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Left symmetric left distributive operations on a group

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Abstract. We study equational theories of several left symmetric left distributive operations on groups. Normal forms of terms in the variety of LSLD groupoids, LSLD medial groupoids, LSLD idempotent groupoids and LSLD medial idempotent groupoids are found.

1. Introduction

Let G be a group. We define a new operation $*$ on G by

$$a * b = ab^{-1}a.$$

It is easy to check that the resulting groupoid $G(*)$, called *the core of* G, satisfies the following equations:

$$x(xy) \approx y, \qquad \text{(left symmetry)}$$
$$x(yz) \approx (xy)(xz), \qquad \text{(left distributivity)}$$
$$xx \approx x. \qquad \text{(idempotency)}$$

Actually, the variety LSLDI of left symmetric left distributive idempotent groupoids is generated by all $G(*)$, G a group, according to R.S. Pierce [8]. An important particular case is that of cores of abelian groups. They satisfy an additional equation

$$(xy)(uv) \approx (xu)(yv), \qquad \text{(mediality)}$$

and, in fact, they generate the variety LSMI of left symmetric medial idempotent groupoids, according to D. Joyce [4] (note that mediality and idempotency imply both left and right distributivity, while LSM non-idempotent groupoids are not necessarily left distributive); LSLDM is the variety of left distributive members of LSM.

LSLDI groupoids were studied since the 1970's and they were also called symmetric sets (N. Nobusawa in [7] and others), symmetric groupoids (R.S. Pierce [8], [9]) or involutory quandles (D. Joyce [4], [5]). A survey of results can be found in [12]. LSMI groupoids were extensively studied in a series of papers by B. Roszkowska,

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see e.g. [10], [11] and references there. The core operation was introduced in the book [1] of R.H. Bruck, more generally for loops, by $a * b = a(b\backslash a)$. One can check that cores of (left) Bol loops are LSLDI. (A Bol loop is a quasigroup with a unit element satisfying $x(y(xz)) \approx (x(yx))z$.) Cores play a significant role in the structure theory of LSLDI groupoids. For instance, D. Joyce proved in [5] that an LSLDI groupoid is simple if and only if it is either the core of a simple group, or a conjugacy class of involutions in a non-abelian simple group under the core operation. Concerning general selfdistributive structures, we recommend the book [3].

The purpose of this note is to look at the non-idempotent case, which seems to be unexplored, except for the short paper [6] of T. Kepka. Let G be a group and g an involutory *antiautomorphism* of G, i.e. an involution of G satisfying $g(xy) = g(y)g(x)$, such that $x^2 g(x^2) = 1$ and $xg(x)$ is in the center of G for every $x \in G$. We define a new operation $*_g$ on G by

$$a *_g b = ag(b)a.$$

It is easy to check that the resulting groupoid $G(*_g)$ is LSLD and $a \in G$ satisfies $a *_g a = a$ iff $g(a) = a^{-1}$. It turns out that the class of all $G(*_g)$ generates the variety of LSLD groupoids. Some other LSLD operations on groups will also be discussed.

In the first part, we briefly reprove the results of Pierce and Joyce, finding a useful normal form for terms in LSLDI and LSMI. In the second part we introduce several non-idempotent LSLD group operations and investigate their equational theories. Normal forms are also found.

Our terminology and notation are rather standard, consult e.g. [2]. Letters in groupoid terms are right associated, e.g. $xyz = x \cdot yz = x(yz)$. The center of a group G is denoted $\mathrm{Z}(G)$. We denote $\mathbf{F}_{\mathcal{V}}(X)$ the free groupoid over a set X in a variety $\mathcal{V}$ and $\mathbf{FG}(X)$ ($\mathbf{FA}(X)$, resp.) the free group (free abelian group, resp.) over X.

2. The idempotent case

Let G be a group. We put for all $a, b \in G$

(a) $a * b = ab^{-1}a$;
(b) $a \circ_f b = af(a^{-1}b)$ for an involutory automorphism f of G;
(c) $a \diamond_f b = af(ba^{-1})$ for an involutory automorphism f of G satisfying $xf(x) \in \mathrm{Z}(G)$ for all $x \in G$.

