Length Preserving Multiresolution Editing of Curves

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Abstract

In this paper a method for multiresolution deformation of planar piecewise linear curves that preserves the curve length is presented. In a wavelet based multiresolution editing framework, the curve can be deformed at any level of resolution through its control points. Enforcing the length constraint is carried out in two steps. In a first step the multiresolution decomposition of the curve is used in order to approximate the initial curve length. In a second step the length constraint is satisfied exactly by iteratively smoothing the deformed curve. Wrinkle generation is an application the paper particularly focuses on. It is shown how the multiresolution definition of the curve allows to explicitly and intuitively control the scale of the generated wrinkles.

1. Introduction

Multiresolution analysis is widely used in various domains including visualization, geometric modeling and computer graphics. Multiresolution representations based on wavelets have been developed for parametric curves [3], and can be generalized to tensor-product surfaces, to surfaces of arbitrary topological type [13], to spherical data [17], and to volume data [4].

General multiresolution editing or deformation techniques for parametric curves have been explored in detail by Finkelstein and Salesin [8], Gortler and Cohen [9], Elber and Gotsman [6] using B-splines. Multiresolution curve editing methods with the constraint of constant enclosed area have been developed in [5, 11].

This paper introduces a multiresolution editing tool for piecewise linear planar curves which allows to satisfy the non-linear constraint of length preservation. Following traditional principles of animation [12], constant length deformation as well as constant volume or area, is generally required when modeling or animating deformable objects naturally. Beside general deformations through select-and-drag control points, the present paper focuses on a particular type of deformation: the generation of wrinkles in case of shrinking or compressing the curve.

The paper is organized as follows. In section 2 the basic mathematical tools are given. Section 3 presents a method of length preserving via optimization, based on a linearization of the length constraint. This method is discussed and improved in a two-step method presented in section 4. It is shown in particular how the generation of wrinkles is controlled in a whole editing process. Section 5 concludes and gives some possible future works.

2. Tools

Let us briefly sketch the notation of the wavelet based multiresolution analysis that will be used in this section. For more details see [14]. Suppose we have a certain functional space E and some nested linear *approximation spaces* $V^j \subset E$ with $V^0 \subset V^1 \subset \cdots \subset V^n$. Let V^n be of dimension N. Let V^j be spanned by a basis of *scaling functions* $\varphi^j = [\varphi^j_1, \ldots, \varphi^j_m]^T$. A *detail space* W^j is the complement of V^j in V^{j+1}. Its basis of *wavelets* $\psi^j = [\psi^j_1, \ldots, \psi^j_{N-m}]^T$ is such that together with φ^j they form a basis of V^{j+1}. The space V^n can therefore be decomposed as follows:

$$V^n = V^{n-1} \oplus W^{n-1} = V^{n-2} \overset{n-1}{\underset{j=n-2}{\oplus}} W^j = \cdots = V^0 \overset{n-1}{\underset{j=0}{\oplus}} W^j. \tag{1}$$

A *multiresolution curve* is then defined as a planar parametric curve $c(t) = (c^n)^T(\varphi^n)$, element of V^n, where c^n is a column of its N control points in $\mathbb{R}^2$. Due to property (1) the same curve can be expressed at any *level of resolution* $e \in [0, n]$, i.e. as an element of $V_e \bigoplus_{j=e}^{n-1} W_j$. In terms of the basis functions, some coarse control points c^e form approximations (global shape) of the initial control polygon and the detail coefficients $d^e, \ldots, d^{n-1}$ encode the character of the object:

$$c(t) = (c^e)^T(\varphi^e) + (d^e)^T(\psi^e) + \cdots + (d^{n-1})^T(\psi^{n-1}), \quad e = 0, \ldots, n.$$

The *filter bank* algorithm [14, 8] is used to compute the coefficients of all levels of resolutions from the initial coefficients c^n and vice versa.

The length of a parametric curve $c(t) = (x(t), y(t))$ is given by $L = \int \sqrt{x'(t)^2 + y'(t)^2} dt$. The curves the present paper is dealing with are continuous and piecewise linear. In that case the length simplifies to $L = \sum_{i=0}^{N-2} \|c_{i+1} - c_i\|_2$, where c_i denote the control points at level n.

For the multiresolution representation of piecewise linear curves the scheme based on the Lazy wavelets [18] is used:

$$\begin{cases} c^j_i = c^{j+1}_{2i} \\ d^j_i = c^{j+1}_{2i+1} - \frac{1}{2}(c^{j+1}_{2i} + c^{j+1}_{2i+2}). \end{cases}$$

The reasons for the particular choice of working only with piecewise linear curve are twofold. First, the length preserving deformation method of chapter 3 is intended to work dynamically. Linear curves in contrast to higher order polynomial curves simplify and accelerate the computations. Second, the particular

effect of wrinkle creation in response to the motion of a control point is a purely visual effect. Therefore, on a computer screen no visual difference can be observed between displaying a dense polygonal curve or a rasterized polynomial curve.

