrix is of the form*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$$

where $0 < \mu \le |c|$ *and* $\nu > 0$*; and the scalar curvature is always strictly negative.*

Proof. This follows immediately from Lemma 5.3. □

Theorem 5.6 (Case: $c = 0$) *The Ricci transformation for* G_0 *has signature either* $(+,-,-)$ *or* $(0,-,-)$ *depending on the choice of left invariant metric which is equivalent up to automorphism to a metric whose associated matrix is of the form*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix}$$

where $\mu, \nu > 0$*; and the scalar curvature is always strictly negative.*

Lie algebra	Associated simply connected Lie group	Automorphisms of a Lie algebra		Left invariant metrics	Signature of Ricci transformation
$\mathfrak{g}_I \cong \mathbb{R}^2 \rtimes_{\sigma_I} \mathbb{R}$, where $\sigma_I(t) = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}$	$G_I \cong \mathbb{R}^2 \rtimes_{\varphi_I} \mathbb{R}$, where $\varphi_I(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix}$	$\left\{ \begin{bmatrix} \mathrm{GL}(2,\mathbb{R}) & * \\ 0 & 1 \end{bmatrix} \middle\vert\, * \in \mathbb{R}^2 \right\}$		$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix}$ $\nu > 0$	$(-,-,-)$
$\mathfrak{g}_c \cong \mathbb{R}^2 \rtimes_{\sigma_c} \mathbb{R}$, where $\sigma_c(t) = \begin{bmatrix} 0 & -ct \\ t & 2t \end{bmatrix}$	$G_c \cong \mathbb{R}^2 \rtimes_{\varphi_c} \mathbb{R}$, where $\varphi_c(t) = e^t \frac{e^{zt}+e^{-zt}}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + e^t \frac{e^{zt}-e^{-zt}}{2z} \begin{bmatrix} -1 & -c \\ 1 & 1 \end{bmatrix}$ $(z = \sqrt{1-c} \neq 0)$, $\varphi_1(t) = e^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + e^t t \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$	$\left\{ \begin{bmatrix} \beta-\alpha & -c\alpha & * \\ \alpha & \beta+\alpha & * \\ 0 & 0 & 1 \end{bmatrix} \middle\vert\, \begin{matrix} \alpha,\beta,* \in \mathbb{R}, \\ \beta^2+(c-1)\alpha^2 \neq 0 \end{matrix} \right\}$	$c<0$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$ $0<\mu\leq\vert c\vert$, $\nu>0$	$(+,-,-)$
			$c=0$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$ $\mu,\nu>0$	$(+,-,-)$
				$\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix}$ $\nu>0$	$(0,-,-)$
			$0<c<1$	$P^t \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix} P$ $0\leq\mu<1$, $\nu>0$	$(+,-,-)$ if $\mu<\sqrt{2-(1-c)(c+\sqrt{c^2+4})}/\sqrt{2}$ $(0,-,-)$ if $\mu=\sqrt{2-(1-c)(c+\sqrt{c^2+4})}/\sqrt{2}$ $(-,-,-)$ if $\mu>\sqrt{2-(1-c)(c+\sqrt{c^2+4})}/\sqrt{2}$
			$c=1$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$ $0<\mu\leq 1$, $\nu>0$	$(+,-,-)$
				$\begin{bmatrix} 1 & \lambda & 0 \\ \lambda & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix}$ $0<\lambda<1$, $\nu>0$	$(+,-,-)$ if $\lambda<\sqrt{5}-2$ $(0,-,-)$ if $\lambda=\sqrt{5}-2$ $(-,-,-)$ if $\lambda>\sqrt{5}-2$
			$c>1$	$\begin{bmatrix} 1 & 1 & 0 \\ 1 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$ $1<\mu\leq c$, $\nu>0$	$(+,-,-)$ if $\mu>\sqrt{c^2+4}-\sqrt{4+2c-2\sqrt{c^2+4}}$ $(0,-,-)$ if $\mu=\sqrt{c^2+4}-\sqrt{4+2c-2\sqrt{c^2+4}}$ $(-,-,-)$ if $\mu<\sqrt{c^2+4}-\sqrt{4+2c-2\sqrt{c^2+4}}$