One can check that the defined operations are LSLDI. Moreover, the operation $\diamond_f$ is medial. Let $\mathcal{A}_{\mathrm{I}}$ ($\mathcal{B}_{\mathrm{I}}$, $\mathcal{C}_{\mathrm{I}}$ resp.) be the variety generated by all $G(*)$ ($G(\circ_f)$, $G(\diamond_f)$, resp.) and $\mathcal{A}_{\mathrm{Iab}}$ the variety generated by all $G(*)$, G an *abelian* group.

Theorem 2.1. *(1) The varieties* $\mathcal{A}_{\mathrm{I}}$, $\mathcal{B}_{\mathrm{I}}$ *and* LSLDI *coincide.*
(2) The varieties $\mathcal{A}_{\mathrm{Iab}}$, $\mathcal{C}_{\mathrm{I}}$ *and* LSMI *coincide.*

We are going to prove the theorem in a series of lemmas.

Lemma 2.2. *Let* X *be a non-empty set.*

(1) *Each element of the free LSLDI groupoid* $\mathbf{F}_{\mathrm{LSLDI}}(X)$ *can be written in the form* $x_1x_2\dots x_{n-1}x_n$, *where*

$$x_1,\dots,x_n \in X \text{ and } x_i \neq x_{i+1},\ i=1,\dots,n-1. \qquad (\dagger)$$

(2) *Each element of the free LSMI groupoid* $\mathbf{F}_{\mathrm{LSMI}}(X)$ *can be written in the form* $x_1x_2\dots x_{n-1}x$, *where*

$$\begin{gathered} x_1,\dots,x_{n-1},x\in X, \quad \{x_1,x_3,\dots\}\cap\{x_2,x_4,\dots\}=\emptyset, \\ x\notin\{x_{n-1},x_{n-3},\dots\} \text{ and } x_1\preceq x_3\preceq\dots,\ x_2\preceq x_4\preceq\dots, \end{gathered} \qquad (\ddagger)$$

where $(X,\preceq)$ *is a linear order.*

Proof. Firstly, observe that (using left symmetry and left distributivity)

$$xy\cdot z \approx xy\cdot xxz \approx xyxz. \qquad (\S)$$

(1) We prove by induction that each term $t\in\mathbf{F}_{\mathrm{LSLDI}}(X)$ can be written in the form (†). The statement is trivial for $t\in X$. Given two terms $t=x_1x_2\dots x_{n-1}x_n$, $s=y_1y_2\dots y_{m-1}y_m$ in the form (†), using (§) n-times we get

$$\begin{aligned} ts &= (x_1x_2\dots x_{n-1}x_n)s \approx x_1(x_2\dots x_{n-1}x_n)x_1s \\ &\approx x_1x_2(x_3\dots x_{n-1}x_n)x_2x_1s \approx \dots \approx x_1x_2\dots x_{n-1}x_nx_{n-1}\dots x_2x_1s. \end{aligned}$$

If $x_1\neq y_1$, we have ts in the form (†). Otherwise, using left symmetry, we get rid of identical variables and we obtain ts in the form (†).

(2) Note that in LSLD mediality is equivalent to the equation $xyzt\approx zyxt$. Consequently, the statement follows from (1). □

Let Z be a countable infinite set. Put $\widetilde{Z}=\{\widetilde{x}:x\in Z\}$, a disjoint copy of Z, and $Y=Z\cup\widetilde{Z}$.

(a) Let A be the subgroupoid of $\mathbf{FG}(Z)(*)$ generated by the set Z.
(b) Let B be the subgroupoid of $\mathbf{FG}(Y)(\circ_f)$ generated by the set Z, where $f:\mathbf{FG}(Y)\to\mathbf{FG}(Y)$ is the automorphism determined by $f(x)=\widetilde{x}$, $f(\widetilde{x})=x$, $x\in Z$.
(c) Let C be the subgroupoid of $\mathbf{FA}(Z)(*)=\mathbf{FA}(Z)(\diamond_\iota)$ generated by the set Z (where $\iota(x)=-x$).

Lemma 2.3. *(1) All terms of the form (†) represent in the groupoids A and B pairwise different elements (identifying term variables $x_1, x_2, \dots$ with elements of Z).*

(2) All terms of the form (‡) represent in the groupoid C pairwise different elements (identifying term variables $x_1, x_2, \dots$ with elements of Z).

