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LENGTHS OF CIRCULAR TRAJECTORIES ON GEODESIC SPHERES IN A COMPLEX PROJECTIVE SPACE

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Abstract

We study trajectories for Sasakian magnetic fields which are also circles of positive geodesic curvature on geodesic spheres in a complex projective space. Investigating their extrinsic shapes we give a condition for them to be closed. By use of information on lengths of circles on a complex projective space, we give their lengths, and estimate the bottom of the length spectrum of circular trajectories.

1. Introduction

Let M be a real hypersurface in a Kähler manifold $\tilde{M}$ with complex structure J and Riemannian metric $\langle\ ,\ \rangle$. This hypersurface admits a canonical almost contact metric structure $(\phi, \xi, \eta, \langle\ ,\ \rangle)$ induced by J. The characteristic vector field ξ is given by $\xi = -J\mathcal{N}$ with a unit normal $\mathcal{N}$ on M in $\tilde{M}$, and the characteristic tensor ϕ is given by $\phi(v) = Jv - \langle v, \xi\rangle\mathcal{N}$. Associated with this structure we have a canonical closed 2-form $\mathbf{F}_\phi$ on M defined by $\mathbf{F}_\phi(v, w) = \langle v, \phi(w)\rangle$. Given a constant κ we set $\mathbf{F}_\kappa = \kappa\mathbf{F}_\phi$ and call it a *Sasakian magnetic field* on M (c.f. [10]). The motion of an electric charged particle of unit speed under this magnetic field is a smooth curve γ which is parameterized by its arclength and satisfies the equation $\nabla_{\dot\gamma}\dot\gamma = \kappa\phi\dot\gamma$. We call it a trajectory for $\mathbf{F}_\kappa$. When $\kappa = 0$, trajectories are geodesics. We can hence consider trajectories for Sasakian magnetic fields are generalization of geodesics which are associated with the almost contact metric structure. But as the characteristic tensor ϕ is not parallel, trajectories for Sasakian magnetic fields are not simple objects in general from curve-theoretic point of view.

In this paper, we focus our mind on trajectories which are also circles on typical homogeneous real hypersurfaces in a complex projective space and study a

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condition for them to be closed. If we consider circles on a standard sphere or on a Euclidean space, we see they are closed. But the situation is not the same on geodesic spheres in a complex projective space even if we restrict ourselves to circular trajectories. We show that there are infinitely many open circular trajectories and infinitely many closed circular trajectories. The reasons why we focus our mind on trajectories which are also circles are one is that circles are simplest curves next to geodesics and the other is that trajectories for Kähler magnetic fields are always circles. A constant multiple $\mathbf{B}_\kappa = \kappa \mathbf{B}_J$ of the Kähler form $\mathbf{B}_J$ on a Kähler manifold $(\tilde{M}, J)$ is said to be a Kähler magnetic field. A trajectory for $\mathbf{B}_\kappa$ is a smooth curve which is parameterized by its arclength and satisfies the equation $\nabla_{\dot\gamma}\dot\gamma = \kappa J\dot\gamma$ (see [1] and its sequels for more detail). Since some homogeneous real hypersurfaces in a non-flat complex space form can be regarded as odd dimensional objects corresponding to complex space forms, which are called Sasakian space forms, we are interested in some similarity and difference between Sasakian magnetic fields and Kähler magnetic fields.

In this paper we also study the distribution of lengths of circular trajectories. It is known that geodesic spheres of sufficiently large radius in a complex projective space are examples of "Berger spheres". Borrowing an idea on the study on lengths of circles given in [4], we give an expression of lengths of closed circular trajectories and estimate the length of shortest closed circular trajectories. We here note that every trajectory for our Sasakian magnetic fields is a homogeneous curve on a geodesic sphere in a complex projective space, that is, it is an orbit of a one-parameter subgroup of the isometry group of a geodesic sphere ([5]). Though trajectories for Sasakian magnetic fields essentially are not circles, from the viewpoint of curve-theory, we may say that our study gives a clue to the study of homogeneous curves on Sasakian space forms.

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2. Main results

A smooth curve γ on a Riemannian manifold which is parameterized by its arclength is called a *circle* if it satisfies the differential equation $\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma + k^2\dot\gamma = 0$ with some nonnegative constant k. This constant k is called the geodesic curvature of a circle. When $k = 0$ it is a geodesic. We may hence say that circles of positive geodesic curvature are simplest curves next to geodesics. In this paper we study trajectories for Sasakian magnetic fields which are also circles of positive geodesic curvature. We call such a trajectory *circular*.

A smooth curve γ parameterized by its arclength is said to be closed if there is a positive constant t_c satisfying $\gamma(t + t_c) = \gamma(t)$ for all t. The minimum positive t_c with this property is called the length of γ and is denoted by length(γ). For a smooth curve which is not closed we say it is open and set length$(\gamma) = \infty$. We here give our main results. On geodesic spheres in a complex projective space $\mathbf{C}P^n$ we find trajectories are closed if and only if their radii and strengths of Sasakian magnetic fields satisfy a relation.

THEOREM 1. *Let γ be a circular trajectory for a Sasakian magnetic field $\mathbf{F}_\kappa$ on a geodesic sphere $G(r)$ of radius r $(0 < r < \pi/2)$ in a complex projective space $\mathbf{C}P^n(4)$ of constant holomorphic sectional curvature 4.*

(1) *When $\pi/4 < r < \pi/2$ and $\kappa^2 = (3\sqrt{2(\cot^2 r + 1)} - 4)/2$, it is closed and its length is $2\pi\sqrt{2 \sin r(3\sqrt{2} - 4 \sin r)}$.*

(2) *Otherwise, γ is closed if and only if*

$$\frac{(\kappa^2+2)|2\kappa^4 + 8\kappa^2 - 9\cot^2 r - 1|}{2(\kappa^4 + 4\kappa^2 - 3\cot^2 r + 1)^{3/2}} = \frac{q(9p^2 - q^2)}{(3p^2 + q^2)^{3/2}} \tag{2.1}$$

holds with some relatively prime positive integers p, q satisfying $p > q$. In this case its length is given as

$$\pi\delta(p,q)|\kappa|\sqrt{(3p^2+q^2)/(\kappa^4 + 4\kappa^2 - 3\cot^2 r + 1)},$$

where $\delta(p,q) = 1$ when pq is odd and $\delta(p,q) = 2$ when pq is even.