Table 2 Non-unimodular case

In Table 2, $P = \begin{bmatrix} \frac{1+z}{-2cz} & \frac{1}{-2z} & 0 \\ \frac{1-z}{2cz} & \frac{1}{2z} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Proof. Suppose that the metric $\langle\, , \rangle$ is associated with the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$ with $\mu,\ \nu > 0$. Then by Lemma 5.3, the signature of the Ricci transformation $\hat{r}$ is $(+, -, -)$ and the scalar curvature $\rho = -\frac{\mu+16}{2\nu}$ is strictly negative.

Suppose that the metric $\langle , \rangle$ is associated with the matrix $\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix}$ with $\nu > 0$. Note that

$$\langle X, X\rangle = \langle Y, Y\rangle = 1,\ \langle X, Y\rangle = \frac{1}{2}, \quad \langle Z, Z\rangle = \nu, \quad \langle X, Z\rangle = \langle Y, Z\rangle = 0.$$

Taking $Y_1 = \frac{1}{\sqrt{\nu}}Z,\ Y_2 = X,\ Y_3 = \frac{1}{\sqrt{3}}X - \frac{2}{\sqrt{3}}Y$, they form an orthonormal basis and satisfy

$$[Y_1, Y_2] = \frac{1}{2\sqrt{\nu}}Y_2 - \frac{\sqrt{3}}{2\sqrt{\nu}}Y_3, \quad [Y_1, Y_3] = -\frac{\sqrt{3}}{2\sqrt{\nu}}Y_2 + \frac{3}{2\sqrt{\nu}}Y_3, \quad [Y_2, Y_3] = 0.$$

With respect to the basis $\{Y_1, Y_2, Y_3\}$, the associated matrix of the Ricci transformation $\hat{r}$ is of the form

$$[\hat{r}] = \begin{bmatrix} -\dfrac{4}{\nu} & 0 & 0 \\ 0 & -\dfrac{1}{\nu} & \dfrac{\sqrt{3}}{\nu} \\ 0 & \dfrac{\sqrt{3}}{\nu} & -\dfrac{3}{\nu} \end{bmatrix}.$$

Thus the principal Ricci curvatures are

$$r_1 = 0, \quad r_2 = r_3 = -\frac{4}{\nu}.$$

Therefore the signature of the Ricci transformation $\hat{r}$ is $(0, -, -)$, and the scalar curvature

$$\rho = r(Y_1) + r(Y_2) + r(Y_3) = -\frac{8}{\nu}$$

is strictly negative. □

Corollary 5.7 *The sectional curvatures on G_0 for the metric* $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$ *or* $\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix}$ *with $\mu, \nu > 0$ are*

$$\kappa(X, Y) = \begin{cases} \dfrac{\mu}{4\nu}, \\ 0, \end{cases} \qquad \kappa(Y, Z) = \begin{cases} \dfrac{\mu - 16}{4\nu}, \\ \dfrac{-5}{2\nu}, \end{cases} \quad \kappa(Z, X) = \begin{cases} \dfrac{-3\mu}{4\nu}, \\ \dfrac{-1}{\nu}. \end{cases}$$

Theorem 5.8 (Case: $c = 1$) *The Ricci transformation for G_1 has signature $(+, -, -)$, $(0, -, -)$ or $(-, -, -)$ depending on the choice of left invariant metric which is equivalent up to automorphism to a metric whose associated matrix is of the form*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix} \quad or \quad \begin{bmatrix} 1 & \lambda & 0 \\ \lambda & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix}$$

where $0 < \lambda < 1,\ 0 < \mu \le 1$ and $\nu > 0$; and the scalar curvature is always strictly negative.

