Left-right Browder and Left-right Fredholm Operators

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Abstract. We consider left and right Browder operators, left and right Fredholm operators, spectra related with these operators, and various operator quantities.

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1. Introduction

Let X denote an infinite dimensional Banach space. We use $B(X)$ to denote the set of all linear bounded operators on X. Also, $K(X)$ and $F(X)$, respectively, denote the set of all compact and finite rank operators on X. For $A \in B(X)$ we use $N(A)$ and $R(A)$, respectively, to denote the null-space and the range of A.

We use $\mathcal{G}_l(X)$ and $\mathcal{G}_r(X)$, respectively, to denote the set of all left and right invertible operators on X. It is well-known that $A \in \mathcal{G}_l(X)$ if and only if A is injective and $R(A)$ is a closed and complemented subspace of X. Also, $A \in \mathcal{G}_r(X)$ if and only if A is onto and $N(A)$ is a complemented subspace of X. The set of all invertible operators on X is denoted by $\mathcal{G}(X)$.

Let $\alpha(A) = \dim N(A)$ if $N(A)$ is finite dimensional, and let $\alpha(A) = \infty$ if $N(A)$ is infinite dimensional. Similarly, let $\beta(A) = \dim X/R(A) = \operatorname{codim} R(A)$ if $X/R(A)$ is finite dimensional, and let $\beta(A) = \infty$ if $X/R(A)$ is infinite dimensional.

Sets of upper and lower Fredholm operators, respectively, are defined as

$$\Phi_+(X) = \{A \in B(X) : \alpha(A) < \infty \text{ and } R(A) \text{ is closed}\},$$

and

$$\Phi_-(X) = \{A \in B(X) : \beta(A) < \infty\}.$$

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Operators in $\Phi_\pm(X) = \Phi_+(X) \cup \Phi_-(X)$ are called semi-Fredholm operators. For such operators the index is defined by $i(A) = \alpha(A) - \beta(A)$. Let $\Phi_+^-(X) = \{A \in \Phi_+(X) : i(A) \leq 0\}$ and $\Phi_-^+(X) = \{A \in \Phi_-(X) : i(A) \geq 0\}$. The set of Fredholm operators is defined as

$$\Phi(X) = \Phi_+(X) \cap \Phi_-(X).$$

The set of Weyl operators is defined as $\Phi_0(X) = \{A \in \Phi(X) : i(A) = 0\}$.

Let S be a subset of a Banach space A. The perturbation class of S, denoted by $P(S)$, is the set

$$P(S) = \{a \in \mathcal{R} : a + s \in S \text{ for every } s \in S\}.$$

The Calkin algebra over X is the quotient algebra $C(X) = B(X)/K(X)$, and $\pi : B(X) \to C(X)$ denotes the natural homomorphism. Let $r_e(A)$ denote spectral radius of the element $\pi(A)$ in $C(X)$, $A \in B(X)$, i.e. $r_e(A) = \lim\limits_{n\to\infty} (\|\pi(A^n)\|)^{\frac{1}{n}}$ and it is called essential spectral radius of A. An operator $A \in B(X)$ is Riesz if $\{\lambda \in \mathbb{C} : A - \lambda \in \Phi(X)\} = \mathbb{C}\backslash\{0\}$, i.e. $r_e(A) = 0$. For $A \in B(X)$ set

$$\|A\|_{P\Phi} = \inf\{\|A - P\| : P \in P(\Phi(X))\}.$$

It is known that $r_e(A) = \lim\limits_{n\to\infty} (\|A^n\|_{P\Phi})^{\frac{1}{n}}$.

An operator $A \in B(X)$ is relatively regular (or g-invertible) if there exists $B \in B(X)$ such that $ABA = A$. It is well-known that A is relatively regular if and only if $R(A)$ and $N(A)$ are closed and complemented subspaces of X.

Sets of left and right Fredholm operators, respectively, are defined as

$$\Phi_l(X) = \{A \in B(X) : R(A) \text{ is a closed and complemented subspace of } X \text{ and } \alpha(A) < \infty\},$$

and

$$\Phi_r(X) = \{A \in B(X) : N(A) \text{ is a complemented subspace of } X \text{ and } \beta(A) < \infty\}.$$

It is known that the sets $\Phi_l(X)$ and $\Phi_r(X)$ are open [1] (Chapter 5.2, Theorem 6), and $P(\Phi_l(X)) = P(\Phi(X)) = P(\Phi_r(X))$ [1] (Chapter 5.2, Corollary 3).

An operator $A \in B(X)$ is left (right) Weyl if A is left (right) Fredholm operator and $i(A) \leq 0$ ($i(A) \geq 0$). We use $\mathcal{W}_l(X)$ ($\mathcal{W}_r(X)$) to denote the set of all left (right) Weyl operators.

The ascent of $A \in B(X)$, denoted by $\text{asc}(A)$, is the smallest $n \in \mathbb{N}$ such that $N(A^n) = N(A^{n+1})$. If such n does not exist, then $\text{asc}(A) = \infty$. The descent of A, denoted by $\text{dsc}(A)$, is the smallest $n \in \mathbb{N}$ such that $R(A^n) = R(A^{n+1})$. If such n does not exist, then $\text{dsc}(A) = \infty$.

An operator $A \in B(X)$ is upper semi-Browder if it is upper semi-Fredholm of finite ascent, and A is lower semi-Browder if it is lower semi-Fredholm of finite descent. Let $\mathcal{B}_+(X)$ ($\mathcal{B}_-(X)$) denote the set of all upper (lower) semi-Browder operators. The set of Browder operators is defined as $\mathcal{B}(X) = \mathcal{B}_+(X) \cap \mathcal{B}_-(X)$.

The operator $A \in B(X)$ is left Browder if it is left Fredholm of finite ascent, and A is right Browder if it is right Fredholm of finite descent. Let $\mathcal{B}_l(X)$ ($\mathcal{B}_r(X)$) denote the set of all left (right) Browder operators.

From [4] (Theorem 7.9.2) and [1] (Chapter 5.2, Theorem 7), for $A \in B(X)$ and $K \in K(X)$ which commutes with A, it follows that

$$A \text{ is left Browder } \iff A + K \text{ is left Browder,} \tag{1.1}$$

$$A \text{ is right Browder} \iff A + K \text{ is right Browder.} \tag{1.2}$$

The following assertions [13] (Theorem 7 and Theorem 8) tell us that it holds more generally. If $A \in B(X)$, and if E Riesz which commutes with A, then

$$A \text{ is left Browder} \iff A + E \text{ is left Browder },$$
$$A \text{ is right Browder } \iff A + E \text{ is right Browder.}$$

Moreover, the following hold.

Theorem 1.1. *If $A \in B(X)$ and $E \in B(X)$ is Riesz, then*

$$AE - EA \in P(\Phi(X)) \Longrightarrow \sigma_w^{left}(A) = \sigma_w^{left}(A + E), \tag{1.3}$$

$$AE - EA \in P(\Phi(X)) \Longrightarrow \sigma_w^{right}(A) = \sigma_w^{right}(A + E), \tag{1.4}$$

$$AE = EA \Longrightarrow \sigma_b^{left}(A) = \sigma_b^{left}(A + E), \tag{1.5}$$

$$AE = EA \Longrightarrow \sigma_b^{right}(A) = \sigma_b^{right}(A + E). \tag{1.6}$$

The following theorem gives a characterization of left and right Browder operators [13] (Theorem 5 and Theorem 6).

