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Leray endomorphisms and cone maps

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Leray Endomorphisms and Cone Maps (*).

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dedicated to Jean Leray

Introduction.

Computations of the fixed point index of a map which is not necessarily compact have proved to lead to interesting applications (cf. [16, 17, 19]). In this paper we shall try to generalize to some non compact maps the index computations due to C. C. Fenske and H.-O. Peitgen [5], G. Fournier and H.-O. Peitgen [8] and R. D. Nussbaum [16].

The notion of fixed point index used in this paper shall be the one defined by R. D. Nussbaum [18]. As an alternative, the one defined by J. Eells and G. Fournier in [4] generalized to convex sets would be sufficient. Our methods of proof strongly rely on the calculation of the generalized Lefschetz number and the generalized trace due to J. Leray [13].

0. – Preliminaries.

0.1. *Compact attractors and ejective sets.*

An extensive use of the notion of « compact attractor » which is due to Nussbaum [15] shall be made in the following.

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(0.1.1) DEFINITION. Let X be a topological space and $f\colon X \to X$ a continuous map. A compact, nonempty set $M \subset X$ such that M is f-invariant i.e., $f(M) \subset M$, will be called a compact attractor for f if, given any open neighbourhood U of M and any compact set $K \subset X$, there exists an integer $n = n(K, U)$ such that $f^m(K) \subset U$ for $m \geq n$.

In the above situation we say that M « attracts » the compact subsets of X. If $f^m(K) \subset M$ for $m \geq n(K)$, then we say that M « absorbs » the compact subsets of X under f. (f^m is the m-th iterate of f).

(0.1.2) PROPOSITION. *Let X be a topological space and $f\colon X \to X$ a continuous map. Let V be an open subset of X such that there exists $n \in N$ such that*

$$f^m(X \setminus V) \subset X \setminus V$$

for all $m \geq n$. Then $X \setminus \overline{V}$ absorbs the compact subsets of

$$U_V = \bigcup_{i=1}^{\infty} f^{-i}(X \setminus \overline{V})$$

under f. ($\overline{V}$ denotes the closure of V).

PROOF. Observe that $X \setminus V \subset U_V$ whence $f(U_V) \subset U_V$. Now, let $K \subset U_V$ be compact. Then there exists $j = j(K)$ such that

$$K \subset \bigcup_{i=1}^{j} f^{-i}(X \setminus \overline{V}),$$

and hence, for all $m \geq n + j$, we have that

$$f^m(K) \subset \bigcup_{i=1}^{j} f^{m-i}(X \setminus \overline{V}) \subset X \setminus \overline{V}.$$

The notion of an « ejective » point is due to F. E. Browder [1, 2] and plays a fundamental role in recent studies of the existence of periodic solutions of certain nonlinear functional differential equations.

(0.1.3) DEFINITION. Let X be a topological space and $f\colon X \to X$ a continuous map. A closed subset F of X is said to be ejective for f relative to an open neighbourhood U of F provided that, for all $x \in \overline{U} \setminus F$ there exists $n = n(x)$ such that $f^n(x) \in X \setminus \overline{U}$.

The relation between compact attractors and ejective sets (to be made precise in the next proposition) is fundamental for our considerations.

A similar observation has been used in [5], but there, however, the mappings are always assumed to be locally compact and this is a restriction which, in view of the class of mappings we shall consider, has to be eliminated. Unfortunately, this programm requires some very technical and elaborate arguments.

(0.1.4) PROPOSITION. *Let X be a topological space and $f: X \to X$ a continuous map which has a compact attractor M. Let F be an ejective set for f, and assume that*

$$f(X \setminus F) \subset X \setminus F. \tag{*}$$

Then $f: X \setminus F \to X \setminus F$ has a compact attractor M'.