P r o o f. Suppose that the metric $\langle,\rangle$ is associated with the matrix $\begin{bmatrix}1&0&0\\0&\mu&0\\0&0&\nu\end{bmatrix}$ with $0<\mu\le 1,\ \nu>0$. Then by Lemma 5.3, the signature of the Ricci transformation $\hat{r}$ is $(+,-,-)$ and the scalar curvature $\rho=-\frac{(\mu-1)^2+16\mu}{2\mu\nu}$ is strictly negative.

Suppose that the metric $\langle,\rangle$ is associated with the matrix $\begin{bmatrix}1&\lambda&0\\\lambda&1&0\\0&0&\nu\end{bmatrix}$ with $0<\lambda<1,\ \nu>0$. Note that

$$\langle X,X\rangle=\langle Y,Y\rangle=1,\quad \langle X,Y\rangle=\lambda,\quad \langle Z,Z\rangle=\nu,\quad \langle X,Z\rangle=\langle Y,Z\rangle=0.$$

Taking $Y_1=\frac{1}{\sqrt{\nu}}Z,\ Y_2=X,\ Y_3=-\frac{\lambda}{\sqrt{1-\lambda^2}}X+\frac{1}{\sqrt{1-\lambda^2}}Y$, they form an orthonormal basis and satisfy

$$[Y_1,Y_2]=\frac{\lambda}{\sqrt{\nu}}Y_2+\frac{\sqrt{1-\lambda^2}}{\sqrt{\nu}}Y_3,\quad [Y_1,Y_3]=-\frac{(1-\lambda)^2}{\sqrt{(1-\lambda^2)\nu}}Y_2+\frac{2-\lambda}{\sqrt{\nu}}Y_3,\quad [Y_2,Y_3]=0.$$

With respect to the basis $\{Y_1,Y_2,Y_3\}$, the associated matrix of the Ricci transformation $\hat{r}$ is of the form

$$[\hat{r}]=\begin{bmatrix}-\dfrac{4}{(1+\lambda)\nu}&0&0\\ 0&-\dfrac{4}{(1+\lambda)\nu}&-\dfrac{2(1-\lambda)}{\sqrt{1-\lambda^2}\nu}\\ 0&-\dfrac{2(1-\lambda)}{\sqrt{1-\lambda^2}\nu}&-\dfrac{4}{(1+\lambda)\nu}\end{bmatrix}.$$

Thus the principal Ricci curvatures are

$$r_1=-\frac{2\big((1+\lambda)-\sqrt{2(1-\lambda)}\,\big)}{(1+\lambda)\nu},\quad r_2=-\frac{2\big((1+\lambda)+\sqrt{2(1-\lambda)}\,\big)}{(1+\lambda)\nu},\quad r_3=-\frac{4}{(1+\lambda)\nu}.$$

Note that $r_1>0,=0,<0$ if and only if $\lambda<\sqrt{5}-2,=\sqrt{5}-2,>\sqrt{5}-2$, respectively, and $r_2<0,\ r_3<0$. Therefore the signature of the Ricci transformation $\hat{r}$ is $(+,-,-)$,$(0,-,-)$ or $(-,-,-)$. In particular, the scalar curvatures

$$\rho=r(Y_1)+r(Y_2)+r(Y_3)=-\frac{4(2+\lambda)}{(1+\lambda)\nu}$$

is strictly negative. □

Corollary 5.9 *The sectional curvatures on G_1 for the metric* $\begin{bmatrix}1&0&0\\0&\mu&0\\0&0&\nu\end{bmatrix}$ *or* $\begin{bmatrix}1&\lambda&0\\\lambda&1&0\\0&0&\nu\end{bmatrix}$ *with $0<\lambda<1, 0<\mu\le 1,\nu>0$ are*

$$\kappa(X,Y)=\begin{cases}\dfrac{(1-\mu)^2}{4\mu\nu}\\[2mm] -\dfrac{2\lambda(1-\lambda)}{\nu\sqrt{1-\lambda^2}},\end{cases}\quad \kappa(Y,Z)=\begin{cases}\dfrac{\mu^2-14\mu-3}{4\mu\nu}\\[2mm] -\dfrac{4-2\lambda-4\lambda^2+4\lambda^3}{(1+\lambda)\nu},\end{cases}\quad \kappa(Z,X)=\begin{cases}\dfrac{(1-\mu)(1+3\mu)}{4\mu\nu}\\[2mm] -\dfrac{2\lambda}{\nu(1+\lambda)}.\end{cases}$$

