LEGENDRE FUNCTIONS OF FRACTIONAL ORDER*

BY

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Introduction. In the theory of the propagation of spherical waves in free space the angular wave functions are Legendre polynomials, $P_n(\cos\theta)$, or associated Legendre polynomials, $P_n^m(\cos\theta)$, where n and m are restricted to integral values. These functions are polynomials in $\cos\theta$, their properties have been widely studied, numerical values have been tabulated, and in general they may be regarded as known functions.

In more recent years, however, Legendre functions of non-integral order, which we shall denote by $P_\nu(\cos\theta)$, have also occurred in physical problems. Thus, for wave propagation inside a circular horn of given angle, the boundary conditions introduce a characteristic equation which is actually an equation in the parameter ν. It has been customary to simplify the problem by choosing horn angles corresponding to integral values of ν, but a complete solution should include a study of the behavior of $P_\nu(\cos\theta)$ as a function of ν.

Similarly, in the mode theory of antennas developed by Schelkunoff the appropriate angular wave functions in the antenna region are Legendre functions of order $n + 120/K$, where n is an integer and K is the characteristic impedance of the biconical antenna to the principal wave. For thin cones K is large and the order of the Legendre functions is nearly, but not quite, integral. Further, when the cone angle is large, ν may have quite general real values.

Another application has appeared early this year, when P. Grivet† used Legendre functions of fractional order in the approximate solution of an electron lens problem, with particular emphasis on small values of ν.

Thus it appears that the properties of Legendre functions of non-integral order are of quite general interest, and it may be worth while to put on record some formulas that were developed a few years ago in connection with Schelkunoff's antenna theory. At that time the formulas were used to compute values of $P_\nu(\cos\theta)$, $0 \leqq \theta < \pi$, for values of ν between 0 and 2 at intervals of 0.1, and curves based on these computations have already been published.** Those curves show $P_\nu(\cos\theta)$ as a function of θ for the fractional values of ν; in this memorandum we include a table of numerical values (Appendix, Table I), and also a new set of curves (Figure 1) showing $P_\nu(\cos\theta)$ as a function of ν for values of θ between 0° and 175°. We have confined our computations to real values of ν, but it might be worth noting that the approximate formulas, and in particular the fundamental series expansions (3) and (17), are also valid for complex values of ν, in all regions in which they converge.

The function $P_\nu(\cos\theta)$ has a logarithmic singularity at $\theta = \pi$ for all non-integral values of ν; and it may be expressed in closed form at $\theta = \pi/2$ for all values of ν; hence

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†P. Grivet and M. Bernard, *Théorie de la lentille électrostatique constituée par deux cylindres coaxiaux*, Ann. de Radioélect., **6**, 1-9 (1952); P. Grivet, *Un nouveau modèle mathématique de lentille électronique*, Jour. de Phys. et le Rad., **13**, 1A-9A (1952).