Theorem 1.2. *Let $A \in B(X)$. Then A is left (right) Browder iff there exist closed subspaces X_1 and X_2 invariant with respect to A such that $X = X_1 \oplus X_2$, $\dim X_1 < \infty$, the reduction $A_1 = A_{|X_1} : X_1 \to X_1$ is nilpotent and the reduction $A_2 = A_{|X_2} : X_2 \to X_2$ is left (right) invertible.*

Corresponding spectra of $A \in B(X)$ are defined as:

$\sigma_l(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin \mathcal{G}_l(X)\}$-the left spectrum,

$\sigma_r(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin \mathcal{G}_r(X)\}$-the right spectrum,

$\sigma_a(A) = \{\lambda \in \mathbb{C} : A - \lambda$ is not bounded below $\}$-the approximate point spectrum,

$\sigma_d(A) = \{\lambda \in \mathbb{C} : A - \lambda$ is not onto $\}$-the defect spectrum,

$\sigma_b(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin \mathcal{B}(X)\}$-the Browder spectrum,

$\sigma_b^+(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin \mathcal{B}_+(X)\}$-the Browder essential approximate point spectrum,

$\sigma_b^+(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin \mathcal{B}_-(X)\}$-the Browder essential defect spectrum,

$\sigma_b^{left}(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin \mathcal{B}_l(X)\}$-the left Browder spectrum,

$\sigma_b^{right}(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin \mathcal{B}_r(X)\}$-the right Browder,

$\sigma_w(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin \Phi_0(X)\}$-the Weyl spectrum,

$\sigma_w^{left}(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin \mathcal{W}_l(X)\}$-the left Weyl spectrum,

$\sigma_w^{right}(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin \mathcal{W}_r(X)\}$-the right Weyl spectrum,

$\sigma_w^+(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin \Phi_+^-(X)\}$-the essential approximate point spectrum,

$\sigma_w^-(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin \Phi_-^+(X)\}$-the essential defect spectrum,

$\sigma_e(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin \Phi(X)\}$-the Fredholm spectrum,

$\sigma_e^{left}(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin \Phi_l(X)\}$-the left Fredholm spectrum,

$\sigma_e^{right}(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin \Phi_r(X)\}$-the right Fredholm spectrum,

$\sigma_e^+(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin \Phi_+(X)\}$-the upper semi-Fredholm spectrum,

$\sigma_e^-(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin \Phi_-(X)\}$-the lower semi-Fredholm spectrum.

2. Properties of corresponding spectra

We prove the following auxiliary assertion.

Lemma 2.1. *Let $A \in B(X)$ and let X be a direct sum of closed subspaces X_1 and X_2 which are A-invariant. If $A_1 = A_{|X_1} : X_1 \to X_1$ and $A_2 = A_{|X_2} : X_2 \to X_2$, then the following statements hold:*

(2.1.1) *The operator A is g-invertible if and only if A_1 and A_2 are g-invertible.*

(2.1.2) *The operator $A \in \Phi_l(X)$ if and only if $A_1 \in \Phi_l(X_1)$ and $A_2 \in \Phi_l(X_2)$, and in that case $i(A) = i(A_1) + i(A_2)$.*

(2.1.3) *The operator $A \in \Phi_r(X)$ if and only if $A_1 \in \Phi_r(X_1)$ and $A_2 \in \Phi_r(X_2)$, and in that case $i(A) = i(A_1) + i(A_2)$.*

(2.1.4) *The operator $A \in \mathcal{B}_l(X)$ if and only if $A_1 \in \mathcal{B}_l(X_1)$ and $A_2 \in \mathcal{B}_l(X_2)$, and in that case $i(A) = i(A_1) + i(A_2)$.*

(2.1.5) *The operator $A \in \mathcal{B}_r(X)$ if and only if $A_1 \in \mathcal{B}_r(X_1)$ and $A_2 \in \mathcal{B}_r(X_2)$, and in that case $i(A) = i(A_1) + i(A_2)$.*

Proof. (2.1.1): The operator A has the following matrix form with respect to the decomposition $X = X_1 \oplus X_2$:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \to \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

Suppose that A is g-invertible. Then there exists $B \in B(X)$ such that $ABA = A$, and B has the following matrix form:

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \to \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},$$

and we get

$$\begin{bmatrix} A_1B_{11}A_1 & A_1B_{12}A_2 \\ A_2B_{21}A_1 & A_2B_{22}A_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},$$

which implies $A_1B_{11}A_1 = A_1$ and $A_2B_{22}A_2 = A_2$. Thus, A_1 and A_2 are g-invertible operators.

Conversely, suppose that $A_1 \in B(X_1)$ and $A_2 \in B(X_2)$ are g-invertible operators. Then there exist $B_1 \in B(X_1)$ and $B_2 \in B(X_2)$ such that $A_1B_1A_1 = A_1$ and $A_2B_2A_2 = A_2$. Let

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}.$$

Then we have $B \in B(X)$ and $ABA = A$, so A is a g-invertible operator.

(2.1.2): Since $N(A) = N(A_1) \oplus N(A_2)$ and $R(A) = R(A_1) \oplus R(A_2)$, it follows that $\alpha(A) = \alpha(A_1) + \alpha(A_2)$ and $\beta(A) = \beta(A_1) + \beta(A_2)$. Therefore, $\alpha(A) < \infty$ if and only if $\alpha(A_1) < \infty$ and $\alpha(A_2) < \infty$. Hence, according to (2.1.1), $A \in \Phi_l(X)$ if and only if $A_1 \in \Phi_l(X_1)$ and $A_2 \in \Phi_l(X_2)$, and in that case $i(A) = \alpha(A) - \beta(A) = (\alpha(A_1) + \alpha(A_2)) - (\beta(A_1) + \beta(A_2)) = i(A_1) + i(A_2)$.

(2.1.3): Similarly to (2.1.2).

(2.1.4): Since $N(A^n) = N(A_1^n) \oplus N(A_2^n)$ for $n \in \mathbb{N}$, we conclude that $\mathrm{asc}(A) < \infty$ if and only if $\mathrm{asc}(A_1) < \infty$ and $\mathrm{asc}(A_2) < \infty$. Now the statements follows from (2.1.2).

(2.1.5): From $R(A^n) = R(A_1^n) \oplus R(A_2^n)$ for $n \in \mathbb{N}$, we see that $\mathrm{dsc}(A) < \infty$ if and only if $\mathrm{dsc}(A_1) < \infty$ and $\mathrm{dsc}(A_2) < \infty$. Then the conclusion follows from (2.1.3). □

Let $\mathcal{P}(X)$ denote the set of all projections $P \in B(X)$ such that $\operatorname{codim} R(P) < \infty$. For $A \in B(X)$ and $P \in \mathcal{P}(X)$, the compression $A_P : R(P) \to R(P)$ is defined by $A_P y = PAy$, $y \in R(P)$, i.e. $A_P = PA_{|R(P)}$, where $A_{|R(P)} : R(P) \to X$ is the restriction of A. Clearly, $R(P)$ is a Banach space and $A_P \in B(R(P))$.

Zemánek [12] gave the proof of the fact that if $P \in \mathcal{P}(X)$, then A is semi-Fredholm if and only if A_P is semi-Fredholm and $i(A) = i(A_P)$.

We prove the following result in that case.