In the case of polylines one can choose either to keep the total length constant or to preserve the length of each segment. We choose the second way because of two main reasons: first it ensures the balance between segment's length that is to say the control points don't gather in a small part of the curve. Secondly it allows the length constraints to be expressed in such a way that computationally inefficient square roots evaluations can be avoided.

The length constraints on a deformed curve C with control points $c_i = (x_i, y_i)$ can now be reformulated as follows:

$$f_i(C) = \Delta x_i^2 + \Delta y_i^2 - l_i^2 = 0 \quad \text{for} \quad i = 0, \ldots, N - 2 \,,$$

where l_i is the reference length of segment $[c_i; c_{i+1}]$ before deformation.

3. Length Preserving via Optimization

In a classical MR editing environment the user chooses a resolution level in which the curve is modified by displacing one coarse control point. The shape of the curve changes more or less locally depending on whether a low or a high resolution level has been chosen. The same procedure can be applied when integrating the additional constraint of keeping the curve length constant during deformation. We introduce here a method of length enforcing via optimization. Wrinkle generation is a particular application, which should occur naturally in the case of shrinking the curve or parts of it. The cost function to be minimize will take care of this feature and simultaneously produce smooth shapes. After presenting the basic ideas and the calculus we show some examples and discuss them.

Overview

Let C_R be a given reference curve in V^n. Its length is denoted by L_R. Let C_A be another curve in V^n (resulting from the edition of C_R). The (x, y)-coordinate vectors of its control points are denoted by (X_A, Y_A) and its length L_A is assumed to be different from L_R. The problem to solve consists in *length enforcing* C_A, i.e. in finding a curve C_F with control points (X, Y) whose length is equal to L_R and that is close to C_A. This is why C_A is also named an *attracting curve*.

In the present context (see section 2), there are in total $N - 1$ length constraints to satisfy and N vector degrees of freedom available, that is to say $2N$ scalar degrees of freedom. Hence a lot of length enforcing deformations would be possible. Let us illustrate in the following the use of a broad optimization method, i.e. minimization of a cost function which integrates a smoothness term and a distance term subject to the length constraints. The smoothness term prevents the resulting curve to have unwanted wiggles. The distance term is needed in order to minimize the distance to the attracting curve C_A as much as possible.

Smoothness Criterion

In variational design a physical model is used for the description of a "smooth" or a "fair" curve or surface [15, 19, 2]. The most widely used fairness criterion originate from the observation that the shape of a thin elastic beam or a thin plate under deformation which minimizes the bending energy is always smooth, i.e. has a visual pleasing shape.

Since the bending energy for a planar parametric curve, $E = \int \kappa^2(t)dt$, is a nonlinear functional, it is common to use instead the linearized version [7, 1] (both expressions are identical if $|c'(t)| \equiv 1$):

$$E = \int |c''(t)|^2 dt = \int x''(t)^2 + y''(t)^2 dt .$$

In the case of a polyline, a discrete version is derived from a finite difference approximation of the derivatives:

$$E(X, Y) = \sum_{i=1}^{N-2} \| \frac{1}{4}(c_{i-1} - 2c_i + c_{i+1}) \|^2 = \frac{1}{2}(X^T H X + Y^T H Y)$$

where

$$H = \frac{1}{8} \begin{bmatrix} 1 & -2 & 1 & & & & \\ -2 & 5 & -4 & 1 & & 0 & \\ 1 & -4 & 6 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 6 & -4 & 1 \\ & 0 & & 1 & -4 & 5 & -2 \\ & & & & 1 & -2 & 1 \end{bmatrix} , \quad X = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix} , \quad Y = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{bmatrix} .$$

Length Preserving Deformation

The flexibility of an energy based optimization can be enhanced by introducing an additional distance term: $D(X, Y) = \|X - X_A\|^2 + \|Y - Y_A\|^2$.

The problem we aim to solve is now the following optimization problem:

$$\min_{X,Y} \{ (1 - \beta)E(X, Y) + \beta D(X, Y) \} \quad \text{subject to } f_i = 0 , \ i = 0, \dots, N - 2 , \quad (2)$$

where the constraints f_i are defined in section 2.

$0 \leq \beta \leq 1$ is a scalar value which balances between a smoother curve and a curve closer to C_A.