Though the above theorem shows the explicit formula on lengths of closed circular trajectories, as the relation between radii and strengths of Sasakian magnetic fields is given by cubic equations, it is not easy to get how lengths of circular trajectories are distributed on the real line. We hence consider properties of the length spectrum. We say two smooth curves γ_1, γ_2 on $G(r)$ are congruent to each other if there exist an isometry φ of $G(r)$ and a constant t_0 with $\gamma_2(t+t_0) = \varphi \circ \gamma_1(t)$ for all t. We denote by $\mathscr{T}_\phi(G(r))$ the set of all congruence classes of circular trajectories on $G(r)$ in $\mathbf{C}P^n(4)$. We define $\mathscr{L} : \mathscr{T}_\phi(G(r)) \to \mathbf{R} \cup \{\infty\}$ by $\mathscr{L}([\gamma]) = \text{length}(\gamma)$ and call it the length spectrum of circular trajectories. Here $[\gamma]$ denotes the congruence class containing a circular trajectory γ. We set $\mathrm{LSpec}_\phi(G(r)) = \mathscr{L}(\mathscr{T}_\phi(G(r))) \cap \mathbf{R}$ and call it also the length spectrum of circular trajectories.

THEOREM 2. *The length spectrum* $\mathrm{LSpec}_\phi(G(r))$ *of circular trajectories on a geodesic sphere $G(r)$ in $\mathbf{C}P^n$ is not bounded.*

With Proposition 1 and Lemma 1 which will be mentioned below, this theorem guarantees that there exist infinitely many closed circular trajectories. On the other hand, as the set of all the solutions for (2.1) is a discrete subset of a real line even if (p,q) runs over all pairs of relatively prime positive integers satisfying $p > q$, we find there also exist infinitely many open circular trajectories.

COROLLARY 1. *On a geodesic sphere $G(r)$ in $\mathbf{C}P^n$, there exist infinitely many closed circular trajectories and infinitly many open circular trajectories.*

It is known that every trajectory for Kähler magnetic fields on a complex projective space is closed. From the viewpoint of shapes of trajectories, Sasakian magnetic fields have different aspects from Kähler magnetic fields.

3. Circular trajectories on geodesic spheres

We shall start by reviewing the circular condition on trajectories for Sasakian magnetic fields on geodesic spheres in a complex projective space $\mathbf{C}P^n$. A smooth curve γ parameterized by its arclength on a Riemannian manifold N is said to be a *helix of proper order d* if it satisfies the following system of ordinary differential equations $\nabla_{\dot\gamma} Y_j = -\kappa_{j-1} Y_{j-1} + \kappa_j Y_{j+1}$ $(1 \le j \le d)$ with positive constants $\kappa_1, \dots, \kappa_{d-1}$ and an orthonormal system $\{Y_1 = \dot\gamma, Y_2, \dots, Y_d\}$ of vector fields along γ. Here we put $\kappa_0 = \kappa_d = 0$ and choose Y_0, Y_{d+1} to be null vector fields along γ. We call these constants $\kappa_1, \dots, \kappa_{d-1}$ and the frame $\{Y_1, \dots, Y_d\}$ the geodesic curvatures and the Frenet frame of γ, respectively. A helix of order 1 is a geodesic and a helix of proper order 2 is a circle of positive geodesic curvature.

For a trajectory γ for a Sasakian magnetic field $\mathbf{F}_\kappa$ on a geodesic sphere $G(r)$ of radius r in $\mathbf{C}P^n$, we define its structure torsion ρ_γ by $\rho_\gamma = \langle \dot\gamma, \xi \rangle$. Though the structure torsion of a trajectory is not necessarily constant along this on general real hypersurfaces, as the shape operator A and ϕ of $G(r)$ in $\mathbf{C}P^n$ are simultaneously diagonalizable, each trajectory for a Sasakian magnetic field on $G(r)$ in $\mathbf{C}P^n$ has constant structure torsion (see [9]). For the trivial magnetic field $\mathbf{F}_0$, its trajectories are geodesics. In terms of helices, the feature of trajectories for non-trivial Sasakian magnetic fields are as follows.

PROPOSITION 1 ([9]). *Let γ be a trajectory for a non-trivial Sasakian magnetic field $\mathbf{F}_\kappa$ on a geodesic sphere $G(r)$ of radius r $(0 < r < \pi/2)$ in a complex projective space $\mathbf{C}P^n(4)$ of constant holomorphic sectional curvature* 4.

(i) *If $\rho_\gamma = \pm 1$, it is a geodesic on $G(r)$.*
(ii) *If it satisfies $\kappa\rho_\gamma = \cot r$, it is a circle of geodesic curvature $|\kappa|\sqrt{1-\rho_\gamma^2}$.*
(iii) *Otherwise, it is a helix of proper order* 3.

In particular, a trajectory γ on $G(r)$ is circular if and only if $\kappa\rho_\gamma = \cot r$ holds.

If the structure torsion of a trajectory γ for a Sasakian magnetic field $\mathbf{F}_\kappa$ on a geodesic sphere is equal to ± 1, its equation turns to $\nabla_{\dot\gamma}\dot\gamma = 0$, hence is a geodesic. Since the length of geodesics on geodesic spheres in $\mathbf{C}P^n$ are well-known (see for example [7]), we have the following.

PROPOSITION 2. *If a trajectory for a Sasakian magnetic field $\mathbf{F}_\kappa$ on $G(r)$ in $\mathbf{C}P^n(4)$ has structure torsion ± 1, then it is closed of length $\pi \sin 2r$.*

On contrary, as we do not have sufficient information on circles on geodesic spheres, we need to study lengths of circular trajectories. Our study also gives a clue to get properties of circles on geodesic spheres.

4. Extrinsic shapes of circular trajectories on geodesic spheres

It is one of natural ways to study curves on a geodesic sphere $G(r)$ in $\mathbf{C}P^n$ through an immersion $\iota : G(r) \to \mathbf{C}P^n$. For a smooth curve γ on $G(r)$ we call the curve $\iota \circ \gamma$ its *extrinsic shape* in $\mathbf{C}P^n$. In order to show our results we investigate some properties of circular trajectories on $G(r)$ by observing their extrinsic shapes in $\mathbf{C}P^n$.

We here give one more terminology. We call a helix on a Riemannian manifold N *Killing* if it is generated by some Killing vector field on N. Being different from helices on Euclidean spaces, a helix on $\mathbf{C}P^n$ is not necessarily Killing. For a helix σ of proper order d with Frenet frame $\{Y_1, \dots, Y_d\}$ on $\mathbf{C}P^n$, we define its complex torsions τ_{ij} $(1 \leq i < j \leq d)$ by $\tau_{ij} = \langle Y_i, JY_j \rangle$. According to the result in [12] a helix σ on $\mathbf{C}P^n$ is Killing if and only if all its complex torsions are constant along σ. Since the set of congruence classes of helices is complicated, we give more classification on helices. We say a helix of proper order either $2d-1$ or $2d$ on $\mathbf{C}P^n$ to be *essential* if it lies on some totally geodesic $\mathbf{C}P^d$ (c.f. [3]). We then have the following ([11]):