Theorem 2.2. *Let $A \in B(X)$, $P \in \mathcal{P}(X)$. Then*
(2.2.1) *$A \in \Phi_l(X)$ if and only if $A_P \in \Phi_l(R(P))$, and in that case $i(A_P) = i(A)$.*
(2.2.2) *$A \in \Phi_r(X)$ if and only if $A_P \in \Phi_r(R(P))$, and in that case $i(A_P) = i(A)$.*
(2.2.3) *If $AP = PA$, then $A \in \mathcal{B}_l(X)$ if and only if $A_P \in \mathcal{B}_l(R(P))$, and in that case $i(A_P) = i(A)$.*
(2.2.4) *If $AP = PA$, then $A \in \mathcal{B}_r(X)$ if and only if $A_P \in \mathcal{B}_r(R(P))$, and in that case $i(A_P) = i(A)$.*

Proof. (2.2.3), (2.2.4): Suppose that $P \in \mathcal{P}(X)$, $A \in B(X)$ and $AP = PA$. Then $X = R(P) \oplus N(P)$ and subspaces $R(P)$ and $N(P)$ are invariant for $PAP \in B(X)$. The operator PAP has the following matrix form:

$$PAP = \begin{bmatrix} A_P & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(P) \\ N(P) \end{bmatrix} \to \begin{bmatrix} R(P) \\ N(P) \end{bmatrix}.$$

Since $\dim N(P) < \infty$, from (2.1.4) and (2.1.5) it follows that PAP is left (right) Browder if and only if A_P is left (right) Browder and $i(PAP) = i(A_P) + i(0) = i(A_P)$. Since

$$A = PA + (I - P)A = PAP + PA(I - P) + (I - P)A,$$

and since $PA(I-P)+(I-P)A$ is a finite rank operator, which commutes with PAP, by (1.1) and (1.2) it follows that PAP is a left (right) Browder operator if and only if A is left (right) Browder, and in that case $i(PAP) = i(A)$.

(2.2.1) and (2.2.2) can be proved similarly, using (2.1.2) and (2.1.3). □

It is known [12] that

$$\sigma_w^+(A) = \bigcap_{P \in \mathcal{P}(X)} \sigma_a(A_P), \quad \sigma_w^-(A) = \bigcap_{P \in \mathcal{P}(X)} \sigma_d(A_P),$$
$$\sigma_b^+(A) = \bigcap_{P \in \mathcal{P}(X),\ AP=PA} \sigma_a(A_P), \quad \sigma_b^-(A) = \bigcap_{P \in \mathcal{P}(X),\ AP=PA} \sigma_d(A_P).$$

We prove analogous assertion for the left and right Browder and Weyl spectra.

Theorem 2.3. *Let $A \in B(X)$. Then*

$$\sigma_b^{left}(A) = \bigcap_{P \in \mathcal{P}(X),\ AP=PA} \sigma_l(A_P), \tag{2.1}$$

$$\sigma_b^{right}(A) = \bigcap_{P \in \mathcal{P}(X),\ AP=PA} \sigma_r(A_P). \tag{2.2}$$

Proof. To prove the inclusion "$\subset$" in (2.1) (or (2.2)), suppose that $\lambda \notin \sigma_l(A_P)$ ($\lambda \notin \sigma_r(A_P)$) for some $P \in \mathcal{P}(X)$ such that $AP = PA$. Then $A_P - \lambda I_P = (A-\lambda)_P$ is left (right) invertible, and so $(A-\lambda)_P$ is left (right) Browder. By Theorem 2.2 it follows that $A - \lambda$ is left (right) Browder, i.e. $\lambda \notin \sigma_b^{left}(A)$ ($\lambda \notin \sigma_b^{right}(A)$).

To prove the converse inclusion, suppose that $\lambda \notin \sigma_b^{left}(A)$ ($\lambda \notin \sigma_b^{right}(A)$). Then $A-\lambda \in \mathcal{B}_l(X)$ ($A-\lambda \in \mathcal{B}_r(X)$). By Theorem 1.2, X is a direct sum of closed subspaces X_1 and X_2, which are $A-\lambda$-invariant. Consequently, they are A-invariant, and they have the following properties: $\dim X_1 < \infty$ and $A_1 - \lambda$ is nilpotent on X_1, where $A_1 = A_{|X_1} : X_1 \to X_1$ and if $A_2 = A_{|X_2} : X_2 \to X_2$, then $A_2 - \lambda$ is left (right) invertible. Let P be the projection of X onto X_2 along X_1. Clearly, $P \in \mathcal{P}(X)$. Since the subspaces X_1 and X_2 are invariant for A, we see that $AP = PA$ and $(A-\lambda)_P = A_2 - \lambda$. Thus $A_P - \lambda I_P$ is left (right) invertible and so $\lambda \notin \sigma_l(A_P)$ ($\lambda \notin \sigma_r(A_P)$). □

Combining (2.2.1) and (2.2.2) with the proof of Theorem 2 in [12] we get the following theorem.

Theorem 2.4. *Let $A \in B(X)$. Then*

$$\sigma_w^{left}(A) = \bigcap_{P \in \mathcal{P}(X)} \sigma_l(A_P), \tag{2.3}$$

$$\sigma_w^{right}(A) = \bigcap_{P \in \mathcal{P}(X)} \sigma_r(A_P). \tag{2.4}$$

Proof. To prove the inclusion "$\subset$" in (2.3) (or (2.4)), suppose that $\lambda \notin \sigma_l(A_P)$ ($\lambda \notin \sigma_r(A_P)$) for some $P \in \mathcal{P}(X)$, then $A_P - \lambda I_P = (A-\lambda)_P$ is left (right) invertible, and so $(A-\lambda)_P$ is left (right) Weyl. By (2.2.1) (or (2.2.2)) it follows that $A - \lambda$ is left (right) Weyl, i.e. $\lambda \notin \sigma_w^{left}(A)$ ($\lambda \notin \sigma_w^{right}(A)$).

To prove the converse in (2.3), suppose that $\lambda \notin \sigma_w^{left}(A)$. Then $A-\lambda \in \Phi_l(X)$ and $i(A-\lambda) \le 0$. Since $\alpha(A-\lambda) \le \beta(A-\lambda)$, there exists a subspace V such that $\dim V = \dim N(A-\lambda) < \infty$ and $V \cap R(A-\lambda) = \{0\}$. There exists a joint closed complement W of V and $N(A-\lambda)$, that is $X = V \oplus W = N(A-\lambda) \oplus W$. Let P be the projection such that $R(P) = W$ and $N(P) = V$. Then $P \in \mathcal{P}(X)$ and we show that $(A-\lambda)_P$ is left invertible. From $A-\lambda \in \Phi_l(X)$ it follows that $(A-\lambda)_P$ is a left Fredholm operator on $R(P)$, by (2.2.1). To prove that $(A-\lambda)_P$ is injective, suppose that $w \in W$ and $(A-\lambda)_P w = 0$. Then $P(A-\lambda)w = 0$ and hence $(A-\lambda)w \in N(P) \cap R(A-\lambda) = V \cap R(A-\lambda) = \{0\}$, which implies $w \in W \cap N(A-\lambda) = \{0\}$. Therefore, $(A-\lambda)_P$ is injective. This proves that $(A-\lambda)_P$ is left invertible, and hence $\lambda \notin \sigma_l(A_P)$.