**S. A. Schelkunoff, *Applied mathematics for engineers and scientists*, D. Van Nostrand, New York, 1948, pp. 423-424.

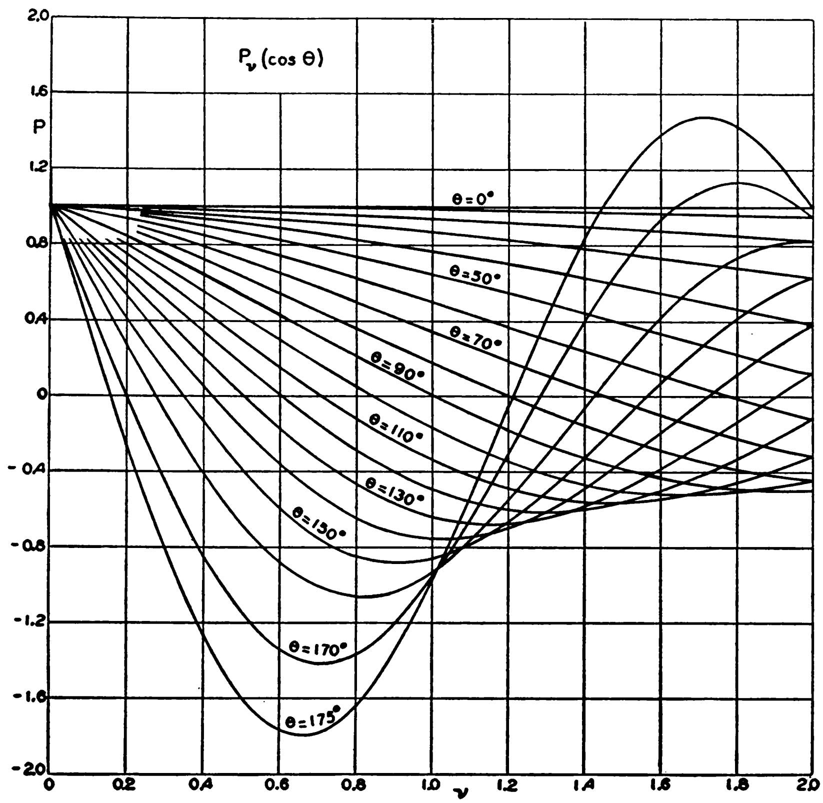

FIG. 1.

it is convenient to consider separately appropriate expansions in the neighborhoods of $\theta = 0, \pi/2, \pi$ respectively.

It may also be pointed out that for nearly integral values of ν, say $\nu = n + \delta$, it is sufficient to consider the case of $n = 0$ and small (positive or negative) values of δ. Then the recurrence formulas

$$P_{1+\delta}(\cos\theta) = \frac{1+2\delta}{1+\delta}\cos\theta P_\delta(\cos\theta) - \frac{\delta}{1+\delta}P_{-\delta}(\cos\theta),$$
$$P_{n+\delta}(\cos\theta) = \frac{2n-1+2\delta}{n+\delta}\cos\theta P_{n-1+\delta}(\cos\theta) - \frac{n-1+\delta}{n+\delta}P_{n-2+\delta}(\cos\theta), \tag{1}$$

may be used to obtain the required values for all values of n.

1. Small values of θ. When θ is not too large, the usual series expansion,*

$$P_\nu(\cos\theta) = \sum_{s=0}^{\infty} \frac{(-)^s(\nu+s)!}{(\nu-s)!s!s!}\sin^{2s}\frac{1}{2}\theta, \tag{2}$$

converges quite rapidly and tables of the factorial function are available. This series, however, does not exhibit the analytic nature of $P_\nu(\cos\theta)$ as a function of ν. Hence,

*S. A. Schelkunoff, *loc. cit.*, p. 420.

for small values of ν, we developed a series expansion in powers of ν,

$$P_\nu(\cos\theta) = 1 + \sum_{n=1}^{\infty} a_n \nu^n, \tag{3}$$

where the a's are functions of θ which can be computed from a set of recurrence relations. If we write $z = \sin^2(\theta/2)$ we can express the a's in the following form:

$$a_{2n+1} = \frac{(-)^{n+1}}{n!} \sum_{s=n+1}^{\infty} k_{n,s} \frac{z^s}{s}, \qquad a_{2n+2} = \frac{(-)^{n+1}}{n!} \sum_{s=n+1}^{\infty} k_{n,s} \frac{z^s}{s^2}, \tag{4}$$

where

$$\begin{aligned} k_{0,s} &= 1, \qquad s = 1, 2, \cdots \\ k_{n,s} &= 0, \qquad s \leqq n \\ k_{n,n+1} &= \frac{1}{n!}, \end{aligned} \tag{5}$$

$$k_{n,s+1} = k_{n,s} + \frac{n}{s^2} k_{n-1,s}, \qquad s = n+1, n+2, \cdots$$

It can be seen that the values k can be tabulated very rapidly, and then multiplied by the appropriate factors involving z to obtain the values a.