1) A helix of proper order 2 is essential if and only if $\tau_{12} = \pm 1$;
2) A helix of proper order 3 with geodesic curvatures κ_1, κ_2 is essential and Killing if and only if its complex torsions satisfy $\tau_{12} = \pm\kappa_1/\sqrt{\kappa_1^2 + \kappa_2^2}$, $\tau_{13} = 0$ and $\tau_{23} = \pm\kappa_2/\sqrt{\kappa_1^2 + \kappa_2^2}$, where double signs take the same signatures.
3) A helix of proper order 4 with geodesic curvatures κ_1, κ_2, κ_3 is essential and Killing if and only if its complex torsions satisfy either

   I) $\tau_{12} = \tau_{34} = \pm(\kappa_1 + \kappa_3)/\sqrt{\kappa_2^2 + (\kappa_1 + \kappa_3)^2}$, $\tau_{13} = \tau_{24} = 0$,
   $\tau_{23} = \tau_{14} = \pm\kappa_2/\sqrt{\kappa_2^2 + (\kappa_1 + \kappa_3)^2}$,

   II) $\tau_{12} = -\tau_{34} = \pm(\kappa_1 - \kappa_3)/\sqrt{\kappa_2^2 + (\kappa_1 - \kappa_3)^2}$, $\tau_{13} = \tau_{24} = 0$,
   $\tau_{23} = -\tau_{14} = \pm\kappa_2/\sqrt{\kappa_2^2 + (\kappa_1 - \kappa_3)^2}$.

   In each of the above conditions double signs take the same signatures.

We now study extrinsic shapes of circular trajectories on a geodesic sphere $G(r)$ of radius r $(0 < r < \pi/2)$ in $\mathbf{C}P^n(4)$. Let A denote the shape operator of $G(r)$ with respect to a unit normal $\mathcal{N}$ on $G(r)$ in $\mathbf{C}P^n$. It is known that it has two distinct principal curvatures. The tangent space splits orthogonally into $TG(r) = T^0G(r) \oplus \mathbf{R}\xi$, where $T^0G(r)$ is a bundle of principal curvature vectors associated with the principal curvature $\cot r$ and ξ satisfies $A\xi = 2\cot 2r\xi$. We denote by ∇ and $\tilde{\nabla}$ the Riemannian connections on $G(r)$ and $\mathbf{C}P^n(4)$, respectively. They are related by the Gauss formula $\tilde{\nabla}_X Y = \nabla_X Y + \langle AX, Y \rangle \mathcal{N}$ and Weingarten formula $\tilde{\nabla}_X \mathcal{N} = -AX$ for vector fields X, Y tangent to $G(r)$. Extrinsic shapes of circular trajectories on geodesic spheres are as follows.

Proposition 3. *Let $G(r)$ be a geodesic sphere of radius r in $\mathbf{C}P^n(4)$.*

(1) *When $\pi/4 < r < \pi/2$, the extrinsic shape of a circular $\mathbf{F}_{\pm 1}$-trajectory is a circle of geodesic curvature $\sqrt{1 - \cot^2 r}$ and of complex torsion $\tau_{12} = \mp\sqrt{1 - \cot^2 r}$.*

(2) *Otherwise, the extrinsic shape of circular* $\mathbf{F}_\kappa$*-trajectory is an essential Killing helix of proper order* 4 *which satisfies the condition* (II) *whose geodesic curvatures* κ_1, κ_2, κ_3 *are given as*

$$\frac{1}{\kappa^2}\sqrt{\kappa^6+(1-2\kappa^2)\cot^2 r},\quad \frac{|\kappa^2-1|\cot r\sqrt{\kappa^2-\cot^2 r}}{\kappa^2\sqrt{\kappa^6+(1-2\kappa^2)\cot^2 r}},\quad \frac{\kappa^2-\cot^2 r}{\sqrt{\kappa^6+(1-2\kappa^2)\cot^2 r}}.$$

Proof. Since a circular $\mathbf{F}_\kappa$-trajectory γ satisfies $\kappa\rho_\gamma=\cot r$ and $\rho_\gamma\neq\pm 1$, we see

$$A\dot\gamma=(\cot r)\dot\gamma-(\rho_\gamma\tan r)\xi=\kappa\rho_\gamma\dot\gamma-\kappa^{-1}\xi,$$

hence we have $\langle A\dot\gamma,\dot\gamma\rangle=\cot r-\rho_\gamma^2\tan r=\rho_\gamma(\kappa-\kappa^{-1})$. By use of the Gauss formula we get

$$\tilde\nabla_{\dot\gamma}\dot\gamma=\kappa\phi\dot\gamma+\langle A\dot\gamma,\dot\gamma\rangle\mathscr{N}=\kappa J\dot\gamma-\rho_\gamma\kappa^{-1}\mathscr{N}.$$

Thus we obtain

$$\kappa_1=\sqrt{\kappa^2(1-\rho_\gamma^2)+\rho_\gamma^2(\kappa-\kappa^{-1})^2}=\sqrt{\kappa^2-2\rho_\gamma^2+\kappa^{-2}\rho_\gamma^2}\ \ (\neq 0),$$
$$Y_2=\frac{1}{\kappa_1}(\kappa J\dot\gamma-\rho_\gamma\kappa^{-1}\mathscr{N}).$$

Continuing calculation by use of Gauss and Weingarten formulas we have

$$\begin{aligned}\tilde\nabla_{\dot\gamma}(\kappa J\dot\gamma-\rho_\gamma\kappa^{-1}\mathscr{N})&=-\kappa^2\dot\gamma+\rho_\gamma\kappa^{-1}A\dot\gamma+\rho_\gamma\xi\\&=-(\kappa^2-2\rho_\gamma^2+\kappa^{-2}\rho_\gamma^2)\dot\gamma+\rho_\gamma(\kappa^{-2}-1)(\rho_\gamma\dot\gamma-\xi).\end{aligned}$$

When $\kappa=\pm 1$, which is the case that $\pi/4<r<\pi/2$ and $\rho=\pm\cot r$ because $|\rho_\gamma|<1$, the extrinsic shape of γ is a circle of positive geodesic curvature. Otherwise we have

$$\kappa_2=\kappa_1^{-1}|\rho_\gamma(\kappa^{-2}-1)|\sqrt{1-\rho_\gamma^2},\quad Y_3=(\mathrm{sgn}(\rho_\gamma(\kappa^{-2}-1))/\sqrt{1-\rho_\gamma^2})(\rho_\gamma\dot\gamma-\xi),$$

where $\mathrm{sgn}(\alpha)$ denotes the signature of a number α. As we have

$$\begin{aligned}\tilde\nabla_{\dot\gamma}(\rho_\gamma\dot\gamma-\xi)&=\rho_\gamma(\kappa J\dot\gamma-\rho_\gamma\kappa^{-1}\mathscr{N})-JA\dot\gamma=(1-\rho_\gamma^2)\kappa^{-1}\mathscr{N}\\&=-\frac{\rho_\gamma(\kappa^{-2}-1)(1-\rho_\gamma^2)}{\kappa_1}\cdot\frac{1}{\kappa_1}(\kappa J\dot\gamma-\rho_\gamma\kappa^{-1}\mathscr{N})\\&\quad+\frac{1-\rho_\gamma^2}{\kappa_1^2}\{-\rho_\gamma(\kappa-\kappa^{-1})\phi\dot\gamma+\kappa(1-\rho_\gamma^2)\mathscr{N}\},\end{aligned}$$

we obtain

$$\kappa_3=\kappa_1^{-1}(1-\rho_\gamma^2)\ \ (>0),\quad Y_4=\frac{\mathrm{sgn}(\rho_\gamma(\kappa^{-2}-1))}{\kappa_1\sqrt{1-\rho_\gamma^2}}\{-\rho_\gamma(\kappa-\kappa^{-1})\phi\dot\gamma+\kappa(1-\rho_\gamma^2)\mathscr{N}\}.$$