Proof. (a) Given $x_1, \dots, x_n \in Z$ such that $x_i \neq x_{i+1}$, $i = 1, \dots, n-1$, put $\varepsilon = (-1)^n$. Then

$$x_1 * x_2 * \dots x_{n-1} * x_n = x_1 x_2^{-1} \dots x_{n-1}^{\varepsilon} x_n^{-\varepsilon} x_{n-1}^{\varepsilon} \dots x_2^{-1} x_1.$$

Clearly, this word is not reducible and thus all such words must be pairwise different.

(b) Given $x_1, \dots, x_n \in Z$ such that $x_i \neq x_{i+1}$, $i = 1, \dots, n-1$, put $\varepsilon = 0$ for n odd and $\varepsilon = 1$ for n even. Then

$$x_1 \circ_f x_2 \circ_f \dots x_{n-1} \circ_f x_n = x_1 f(x_1^{-1}) f(x_2) x_2^{-1} x_3 f(x_3^{-1}) \dots f^{\varepsilon}(x_{n-1}^{-1}) f^{\varepsilon}(x_n).$$

Clearly, this word is not reducible and thus all such words must be pairwise different.

(c) Given $x_1, \dots, x_n \in Z$ such that $\{x_1, x_3, \dots\} \cap \{x_2, x_4, \dots\} = \emptyset$, we have (denoting the free group operation additively)

$$x_1 * x_2 * \dots x_{n-1} * x_n = 2x_1 + 2x_3 + \dots - 2x_2 - 2x_4 - \dots \pm x_n.$$

Clearly, the value uniquely determines the term up to the order of $x_1, x_3, \dots$ and $x_2, x_4, \dots$. □

Corollary 2.4. *(1) For every binary term over X there is a unique LSLDI-equivalent term of the form (†).*

(2) For every binary term over X there is a unique LSMI-equivalent term of the form (‡).

Now, we are ready to prove Theorem 2.1. As noted above, $\mathcal{A}_{\mathrm{I}}, \mathcal{B}_{\mathrm{I}} \subseteq$ LSLDI and $\mathcal{C}_{\mathrm{I}}, \mathcal{A}_{\mathrm{Iab}} \subseteq$ LSMI. According to Lemmas 2.3 and 2.4, the groupoids A, B and $\mathbf{F}_{\mathrm{LSLDI}}(\omega)$ are isomorphic and also the groupoids C and $\mathbf{F}_{\mathrm{LSMI}}(\omega)$ are isomorphic. Thus $\mathcal{A}_{\mathrm{I}} = \mathcal{B}_{\mathrm{I}} =$ LSLDI and $\mathcal{C}_{\mathrm{I}} = \mathcal{A}_{\mathrm{Iab}} =$ LSMI.

Remark 2.5. R.S. Pierce found another core representation of the countably generated free LSLDI groupoid: it is isomorphic to the subgroupoid generated by all involutions in the core of the countable infinite free power of the two-element group.

Remark 2.6. The quasivariety generated by group cores (or even by cores of Bol loops) is strictly smaller than the (quasi)variety of LSLDI groupoids, since it satisfies the quasiequation $xy \approx y \Rightarrow yx \approx x$, which does not hold e.g. in the following LSLDI groupoid:

	0	1	2
0	0	1	2
1	0	1	2
2	1	0	2

The same holds for operations $\circ_f$ and $\diamond_f$ as well. It is an open question to find a basis of the quasivariety generated by cores of groups.

3. The non-idempotent case

We present non-idempotent generalizations of the above operations and we introduce a new operation. Let G be a group. We put for all $a, b \in G$

(a) $a *_g b = ag(b)a$ for an involutory *antiautomorphism* g of G satisfying $x^2 g(x^2) = 1$ and $xg(x) \in \mathrm{Z}(G)$ for all $x \in G$;
(b) $a \circ_{f,e} b = af(a^{-1}be)$ for an involutory automorphism f of G and $e \in G$ satisfying $e^2 = 1$, $f(e) = e$;
(c) $a \diamond_{f,e} b = af(ba^{-1}e)$ for an involutory automorphism f of G and $e \in G$ satisfying $e^2 = 1$, $ex = xf(e)$ and $xf(x) \in \mathrm{Z}(G)$ for all $x \in G$;
(d) $a \odot_e b = aea^{-1}b$ for $e \in G$ such that $e^2 = 1$.