In particular

$$\begin{aligned} a_1 &= -\sum_{s=1}^{\infty} \frac{z^s}{s} = \log(1-z) = 2\log\cos\frac{1}{2}\theta, \\ a_2 &= -\sum_{s=1}^{\infty} \frac{z^s}{s^2}, \\ a_3 &= \sum_{s=2}^{\infty} \sigma_{2,s}\frac{z^s}{s}, \qquad a_4 = \sum_{s=2}^{\infty} \sigma_{2,s}\frac{z^s}{s^2}, \qquad \sigma_{2,3} = \sum_{r=1}^{s-1}\frac{1}{r^2}. \end{aligned} \tag{6}$$

When $\theta \leqq \pi/2$ we have $z \leqq 1/2$ and the series (4) converge quite rapidly, while the successive a's become smaller. The series (3) is valid for either positive or negative values of ν, and can be used very conveniently to compute $P_\nu(\cos\theta)$ for values of ν such that $|\nu| < 1/2$. Then the recurrence relations (1) may be used for larger values of ν, using also the general relation $P_{-\nu-1} = P_\nu$.

At $\nu = \pm 1/2$ the convergence is rather slow, except for small values of θ, but the Legendre functions can be expressed in terms of elliptic integrals,

$$\begin{aligned} P_{1/2}(\cos\theta) &= \frac{2}{\pi}\left[2E\left(\sin\frac{1}{2}\theta\right) - K\left(\sin\frac{1}{2}\theta\right)\right], \\ P_{-1/2}(\cos\theta) &= \frac{2}{\pi}K\left(\sin\frac{1}{2}\theta\right), \end{aligned} \tag{7}$$

where K and E are the complete elliptic integrals of the first and second kind, respectively. These functions have been frequently tabulated, and may be regarded as known. The recurrence formulas enable us to express the Legendre functions of order $\nu = n + 1/2$ in terms of K and E.

We shall see later that, for small values of ν, the first approximation

$$P_\nu(\cos\theta) = 1 + 2\nu \log \cos \frac{1}{2}\theta \tag{8}$$

is good throughout the range $0 \leqq \theta < \pi$.

2. The neighborhood of $\theta = \pi/2$. When $\theta = \pi/2$ the value of the Legendre function is

$$P_\nu\left(\cos \frac{1}{2}\pi\right) = \frac{\cos \frac{1}{2}\nu\pi(\frac{1}{2}\nu - \frac{1}{2})!}{\sqrt{\pi}\,(\frac{1}{2}\nu)!} \tag{9}$$

for all values of ν. In the neighborhood of $\pi/2$ we write $\theta = \pi/2 - \alpha$ and a series expansion which converges rapidly for small values of α is

$$\begin{aligned} P_\nu\left[\cos\left(\frac{1}{2}\pi - \alpha\right)\right] &= P_\nu(\sin\alpha) = \sum_{r=0}^{\infty} \frac{(\frac{1}{2}\nu + \frac{1}{2}r - \frac{1}{2})!}{\sqrt{\pi}\,(\frac{1}{2}\nu - \frac{1}{2}r)!} \cos\frac{\nu - r}{2}\pi \,\frac{(2\sin\alpha)^r}{r!} \\ &= \cos\frac{1}{2}\nu\pi \frac{(\frac{1}{2}\nu - \frac{1}{2})!}{\sqrt{\pi}\,(\frac{1}{2}\nu)!} F\left(\frac{1}{2}\nu + \frac{1}{2}, -\frac{1}{2}\nu; \frac{1}{2}; \sin^2\alpha\right) \\ &+ 2\sin\alpha \sin\frac{1}{2}\nu\pi \frac{(\frac{1}{2}\nu)!}{\sqrt{\pi}\,(\frac{1}{2}\nu - \frac{1}{2})!} F\left(\frac{1}{2}\nu + 1, -\frac{1}{2}\nu + \frac{1}{2}; \frac{3}{2}; \sin^2\alpha\right). \end{aligned} \tag{10}$$