Since we see

$$\begin{aligned}\tilde{\nabla}_{\dot\gamma}\{-\rho_\gamma(\kappa-\kappa^{-1})\phi\dot\gamma+\kappa(1-\rho_\gamma^2)\mathcal{N}\} &= \tilde{\nabla}_{\dot\gamma}\{-\rho_\gamma(\kappa-\kappa^{-1})J\dot\gamma+(\kappa-\rho_\gamma^2\kappa^{-1})\mathcal{N}\}\\ &= -(1-\rho_\gamma^2)(\rho_\gamma\dot\gamma-\xi),\end{aligned}$$

the extrinsic shape is a helix of proper order 4. In view of the Frenet frame of the extrinsic shape, we find that it lies on some totally geodesic $\mathbf{C}P^2$ and is essential. Moreover, we have

$$\tau_{12}=\frac{1}{\kappa_1}\langle\dot\gamma,-\kappa\dot\gamma+\rho_\gamma\kappa^{-1}\xi\rangle=\frac{\mathrm{sgn}(\kappa)\cdot(\rho_\gamma^2-\kappa^2)}{\sqrt{\kappa^4-2\kappa^2\rho_\gamma^2+\rho_\gamma^2}}=\frac{-\mathrm{sgn}(\kappa-\kappa^{-1})\cdot(\kappa_1-\kappa_3)}{\sqrt{\kappa_2^2+(\kappa_1-\kappa_3)^2}},$$

$$\tau_{34}=\frac{1}{\kappa_1(1-\rho_\gamma^2)}\langle\rho_\gamma\dot\gamma-\xi,\rho_\gamma(\kappa-\kappa^{-1})\dot\gamma-(\kappa-\rho_\gamma^2\kappa^{-1})\xi\rangle=\frac{\mathrm{sgn}(\kappa)\cdot(\kappa^2-\rho_\gamma^2)}{\sqrt{\kappa^4-2\kappa^2\rho_\gamma^2+\rho_\gamma^2}},$$

hence the extrinsic shape satisfies the condition (II) if it is of proper order 4. □

Remark 1. When $\pi/4<r<\pi/2$, the extrinsic shape of a circular $\mathbf{F}_{\pm\sqrt{\cot r}}$-trajectory is a moderate Killing helix. That is, its complex torsions satisfy $\tau_{12}=\tau_{13}=\tau_{24}=\tau_{34}=0$ and $\tau_{23}=-\tau_{14}=1$.

From the viewpoint of geometry of helices, Proposition 3 gives geometrically a family of essential Killing helices of proper order 4 on $\mathbf{C}P^n$ with two parameters.

5. Lengths of circular trajectories on geodesic spheres

In this section, by making use of a Hopf fibration $\varpi: S^{2n+1}(1)\to\mathbf{C}P^n(4)$ of a standard sphere, which connects the geometry of $\mathbf{C}P^n$ with that of a complex Euclidean space $\mathbf{C}^{n+1}$, we show Theorem 1. For this sake we quickly recall some properties of circles on $\mathbf{C}P^n$ (see [6] for detail). For a circle σ on $\mathbf{C}P^n(4)$ we take its horizontal lift $\hat\sigma$ with respect to a Hopf fibration. When the geodesic curvature of σ is $1/\sqrt{2}$ and its complex torsion is τ $(0\le|\tau|<1)$, its horizontal lift satisfies the differential equation $\hat\sigma'''+(3/2)\hat\sigma'-\sqrt{-1}(\tau/\sqrt{2})\hat\sigma=0$ as a curve in $\mathbf{C}^{n+1}$. Its characteristic equation with variable Λ turns to

$$\Theta^3-(3/2)\Theta+\tau/\sqrt{2}=0 \tag{5.1}$$

if we set $\Theta=-\sqrt{-1}\Lambda$. This cubic equation should have three distinct real solutions. If we denote them by a, b, c $(a<b<c)$, then we see that $\hat\sigma$ is of the form $\hat\sigma(t)=Ae^{\sqrt{-1}at}+Be^{\sqrt{-1}bt}+Ce^{\sqrt{-1}ct}$ with some $A,B,C\in\mathbf{C}^{n+1}$. By this expression we find that σ is closed if and only if one of (hence all of) the ratios b/a, c/b, a/c is (are) rational and that the length of σ in this case is $2\pi\times\mathrm{L.C.M.}\{(b-a)^{-1},(c-b)^{-1}\}$. On the other hand, by a parallel isometric embedding $S^1\times S^{n-1}/{\sim}\to\mathbf{C}P^n(4)$ defined in [13], geodesics on $S^1\times S^{n-1}/{\sim}$

are mapped to circles of geodesic curvature $1/\sqrt{2}$ on $\mathbf{C}P^n(4)$. This shows a condition for these circles to be closed and their lengths. As we get these information by two ways, arithmetically and geometrically, we can combine them.

PROPOSITION 4 ([6]). *Let σ be a circle of geodesic curvature κ and of complex torsion τ on $\mathbf{C}P^n(4)$.*

(1) *If $\tau = \pm 1$, it is closed of length $2\pi/\sqrt{\kappa^2+4}$.*
(2) *If $\tau = 0$, it is closed of length $2\pi/\sqrt{\kappa^2+1}$.*
(3) *If $0 < |\tau| < 1$, it is closed if and only if*

$$(3\sqrt{3}/2)\kappa\tau(\kappa^2+1)^{-3/2} = \pm q(9p^2-q^2)(3p^2+q^2)^{-3/2}$$

holds with some relatively prime positive integers p, q $(p > q)$. In this case, its length is given as $\pi\delta(p,q)\sqrt{(3p^2+q^2)/3(\kappa^2+1)}$, where $\delta(p,q) = 1$ when pq is odd and $\delta(p,q) = 2$ when pq is even.