PROOF. Since F is ejective, we can choose an open neighbourhood V of F such that

$$X \setminus F \subset \bigcup_{i=0}^{\infty} f^{-i}(X \setminus \overline{V}).$$

According to (0.1.5), there exists a compact, f-invariant set M' such that

$$M \setminus V \subset M' \subset M \setminus F \subset X \setminus F.$$

It remains to show that M' attracts the compact subsets of $X \setminus F$ under f: let $K \subset X \setminus F$ be compact and let U be an open neighbourhood of M' in $X \setminus F$. Consider

$$M_K = \bigcup_{i=0}^{\infty} f^i(K) \cup M.$$

Obviously, M_K is f-invariant and, since M attracts the compact subsets of X and K and M are compact, it follows that M_K is compact. According to (0.1.5), there exists a compact, f-invariant set M'_K such that

$$M_K \setminus V \subset M'_K \subset M_K \setminus F.$$

Moreover, since K is compact, there exists $m_K \in \boldsymbol{N}$ such that

$$K \subset \bigcup_{i=0}^{m_K} f^{-i}(M_K \setminus \overline{V})$$

and hence

$$f^{m_K}(K) \subset \bigcup_{i=0}^{m_K} f^i(M_K \setminus \overline{V}) \subset \bigcup_{i=0}^{m_K} f^i(M'_K) \subset M'_K.$$

Thus, it remains to show that there exists $n_K \in N$ such that

$$f^n(M'_K) \subset U \quad \text{for all } n \geqslant n_K .$$

Again, since M'_K is compact, there exists $j_K \in N$ such that

$$M'_K \subset \bigcup_{i=0}^{j_K} f^{-i}(X \setminus \overline{V}) .$$

Note that $M' \subset U$ and M' is f-invariant. Hence

$$M' \subset \bigcap_{i=0}^{j_K} f^{-i}(U) = W ,$$

and, since $M \setminus V \subset M' \subset W$, we have that $M \subset V \cup (M \setminus V) \subset V \cup W$. Moreover, since M is f-invariant, it follows that

$$M \subset \bigcap_{i=0}^{j_K} f^{-i}(V \cup W) .$$

Hence, since M attracts M'_K, there exists $n_K > j_K$ such that

$$f^j(M'_K) \subset \bigcap_{i=0}^{j_K} f^{-i}(V \cup W)$$

for all $j \geqslant n_K - j_K$.

We now have that, for all $j \geqslant n_K - j_K$,

$$f^j(M'_K) \subset M'_K \cap f^j(M'_K) \subset \left\{\bigcup_{i=0}^{j_K} f^{-i}(X \setminus \overline{V})\right\} \cap \left\{\bigcap_{i=0}^{j_K} f^{-i}(V \cup W)\right\}$$
$$\subset \bigcup_{i=0}^{j_K} f^{-i}(\{X \setminus \overline{V}\} \cap \{V \cup W\}) \subset \bigcup_{i=0}^{j_K} f^{-i}(W) .$$

Finally, we have that, for all $j \geqslant n_K$,

$$f^j(M'_K) \subset f^{j_K}\left(\bigcup_{i=0}^{j_K} f^{-i}(W)\right) \subset \bigcup_{i=0}^{j_K} f^i(W) \subset \bigcup_{i=0}^{j_K} f^i(f^{-i}(U)) \subset U .$$

(0.1.5) LEMMA. *Let X be a topological space and $f\colon X \to X$ a continuous map. Let F be closed and V be open in X such that $f(X \setminus F) \subset X \setminus F$, $F \subset V$ and*

$$X \setminus F \subset \bigcup_{i=0}^{\infty} f^{-i}(X \setminus \overline{V}) .$$

If M is a compact, f-invariant subset, then there exists a compact, f-invariant subset M' such that

$$M\setminus V \subset M' \subset M\setminus F\,.$$

PROOF. Since $M\setminus V$ is compact, we have that $f^i(M\setminus V)$ is compact for all $i \in \boldsymbol{N}$. Moreover,

$$f^i(M\setminus V) \subset f^i(M\setminus F) \subset M\setminus F \subset \Big\{\bigcup_{i=0}^{\infty} f^{-i}(X\setminus \overline{V})\Big\} \cap \Big\{\bigcap_{i=0}^{\infty} f^{-i}(M)\Big\} \subset \bigcup_{i=0}^{\infty} f^{-i}(M\setminus \overline{V}) \quad \text{for all } i \geqslant 0\,.$$