Theorem 5.10 (Case: $c>1$) *The Ricci transformation for G_c with $c>1$ has signature $(+,-,-)$, $(0,-,-)$ or $(-,-,-)$ depending on the choice of left invariant metric which is equivalent up to automorphism to a metric whose associated matrix is of the form*

$$\begin{bmatrix}1&1&0\\1&\mu&0\\0&0&\nu\end{bmatrix}$$

where $1<\mu\le c$ and $\nu>0$; and the scalar curvature is always strictly negative.

Proof. Assume that the metric $\langle\, ,\rangle$ is associated with the matrix $\begin{bmatrix}1 & 1 & 0\\ 1 & \mu & 0\\ 0 & 0 & \nu\end{bmatrix}$ with $1 < \mu \leq c,\ \nu > 0$. Note that

$$\langle X, X\rangle = \langle X, Y\rangle = 1, \quad \langle Y, Y\rangle = \mu, \quad \langle Z, Z\rangle = \nu, \quad \langle X, Z\rangle = \langle Y, Z\rangle = 0.$$

Taking $Y_1 = \frac{1}{\sqrt{\nu}}Z,\ Y_2 = X,\ Y_3 = -\frac{1}{\sqrt{\mu-1}}X + \frac{1}{\sqrt{\mu-1}}Y$, they form an orthonormal basis and satisfy

$$[Y_1, Y_2] = \frac{1}{\sqrt{\nu}}Y_2 + \frac{\sqrt{\mu-1}}{\sqrt{\nu}}Y_3, \quad [Y_1, Y_3] = -\frac{c-1}{\sqrt{(\mu-1)\nu}}Y_2 + \frac{1}{\sqrt{\nu}}Y_3, \quad [Y_2, Y_3] = 0.$$

With respect to the basis $\{Y_1, Y_2, Y_3\}$, the associated matrix of the Ricci transformation $\hat{r}$ is of the form

$$[\hat{r}] = \begin{bmatrix} -\dfrac{(c-\mu)^2 + 4(\mu-1)}{2(\mu-1)\nu} & 0 & 0\\ 0 & -\dfrac{\mu^2 + 2\mu - c^2 + 2c - 4}{2(\mu-1)\nu} & \dfrac{c-\mu}{\sqrt{\mu-1}\nu}\\ 0 & \dfrac{c-\mu}{\sqrt{\mu-1}\nu} & \dfrac{\mu^2 - 6\mu - c^2 + 2c + 4}{2(\mu-1)\nu} \end{bmatrix}.$$

Thus the principal Ricci curvatures are

$$r_1 = -\frac{4(\mu-1) - (c-\mu)\sqrt{(c-\mu)^2 + 2c(\mu-1)}}{2(\mu-1)\nu},$$

$$r_2 = -\frac{4(\mu-1) + (c-\mu)\sqrt{(c-\mu)^2 + 2c(\mu-1)}}{2(\mu-1)\nu},$$

$$r_3 = -\frac{(c-\mu)^2 + 4(\mu-1)}{2(\mu-1)\nu}.$$

Note that $r_1 > 0, = 0, < 0$ if and only if

$$\mu > \sqrt{c^2+4} - \sqrt{4 + 2c - 2\sqrt{c^2+4}}, \quad = \sqrt{c^2+4} - \sqrt{4 + 2c - 2\sqrt{c^2+4}}, \quad < \sqrt{c^2+4} - \sqrt{4 + 2c - 2\sqrt{c^2+4}},$$

respectively, and $r_2 < 0,\ r_3 < 0$. Therefore the signature of the Ricci transformation $\hat{r}$ is $(+, -, -)$, $(0, -, -)$ or $(-, -, -)$, and the scalar curvature