To prove the converse in (2.4), suppose that $\lambda \notin \sigma_w^{right}(A)$. Then $A-\lambda \in \Phi_r(X)$ and $i(A-\lambda) \ge 0$. Hence $\alpha(A-\lambda) \ge \beta(A-\lambda)$ and $\beta(A-\lambda) < \infty$. Let M be a subspace of $N(A-\lambda)$ such that $\dim M = \operatorname{codim} R(A-\lambda) < \infty$. There exists a closed subspace V of X such that $X = M \oplus V$. Since $\operatorname{codim} V =$

codim $R(A-\lambda) < \infty$, there exists a joint complement W of V and $R(A-\lambda)$, that is

$$X = W \oplus V = W \oplus R(A-\lambda). \tag{2.5}$$

Let P be the projection such that $R(P) = V$ and $N(P) = W$, clearly, $P \in \mathcal{P}(X)$. Since $X = M \oplus V$ and $M \subset N(A-\lambda)$, we see that $(A-\lambda)V = (A-\lambda)X$, so $R((A-\lambda)_P) = P((A-\lambda)V) = P(R(A-\lambda))$. From (2.5) we get $P(R(A-\lambda)) = V$. Therefore, $R((A-\lambda)_P) = V$, i.e. $(A-\lambda)_P$ is onto. By (2.2.2) it follows that $(A-\lambda)_P$ is right Fredholm, and so $(A-\lambda)_P$ is right invertible. Hence $\lambda \notin \sigma_r(A_P)$. □

The following example shows that in general $\sigma_w^{left}(A) \neq \sigma_b^{left}(A)$ and $\sigma_w^{right}(A) \neq \sigma_b^{right}(A)$. This example was used in [9] and [7].

Example. Let H be a separable Hilbert space, let V be the right shift on H and let $N \in B(H)$ be quasinilpotent. If $A = V \oplus V^* \oplus N$, then $\sigma_b^{left}(A) = \sigma_b^{right}(A) = D$ and $\sigma_e^{left}(A) = \sigma_e^{right}(A) = \sigma_w^{left}(A) = \sigma_w^{right}(A) = \partial D \cup \{0\}$, where D is the closed unit ball.

Proof. Since $\sigma_b(A) = D$ [9] and $\sigma_b^+(A) = \sigma_b^-(A) = D$ [7], from $\sigma_b^+(A) \subset \sigma_b^{left}(A) \subset \sigma_b(A)$ and $\sigma_b^-(A) \subset \sigma_b^{right}(A) \subset \sigma_b(A)$ we get $\sigma_b^{left}(A) = \sigma_b^{right}(A) = D$.

From $\sigma_w(A) = \partial D \cup \{0\}$ [9], $\partial\sigma_w(A) \subset \sigma_e^+(A) \subset \sigma_e(A) \subset \sigma_w(A)$ and $\partial\sigma_w(A) \subset \sigma_e^-(A) \subset \sigma_w(A)$ we obtain $\sigma_e^+(A) = \sigma_e^-(A) = \sigma_e(A) = \partial D \cup \{0\}$. Since $\sigma_e^+(A) \subset \sigma_e^{left}(A) \subset \sigma_e(A)$ and $\sigma_e^-(A) \subset \sigma_e^{right}(A) \subset \sigma_e(A)$, it follows that $\sigma_e^{left}(A) = \sigma_e^{right}(A) = \partial D \cup \{0\}$. As $\sigma_e^{left}(A) \subset \sigma_w^{left}(A) \subset \sigma_w(A)$ and $\sigma_e^{right}(A) \subset \sigma_w^{right}(A) \subset \sigma_w(A)$, we get $\sigma_w^{left}(A) = \sigma_w^{right}(A) = \partial D \cup \{0\}$. □

Recall that for $A,\ B \in B(X)$ the following hold:
If $A,\ B \in \Phi_l(X)$ $(\Phi_r(X))$, then $BA \in \Phi_l(X)$ $(\Phi_r(X))$;
If $BA \in \Phi_l(X)$, then $A \in \Phi_l(X)$; If $BA \in \Phi_r(X)$, then $B \in \Phi_r(X)$.

Also recall that for $A,\ B \in B(X)$ it holds [4] (Theorem 7.9.2):
If $AB = BA$, then $A, B \in \mathcal{B}_+(X)$ $(\mathcal{B}_-(X))$ if and only if $AB \in \mathcal{B}_+(X)$ $(\mathcal{B}_-(X))$.

Now it is easy to see that the next statements hold.

Lemma 2.5. *Let $A,\ B \in B(X)$ and $AB = BA$. Then*
(2.5.1) $A, B \in \mathcal{B}_l(X) \iff AB \in \mathcal{B}_l(X)$,
(2.5.2) $A, B \in \mathcal{B}_r(X) \iff AB \in \mathcal{B}_r(X)$.

Theorem 2.6. *Let $A \in B(X)$ and let f be an analytic function defined in a neighborhood of $\sigma(A)$. Then*

$$f(\sigma_b^{left}(A)) = \sigma_b^{left}(f(A)),$$
$$f(\sigma_b^{right}(A)) = \sigma_b^{right}(f(A)).$$

Proof. Follows from Lemma 2.5, [6] (Chapter I, Theorem 6.4 and Theorem 6.8), (2.1.4), (2.1.5) and the fact that the left (right) Browder spectrum of any operator is non-empty set. □

Let us remark that previous theorem can be proved also in the following way: for $A \in B(X)$, $\sigma_b^{left}(A) = \sigma_e^{left}(A) \cup \sigma_b^+(A)$ and it is well-known that $f(\sigma_e^{left}(A)) = \sigma_e^{left}(f(A))$ [3] and $f(\sigma_b^+(A)) = \sigma_b^+(f(A))$ [7] (Theorem 3.4) for every analytic function f defined in a neighborhood of $\sigma(A)$. Thus,

$$\begin{aligned} f(\sigma_b^{left}(A)) &= f(\sigma_e^{left}(A) \cup \sigma_b^+(A)) = f(\sigma_e^{left}(A)) \cup f(\sigma_b^+(A)) \\ &= \sigma_e^{left}(f(A)) \cup \sigma_b^+(f(A)) = \sigma_b^{left}(f(A)). \end{aligned}$$

Similarly for the right Browder spectrum.

Let (G_n) be a sequence of compact subsets of $\mathbb{C}$. The limit superior, $\limsup G_n$, is the set of all λ in $\mathbb{C}$ such that every neighborhood of λ intersects infinitely many G_n.

It is known that $\mathcal{B}_+(X)$ and $\mathcal{B}_-(X)$ are open subsets in $B(X)$ [5] (Satz 4). Since the sets $\Phi_l(X)$ and $\Phi_r(X)$ are open, we conclude that $\mathcal{B}_l(X)$ and $\mathcal{B}_r(X)$ are open subsets in $B(X)$ and consequently, for $A \in B(X)$ the mapping $A \mapsto \sigma_b^{left}(A)$ is upper semi-continuous, i.e. if $A_n \in B(X)$ and $A_n \to A$, then $\limsup \sigma_b^{left}(A_n) \subset \sigma_b^{left}(A)$. Analogously, the mapping $A \mapsto \sigma_b^{right}(A)$ is upper semi-continuous.

If X and Y are infinite dimensional Banach spaces, $A \in B(X)$, $B \in B(Y)$ and $C \in B(Y, X)$, we denote

$$M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \in B(X \oplus Y).$$

Theorem 2.7. *For each $j \in \{e, w, b\}$ and $* \in \{+, -, left, right\}$ there is inclusion*

$$\sigma_j^*(M_C) \subset \sigma_j^*(A) \cup \sigma_j^*(B).$$

Particulary, if A and B are left (resp. right, upper, lower) Browder (Weyl), then M_C is left (resp. right, upper, lower) Browder (Weyl).