Hence, for $i = 1$, there exists $n \in \boldsymbol{N}$ such that

$$f(M\setminus V) \subset \bigcup_{i=0}^{n} f^{-i}(M\setminus \overline{V})\,.$$

Set $M' = \bigcup_{i=0}^{n} f^i(M\setminus V)$. Then M' is compact, $M\setminus V \subset M' \subset M\setminus F \subset X\setminus F$, and

$$f^{n+1}(M\setminus V) \subset \bigcup_{i=0}^{n} f^i(M\setminus V) = M'\,; \qquad \text{i.e., } f(M') \subset M'\,.$$

(0.1.6) COROLLARY. *Let X be a topological space and $f\colon X \to X$ a continuous map which has a compact attractor M. Let V be open in X and such that, for all $x \in X\setminus V$ there is $n = n(x) \in \boldsymbol{N}$ such that $f^n(x) \in X\setminus \overline{V}$.*

Then, if $U_V = \bigcup_{i=1}^{\infty} f^{-i}(X\setminus \overline{V})$, one has that U_V is f-invariant and has a compact attractor.

PROOF. Since $X\setminus \overline{V} \subset X\setminus V \subset U_V$, we have that U_V is f-invariant. Since U_V is open, we have that $F = X\setminus U_V$ is closed, $F \subset V$, and, for all $x \in V\setminus F \subset U_V$ there exists $n = n(x) \in \boldsymbol{N}$ such that

$$f^n(x) \in X\setminus \overline{V}\,;$$

i.e., F is ejective. According to (0.1.4), $f\colon U_V \to U_V$ has a compact attractor.

0.2. *Leray endomorphisms and generalized Lefschetz numbers.*

The notions of this paragraph are due to J. Leray [13]. They have proved to be of great importance in fixed point theory (cf. [9]).

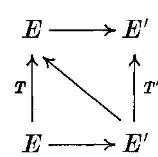
Let E be a vector space and let T be an endomorphism. Set $N(T) = \{e \in E\colon \text{there exists } n \in \boldsymbol{N} \text{ with } T^n(e) = 0\}$. One observes that $T^{-1}(N(T)) \subset N(T)$ and $T(N(T)) \subset N(T)$, and hence T induces an endomorphism $\tilde{T}\colon \tilde{E} \to \tilde{E}$ where $\tilde{E} := E/N(T)$. If $\tilde{E}$ is finite-dimensional, then, since T is injective, $\tilde{T}$ is an isomorphism. We define the « Leray trace » $\mathrm{Tr}(T)$ of T to be the ordinary trace $\mathrm{tr}(\tilde{T})$ of $\tilde{T}$.

Let $E = \{E_q\}$ be a graded vector space and $T = \{T_q\}$ be an endomorphism of degree zero. If $\tilde{E} = \{\tilde{E}_q\}$ is of finite type, then we say that T is a « Leray endomorphism » and we define the « generalized Lefschetz number » $\Lambda(T)$ of T by

$$\Lambda(T) = \sum_q (-1)^q \, \mathrm{Tr}(T).$$

We have the following properties.

(0.2.1) (cf. [10]) Assume that the following diagram of graded vector spaces and morphisms is commutative.



Then, if T or T' is a Leray endomorphism, so is the other and, in that case, $\Lambda(T) = \Lambda(T')$.