When we consider $P_\nu(\cos \pi/2)$ as a function of ν, it can be shown that the first few terms in the expansion of the function (9) in powers of ν are

$$\begin{aligned} P_\nu\left(\cos\frac{\pi}{2}\right) &= 1 - \nu\log 2 - \frac{\nu^2}{2!}\left[\frac{1}{2}\pi^2 - (\log 2)^2\right] \\ &+ \frac{\nu^3}{3!}\left[\frac{\pi^2}{2}\log 2 - (\log 2)^3 - \frac{3}{2}\sum_{n=1}^{\infty}\frac{1}{n^3}\right] + \cdots . \end{aligned} \tag{11}$$

These terms, however, check with those obtained in Section 1, since, at $\theta = \pi/2$,

$$2\log\cos\frac{1}{2}\theta = 2\log\cos\frac{1}{4}\pi = -\log 2;$$

the relation

$$\sum_{s=1}^{\infty}\frac{1}{2^s s^2} = \frac{\pi^2}{12} - \frac{1}{2}(\log 2)^2$$

is given in Smithsonian Mathematical Tables, p. 142; and

$$\sum_{s=2}^{\infty}\frac{\sigma_{2,s}}{2^s s} = \frac{\pi^2}{12}(\log 2) - \frac{1}{2}(\log 2)^3 - \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^3}$$

can be verified numerically. If we try to expand the series in equation (10) in powers of ν we find that when all terms have been collected we simply return to the series (3).

3. The neighborhood of $\theta = \pi$. In this region the most useful expansion for all values of ν seems to be that first obtained by E. Hille.* If we write $\theta = \pi - \varphi$ Hille's

*See E. W. Hobson, *Spherical and ellipsoidal harmonics*, University Press, Cambridge, 1931, p. 225.

formula may be written in the alternative forms

$$P_\nu[\cos(\pi - \varphi)] = P_\nu(-\cos\varphi)$$

$$= \frac{\sin\nu\pi}{\pi}\sum_{r=0}^{\infty}\frac{(-)^r(\nu+r)!}{(\nu-r)!}\left[2\log\sin\frac{1}{2}\varphi + \psi(\nu+r) + \psi(-\nu+r-1) - 2\psi(r)\right]\frac{z^r}{r!r!} \quad (12)$$

$$= \left[\frac{2\sin\nu\pi}{\pi}\log\sin\frac{1}{2}\varphi + \cos\nu\pi\right]P_\nu(\cos\varphi)$$

$$+ \frac{\sin\nu\pi}{\pi}\sum_{r=0}^{\infty}\frac{(-)^r(\nu+r)!}{(\nu-r)!}[\psi(\nu+r) + \psi(\nu-r) - 2\psi(r)]\frac{z^r}{r!r!} \quad (13)$$

where $z = \sin^2\varphi/2$ and $\psi(x)$ is the logarithmic derivative of the factorial, as used by Hobson,

$$\psi(x) = \frac{d}{dx}(\log x!).$$

When φ is small the series in equation (13) converges rapidly and only a few terms are significant. Further, the first term of the series vanishes with ν for small ν and thus the dominant terms in the expansion in powers of ν are those obtained from the first term in equation (13),

$$P_\nu(\cos\theta) = P_\nu(-\cos\varphi) = P_\nu(\cos\varphi)\left(1 + 2\nu\log\sin\frac{1}{2}\varphi\right)$$
$$\simeq 1 + 2\nu\log\cos\frac{1}{2}\theta. \quad (14)$$