Proof. Let $M = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. By Lemma 2.1 it follows that $\sigma_j^*(M) = \sigma_j^*(A) \cup \sigma_j^*(B)$. Observe that

$$\begin{bmatrix} I & 0 \\ 0 & kI \end{bmatrix} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \frac{1}{k}I \end{bmatrix} = \begin{bmatrix} A & \frac{1}{k}C \\ 0 & B \end{bmatrix} = M_C^k \to M \text{ as } k \to \infty.$$

Since M_C^k and M_C are similar, it follows that $\sigma_j^*(M_C^k) = \sigma_j^*(M_C)$. By openess of all the relevant semigroups the mappings σ_j^* are each upper semicontinuous: thus indeed

$$\sigma_j^*(M_C) = \limsup \sigma_j^*(M_C^k) \subset \sigma_j^*(A) \cup \sigma_j^*(B).$$

□

3. Geometric characteristics

For $A \in B(X)$, the injectivity radius of A, denoted by $s_{inj}(A)$, is defined as follows:

$$\begin{aligned} s_{inj}(A) &= \inf\{|\lambda| \,:\, \lambda \in \sigma_a(A)\} \\ &= \max\{\epsilon \geq 0 \,:\, |\lambda| < \epsilon \Longrightarrow A - \lambda \text{ is bounded below}\}. \end{aligned}$$

The surjectivity radius of the operator A, denoted by $s_{sur}(A)$, is defined as follows:

$$\begin{aligned} s_{sur}(A) &= \inf\{|\lambda| \,:\, \lambda \in \sigma_d(A)\} \\ &= \max\{\epsilon \geq 0 \,:\, |\lambda| < \epsilon \Longrightarrow A - \lambda \text{ is onto}\}. \end{aligned}$$

The semi-Fredholm radius of A is

$$\begin{aligned} s(A) &= \inf\{|\lambda| : A - \lambda \text{ is not semi} - \text{Fredholm}\} \\ &= \max\{\epsilon \geq 0 \,:\, |\lambda| < \epsilon \Longrightarrow A - \lambda \text{ is semi} - \text{Fredholm}\}. \end{aligned}$$

Zemánek [11] proved the following results: If $A \in B(X)$ is bounded below, then

$$s(A) = \sup_{F \in F(X)} s_{inj}(A + F).$$

If $A \in B(X)$ is surjective, then

$$s(A) = \sup_{F \in F(X)} s_{sur}(A + F).$$

For $A \in B(X)$ we define the $\mathcal{G}_l$-radius $s_l(A)$ and $\mathcal{G}_r$-radius $s_r(A)$:

$$\begin{aligned} s_l(A) &= \inf\{|\lambda| : \lambda \in \sigma_l(A)\} = \max\{\epsilon \geq 0 \,:\, |\lambda| < \epsilon \Longrightarrow A - \lambda \in \mathcal{G}_l(X)\}, \\ s_r(A) &= \inf\{|\lambda| : \lambda \in \sigma_r(A)\} = \max\{\epsilon \geq 0 \,:\, |\lambda| < \epsilon \Longrightarrow A - \lambda \in \mathcal{G}_r(X)\}. \end{aligned}$$

Analogously, we define left and right Fredholm, Weyl and Browder radius of A:

$$s_\omega^*(A) = \text{dist}(0, \sigma_\omega^*(A)),$$

where $\omega = e, w, b$, and $* = left, right$, and also upper and lower semi-Browder radius of A:

$$\begin{aligned} s_b^+(A) &= \text{dist}(0, \sigma_b^+(A)), \\ s_b^-(A) &= \text{dist}(0, \sigma_b^-(A)). \end{aligned}$$

Using Zemánek's method of removing jumping points, we prove the following result.

Theorem 3.1. (3.1.1) *Let $A \in B(X)$ be a left invertible operator. Then*

$$s_b^{left}(A) = \sup_{AF=FA, F\in F(X)} s_l(A+F) = \sup_{AE=EA, E\in R(X)} s_l(A+E).$$

(3.1.2) *Let $A \in B(X)$ be a right invertible operator. Then*

$$s_b^{right}(A) = \sup_{AF=FA, F\in F(X)} s_r(A+F) = \sup_{AE=EA, E\in R(X)} s_r(A+E).$$

Proof. (3.1.1): Let $A \in \mathcal{G}_l(X)$, and let $D = \{\lambda \in \mathbb{C} : |\lambda| < s_b^{left}(A)\}$. Then $A - \lambda \in \Phi_+(X)$ for every $\lambda \in D$. According [2] (Theorem 3.2.20), $\alpha(A-\lambda)$ is equal to 0 everywhere in the disk D, except possibly in the set which is at most countable, and all points of this set are isolated. These points are called jumping points. The set of all accumulation points of the set of all jumping points can only be a subset of the boundary of D.

From (1.5) we obtain

$$s_b^{left}(A) = \operatorname{dist}(0, \sigma_b^{left}(A)) = \operatorname{dist}(0, \sigma_b^{left}(A+E)) \geq s_l(A+E)$$

for every $E \in R(X)$ which commute with A. Hence,

$$s_b^{left}(A) \geq \sup_{AE=EA, E\in R(X)} s_l(A+E) \geq \sup_{AF=FA, F\in F(X)} s_l(A+F). \tag{3.1}$$

If A does not have any jumping point in D, then

$$s_b^{left}(A) = s_l(A) \leq \sup_{AF=FA, F\in F(X)} s_l(A+F). \tag{3.2}$$

From (3.1) and (3.2) we get (3.1.1).

Suppose that A has the jumping points in D. Denote the jumping points such that

$$|\lambda_1| \leq |\lambda_2| \leq \dots |\lambda_n| \leq \dots < s_b^{left}(A).$$

Therefore, $s_l(A) = |\lambda_1|$.

Since $A - \lambda_1 \in \mathcal{B}_l(X)$, from [6] (Theorem 20.10) it follows that X is a direct sum of closed subspaces X_1 and X_2 in X, which are invariant for $A - \lambda_1$, i.e. they are invariant for A, $\dim X_1 < \infty$, $A - \lambda_1$ is nilpotent on X_1, and for the reduction $A_2 = A_{|X_2} : X_2 \to X_2$ we have $A_2 - \lambda_1$ is injective.

Let $\mu \in \mathbb{C}$ such that $|\mu| > \|A\| + s_b^{left}(A)$ and $F = \mu P$, where P is the projection from X onto X_1 along X_2. Let $\lambda \in D$. Then $\|A - \lambda\| \leq \|A\| + s_b^{left}(A) < |\mu|$, so $A - \lambda + \mu$ is invertible. Hence the reduction $(A + \mu - \lambda)_{|X_1} = (A + F - \lambda)_{|X_1} : X_1 \to X_1$ is invertible on X_1 and $N(A + F - \lambda) = N((A + F - \lambda)_{|X_1}) \oplus N((A + F - \lambda)_{|X_2}) = \{0\} \oplus N((A_2 - \lambda)_{|X_2}) = \{0\}$, for all $\lambda \in D \backslash \{\lambda_2, \dots, \lambda_n, \dots\}$. For all $\lambda \in D$ it holds $A - \lambda \in \mathcal{B}_l(X)$. Since $F \in F(X)$ and $AF = FA$, by (1.1) it follows that $A + F - \lambda \in \mathcal{B}_l(X)$. Therefore, $A + F - \lambda$ is left invertible for all $\lambda \in D \backslash \{\lambda_2, \dots, \lambda_n, \dots\}$.

Let $\epsilon > 0$. Then there exist only finitely many jumping points λ_i such that $|\lambda_i| < s_b^{left}(A) - \epsilon$. Therefore, applying the previous method finitely many times, we obtain the operator $F_1 \in F(X)$ such that $A + F_1 - \lambda$ is left invertible for $|\lambda| < s_b^{left}(A) - \epsilon$, i.e.

$$s_l(A + F_1) \geq s_b^{left}(A) - \epsilon. \tag{3.3}$$

From (3.3) and (3.1) we get (3.1.1).