The length constraints are quadratic expressions. Since minimizing a quadratic cost function subject to quadratic constraints is costly and since one of our

objectives is to provide a fast algorithm, let us approach the length constraint by using linearized constraints instead.

Linearizing the Length Constraints

With the technique of Lagrange multipliers [10], the minimization problem is restated to finding a stationary point of the following function with respect to X, Y and Λ:

$$(1 - \beta)E(X, Y) + \beta D(X, Y) + \sum_{i=0}^{N-2} \lambda_i f_i$$

where $\Lambda = (\lambda_0, \ldots, \lambda_{N-2})^T$ is the vector of Lagrange multipliers. In order to linearize each $f_i(X, Y) = (\Delta x_i)^2 + (\Delta y_i)^2 - l_i$, let us approach it by the linear part $\tilde{f}_i$ of its Taylor expansion with respect to Δx_i and Δy_i in the neighborhood of Δx_i^A and Δy_i^A. Hence one gets

$$\tilde{f}_i = 2\Delta x_i^A \Delta x_i - (\Delta x_i^A)^2 + 2\Delta y_i^A \Delta y_i - (\Delta y_i^A)^2 - l_i^2.$$

Remark. In order to replace f_i by the linearized expression $\tilde{f}_i$ we have to ensure that $\Delta x_i, \Delta y_i$ are close to $\Delta x_i^A, \Delta y_i^A$. In other words the curve C_A must nearly satisfy the length constraints.

Assuming that the approximation of f_i by $\tilde{f}_i$ holds, the problem can be restated as solving:

$$\overrightarrow{\nabla} g = 0 \quad \text{where} \quad g(X, Y, \Lambda) = (1 - \beta)E(X, Y) + \beta D(X, Y) + \sum \lambda_i \tilde{f}_i . \quad (3)$$

A symmetric square sparse system of linear equations has to be solved:

$$\begin{cases} (1 - \beta)HX + 2\beta X + 2\Delta_X \Lambda = 2\beta X_A \\ (1 - \beta)HY + 2\beta Y + 2\Delta_Y \Lambda = 2\beta Y_A \\ 2\Delta_X^T X + 2\Delta_Y^T Y = b \end{cases} \quad (4)$$

where where $b = (l_i^2 + (\Delta x_i^A)^2 + (\Delta y_i^A)^2)_{i=0\ldots N-2}^T$, and where Δ_X (resp. Δ_Y with y instead of x) is a $N \times (N - 1)$ matrix:

$$\Delta_X = \begin{bmatrix} -\Delta x_0^A & & & \\ \Delta x_0^A & -\Delta x_1^A & & 0 \\ & \Delta x_1^A & \ddots & \\ & & \ddots & \ddots \\ & 0 & \ddots & -\Delta x_{N-2}^A \\ & & & \Delta x_{N-2}^A \end{bmatrix} .$$

Let us note that due to the linearization the result (X, Y) is not exact for the length constraints. In order to increase the precision, the solving can be encapsulated in a loop which iterates the system solving.

Examples and Discussion

Figure 1 illustrates the role of β : a deformation is applied to the initial curve by editing it at the coarsest level of resolution. Fig. 1(a) shows the initial curve together with its coarsest approximation consisting of 2 control points. Fig. 1(b) shows the deformed coarse polygon and the corresponding reconstructed curve without length preservation. This is the attracting curve C_A. The problem is then solved for different values of β, see Figures 1(c), 1(d), and 1(e). As expected, the role of β is to control the shape of the resulting curve. A small value gives preference to a smooth curve in the sense of minimal bending energy. A higher value keeps the mean distance to C_A (gray) as small as possible. In the particular case of shrinking the initial curve, as illustrated in that figure, β controls the creation of wrinkles. A high value produces small and frequent wrinkles, while a small value stands for larger and less frequent wrinkle.

These results suggest that this method could be used as it is, because it allows the curve to be deformed at a given scale (controlled through the value of β) while preserving its length. But two reasons make it impractical to use:

- If the edited curve C_A is far from satisfying the length constraint, then the error due to linearization is important. We hence have to iterate the solving in order to bound the error below some threshold which may be expensive (tens of solvings are necessary for Fig. 1).
- As shown in Fig. 1, β controls the appearance of the curve. However the choice of β must take into account a lot of parameters including the scale of the wrinkles and the number of control points. Hence it is quite impractical to choose the wrinkling scale. We aim to have an easy-to-use editing tool.