If we include the series terms, the expansion in powers of ν for small φ found most convenient for computation may be written

$$P_\nu(-\cos\varphi) = P_\nu(\cos\varphi)\left[\cos\nu\pi + \frac{2\sin\nu\pi}{\pi}\left(\log\sin\frac{1}{2}\varphi + \psi(\nu) - \psi(0)\right)\right]$$
$$+ \frac{\sin\nu\pi}{\pi}(c_0 + c_1\nu + c_2\nu^2 + \cdots), \quad (15)$$

where

$$c_0 = -a_1,$$
$$c_1 = -2a_2,$$
$$c_2 = -a_3 + 2\sum_{s=1}^{\infty}\frac{z^s}{s^3},$$
$$c_4 = -2a_4 - 2\sum_{s=2}^{\infty}\sigma_{2,s}\frac{z^s}{s},$$
$$c_5 = -a_4 - 2\sum_{s=2}^{\infty}\sigma_{2,s}\frac{z^s}{s^3} - 2\sum_{s=2}^{\infty}\sigma_{3,s}\frac{z^s}{s^2} \quad (16)$$

and where the values a are the constants of $P_\nu(\cos\varphi)$, $z = \sin^2\varphi/2$, and

$$\sigma_{n,s} = \sum_{r=1}^{s-1} \frac{1}{r^n}.$$

When φ is not too small it is also possible to use the series expansion

$$P_\nu(-\cos\varphi) = 1 + \sum_{n=1}^{\infty} b_n \nu^n, \tag{17}$$

with

$$\begin{aligned} b_1 &= 2 \log \sin \frac{1}{2}\varphi, \\ b_2 &= a_1 b_1 - a_2 - \frac{\pi^2}{6}, \\ b_3 &= b_1\left(a_2 - \frac{\pi^2}{6}\right) + 2\sum_{s=1}^{\infty} \frac{z^s}{s^3} - 2\sum_{s=1}^{\infty}\frac{1}{s^3}, \\ b_4 &= b_1\left(a_3 - \frac{\pi^2}{6}a_1\right) - 2a_1 \sum_{s=1}^{\infty}\frac{1}{s^3} + \frac{\pi^2}{6}a_2 - a_4 + \frac{\pi^4}{120} - 2\sum_{s=2}^{\infty}\sigma_{3,s}\frac{z^s}{s}. \end{aligned} \tag{18}$$

The coefficients are of course more complicated than those of equation (3), but if the a's have already been computed the remaining terms may be evaluated without too much labor.

4. Zeros of $P_\nu(\cos\theta)$. In Grivet's electron lens theory certain focal distances are determined from the roots θ_0 of $P_\nu(\cos\theta) = 0$ and from the values of $dP_\nu/d\theta$ at $\theta = \theta_0$. From the values of Table I the curve of Figure 2 has been drawn, showing the roots

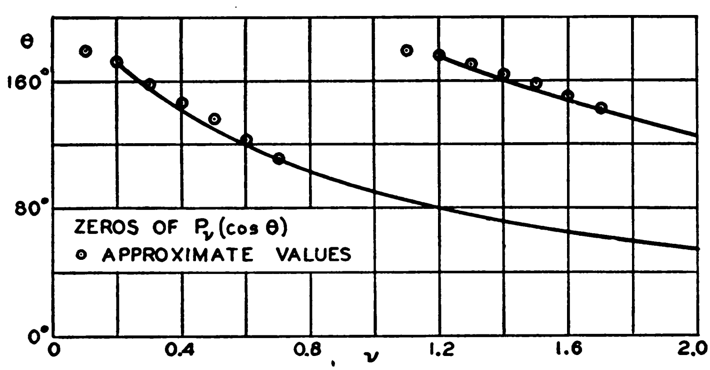

FIG. 2.

of P_ν for values of ν between 0 and 2. However, approximate values of the roots can be found from the approximate formulas of the preceding sections, and it is surprising that the simple formula (8) gives values very near the true values except in the neighborhood of $\nu = 1/2$. At this point the root can be obtained from elliptic integral tables. For smaller values of ν we find from (8)

$$\cos\frac{1}{2}\theta = \exp\left(-\frac{1}{2\nu}\right), \tag{19}$$

and for values of ν between 0.5 and 1.5 we combine equation (8) with the recurrence relation (1) to derive the equation

$$\log \cos \frac{1}{2}\,\theta = -\frac{1}{2\delta}\frac{(1 + 2\delta)\cos\theta - \delta}{(1 + 2\delta)\cos\theta + \delta}, \qquad \nu = 1 + \delta. \tag{20}$$

Similar equations can be obtained for larger values of ν by repeated use of the recurrence formula, and it is always possible to obtain an equation for the root which expresses $\log \cos \theta/2$ as the ratio of two polynomials in $\cos\theta$.