The statement (3.1.2) can be proved similarly. □

Theorem 3.2. *Let $A \in B(X)$.*

(3.2.1) *If A is bounded below, then*

$$s_b^+(A) = \sup_{AF=FA, F\in F(X)} s_{inj}(A+F) = \sup_{AE=EA, E\in R(X)} s_{inj}(A+E).$$

(3.2.2) *If A is surjective, then*

$$s_b^-(A) = \sup_{AF=FA, F\in F(X)} s_{sur}(A+F) = \sup_{AE=EA, E\in R(X)} s_{sur}(A+E).$$

Proof. Analogously to Theorem 3.1, using [8] (Theorem 7). □

Theorem 3.3. *Let $J(X)$ be any non zero ideal of Riesz operators.*

(3.3.1) *If $A \in B(X)$ is left invertible, then*

$$\begin{aligned} s_w^{left}(A) &= s_e^{left}(A) = \sup_{F\in F(X)} s_l(A+F) = \sup_{E\in J(X)} s_l(A+E) \\ &= \sup_{E\in R(X), AE-EA\in P(\Phi(X))} s_l(A+E). \end{aligned}$$

(3.3.2) *If $A \in B(X)$ is right invertible, then*

$$\begin{aligned} s_w^{right}(A) &= s_e^{right}(A) = \sup_{F\in F(X)} s_r(A+F) = \sup_{E\in J(X)} s_r(A+E) \\ &= \sup_{E\in R(X), AE-EA\in P(\Phi(X))} s_r(A+E). \end{aligned}$$

Proof. The first equality in (3.3.1) and (3.3.2) follows from the continuity of the index. The other equalities in (3.3.1) and (3.3.2) follow from (1.3), (1.4), [6] (Theorem 16.21 and Corollary 12.4), and [1] (Chapter 5.2, Theorem 7), analogously to the proof of Theorem 3.1. □

For $A \in B(X)$, set

$$m_e^{left}(A) = \mathrm{dist}(A, B(X)\backslash\Phi_l(X)),$$
$$m_e^{right}(A) = \mathrm{dist}(A, B(X)\backslash\Phi_r(X)).$$

We extend some results from [14] to left and right Fredholm operators.

Theorem 3.4. *Let $A,\ B \in B(X)$. Then:*
(3.4.1) $m_e^{left}(A) > 0 \Longleftrightarrow A \in \Phi_l(X)$,
(3.4.2) $m_e^{left}(A+B) = m_e^{left}(A)$ for each $A \in B(X) \Longleftrightarrow B \in P(\Phi(X))$,
(3.4.3) $m_e^{left}(A+B) \leq m_e^{left}(A) + \|B\|$,
(3.4.4) $m_e^{left}(A+B) \leq m_e^{left}(A) + \|B\|_{P\Phi}$.
(3.4.5) *If* $\|B\|_{P\Phi} < m_e^{left}(A)$*, then* $A,\ A+B \in \Phi_l(X)$ *and* $i(A) = i(A+B)$.
(3.4.6) *If* $\|A\|_{P\Phi} < m_e^{left}(I)$*, then* $I - A \in \Phi(X)$ *and* $i(I-A) = 0$.
(3.4.7) *If* $\|A^n\|_{P\Phi} < m_e^{left}(I)$ *for some* $n > 1$*, then* $I - A \in \Phi(X)$ *and* $i(I-A) = 0$.
(3.4.8) *If* $AB - BA \in P(\Phi(X))$ *and*

$$r_e(B) < \overline{\lim_{n\to\infty}} (m_e^{left}(A^n))^{\frac{1}{n}},$$

then $A,\ A+B \in \Phi_l(X)$ *and* $i(A+B) = i(A)$.
(3.4.9) $s_e^{left}(A) \geq \overline{\lim_{n\to\infty}} (m_e^{left}(A^n))^{\frac{1}{n}}$.

Proof. (3.4.1): Clearly, since $\Phi_l(X)$ is open.

(3.4.2): ($\Longrightarrow$) Suppose that $m_e^{left}(A+B) = m_e^{left}(A)$ for each $A \in B(X)$. If $A \in \Phi_l(X)$, then $m_e^{left}(A) > 0$ by (3.4.1), and so $m_e^{left}(A+B) > 0$. It follows that $A+B \in \Phi_l(X)$. Therefore, $B \in P(\Phi_l(X)) = P(\Phi(X))$.

($\Longleftarrow$) Let $B \in P(\Phi(X)) = P(\Phi_l(X))$. Then $-B \in P(\Phi_l(X))$ and we have that $C \in \Phi_l(X)$ if and only if $C + B \in \Phi_l(X)$. Thus, $C \in B(X)\backslash\Phi_l(X)$ if and only if $C \in -B + B(X)\backslash\Phi_l(X)$. Consequently,

$$\begin{aligned} m_e^{left}(A) &= \inf\{\|A - C\| : C \in B(X)\backslash\Phi_l(X)\} \\ &= \inf\{\|A - (-B + C_1\| : C_1 \in B(X)\backslash\Phi_l(X)\} \\ &= \inf\{\|(A+B) - C_1\| : C_1 \in B(X)\backslash\Phi_l(X)\} \\ &= m_e^{left}(A+B). \end{aligned}$$

(3.4.3): Clearly.

(3.4.4): Let $P \in P(\Phi(X))$. According to (3.4.2) and (3.4.3) we have

$$m_e^{left}(A+B) = m_e^{left}(A+B+P) \leq m_e^{left}(A) + \|B+P\|,$$

which implies $m_e^{left}(A+B) \leq m_e^{left}(A) + \inf\{\|B+P\| : P \in P(\Phi(X))\} = m_e^{left}(A) + \|B\|_{P\Phi}$.

(3.4.5): Suppose that $\|B\|_{P\Phi} < m_e^{left}(A)$ and let $\lambda \in [0,1]$. From (3.4.4) it follows that

$$\begin{aligned} m_e^{left}(A) &= m_e^{left}(A + \lambda B + (-\lambda B)) \leq m_e^{left}(A + \lambda B) + \| - \lambda B\|_{P\Phi} \\ &= m_e^{left}(A+\lambda B) + \lambda\|B\|_{P\Phi} < m_e^{left}(A+\lambda B) + m_e^{left}(A), \end{aligned}$$

and so $m_e^{left}(A + \lambda B) > 0$. It follows that $A + \lambda B \in \Phi_l(X)$, and hence $A,\ A+B \in \Phi_l(X)$. Since the index is locally constant, we obtain $i(A+B) = i(A)$.

(3.4.6): Let $\|A\|_{P\Phi} < m_e^{left}(I)$. From (3.4.5) we get $I - A \in \Phi_l(X)$ and $i(I-A) = i(I) = 0$. Thus, $I - A \in \Phi(X)$.

(3.4.7): Let $\|A^n\|_{P\Phi} < m_e^{left}(I)$ for some $n > 1$ and let $\lambda \in [0,1]$. Then $\|(\lambda A)^n\|_{P\Phi} = \lambda^n \|A^n\|_{P\Phi} \le \|A^n\|_{P\Phi} < m_e^{left}(I)$ and from (3.4.6) it follows that $I - (\lambda A)^n \in \Phi(X)$. Since

$$\begin{aligned} I - (\lambda A)^n &= (I - \lambda A)(I + \lambda A + \ \dots\ + \lambda^{n-1}A^{n-1}) \\ &= (I + \lambda A + \ \dots\ + \lambda^{n-1}A^{n-1})(I - \lambda A), \end{aligned}$$

we conclude $I - \lambda A \in \Phi(X)$. Consequently, $I - A \in \Phi(X)$ and $i(I - A) = i(I) = 0$.