One can check that the introduced operations are LSLD. Moreover, the operation $\diamond_{f,e}$ is medial. They are idempotent, if and only if

(a) $g(x) = x^{-1}$ for all $x \in G$; in which case $G(*_g) = G(*)$.
(b,c) $e = 1$; in which case $G(\circ_{f,1}) = G(\circ_f)$ and $G(\diamond_{f,1}) = G(\diamond_f)$.
(d) $e = 1$; in which case $G(\odot_1)$ is a right zero band (it satisfies $xy \approx y$).

Further,

(a) $a *_g a = a$ iff $g(a) = a^{-1}$;
(b,c,d) $G(\circ_{f,e})$, $G(\diamond_{f,e})$, $G(\odot_e)$ are idempotent-free, whenever $e \neq 1$.

Again, we consider $\mathcal{A}$ ($\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ resp.) to be the variety generated by all $G(*_g)$ ($G(\circ_{f,e})$, $G(\diamond_{f,e})$, $G(\odot_e)$, resp.) and A_{ab} the variety generated by all $G(*_g)$, G an *abelian* group. Obviously, $\mathcal{A}_\mathrm{I} \subset \mathcal{A}$, $\mathcal{A}_{\mathrm{Iab}} \subset \mathcal{A}_{\mathrm{ab}}$, $\mathcal{B}_\mathrm{I} \subset \mathcal{B}$, $\mathcal{C}_\mathrm{I} \subset \mathcal{C}$.

Theorem 3.1. *(1) The varieties $\mathcal{A}$, $\mathcal{B}$ and* LSLD *coincide.*
(2) The varieties $\mathcal{A}_{\mathrm{ab}}$, $\mathcal{C}$ and LSLDM *coincide.*
(3) The varieties $\mathcal{D}$ and LSLD *coincide.*

Firstly, we state several general structural results (following [6]). For an LSLD groupoid H we put $ip_H = \{(x, xx) : x \in H\} \cup id_H$. It is easy to see that $\mathbf{F}_{\mathrm{LSLD}}(1) = \mathbf{F}_{\mathrm{LSLDM}}(1)$ is the (unique up to an isomorphism) two-element LSLD groupoid; we denote it $\mathbf{T}$.

$\mathbf{T}$	0	1
0	1	0
1	1	0

Lemma 3.2. *Let H be an LSLD groupoid. Then ip_H is a congruence on H, H/ip_H is idempotent and ip_H is the smallest congruence such that the corresponding factor is idempotent. Moreover, if A is a non-trivial block of ip_H, then $A \simeq \mathbf{T}$.*

Proof. Easy. □

Lemma 3.3. *Let H be an LSLD groupoid. Then $\mathbf{T}$ is a homomorphic image of H if and only if H is isomorphic to $\mathbf{T} \times (H/ip_H)$.*

Proof. Choose a projection $g : H \to \mathbf{T}$ and denote $h : H \to H/ip_H$ the natural projection. Then $f : H \to \mathbf{T} \times (H/ip_H)$, defined by $f(x) = (g(x), h(x))$ for every $x \in H$, is a homomorphism. Since $\mathbf{T}$ is idempotent-free, so is H. Thus f is onto, because h is so, and $f(x) = (0, h(x))$ iff $f(xx) = (1, h(x))$. Further, if $g(x) = g(y)$, then $(x, y) \notin ip_H$, since $\mathbf{T}$ is idempotent-free, and therefore $h(x) \neq h(y)$. Hence f is injective and thus an isomorphism. The converse is obvious. □

Proposition 3.4. *Let $\mathcal{V}$ be a non-idempotent subvariety of* LSLD *and $\mathcal{W}$ the subvariety of idempotent groupoids from $\mathcal{V}$. Let X be a non-empty set. Then $\mathbf{F}_\mathcal{V}(X)$ is isomorphic to $\mathbf{T} \times \mathbf{F}_\mathcal{W}(X)$. Consequently, the variety $\mathcal{V}$ is generated by $\mathcal{W} \cup \{\mathbf{T}\}$.*