When ν is small the root θ_0 is near π, and a somewhat more accurate formula is obtained from equation (15):

$$\log \sin \frac{1}{2}\,\varphi = \frac{\log \cos \frac{1}{2}\varphi}{1 + 2\nu \log \cos \frac{1}{2}\varphi} - \frac{\pi}{2}\cot \nu\pi - \psi(\nu) + \psi(0), \tag{21}$$

where $\theta = \pi - \varphi$. Similarly for $\nu = 1 + \delta$, where δ is small, the smallest root can be found from equation (20), but there is a second root near π which is determined more accurately from the equation

$$\log \cos \frac{1}{2}\,\theta = -\frac{\pi}{2}\cot \delta\pi\left[\frac{(1 + 2\delta)\cos\theta - \delta}{(1 + 2\delta)\cos\theta + \delta}\right] - \frac{\cos\theta[\psi(\delta) - \psi(0) - \log\sin\frac{1}{2}\theta]}{(1 + 2\delta)\cos\theta + \delta}. \tag{22}$$

In Figure 2 we have indicated by circles the values of the roots obtained from equations (19) and (20) where these may be distinguished from the curve values.

For the derivative of P_ν at $\theta = \theta_0$ we can find simple formulas by differentiating the approximate formulas (3) and (15). Thus retaining the first three terms in (3) and using the approximation (19) for θ_0 we find

$$\left.\frac{dP_\nu}{d\theta}\right|_{\theta=\theta_0} = -\frac{2\nu}{\sin\theta_0}. \tag{23}$$

When ν is small, so that θ_0 is near π the asymtotic approximation is

$$\left.\frac{dP_\nu}{d\theta}\right|_{\theta=\theta_0} = -\frac{2\sin\nu\pi}{\pi\sin\theta_0}\,\frac{1}{1 + 2\nu\log\sin\theta_0/2}. \tag{24}$$

At $\nu = 0.5$ the derivatives of the elliptic integrals give

$$\left.\frac{d}{d\theta}P_{1/2}(\cos\theta)\right|_{\theta=\theta_0} = -\frac{K(\sin\theta_0/2)}{\pi\sin\theta_0}, \tag{25}$$

as shown by Grivet. For $\nu > 0.5$ we may combine (20) and (3) with the recurrence formula to find approximately

$$\left.\frac{d}{d\theta}P_{1+\delta}(\cos\theta)\right|_{\theta=\theta_0} = \frac{-2\delta(1 + \delta)}{\sin\theta_0[(1 + 2\delta)\cos\theta_0 + \delta]}. \tag{26}$$

Note that this equation is valid as $\delta \to 0$. For we have $P_{1+\delta} \to \cos\theta$, and the limit is approached in such a way that $\cos\theta_0 = \lim_{\delta\to 0}\delta/(1 + 2\delta)$. Thus in equation (26) the limiting value is

$$\left.\frac{d}{d\theta}P_1(\cos\theta)\right|_{\theta=\theta_0} = -\frac{1}{\sin\theta_0} = -1,$$

which is correct.