Proof of Theorem 1. Let γ be a circular trajectory on $G(r)$ in $\mathbf{C}P^n(4)$. We take a horizontal lift $\hat{\gamma}$ of the extrinsic shape of γ with respect to a Hopf fibration $\varpi : S^{2n+1}(1) \to \mathbf{C}P^n(4)$. The connections $\overline{\nabla}$ on $\mathbf{C}^{n+1}$ and $\tilde{\nabla}$ on $\mathbf{C}P^n(4)$ are related as

$$\overline{\nabla}_X Y = \tilde{\nabla}_X Y - \langle X, Y\rangle\hat{\mathcal{N}} + \langle X, JY\rangle J\hat{\mathcal{N}} \tag{5.2}$$

with a unit normal $\hat{\mathcal{N}}$ of $S^{2n+1}(1)$ in $\mathbf{C}^{n+1}$ and the complex structure J on $\mathbf{C}^{n+1}$. As we studied in Proposition 3, the extrinsic shape of γ, which is also denoted by γ for simplicity, is either a circle or an essential Killing helix of proper order 4 on $\mathbf{C}P^n$. We first consider the latter case. As it lies on some totally geodesic $\mathbf{C}P^2$, we find it is determined by the following system of differential equations ([2]):

$$\begin{cases} \tilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = \kappa_1 Y_2, \\ \tilde{\nabla}_{\dot{\gamma}} Y_2 = -\kappa_1\dot{\gamma} + \{(\kappa_1-\kappa_3)\dot{\gamma} + \mathrm{sgn}(\kappa-\kappa^{-1})\sqrt{\kappa_2^2+(\kappa_1-\kappa_3)^2}JY_2\}. \end{cases}$$

Regarding $\hat{\gamma}$ as a curve in $\mathbf{C}^{n+1}$, we obtain by use of (5.2) that

$$\begin{cases} \overline{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}} = \kappa_1 Y_2 - \hat{\mathcal{N}}, \\ \overline{\nabla}_{\dot{\gamma}} Y_2 = -\kappa_3\dot{\gamma} + \mathrm{sgn}(\kappa-\kappa^{-1})\sqrt{\kappa_2^2+(\kappa_1-\kappa_3)^2}JY_2 + \tau_{12}J\hat{\mathcal{N}}. \end{cases}$$

Rewriting this system of differential equations, we find it satisfies the following differential equation

$$\hat{\gamma}''' - \sqrt{-1}(\kappa-\kappa^{-1})\hat{\gamma}'' + (2-\rho_\gamma^2)\hat{\gamma}' + \sqrt{-1}(1-\rho_\gamma^2)\kappa^{-1}\hat{\gamma} = 0. \tag{5.3}$$

If we consider the former case that the extrinsic shape of γ is a circle in $\mathbf{C}P^n(4)$, by use of (5.2) we find (5.3) holds even in this case.

We now consider the characteristic equation

$$\Lambda^3 - \sqrt{-1}(\kappa-\kappa^{-1})\Lambda^2 + (2-\rho_\gamma^2)\Lambda + \sqrt{-1}(1-\rho_\gamma^2)\kappa^{-1} = 0 \tag{5.4}$$

for the differential equation (5.3) on a horizontal lift of the extrinsic shape of γ. This cubic equation should have three distinct pure imaginary solutions $\sqrt{-1}a_\kappa$, $\sqrt{-1}b_\kappa$, $\sqrt{-1}c_\kappa$ $(a_\kappa < b_\kappa < c_\kappa)$. By use of these we have $\hat{\gamma}(t) = A_\kappa e^{\sqrt{-1}a_\kappa t} + B_\kappa e^{\sqrt{-1}b_\kappa t} + C_\kappa e^{\sqrt{-1}c_\kappa t}$ with some $A_\kappa, B_\kappa, C_\kappa \in \mathbf{C}^{n+1}$. Therefore we find that γ is closed if and only if there exists a constant d satisfying that all of the ratios $(a_\kappa - d)/(b_\kappa - d)$, $(b_\kappa - d)/(c_\kappa - d)$, $(c_\kappa - d)/(a_\kappa - d)$ are rational, and that its length in this case is $2\pi \times \text{L.C.M.}\{(b_\kappa - a_\kappa)^{-1}, (c_\kappa - b_\kappa)^{-1}\}$. In order to make use of the arithmetic information obtained by Proposition 4, we put $\theta = (-3\sqrt{-1}\Lambda - \kappa + \kappa^{-1})/\sqrt{2(\kappa^2 + \kappa^{-2} + 4 - 3\rho_\gamma^2)}$. We then find the equation (5.4) turns to

$$\theta^3 - \frac{3}{2}\theta - \frac{\operatorname{sgn}(\kappa) \cdot (\kappa^2 + 2)\{2\kappa^4 + (8 - 9\rho_\gamma^2)\kappa^2 - 1\}}{2\sqrt{2}(\kappa^4 + 4\kappa^2 - 3\rho_\gamma^2\kappa^2 + 1)^{3/2}} = 0. \tag{5.5}$$

Comparing two cubic equations (5.5) and (5.1) we put

$$\tau(\kappa; r) = -\frac{\operatorname{sgn}(\kappa) \cdot (\kappa^2 + 2)\{2\kappa^4 + (8 - 9\rho_\gamma^2)\kappa^2 - 1\}}{2(\kappa^4 + 4\kappa^2 - 3\rho_\gamma^2\kappa^2 + 1)^{3/2}}.$$

By direct computation we can check that $|\tau(\kappa; r)| < 1$. By use of the solutions a, b, c $(a < b < c)$ for (5.1) with $\tau = \tau(\kappa; r)$, we have

$$a_\kappa = (a\sqrt{2(\kappa^2 + \kappa^{-2} + 4 - 3\rho_\gamma^2)} + \kappa - \kappa^{-1})/3,$$

and so on for b_κ and c_κ. Thus we find γ is closed if and only if a circle σ of geodesic curvature $1/\sqrt{2}$ and of complex torsion $\tau(\kappa; r)$ on $\mathbf{C}P^n(4)$ is closed. Moreover, in this case we obtain lengths of γ and σ satisfy the relation

$$\text{length}(\gamma) = 3\ \text{length}(\sigma)/\sqrt{2(\kappa^2 + \kappa^{-2} + 4 - 3\rho_\gamma^2)}.$$

Since $\kappa\rho_\gamma = \cot r$, we have $|\kappa| > \cot r$. Thus we find $\tau(\kappa; r) = 0$ if and only if $\pi/4 < r < \pi/2$ and $\kappa^2 = (3\sqrt{2(\cot^2 r + 1)} - 4)/2$. As lengths of circles of geodesic curvature $1/\sqrt{2}$ and of complex torsion τ $(|\tau| < 1)$ are given as

$$\text{length} = \begin{cases} 2\sqrt{6}\pi/3, & \text{if } \tau = 0, \\ (\sqrt{2}/3)\pi\delta(p, q)\sqrt{3p^2 + q^2}, & \text{if } \tau = \pm q(9p^2 - q^2)(3p^2 + q^2)^{-3/2}, \\ \infty, & \text{otherwise}, \end{cases}$$

we get our conclusion. □

Remark 2. On a standard sphere $S^{2n-1}(1)$ in $\mathbf{C}^n$ a trajectory for $\mathbf{F}_\kappa$ is circular if and only if $\kappa\rho_\gamma = 1$ ([9]). Since its geodesic curvature is $\sqrt{\kappa^2 - 1}$, we find that it is closed of length $2\pi/|\kappa|$.