Proof. By Lemma 3.2, $\mathbf{T}$ is in every non-idempotent subvariety of LSLD. Hence $\mathbf{T}$ is a homomorphic image of $\mathbf{F}_\mathcal{V}(X)$ and, by Lemma 3.3, $\mathbf{F}_\mathcal{V}(X) \simeq \mathbf{T} \times H$, where $H \simeq \mathbf{F}_\mathcal{V}(X)/ip_{\mathbf{F}_\mathcal{V}(X)}$. Clearly, $H \simeq \mathbf{F}_\mathcal{W}(X)$. □

Corollary 3.5. *(1) For every binary term over X there is a unique LSLD-equivalent term of the form $x_1 x_2 \dots x_{n-1} x_n$, where*

$$x_1, \dots, x_n \in X \text{ and } x_i \neq x_{i+1}, \ i = 1, \dots, n-2. \qquad (\dagger\dagger)$$

(2) For every binary term over X there is a unique LSLDM-equivalent term of the form $x_1 x_2 \dots x_{n-1} x$, where

$$x_1, \dots, x_{n-1}, x \in X, \quad \{x_1, x_3, \dots\} \cap \{x_2, x_4, \dots\} = \emptyset,$$
$$x_1 \preceq x_3 \preceq \dots, \ x_2 \preceq x_4 \preceq \dots, \qquad (\ddagger\ddagger)$$

where $(X, \preceq)$ is a linear order.

Proof of Theorem 3.1. (1) With respect to Proposition 3.4 and $\mathcal{A}_\mathrm{I} = \mathcal{B}_\mathrm{I} =$ LSLDI, it suffices to prove that $\mathbf{T} \in \mathcal{A}$ and $\mathbf{T} \in \mathcal{B}$. It is easy to check that, in the first case, $\mathbf{T}$ is isomorphic to the subgroupoid $\{1, 3\}$ of $\mathbb{Z}_4(*_{id})$ and, in the second case, $\mathbf{T}$ is isomorphic to $\mathbb{Z}_2(\circ_{id,1})$.

(2) With respect to Proposition 3.4 and $\mathcal{C}_{\mathrm{I}} = \mathcal{A}_{\mathrm{Iab}} = \mathrm{LSMI}$, it suffices to prove that $\mathbf{T} \in \mathcal{A}_{\mathrm{ab}}$ and $\mathbf{T} \in \mathcal{C}$. Use exactly the same representations (observe that $\mathbb{Z}_2(\circ_{id,1}) = \mathbb{Z}_2(\diamond_{id,1})$).

(3) We construct a groupoid $D \in \mathcal{D}$ isomorphic to $\mathbf{F}_{\mathrm{LSLD}}(\omega)$. Let Z be a countable infinite set and D the subgroupoid of $(\mathbb{Z}_2 \amalg \mathbf{FG}(Z))(\odot_e)$ generated by Z, where $\amalg$ means the free product of two groups and e is the non-unit element of $\mathbb{Z}_2$. Then D is isomorphic to $\mathbf{F}_{\mathrm{LSLD}}(Z)$, because for $x_1, \dots, x_n \in Z$ with $x_i \neq x_{i+1}$, $i = 1, \dots, n-2$, the word

$$x_1 \odot_e x_2 \odot_e \dots x_{n-1} \odot_e x_n = x_1 e x_1^{-1} x_2 e x_2^{-1} \dots x_{n-1} e x_{n-1}^{-1} x_n$$

is reducible if and only if $x_{n-1} = x_n$, and in this case it obviously cannot be obtained by any other term in the form (††). □

Remark 3.6. There is another analogy with the core of a group G. For an involutory *automorphism* f of G such that $x^2 f(x^2) = 1$ and $xf(x) \in \mathrm{Z}(G)$ for all $x \in G$ put, again, $a \star_f b = af(b)a$ for all $a, b \in G$. Then $G(\star_f)$ is an LSLD groupoid, it is idempotent iff $f(x) = x^{-1}$, and in this case $G(\star_f) = G(*)$; but then G must be abelian. The variety generated by all $G(\star_f)$ is a proper subvariety of LSLD, since it satisfies the equation $xyzxyzszyxzyxt \approx st$.

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