APPENDIX, TABLE I

Legendre Functions of Fractional Order, P_ν $(\cos\theta)$

θ	ν = .1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
10°	.999161	.998171	.997028	.995735	.994289	.992693	.990947	.989050	.987003	.984808
20°	.996635	.992665	.988095	.982927	.977168	.970822	.963895	.956393	.948322	.939693
30°	.992387	.983428	.973140	.961544	.948665	.934528	.919163	.902601	.884877	.866025
40°	.986362	.970362	.952059	.931521	.908821	.884042	.857275	.828616	.798169	.766044
50°	.978471	.953322	.924694	.892752	.857676	.819665	.778935	.735715	.692387	.642788
60°	.968597	.932102	.890814	.845072	.795249	.741748	.685007	.625480	.563646	.5
70°	.956571	.906416	.850092	.788227	.721505	.650659	.576469	.499745	.421314	.342020
80°	.942171	.875872	.802069	.721889	.636309	.546730	.454374	.360536	.266528	.173648
90°	.925086	.839927	.746089	.645288	.539353	.430189	.319752	.209982	.102787	0
100°	.904886	.797813	.681210	.557670	.430035	.301038	.173516	.050203	−.066312	−.173648
110°	.880955	.748422	.606031	.456324	.307261	.157754	.016269	−.116849	−.237227	−.342020
120°	.852374	.690081	.518406	.342882	.169084	.002434	−.151995	−.289652	−.406673	−.5
130°	.817704	.620144	.414869	.209624	.012012	−.170817	−.332468	−.467557	−.574395	−.642788
140°	.774511	.534092	.289416	.051166	−.170483	−.366364	−.528746	−.651698	−.734616	−.766044
150°	.718190	.423320	.130467	−.145688	−.391682	−.596024	−.749809	−.847187	−.885632	−.866025
160°	.638358	.268268	−.088558	−.411669	−.683193	−.888957	−1.019355	−1.069887	−1.041354	−.939693
170°	.501717	.005894	−.453932	−.847492	−1.150000	−1.343918	−1.420160	−1.378607	−1.227945	−.984808
175°	.365201	−.254581	−.813813	−1.272544	−1.599553	−1.774742	−1.791031	−1.652930	−1.378654	−.996444

θ	ν = 1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
10°	.982463	.979971	.977330	.974544	.971609	.968530	.965306	.961937	.958424	.954769
20°	.930510	.920782	.910521	.899731	.888427	.876616	.864311	.851520	.838256	.824533
30°	.846085	.825099	.803106	.780153	.756288	.731556	.706009	.679698	.652678	.625
40°	.732358	.697230	.660787	.623160	.584483	.544892	.504529	.463536	.422057	.380236
50°	.593260	.542960	.491146	.438457	.385144	.331534	.277906	+.224546	+.171992	+.119764
60°	.435048	.369302	.303277	.237487	+.172439	+.108625	+.046528	−.013395	−.070707	−.125
70°	.262707	+.184212	+.107352	+.032911	−.038356	−.105747	−.168608	−.226363	−.278485	−.324533
80°	+.083160	−.003726	−.085870	−.162209	−.231828	−.293895	−.347694	−.392684	−.428461	−.454769
90°	−.096662	−.185629	−.265506	−.335101	−.393447	−.439820	−.473745	−.495011	−.503662	−.5
100°	−.269688	−.352628	−.421019	−.473792	−.510309	−.530315	−.533992	−.521924	−.495078	−.454769
110°	−.428902	−.496128	−.542575	−.567190	−.571540	−.555254	−.520788	−.468913	−.402560	−.324533
120°	−.567487	−.607957	−.621232	−.608117	−.570350	−.510520	−.431952	−.338567	−.234713	−.125
130°	−.679240	−.680298	−.648801	−.587376	−.500235	−.392577	−.270327	−.139860	−.007413	+.119764
140°	−.756377	−.705113	−.617004	−.498494	−.357352	−.202235	−.042220	+.113682	+.260689	.380236
150°	−.792525	−.672251	−.514788	−.331564	−.135126	+.061648	+.246321	.407651	.536224	.625
160°	−.775450	−.562976	−.319350	−.063159	+.186880	.412909	.599730	.735204	.811308	.824533
170°	−.672324	−.318188	+.047601	+.395130	.696918	.930007	1.077683	1.130627	1.087486	.954769
175°	−.542651	−.058480	.413527	.833237	1.165932	1.385435	1.476543	1.433916	1.265476	.989351