$$\rho = r(Y_1) + r(Y_2) + r(Y_3) = -\frac{(\mu-c)^2 + 12(\mu-1)}{2(\mu-1)\nu}$$

is strictly negative. □

Corollary 5.11 *The sectional curvatures for the metric* $\begin{bmatrix}1 & 1 & 0\\ 1 & \mu & 0\\ 0 & 0 & \nu\end{bmatrix}$ *with* $0 < \mu < c$ *and* $\nu > 0$ *on* G_c *where* $c > 1$ *are*

$$\kappa(X, Y) = \frac{c^2 - 2c\mu + \mu^2 - 4\mu + 4}{4\nu\sqrt{\mu-1}},$$

$$\kappa(Y, Z) = \frac{(4-3\mu)c^2 + 2c(\mu^2 + 2\mu - 4) + \mu(\mu^2 - 12\mu + 12)}{4\mu\nu(\mu-1)},$$

$$\kappa(Z, X) = -\frac{c^2 + 2c\mu - 3\mu^2 - 4c + 4}{4\nu(\mu-1)}.$$

Theorem 5.12 (Case: $0 < c < 1$) *The Ricci transformation for G_c with $0 < c < 1$ has signature $(+,-,-)$, $(0,-,-)$ or $(-,-,-)$ depending on the choice of left invariant metric which is equivalent up to automorphism to a metric whose associated matrix is of the form*

$$\begin{bmatrix} \frac{1+z}{-2cz} & \frac{1}{-2z} & 0 \\ \frac{1-z}{2cz} & \frac{1}{2z} & 0 \\ 0 & 0 & 1 \end{bmatrix}^t \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix} \begin{bmatrix} \frac{1+z}{-2cz} & \frac{1}{-2z} & 0 \\ \frac{1-z}{2cz} & \frac{1}{2z} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $z = \sqrt{1-c}$, $0 \leq \mu < 1$ and $\nu > 0$; and the scalar curvature is always strictly negative.

Proof. Assume that the metric $\langle , \rangle$ is associated with the matrix

$$\begin{bmatrix} \frac{1+z}{-2cz} & \frac{1}{-2z} & 0 \\ \frac{1-z}{2cz} & \frac{1}{2z} & 0 \\ 0 & 0 & 1 \end{bmatrix}^t \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix} \begin{bmatrix} \frac{1+z}{-2cz} & \frac{1}{-2z} & 0 \\ \frac{1-z}{2cz} & \frac{1}{2z} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with $z = \sqrt{1-c}$, $0 \leq \mu < 1$ and $\nu > 0$. Let $\{X_1, X_2, X_3\}$ be the basis for $\mathfrak{g}_c$ given in Remark 3.14, i.e.,

$$X_1 = -cX + (1-z)Y, \quad X_2 = -cX + (1+z)Y, \quad X_3 = Z$$

where $z = \sqrt{1-c}$. Note that

$$[X_1, X_2] = 0, \quad [X_3, X_1] = \big(1 - \sqrt{1-c}\big)X_1, \quad [X_3, X_2] = \big(1 + \sqrt{1-c}\big)X_2.$$

With respect to the basis $\{X_1, X_2, X_3\}$, the metric $\langle\, , \rangle$ is associated with the matrix $\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix}$ with $0 \leq \mu < 1$, $\nu > 0$. Note that

$$\langle X_1, X_1\rangle = \langle X_2, X_2\rangle = 1, \quad \langle X_1, X_2\rangle = \mu, \quad \langle X_3, X_3\rangle = \nu, \quad \langle X_1, X_3\rangle = \langle X_2, X_3\rangle = 0.$$