(a) initial curve

(b) edited curve without length preservation

(c) $\beta = 0.001$ (d) $\beta = 0.01$ (e) $\beta = 0.03$
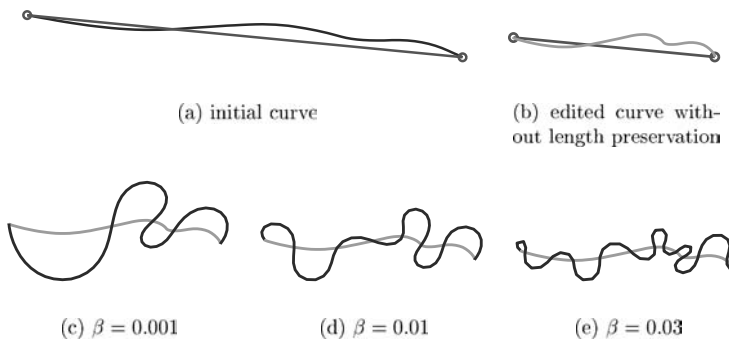
Fig. 1. Role of β: the initial curve (a) with 129 control points is edited at the coarsest level (b) (in gray: reconstructed attracting curve). In (c), (d) and (e): 3 final length preserving curves (in black) are given for different values of β

However, if the attracting curve nearly satisfies the length constraint, this optimization method is very effective in precisely meeting the prescribed length.

4. Control of Wrinkle Generation

In this section we introduce a new step into the editing process taking place before the optimization method presented in the previous section. In order to compensate for the drawbacks of the optimization, it aims to compute in linear time an attracting curve suitable for the optimization step because it sufficiently approximates the lengths constraints while the scale of the deformations induced by these constraints can be explicitly and intuitively controlled by the user.

The main idea is to modify the details of the edited curve at an intermediate level $e \leqslant w < n$ so that the control polygon at level $w + 1$ has the same length as the initial coarse polygon. Hence this step is called *explicit length preserving*. An important advantage of modifying details is that the attractive curve built in this way is close to the edited curve because their coarse control polygons at level w match whenever possible. That is to say this curve follows as much as possible the modification applied by the user.

4.1. Editing Loop with Explicit Length Preservation Step

Let $C_R(t) \in V^n$ be the initial planar piecewise linear curve the user wants to edit while keeping its length constant. Two resolution levels may be specified by the user: level e, at which the user will edit the control polygon, and level w, at which wrinkles will be generated if required by the length constraint. The editing process follows the graph of Fig. 2. Each of the six arrows corresponds to a particular action described in the following paragraphs and illustrated by a real example in Fig. 3:

Decomposition: The curve is decomposed into C_R^e at the level e, chosen by the user (see Fig. 3(a)). The superscripts denote the level of resolution.

Edition: The coarse control polygon C_R^e at level e is modified by the user. In Fig. 3(b) the user "pinched" the right part of the curve. The corresponding de-
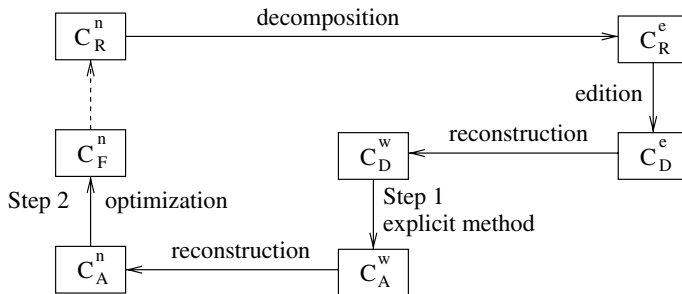

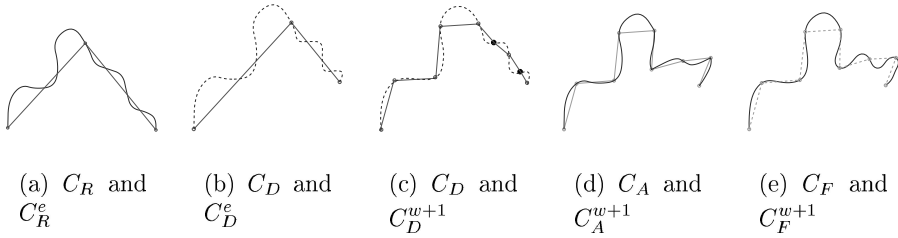**Fig. 2.** Enhanced editing loop for wrinkle generation

(a) C_R and C_R^e (b) C_D and C_D^e (c) C_D and C_D^{w+1} (d) C_A and C_A^{w+1} (e) C_F and C_F^{w+1}

Fig. 3. The editing loop. The curves decomposed at the level e or w are represented by their control polygon

formed curve C_D (doted line in Fig 3(b)) has not the same length as the reference curve C_R. Notice that the choice of level e determines the extent of the deformation.