(0.2.2) Let $T\colon E \to E$ be an endomorphism of a graded vector space of degree zero. Let $A \subset E$ be a graded vector subspace which is T-invariant and such that, for all $e \in E$, there exists $n \in \boldsymbol{N}$ such that $T^n(e) \in A$. Then T is a Leray endomorphism if, and only if, $T\colon A \to A$ is a Leray endomorphism and, if so,

$$\Lambda(T\colon E \to E) = \Lambda(T\colon A \to A).$$

Proof. The assertion can be obtained as a combination of the following facts (cf. [9]):

1) T induces an endomorphism $\hat{T}$ on E/A and $\hat{T}$ is weakly-nilpotent (i.e., for all $f \in E/A$, there is $n \in \boldsymbol{N}$ such that $\hat{T}^n(f) = 0$); i.e., $\hat{T}$ is a Leray endomorphism and $\Lambda(\hat{T}) = 0$.

2) The following diagram with exact rows is commutative.

$$\begin{array}{ccccccccc} 0 & \to & A & \to & E & \to & E/A & \to & 0 \\ & & \uparrow{\scriptstyle T_A} & & \uparrow{\scriptstyle T_E} & & \uparrow{\scriptstyle \hat{T}} & & \\ 0 & \to & A & \to & E & \to & E/A & \to & 0 \end{array}$$

It is easy to check that T_E is a Leray endomorphism if, and only if, T_A and $\hat{T}$ are Leray endomorphisms and

$$\Lambda(T_E) = \Lambda(T_A) + \Lambda(\hat{T}) .$$

Let X be a topological space and $f: X \to X$ a continuous map. Let H denote the singular homology functor with rational coefficients. (Our main reason for choosing this homology here is that it has compact supports.) If $f_* = H(f): H(X) \to H(X)$ is a Leray endomorphism, then we say that f is a « Lefschetz map » and we define the Lefschetz number $\Lambda(f)$ of f by

$$\Lambda(f) = \Lambda(f_*) .$$

Let us recall that a space X is « acyclic » with respect to H if $H_q(X) = 0$ for $q > 0$ and $H_0(X) = \boldsymbol{Q}$ is the field of coefficients. A space X is « contractible » if there exists $x_0 \in X$ and a continuous map $h: X \times [0, 1] \to X$ such that $h(x, 0) = x$ and $h(x, 1) = x_0$ for all $x \in X$. Note that a contractible space is acyclic with respect to H.

We collect a few properties for Lefschetz maps.

(0.2.3) Let $f: X \to X$ be a continuous map and let $Y \subset X$ absorb the compact subsets of X under f. Then

$$A_Y := \{a \in H(X): f_*^n(a) \in i_* H(Y) \text{ for all } n \geqslant 0\}$$

if f_*-invariant and absorbs the elements of $H(X)$ where $i: Y \to X$ denotes the inclusion. Furthermore, f is a Lefschetz map if, and only if, $f_*: A_Y \to A_Y$ is a Leray endomorphism and, in that case,

$$\Lambda(f: X \to X) = \Lambda(f_*: A_Y \to A_Y) .$$

Proof. Evidently, A_Y is f_*-invariant. Choose $a \in H(X)$. Since H has compact supports, there exists $K \subset X$ compact and $b \in H(K)$ such that $j_*(b) = a$ where $j: K \to X$ denotes the inclusion. Now, there exists $n_K \in \boldsymbol{N}$

such that $f^n(K) \subset Y$ for all $n \geqslant n_K$; i.e., $f^n: K \to Y$ is defined. Therefore, $(f^n)_*(b) \in H(Y)$ for all $n \geqslant n_K$. Hence $f_*^n \circ i_* \circ (f^{n_K})_*(b) \in i_* H(Y)$ for all $n \geqslant 0$. Since the following diagramm is commutative,

$$\begin{array}{ccc} Y & \xrightarrow{i} & X \\ {\scriptstyle f^{n_K}}\uparrow & & \uparrow{\scriptstyle f^{n_K}} \\ K & \xrightarrow[j]{} & X \end{array}$$

we obtain that $f_*^{n_K}(a) = f_*^{n_K}(j_*(b)) = i_*((f^{n_K})_*(b)) \in A_Y$.