Taking $Y_1 = \frac{1}{\sqrt{\nu}}X_3$, $Y_2 = X_1$, $Y_3 = \frac{\mu}{\sqrt{1-\mu^2}}X_1 - \frac{1}{\sqrt{1-\mu^2}}X_2$, they form an orthonormal basis and satisfy

$$\begin{aligned}
[Y_1, Y_2] &= \frac{1-\sqrt{1-c}}{\sqrt{\nu}}Y_2, \\
[Y_1, Y_3] &= -\frac{2\sqrt{1-c}\,\mu}{\sqrt{(1-\mu^2)\nu}}Y_2 + \frac{1+\sqrt{1-c}}{\sqrt{\nu}}Y_3, \\
[Y_2, Y_3] &= 0.
\end{aligned}$$

With respect to the basis $\{Y_1, Y_2, Y_3\}$, the associated matrix of the Ricci transformation $\hat{r}$ is of the form

$$[\hat{r}] = \begin{bmatrix} -\frac{2\left(2-c-\mu^2\right)}{\left(1-\mu^2\right)\nu} & 0 & 0 \\ 0 & -\frac{2\left(1-\sqrt{1-c}-(2-\sqrt{1-c}-c)\mu^2\right)}{\left(1-\mu^2\right)\nu} & -\frac{2\mu\left(1-\sqrt{1-c}-c\right)}{\sqrt{1-\mu^2}\,\nu} \\ 0 & -\frac{2\mu\left(1-\sqrt{1-c}-c\right)}{\sqrt{1-\mu^2}\,\nu} & -\frac{2\left(1+\sqrt{1-c}-(\sqrt{1-c}+c)\mu^2\right)}{\left(1-\mu^2\right)\nu} \end{bmatrix}.$$

Thus the principal Ricci curvatures are

$$r_1 = -\frac{2\left(\left(1-\mu^2\right) - \sqrt{(1-c)\left(1-c\mu^2\right)}\right)}{\left(1-\mu^2\right)\nu},$$

$$r_2 = -\frac{2\left(\left(1-\mu^2\right)+\sqrt{\left(1-c\right)\left(1-c\mu^2\right)}\right)}{\left(1-\mu^2\right)\nu},$$

$$r_3 = -\frac{2\left(2-c-\mu^2\right)}{\left(1-\mu^2\right)\nu}.$$

Note that $r_1 > 0, = 0, < 0$ if and only if

$$\mu > \sqrt{\frac{2-(1-c)\left(c+\sqrt{c^2+4}\right)}{2}}, \ = \sqrt{\frac{2-(1-c)\left(c+\sqrt{c^2+4}\right)}{2}}, \ < \sqrt{\frac{2-(1-c)\left(c+\sqrt{c^2+4}\right)}{2}},$$

respectively, and $r_2 < 0$, $r_3 < 0$. Therefore the signature of the Ricci transformation $\hat{r}$ is $(+,-,-)$, $(0,-,-)$ or $(-,-,-)$, and the scalar curvature

$$\rho = r(Y_1) + r(Y_2) + r(Y_3) = -\frac{2(4-c-3\mu^2)}{(1-\mu^2)\nu}$$

is strictly negative. □

Corollary 5.13 *The sectional curvatures on G_c where $0 < c < 1$ for the metric*

$$\begin{bmatrix} \frac{1+z}{-2cz} & \frac{1}{-2z} & 0 \\ \frac{1-z}{2cz} & \frac{1}{2z} & 0 \\ 0 & 0 & 1 \end{bmatrix}^t \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix} \begin{bmatrix} \frac{1+z}{-2cz} & \frac{1}{-2z} & 0 \\ \frac{1-z}{2cz} & \frac{1}{2z} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with $z = \sqrt{1-c}$, $0 \le \mu < 1$ and $\nu > 0$ are

$$\kappa(X,Y) = \frac{\mu^2 - c}{\nu\sqrt{1-\mu^2}},$$

$$\kappa(Y,Z) = \frac{4(1-\sqrt{1-c}-c)\mu^4 + (-1+6\sqrt{1-c}+2c)\mu^2 - (1+\sqrt{1-c})^2}{\nu(1-\mu^2)},$$

$$\kappa(Z,X) = \frac{(3-2\sqrt{1-c}-2c)\mu^2 - (1-\sqrt{1-c})^2}{\nu(1-\mu^2)}.$$

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