(3.4.8): Suppose that $AB - BA \in P(\Phi(X))$ and $r_e(B) < \overline{\lim\limits_{n\to\infty}} (m_e^{left}(A^n))^{\frac{1}{n}}$. Let ϵ be such that $r_e(B) < \epsilon < \overline{\lim\limits_{n\to\infty}} (m_e^{left}(A^n))^{\frac{1}{n}}$. Since $r_e(B) = \lim\limits_{n\to\infty} (\|B^n\|_{P\Phi})^{\frac{1}{n}}$ it follows that $\lim\limits_{n\to\infty} (\|B^n\|_{P\Phi})^{\frac{1}{n}} < \epsilon < \overline{\lim\limits_{n\to\infty}} (m_e^{left}(A^n))^{\frac{1}{n}}$. Thus there is an odd $n \in \mathbb{N}$ such that $(\|B^n\|_{P\Phi})^{\frac{1}{n}} < \epsilon < (m_e^{left}(A^n))^{\frac{1}{n}}$, and so $\|B^n\|_{P\Phi} < m_e^{left}(A^n)$. By (3.4.5) we get $A^n + B^n \in \Phi_l(X)$. Since $P(\Phi(X))$ two-sided ideal of $B(X)$, from $AB - BA \in P(\Phi(X))$ it follows that $A^n + B^n = C(A+B) + P$ where $C = A^{n-1} - BA^{n-2} + \ \dots\ + B^{n-1}$ and $P \in P(\Phi(X))$. Thus, $C(A+B) \in \Phi_l(X)$ and so $A + B \in \Phi_l(X)$. Let us remark that the proof above shows that $A + \lambda B \in \Phi_l(X)$ for $0 \le \lambda \le 1$, which implies that $i(A+B) = i(A)$.

(3.4.9): Suppose that $\overline{\lim\limits_{n\to\infty}} (m_e^{left}(A^n))^{\frac{1}{n}} > 0$. For $\lambda \in \mathbb{C}$, $|\lambda| < \overline{\lim\limits_{n\to\infty}} (m_e^{left}(A^n))^{\frac{1}{n}}$ and $B = \lambda I$ it follows that $r_e(B) = |\lambda| < \overline{\lim\limits_{n\to\infty}} (m_e^{left}(A^n))^{\frac{1}{n}}$. Since $AB = BA$, from (3.4.8) we get $\lambda I - A \in \Phi_l(X)$. Hence $s_e^{left}(A) \ge \overline{\lim\limits_{n\to\infty}} (m_e^{left}(A^n))^{\frac{1}{n}}$.

□

The next theorem is a dual part of Theorem 3.4.

Theorem 3.5. *Let* $A,\ B \in B(X)$. *Then*

(3.5.1) $m_e^{right}(A) > 0 \iff A \in \Phi_r(X)$,

(3.5.2) $m_e^{right}(A + B) = m_e^{right}(A)$ for each $A \in B(X) \iff B \in P(\Phi(X))$,

(3.5.3) $m_e^{right}(A + B) \le m_e^{right}(A) + \|B\|$,

(3.5.4) $m_e^{right}(A + B) \le m_e^{right}(A) + \|B\|_{P\Phi}$.

(3.5.5) *If* $\|B\|_{P\Phi} < m_e^{right}(A)$, *then* $A,\ A + B \in \Phi_r(X)$ *and* $i(A) = i(A+B)$.

(3.5.6) *If* $\|A\|_{P\Phi} < m_e^{right}(I)$, *then* $I - A \in \Phi(X)$ *and* $i(I - A) = 0$.

(3.5.7) *If* $\|A^n\|_{P\Phi} < m_e^{right}(I)$ *for some* $n > 1$, *then* $I - A \in \Phi(X)$ *and* $i(I - A) = 0$.

(3.5.8) *If* $AB - BA \in P(\Phi(X))$ *and*

$$r_e(B) < \overline{\lim_{n\to\infty}} (m_e^{right}(A^n))^{\frac{1}{n}},$$

then $A,\ A + B \in \Phi_r(X)$ *and* $i(A + B) = i(A)$.

(3.5.9) $s_e^{right}(A) \ge \overline{\lim\limits_{n\to\infty}} (m_e^{right}(A^n))^{\frac{1}{n}}$.

Theorem 3.6. *Let* $A,\ B \in B(X)$.
(3.6.1) *If*

$$\|A - B\|_{P\Phi} < m_e^{left}(A) + m_e^{left}(B),$$

then $A,\ B \in \Phi_l(X)$ *and* $i(A) = i(B)$.
(3.6.2) *If*

$$\|A - B\|_{P\Phi} < m_e^{right}(A) + m_e^{right}(B),$$

then $A,\ B \in \Phi_r(X)$ *and* $i(A) = i(B)$.

Proof. (3.6.1): Suppose that $\|A-B\|_{P\Phi} < m_e^{left}(A)+m_e^{left}(B)$. Then $m_e^{left}(A)+m_e^{left}(B) > 0$, and so $m_e^{left}(A)$ and $m_e^{left}(B)$ can not be at the same time equal to zero. If one of them, say $m_e^{left}(B)$, is equal to zero, then $m_e^{left}(A) > 0$. Now from $\|A - B\|_{P\Phi} < m_e^{left}(A)$, by (3.4.5) we conclude $B \in \Phi_l(X)$, that is $m_e^{left}(B) > 0$, which is a contradiction. Therefore, $m_e^{left}(A) > 0$ and $m_e^{left}(B) > 0$, and so $A,\ B \in \Phi_l(X)$. There exists $P \in P(\Phi(X))$ such that

$$\|A - B - P\| < m_e^{left}(A) + m_e^{left}(B).$$

Let $C = B + P$. From (3.4.2) it follows that $m_e^{left}(C) = m_e^{left}(B)$, and so we get

$$\|A - C\| < m_e^{left}(A) + m_e^{left}(C).$$

Therefore, the open ball centered at A with radii $m_e^{left}(A)$ and the open ball centered at C with radii $m_e^{left}(C)$ have a non-empty intersection. Hence their union is linearly connected set contained in $\Phi_l(X) \subset \Phi_+(X)$. Since the index is locally constant, it follows that $i(A) = i(C)$. For $\lambda \in [0, 1]$ we have $\lambda P \in P(\Phi(X))$, which implies $B + \lambda P \in \Phi_l(X)$. Again from local constancy of the index we conclude $i(B) = i(B + P)$. Therefore $i(A) = i(B)$. □

Let us remark that $P(\mathcal{W}_l(X)) = P(\Phi(X))$ and $P(\mathcal{W}_r(X)) = P(\Phi(X))$ and analogous assertions can be formulated for left and right Weyl operators and the quantities:

$$m_w^{left}(A) = \operatorname{dist}(A, B(X)\backslash\mathcal{W}_l(X)),$$
$$m_w^{right}(A) = \operatorname{dist}(A, B(X)\backslash\mathcal{W}_r(X)),\ A \in B(X).$$

Notice that if $m_w^{left}(A) > 0$ ($m_w^{right}(A) > 0$), i.e. if $A \in \mathcal{W}_l(X)$ ($A \in \mathcal{W}_r(X)$), then because of local constancy of the index it holds $m_w^{left}(A) = m_e^{left}(A)$ ($m_w^{right}(A) = m_e^{right}(A)$).