Finally, we obtain the remaining part of the assertion from (0.2.2).

(0.2.4) Let $f: X \to X$ be a continuous map and let $Y \subset X$ be an f-invariant subset which absorbs the compact subsets of X under f. Then, if one of $f: X \to X$ or $f: Y \to Y$ is a Lefschetz map, both are Lefschetz maps and, in that case,

$$\Lambda(f: X \to X) = \Lambda(f: Y \to Y).$$

PROOF. This is a consequence of an argument similar to the one in (0.2.2) and the fact that H has compact supports (cf. [6], II, lemma 1.2).

(0.2.5) Let $f: X \to X$ be a continuous map and let X be acyclic. Then f is a Lefschetz map and

$$\Lambda(f: X \to X) = 1.$$

(0.2.6) Let $f, g: X \to X$ be homotopic maps ($f \sim g$). Then

$$\Lambda(f: X \to X) = \Lambda(g: X \to X)$$

provided one of these numbers is defined.

0.3. *Measure of non-compactness.*

The notion of «measure of non-compactness» is due to Kuratowski [11]. Let (Y, d) be a metric space. We define the « measure of non-compactness » $\gamma(Y)$ of Y to be

$$\gamma(Y) = \inf\{r > 0: \text{there exists a finite covering of } Y \text{ by subsets of diameter at most } r\}.$$

Notice that $\gamma(Y) < \infty$ if, and only if, Y is bounded. Let $f\colon X \to Y$ be a continuous map where (X, d') and (Y, d) are metric spaces. We define

$$\gamma(f) = \inf\{k\colon \gamma_Y(f(A)) \leqslant k\gamma_X(A) \text{ for all } A \subset X\}.$$

One has the following properties (cf. [11, 15]):

(0.3.1) $0 \leqslant \gamma(Y) \leqslant \mathrm{diam}(Y)$ where $\mathrm{diam}(Y)$ is the diameter of Y.

(0.3.2) If $A \subset B \subset Y$, then $\gamma(A) \leqslant \gamma(B)$.

(0.3.3) $\gamma(A \cup B) \leqslant \max\{\gamma(A), \gamma(B)\}$.

(0.3.4) If A is compact, then $\gamma(A) = 0$.

(0.3.5) $\gamma(A) = \gamma(\bar{A})$.

(0.3.6) If (Y, d) is complete and $A_1 \supset A_2 \supset A_3 \supset \ldots$ is a sequence of closed, nonempty subsets of Y such that

$$\lim_{n\to\infty} \gamma(A_n) = 0,$$

then

$$A_\infty = \bigcap_{i=1}^{\infty} A_i$$

is compact, nonempty and, for all neighbourhoods V of A_∞, there exists $n_V \in \boldsymbol{N}$ such that $A_n \subset V$ for all $n \geqslant n_V$.

(0.3.7) If f is a compact map, then $\gamma(f) = 0$.

(0.3.8) If $g\colon Y \to Z$ is a continuous map then $\gamma(g \circ f) \leqslant \gamma(g) \cdot \gamma(f)$.

(0.3.9) If f is a Lipschitz map with Lipschitz constant k, then $\gamma(f) \leqslant k$.

(0.3.10) If $X = Y$ and $\lim_{n\to\infty} \gamma(f^n(X)) = 0$, then f has a compact attractor.

Furthermore, if Y is a linear, normed space, we have the following (cf. [3]):

(0.3.11) $\gamma(A + B) \leqslant \gamma(A) + \gamma(B)$.

(0.3.12) $\gamma(r \cdot A) = |r| \cdot \gamma(A)$ for all $r \in \boldsymbol{R}$.

(0.3.13) $\gamma(\mathrm{co}\, A) = \gamma(A)$, where $\mathrm{co}\, A$ denotes the closed, convex hull of A.

(0.3.14) $\gamma(f + g) \leqslant \gamma(f) + \gamma(g)$.