We here rewrite our main result to the case of geodesic spheres in a complex projective space of constant holomorphic sectional curvature c. For this sake we consider homothetical changes of metrics. Let γ be a trajectory for a Sasakian magnetic field $\mathbf{F}_\kappa$ on a real hypersurface M in a Kähler manifold $\tilde{M}$. If we change the metric $\langle\, ,\rangle$ on $\tilde{M}$ homothetically to the metric $\lambda^2\langle\, ,\rangle$ with some positive λ, then the curve σ given as $\sigma(t) = \gamma(t/\lambda)$ is a curve parameterized by its arclength with respect to the new induced metric and satisfies $\nabla_{\sigma'}\sigma' = (\kappa/\lambda)\phi\sigma'$, hence is a trajectory for $\mathbf{F}_{\kappa/\lambda}$ with respect to the new metric. Under this operation on metrices, sectional curvatures change λ^{-2}-times of the original sectional curvatures, and lengths of closed curves change λ-times of the original lengths. We can hence get the following:

THEOREM 3. *Let $G(r)$ be a geodesic sphere of radius r $(0 < r < \pi/\sqrt{c})$ in a complex projective space $\mathbf{C}P^n(c)$ of constant holomorphic sectional curvature c. A circular trajectory γ for a Sasakian magnetic field $\mathbf{F}_\kappa$ on $G(r)$ satisfies the following*:

(1) *When $\pi/2\sqrt{c} < r < \pi/\sqrt{c}$ and $\kappa^2 = c\{3\sqrt{2\{\cot^2(\sqrt{c}r/2)+1\}} - 4\}/8$, it is closed and its length is $4\pi\sqrt{(2/c)\sin(\sqrt{c}r/2)\{3\sqrt{2} - 4\sin(\sqrt{c}r/2)\}}$.*

(2) *Otherwise, γ is closed if and only if*

$$\frac{(2\kappa^2+c)|32\kappa^4 + 32c\kappa^2 - c^2(9\cot^2(\sqrt{c}r/2)+1)|}{\{16\kappa^4 + 16c\kappa^2 - c^2(3\cot^2(\sqrt{c}r/2)-1)\}^{3/2}} = \frac{q(9p^2-q^2)}{(3p^2+q^2)^{3/2}}$$

holds with some relatively prime positive integers p, q satisfying $p > q$. In this case its length is given as

$$4\pi\delta(p,q)|\kappa|\sqrt{(3p^2+q^2)/\{16\kappa^4 + 16c\kappa^2 - c^2(3\cot^2(\sqrt{c}r/2)-1)\}},$$

where $\delta(p,q) = 1$ when pq is odd and $\delta(p,q) = 2$ when pq is even.

6. Length spectrum of circular trajectories

In this section we study the length spectrum of circular trajectories on geodesic spheres in $\mathbf{C}P^n$. We first recall the congruence-condition on all trajectories for Sasakian magnetic fields.

LEMMA 1 ([9]). *Let γ_i $(i = 1, 2)$ be trajectories for Sasakian magnetic fields $\mathbf{F}_{\kappa_i}$ on $G(r)$ in $\mathbf{C}P^n$. Then they are congruent to each other if and only if one of the following conditions holds*:

i) $|\rho_{\gamma_1}| = |\rho_{\gamma_2}| = 1$,

ii) $\rho_{\gamma_1} = \rho_{\gamma_2} = 0$ *and* $|\kappa_1| = |\kappa_2|$,

iii) $0 < |\rho_{\gamma_1}| = |\rho_{\gamma_2}| < 1$ *and* $\kappa_1\rho_{\gamma_1} = \kappa_2\rho_{\gamma_2}$.

In the rest of this section, we only treat the case $c = 4$. Since the structure torsion ρ_γ of a circular trajectory γ for $\mathbf{F}_\kappa$ on $G(r)$ in $\mathbf{C}P^n(4)$ satisfies $\kappa\rho_\gamma = \cot r$, we find the moduli space $\mathscr{T}_\phi(G(r))$ of circular trajectories on $G(r)$ is set theoretically identified with the subset $(\cot r, \infty)$ in $\mathbf{R}$.

In order to study the length spectrum of circular trajectories, we shall give estimates of lengths of closed circular trajectories. We define two functions $f, g : [\cot^2 r, \infty) \to \mathbf{R}$ by

$$f(s) = \frac{(s+2)(2s^2+8s-9\cot^2 r - 1)}{2(s^2+4s-3\cot^2 r+1)^{3/2}}, \quad g(s) = \frac{s}{s^2+4s-3\cot^2 r+1}.$$

If we put $\mu(p,q) = q(9p^2 - q^2)(3p^2 + q^2)^{-3/2}$ for a pair (p,q) of relatively prime positive integers p, q with $p > q$, by Theorem 1, we see a circular $\mathbf{F}_\kappa$-trajectory γ is closed of length $\pi\delta(p,q)\sqrt{(3p^2+q^2)g(\kappa^2)}$ if κ $(\neq 0)$ satisfies either $f(\kappa^2) = \mu(p,q)$ or $f(\kappa^2) = -\mu(p,q)$. Thus, in order to give estimates of lengths of closed circular trajectories, we need to study those functions. It is easy to find the following:

i) The function f is monotone increasing, hence satisfies

$$1 > f(s) > f(\cot^2 r) = \frac{(\cot^2 r + 2)(2\cot^2 r + 1)(\cot^2 r - 1)}{2(\cot^4 r + \cot^2 r + 1)^{3/2}} > -1;$$

ii) When $\cot^2 r \geq (\sqrt{13} - 3)/2$, the function g is monotone decreasing and $g(s) < \cot^2 r/(\cot^4 r + \cot^2 r + 1)$;

iii) When $\cot^2 r < (\sqrt{13} - 3)/2$, the function g is monotone increasing in the interval $[\cot^2 r, \sqrt{1 - 3\cot^2 r})$ and is monotone decreasing in other part of its domain, hence $g(s) \leq 1/(4 + 2\sqrt{1 - 3\cot^2 r})$.

These show the following estimate on lengths of circular trajectories from above.