Reconstruction: The deformed curve is then partially reconstructed at level w ($e \leq w < n$), chosen by the user (in Fig. 3(c): $w = e + 1$).

Explicit Length Preserving: An attracting curve C_A is constructed (see Fig. 3(d)). This curve is obtained by modifying C_D^w using the details gradually along the control polygon in order to obtain a control polygon C_A^{w+1} whose edges have the same length as the control polygon segments of C_R at level $w + 1$. Some control points at level w may be modified as well in case of stretching. More details can be found in [16]. Modifying details at level w is equivalent to modifying corresponding control points at level $w + 1$. In the example they are highlighted as dark dots in Fig. 3(c). The choice of $w > e$ for length approximation increases the number of coarse control points defining the same portion of the curve and hence determines the number of wrinkles on this portion. The closer to the highest level n is w the higher is the frequency of the wrinkles.

Reconstruction: The attractive curve is then completely reconstructed (Fig. 3d) by re-inserting the details of C_R at the levels $w + 1, \ldots, n$. After reconstruction, the length of C_A is close to satisfy the length constraint thanks to the length preservation at level $w + 1$. Note, that the use of a lazy wavelet scheme leads to some sharp features in the reconstructed curve.

Length Preserving by Smoothing: The optimization method (see section 3) applied to C_A leads to the final curve C_F (Fig. 3(e)). It precisely satisfies the length constraints, has a smooth shape, and it is close to the attracting curve C_A. The main role of this optimization step remains now to smooth the curve. Figure 3(d) illustrates the effect of the smoothing, whose cost dramatically drops.

4.2. Results

We present here an example of MR editing that illustrates the important points we focused on.

The Fig. 4 shows two successive deformations at different levels of decomposition (of the same curve as Fig. 1), i.e. the result of two editing loops. The initial curve 4a with 129 control points is edited at the coarsest level $e = 0$ and its length is
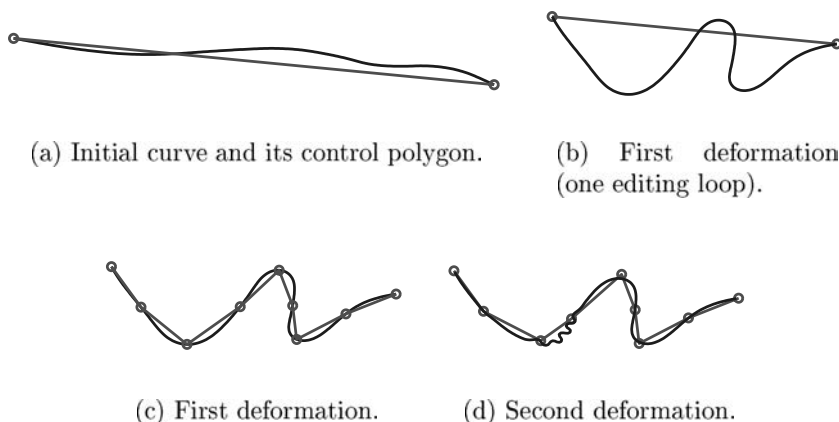
(a) Initial curve and its control polygon.   (b) First   deformation
                                                     (one editing loop).

(c) First deformation.   (d) Second deformation.

Fig. 4. MR editing: the initial curve (a) is at first edited at the coarsest level (b) and then at an intermediate level (c) and (d)

adapted at the scale $w = 1$. Hence it provides large wrinkles (4(b)). Then it is edited at the scale $e = 3$ (Fig. 4(c)): 2 neighboring control points are moved closer. Since the length is preserved at the scale $w = 6$, small wrinkles are created (Fig. 4(d)). Let us note that the curve is modified locally. We also see that multiresolution provides a flexible framework for curve editing. The levels e and w can be chosen in order to get exactly the deformation at the scale you want and on the extent you want.

The whole algorithm works dynamically, i.e. the user selects-and-drags a coarse control point immediately the length preserving curve is displayed. In fact, the first step has linear complexity and the second step needs generally no more than 5 linear system solvings. It is implemented with sparse matrices efficient conjugate gradient methods for reducing the computing time. As an example, the deformations of the curve in Fig. 4 are obtained in less than some tens ms each.

5. Conclusion

A method for length preserving multiresolution editing of planar curves has been introduced. The method combines an explicit length preserving step together with an optimization step in order to provide smooth results with an explicit and intuitive control of the scale of the deformations.

Current and future research concerns the integration of other non-linear constraints into a multiresolution editing environment and the generalization to space curves and surfaces.

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