0.4. *Condensing mappings.*

Let X be a metric space and $\Omega \subset X$ a subset. A continuous map $f\colon \Omega \to X$ is called «condensing» (k-set–contraction, with $k < 1$, in [15]) if $\gamma(f\colon \Omega \to X) < 1$. If $f\colon X \to X$ is a continuous map and $\Omega \subset X$ is a subset, then $f\colon \Omega \to X$ is called « eventually condensing » if there exists $n \in N$ such that $\gamma(f^n\colon \Omega \to X) < 1$. A continuous map $f\colon \Omega \to X$ is called « condensing on bounded subsets » if $\gamma(f\colon A \to X) < 1$ for all bounded subsets $A \subset \Omega$. The notions of maps which are « compact on bounded subsets » or are « eventually condensing on bounded subsets » are defined in a similar manner.

0.5. *Fixed point index.*

The reference for this section is R. D. Nussbaum [18]. First, we fix a class of spaces. We shall write $X \in \mathcal{F}$ if X is a closed subset of a Banach space from which it inherits its metric and if X has a closed, locally finite covering $\{C_\alpha\colon \alpha \in A\}$ by closed, convex sets $C_\alpha \subset X$. We shall write $X \in \mathcal{F}_0$ if $X \in \mathcal{F}$ and if A is finite. Note that if $X \in \mathcal{F}$ than X is an absolute neighbourhood retract ($X \in ANR$).

Suppose that U and Y are open subsets of a space $X \in \mathcal{F}$ such that $U \subset Y$ and $f\colon U \to Y$ is a continuous map. Assume that $\mathrm{Fix}(f) = \{x \in U\colon f(x) = x\}$ is compact (possibly empty). Suppose there exists a bounded open neighbourhood W of $\mathrm{Fix}(f)$, $\overline{W} \subset U$, and a decreasing sequence of spaces $K_n \subset Y$, $K_n \in \mathcal{F}_0$, such that

(1) $K_1 \supset W$;

(2) $f(W \cap K_n) \subset K_{n+1}$;

(3) $\lim_{n\to\infty} \gamma(K_n) = 0$.

(0.5.1) DEFINITION (Nussbaum). If the above conditions are satisfied for some W and some decreasing sequence $\{K_n\}$ we say that « f belongs to the fixed point index class », and we define

$$\mathrm{ind}(f\colon U \to Y) = \lim_{n\to\infty} \mathrm{ind}(f\colon W \cap K_n \to K_n)\,.$$

If K_n is empty for some n, then $\mathrm{ind}(f\colon U \to Y)$ is taken to be zero.

Note that a map $f\colon U \to Y$ which is weakly condensing (in the sense of Eells-Fournier [4]) belongs to the fixed point index class. The fixed point index defined in the above generality satisfies the familiar properties; e.g., the excision, additivity, solution, and commutativity properties. Since we make use of the contraction, normalization, and homotopy properties permanently, we cite them here. We need one more definition.

(0.5.2) DEFINITION. Suppose that $X \in \mathcal{F}$, Y is an open subset of X, and $f\colon Y \to Y$ is a continuous map. Let $M \subset Y$ be a compact, f-invariant set. Assume that there exists an open neighbourhood W of M and a decreasing sequence of sets $K_n \in \mathcal{F}_0$, $K_n \subset Y$, such that $K_1 \supset W$, $f(W \cap K_n) \subset K_{n+1}$ and $\lim_{n\to\infty} \gamma(K_n) = 0$. Then we say that $f\colon Y \to Y$ has property (L) in a neighbourhood of M.

We have the following properties.

(0.5.3) CONTRACTION. Suppose that Y, Z are open in $X \in \mathcal{F}$, U is open in Y, and $Z \subset Y$. Then, if $f(U) \subset Z$,

$$\operatorname{ind}(f\colon U \to Y) = \operatorname{ind}(U \cap Z \to Z)\,.$$

(0.5.4) NORMALIZATION. Suppose that $X \in \mathcal{F}$, Y is open in X, and $f\colon Y \to Y$ is a continuous map which has a compact attractor M.