Lemma 2. *The length of circular $\mathbf{F}_\kappa$-trajectory γ satisfying either $f(\kappa^2) = \mu(p,q)$ or $f(\kappa^2) = -\mu(p,q)$ is roughly estimated from above as*

$$\begin{aligned} \text{length}(\gamma) &< \pi\delta(p.q)\cot r\sqrt{(3p^2+q^2)/(\cot^4 r + \cot^2 r + 1)}, \\ &\qquad \textit{if } \cot^2 r \geq \frac{1}{2}(\sqrt{13} - 3), \\ \text{length}(\gamma) &< \pi\delta(p.q)\sqrt{(3p^2+q^2)/(4 + 2\sqrt{1 - 3\cot^2 r})}, \\ &\qquad \textit{if } \cot^2 r < \frac{1}{2}(\sqrt{13} - 3). \end{aligned}$$

We note that $f(\cot^2 r) > 0$ when $0 < r < \pi/4$. Next we give an estimate on lengths of closed circular trajectories from below. We set

$$\alpha(p,q;r) = \sqrt{\frac{\{3\{2(1+\cot^2 r)\}^{1/2} - 4\}\{(3p^2+q^2)^{3/2} - q(9p^2 - q^2)\}}{3(\cot^2 r + 1)(3p^2+q^2)^{1/2}}}.$$

Lemma 3. *If $f(\kappa^2) = \mu(p,q)$ holds, the length of a circular $\mathbf{F}_\kappa$-trajectory γ is roughly estimated from below as*

$$\text{length}(\gamma) > \pi\delta(p.q)\alpha(p,q;r), \qquad \textit{if } \cot^2 r \geq \frac{1}{2}(\sqrt{13} - 3),$$

$$\text{length}(\gamma) > \pi\delta(p.q) \times \min\left\{\alpha(p,q;r), \sqrt{\frac{(3p^2+q^2)\cot^2 r}{\cot^4 r + \cot^2 r + 1}}\right\}, \quad \textit{if } \cot^2 r < \frac{1}{2}(\sqrt{13} - 3).$$

Lemma 4. *If $f(\kappa^2) = -\mu(p,q)$ holds, the length of a circular $\mathbf{F}_\kappa$-trajectory γ is roughly estimated from below as follows: If $(\sqrt{13}-3)/2 \le \cot^2 r < 1$, we have*

$$\mathrm{length}(\gamma) > \pi\delta(p.q)\sqrt{\frac{(3p^2+q^2)\{3(2\cot^2 r+2)^{1/2}-4\}}{3(\cot^2 r+1)}},$$

and if $\cot^2 r < (\sqrt{13}-3)/2$, *we have*

$$\mathrm{length}(\gamma) > \pi\delta(p.q)\sqrt{3p^2+q^2} \times \min\left\{\sqrt{\frac{3(2\cot^2 r+2)^{1/2}-4}{3(\cot^2 r+1)}}, \frac{\cot r}{\sqrt{\cot^4 r+\cot^2 r+1}}\right\}.$$

Proof of Lemma 3, 4. For given τ with $0 < |\tau| < 1$ we estimate the solution s_τ of the equation $f(s) = \tau$ from above. As f is monotone increasing, in the case $r \le \pi/4$, we have $f(s) \ge f(\cot^2 r) \ge 0$. Thus, when $\tau < 0$, we need $r > \pi/4$ and $2s_\tau^2 + 8s_\tau - 9\cot^2 r - 1 < 0$, hence we see $s_\tau < (3\sqrt{2(\cot^2 r+1)}-4)/2 < 1$. When $\tau > 0$, we need $2s_\tau^2 + 8s_\tau - 9\cot^2 r - 1 > 0$. In the domain $\{s \mid 2s^2+8s-9\cot^2 r-1 > 0\} \cap (\cot^2 r, \infty)$, we have

$$f(s) > \frac{2s^2+8s-9\cot^2 r-1}{2(s^2+4s-3\cot^2 r+1)}.$$

If we set $u_\tau = -2+\sqrt{3(3-2\tau)(1+\cot^2 r)/\{2(1-\tau)\}}$ $(> \cot^2 r)$, which is the positive solution of the equation

$$2s^2+8s-9\cot^2 r-1 = 2\tau(s^2+4s-3\cot^2 r+1),$$

we find $f(u_\tau) > \tau$. As this function f is monotone increasing, we get $s_\tau < u_\tau$.

We now estimate $g(s_\tau)$ from below. When $\cot^2 r \ge (\sqrt{13}-3)/2$, as g is monotone decreasing, we have

$$g(s_\tau) > g((3\sqrt{2(\cot^2 r+1)}-4)/2) = \frac{3\sqrt{2(\cot^2 r+1)}-4}{3(\cot^2 r+1)}.$$

for $\tau < 0$, and

$$g(s_\tau) > g(u_\tau) = \frac{2(1-\tau)u_\tau}{3(\cot^2 r+1)} > \frac{(3\sqrt{2(1+\cot^2 r)}-4)(1-\tau)}{3(\cot^2 r+1)},$$

for $\tau > 0$. When $\cot^2 r < (\sqrt{13}-3)/2$, we have to consider the influence of $g(\cot^2 r)$. As we have $(3p^2+q^2)(1-\mu(p,q)) = (3p^2+q^2)^{-1/2}\{(3p^2+q^2)^{3/2} - q(9p^2-q^2)\}$, we can estimate lengths for the case $\tau = \mu(p,q)$ and get the conclusions. □

Proof of Theorem 2. We consider the number

$$\beta(p,q) = (3p^2+q^2)(1-\mu(p,q)) = \frac{27p^2(p^2-q^2)^2}{\sqrt{3p^2+q^2}\{(3p^2+q^2)^{3/2}+q(9p^2-q^2)\}}.$$

One can easily see that $\lim_{q\to\infty}\beta(q+2,q)=\infty$. Since we have $\lim_{q\to\infty}\mu(q+2,q)=1$, for sufficiently large odd q we have κ with $f(\kappa^2)=\mu(q+2,q)$ and with $|\kappa|>\cot r$. Thus Lemma 3 leads us to the conclusion. $\square$

As was mentioned in section 2, Theorems 1, 2 guarantee that there are infinitely many open circular trajectories and infinitely many closed circular trajectories for Sasakian magnetic fields. This property does not depend on radii of geodesic spheres. It is known that a geodesic sphere $G(\pi/4)$ in $\mathbf{C}P^n(4)$ has constant ϕ-sectional curvature 5 hence is a Sasakian space form (see [8] for example). Thus even if we restrict ourselves on circular trajectories, properties of trajectories for Sasakian magnetic fields and those of Kähler magnetic fields on a complex projective space are different: On $\mathbf{C}P^n(4)$ every trajectory for a Kähler magnetic field $\mathbf{B}_\kappa$ is closed of length $2\pi/\sqrt{\kappa^2+4}$, hence the length spectrum of all trajectories for Kähler magnetic fields is $(0,\pi)$ and is a bounded set ([1]).

We call the infimum $\lambda_0(G(r))$ of the elements of $\mathrm{LSpec}_\phi(G(r))$ the bottom of the length spectrum. We here give a rough estimate of this bottom.