Moreover, the following more general assertions can be proved analogously.

Theorem 3.7. *Let* $\mathcal{U}$ *be an open subset of* $\Phi_\pm(X)$ *such that* $\mu\mathcal{U} \subset \mathcal{U}$ *for every* $\mu \neq 0$. *For* $A \in B(X)$, *set*

$$m_{\mathcal{U}}(A) = \operatorname{dist}(A, B(X)\backslash\mathcal{U}),$$
$$\|A\|_{P(\mathcal{U})} = \inf\{\|A + P\| : P \in P(\mathcal{U})\},$$

where $P(\mathcal{U})$ *is the perturbation class of* $\mathcal{U}$.

Then, for $A,\ B \in B(X)$, the following hold:

(3.7.1) $m_{\mathcal{U}}(A) > 0 \Longleftrightarrow A \in \mathcal{U}$;

(3.7.2) $m_{\mathcal{U}}(A+B) = m_{\mathcal{U}}(A)$ for every $A \in B(X) \Longleftrightarrow B \in P(\mathcal{U})$;

(3.7.3) $m_{\mathcal{U}}(A+B) \leq m_{\mathcal{U}}(A) + \|B\|$;

(3.7.4) $m_{\mathcal{U}}(A+B) \leq m_{\mathcal{U}}(A) + \|B\|_{P(\mathcal{U})}$;

(3.7.5) *If* $\|B\|_{P(\mathcal{U})} < m_{\mathcal{U}}(A)$, *then* $A,\ A+B \in \mathcal{U}$ *and* $i(A) = i(A+B)$.

(3.7.6) *If* $\|A\|_{P(\mathcal{U})} < m_{\mathcal{U}}(I)$, *then* $I - A \in \Phi(X)$ *and* $i(I-A) = 0$.

(3.7.7) *If* $\|A^n\|_{P(\mathcal{U})} < m_{\mathcal{U}}(I)$ *for* $n > 1$, *then* $I - A \in \Phi(X)$ *and* $i(I-A) = 0$.

(3.7.8) *If*

$$\|A - B\|_{P(\mathcal{U})} < m_{\mathcal{U}}(A) + m_{\mathcal{U}}(B),$$

then $A,\ B \in \mathcal{U}$ *and* $i(A) = i(B)$.

Theorem 3.8. *Let $\mathcal{U}$ be an open subset of $\Phi_{\pm}(X)$ such that*
(i) $\mu\mathcal{U} \subset \mathcal{U}$ *for every* $\mu \neq 0$,
(ii) $I \in \mathcal{U}$,
(iii) $K(X) \subset P(\mathcal{U})$.
Then

$$r_e(A) = \lim_{n\to\infty} (\|A^n\|_{P(\mathcal{U})})^{\frac{1}{n}}. \tag{3.4}$$

Proof. Let $\lambda \in \mathbb{C}$ and $|\lambda| > (m_{\mathcal{U}}(I))^{-\frac{1}{n}}(\|A^n\|_{P(\mathcal{U})})^{\frac{1}{n}}$ for some $n \in \mathbb{N}$. Then $m_{\mathcal{U}}(I) > \|(A/\lambda)^n\|_{P(\mathcal{U})}$ and by (3.7.7) it follows that $\lambda I - A \in \Phi(X)$. Therefore $r_e(A) \leq (m_{\mathcal{U}}(I))^{-\frac{1}{n}}(\|A^n\|_{P(\mathcal{U})})^{\frac{1}{n}}$ for all $n \in \mathbb{N}$. Notice that from $I \in \mathcal{U}$ it follows that $m_{\mathcal{U}}(I) > 0$. Therefore,

$$r_e(A) \leq \lim_{n\to\infty} (m_{\mathcal{U}}(I))^{-\frac{1}{n}} \varliminf_{n\to\infty} (\|A^n\|_{P(\mathcal{U})}))^{\frac{1}{n}} = \varliminf_{n\to\infty} (\|A^n\|_{P(\mathcal{U})})^{\frac{1}{n}}.$$

Since $K(X) \subset P(\mathcal{U})$, it follows that $\|A^n\|_{P(\mathcal{U})} \leq \|\pi(A^n)\|$ for every $n \in \mathbb{N}$. Thus

$$r_e(A) \leq \varliminf_{n\to\infty} (\|A^n\|_{P(\mathcal{U})})^{\frac{1}{n}} \leq \varlimsup_{n\to\infty} (\|A^n\|_{P(\mathcal{U})})^{\frac{1}{n}} \leq \lim_{n\to\infty} (\|\pi(A^n)\|)^{\frac{1}{n}} = r_e(A),$$

which implies (3.4). □

Theorem 3.9. *Let $\mathcal{U}$ be an open subset of $\Phi_{\pm}(X)$ such that*
(i) $\mu\mathcal{U} \subset \mathcal{U}$ *for every* $\mu \neq 0$,
(ii) $I \in \mathcal{U}$,
(iii) $K(X) \subset P(\mathcal{U})$,
(iv) $\mathcal{G}\mathcal{U} \subset \mathcal{U}$ *and* $\mathcal{U}\mathcal{G} \subset \mathcal{U}$,
(v) $(\forall A, B \in B(X))(AB \in \mathcal{U} \Longrightarrow A \in \mathcal{U})$ *or* $(\forall A, B \in B(X))(AB \in \mathcal{U} \Longrightarrow B \in \mathcal{U})$.
Then, for $A, B \in B(X)$, the following hold:

(3.9.1) *If* $AB - BA \in P(\mathcal{U})$ *and*

$$r_e(B) < \varlimsup_{n\to\infty} (m_{\mathcal{U}}(A^n))^{\frac{1}{n}},$$

then $A,\ A+B \in \mathcal{U}$ *and* $i(A+B) = i(A)$.

(3.9.2) $s_{\mathcal{U}}(A) \geq \overline{\lim\limits_{n\to\infty}}(m_{\mathcal{U}}(A^n))^{\frac{1}{n}}$, *where* $s_{\mathcal{U}}(A) = \max\{\epsilon \geq 0 : |\lambda| < \epsilon \Rightarrow A - \lambda \in \mathcal{U}\}$.

Proof. Follows from Theorem 3.8 and (3.7.5). □

For $A \in B(X)$, let

$$\|A\|_{P\Phi_+} = \inf\{\|A+P\| : P \in P(\Phi_+(X))\},$$
$$\|A\|_{P\Phi_-} = \inf\{\|A+P\| : P \in P(\Phi_-(X))\},$$

and

$$m_e^+(A) = \mathrm{dist}(A, B(X)\backslash\Phi_+(X)),$$
$$m_e^-(A) = \mathrm{dist}(A, B(X)\backslash\Phi_-(X)).$$

If we take $\Phi_+(X)$ or $\Phi_-(X)$ for $\mathcal{U}$ in (3.7.8), we get:

Corollary 3.10. *Let* $A,\ B \in B(X)$. *If*

$$\|A - B\|_{P\Phi_+} < m_e^+(A) + m_e^+(B),$$

or

$$\|A - B\|_{P\Phi_-} < m_e^-(A) + m_e^-(B),$$

then $i(A) = i(B)$.

From Corollary 3.10 we obtain Theorem 4 in [10]:

Let $T,\ S \in B(X)$. If $\|\pi(T-S)\| < m_e^+(T) + m_e^+(S)$ or $\|\pi(T-S)\| < m_e^-(T) + m_e^-(S)$, then $i(T) = i(S)$.

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