Theorem 4. *The bottom $\lambda_0(G(r))$ of $\mathrm{LSpec}_\phi(G(r))$ on a geodesic sphere $G(r)$ in $\mathbf{C}P^n(4)$ is positive. It is roughly estimated from below as follows*:

$$\begin{cases} \lambda_0(G(r)) > 2\pi\sqrt{(49-10\sqrt{7})\sin r(3\sqrt{2}-4\sin r)/21}, & \text{if } \cot^2 r \geq (\sqrt{13}-3)/2. \\ \lambda_0(G(r)) > 2\pi\cot r\sqrt{7/(\cot^4 r+\cot^2 r+1)}, & \text{if } \cot^2 r < (\sqrt{13}-3)/2. \end{cases}$$

When $r>\pi/4$, it is estimated from above as $\lambda_0(G(r)) \leq 2\pi\sqrt{2\sin r(3\sqrt{2}-4\sin r)}$.

Proof. We first study the estimate of $\lambda_0(G(r))$ from below. We use the same notation as in the proof of Theorem 2. We find that $\beta(p+1,q)>\beta(p,q)$ for arbitrary (p,q) and that $\beta(q+1,q)$ is monotone increasing. In the case $0<r\leq\pi/4$, though $f(\cot^2 r)$ may be larger than $\mu(3,1)$, considering the influence of $\delta(p,q)$, we can conclude that $\lambda_0(G(r))$ is estimated from below by the estimate in Lemma 3 corresponding to $\mu(3,1)$. In the case $(\sqrt{13}-3)/2\leq\cot^2 r<1$, we also have to take in account of lengths of circular trajectories satisfying $f(\kappa^2)=-\mu(p,q)$ and that of trajectories satisfying $f(\kappa^2)=0$. We can get the same estimate as for the case $0<r\leq\pi/4$ by Lemma 4. In the case of $\cot^2 r<(\sqrt{13}-3)/2$, we need to additionally consider $\pi\delta(p,q)\sqrt{(3p^2+q^2)g(\cot^2 r)}$ by Lemmas 3 and 4. Comparing $\sqrt{7g(\cot^2 r)}$, which corresponds to the case $(p,q)=(3,1)$, with the estimate in the case $(\sqrt{13}-3)/2\leq\cot^2 r<1$, we obtain our estimate from below.

We next study an estimate of the bottom $\lambda_0(G(r))$ from above in the case $\pi/4<r<\pi/2$. In this case as $f(\cot^2 r)<0$, we have trajectories satisfying $f(\kappa^2)=\mu(3,1)$. We hence obtain our estimate by Lemma 2. $\square$

When we consider circular trajectories, we omit geodesic trajectories. This is because every trajectory for the trivial magnetic field is a geodesic. But it might be natural to treat geodesics with structure torsion ± 1 as "circular" trajectories because they are trajectories for an arbitrary Sasakian magnetic field. As we mentioned in Proposition 2, their length is $\pi \sin 2r$. It is known that the length of trajectories with structure torsion ± 1 gives the minimal length of geodesics on this sphere. Since our estimate on $\lambda_0(G(r))$ is too rough, we can only conclude that $\pi \sin 2r$ is smaller than $\lambda_0(G(r))$ in the case $\cot^2 r < (\sqrt{13}-3)/2$.

Before closing our paper we make mention of another aspect of trajectories for Sasakian magnetic fields. Let $\psi : \mathbf{C}P^n(c) \to S^{n(n+2)-1}((n+1)c/(2n))$ denote the first standard minimal embedding. We then have a parallel isometric embedding $\psi \circ \imath : G(r) \to S^{n(n+2)-1}((n+1)c/(2n))$ with the inclusion $\imath : G(r) \to \mathbf{C}P^n(c)$. From the viewpoint of submanifold-theory, it is also interesting to consider shapes of trajectories through $\imath$ and $\psi \circ \imath$. According to [2, 11], the extrinsic shape $\imath \circ \gamma$ of a trajectory γ for $\mathbf{F}_\kappa$ on $G(r)$ in $\mathbf{C}P^n(4)$ is a circle of positive geodesic curvature if and only if either $\kappa\rho_\gamma = \cot r - \rho_\gamma^2 \tan r$ or $\kappa = \rho_\gamma \tan r$, and the shape $\psi \circ \imath \circ \gamma$ is a circle of positive geodesic curvature if and only if the former condition holds. Thus shapes of circular trajectories through $\psi \circ \imath$ are not circles of positive geodesic curvature on $S^{n(n+2)-1}$. We may hence say that our study on circular trajectories and that on trajectories from the viewpoint of submanifold-theory give different clues to the study of homogeneous curves on Sasakian space forms.

We also mention that we can discuss the similar argument for circular trajectories on geodesic spheres in a complex hyperbolic space. As these geodesic spheres are not "Berger spheres" and we have other good example of homogeneous submanifolds having two principal curvatures, we shall leave it for the next occasion.

References

[1] T. Adachi, Kähler magnetic flows on a manifold of constant holomorphic sectional curvature, Tokyo J. Math. **18** (1995), 473–483.

[2] T. Adachi, Trajectories on geodesic spheres in a non-flat complex space form, J. Geom. **90** (2008), 163–172.

[3] T. Adachi, Essential Killing helices of order less than five on a non-flat complex space form, to appear in J. Math. Soc. Japan.

[4] T. Adachi and S. Maeda, Length spectrum of circles in a complex projective space, Osaka J. Math. **35** (1998), 553–565.

[5] T. Adachi and S. Maeda, Sasakian curves on hypersurfaces in a nonflat complex space form, Result. Math. **56** (2009), 488–499.

[6] T. Adachi, S. Maeda and S. Udagawa, Circles in a complex projective space, Osaka J. Math. **32** (1995), 709–719.

[7] T. Adachi, S. Maeda and M. Yamagishi, Length spectrum of geodesic spheres in a non-flat complex space form, J. Math. Soc. Japan **54** (2002), 373–408.

[8] D. E. Blair, Riemannian geometry of contact and symplectic manifolds, Progress in Math. **203** (2002), Birkhäuser.

[9] T. Bao and T. Adachi, Circular trajectories on real hypersurfaces in a nonflat complex space form, J. Geom. **96** (2009), 41–55.

[10] O. Ikawa, Motion of electric charged particles in homogeneous Kähler manifolds and homogeneous Sasakian manifolds, Far East J. Math. Sci. **14** (2004), 283–302.
[11] S. Maeda and T. Adachi, Holomorphic helices in a complex space form, Proc. A.M.S. **125** (1997), 1197–1202.
[12] S. Maeda and Y. Ohnita, Helical geodesic immersion into complex space form, Geom. Dedicata **30** (1983), 93–114.
[13] H. Naitoh, Isotropic submanifolds with parallel second fundamental form in $P^m(\mathbf{C})$, Osaka J. Math. **18** (1981), 427–464.

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