Then, if f has property (L) in a neighbourhood of M, f belongs to the fixed point index class, f is a Lefschetz map, and

$$\operatorname{ind}(f\colon Y \to X) = \Lambda(f\colon Y \to Y)\,.$$

(0.5.5) HOMOTOPY. Suppose that $X \in \mathcal{F}$, U, Y are open in X, and $f\colon U \times [0, 1] \to Y$ is a continuous map such that

$$S = \{x \in U\colon \text{there exists } t \text{ such that } f_t(x) = f(x, t) = x\}$$

is compact. Assume there exists a bounded open neighbourhood W of S with $\overline{W} \subset U$ and a decreasing sequence $K_n \in \mathcal{F}_0$, $K_n \subset Y$, such that $K_1 \supset W$, $f\big((W_n \cap K_n) \times [0, 1]\big) \subset K_{n+1}$, and $\lim_{n\to\infty} \gamma(K_n) = 0$. Then

$$\operatorname{ind}(f_t\colon U \to Y) \quad \text{is defined and constant for } 0 \leqslant t \leqslant 1\,.$$

Since it is difficult to tell whether a map belongs to the fixed point index class, we shall select a few examples of those given in [18] and [4]:

(0.5.6) EXAMPLES:

(1) Suppose that $X \in \mathcal{F}$, U is an open subset of X, and $f\colon U \to X$ is a continuous map such that $\operatorname{Fix}(f)$ is compact. Assume that there is an open neighbourhood W of $\operatorname{Fix}(f)$ such that $f\colon W \to X$ is condensing. Then f belongs to the fixed point index class.

(2) Suppose that $X \in \mathcal{F}$, U is an open subset of X, and $f\colon U \to X$ is a continuous map such that $\mathrm{Fix}(f)$ is compact. Assume there exists a compact set $M \supset \mathrm{Fix}(f)$ and a constant k, $0 < k < 1$, such that $d(f(x), M) \leqslant kd(k, M)$ whenever $x \in U$ and $d(x, M) \leqslant r$, r a fixed positive number. Then f belongs to the fixed point index class.

(3) Let U be an open subset of a Banach space X and $f\colon U \to X$ a continuous map such that $\mathrm{Fix}(f)$ is compact. Assume that f is continuously Fréchet differentiable on some open neighbourhood of $\mathrm{Fix}(f)$ and is eventually condensing on some open neighbourhood of $\mathrm{Fix}(f)$. Then f belongs to the fixed point index class.

1. – Main results.

1.1. *Index of ejective sets and fixed points of index zero.*

In this paragraph, we give generalizations and extensions of characterizations due to C. C. Fenske and H. O. Peitgen [5]. First, we give a formula which allows calculation of the index of certain fixed points in terms of generalized Lefschetz numbers.

(1.1.1) THEOREM. *Let Y be an open subset of a space $X \in \mathcal{F}$. Assume that $f\colon Y \to Y$ is a continuous map which has a compact attractor M and has property (L) in a neighbourhood of M. Let $F \subset Y$ be a closed subset, assume that*

$$(*) \qquad f(Y \setminus F) \subset Y \setminus F) ,$$

and $f\colon Y\setminus F \to Y\setminus F$ has a compact attractor. Then

$$\mathrm{ind}(f\colon W \to Y) = \Lambda(f\colon Y \to Y) - \Lambda(f\colon Y\setminus F \to Y\setminus F)$$

for all open subsets W such that

$$\mathrm{Fix}(f\colon W \to Y) = \mathrm{Fix}(f\colon F \to Y) .$$

PROOF. Note that f has no fixed points in $W\setminus F = W \cap (Y\setminus F)$. Hence the additivity property of the index implies that

$$\mathrm{ind}(f\colon Y \to Y) = \mathrm{ind}(f\colon W \to Y) + \mathrm{ind}(f\colon Y\setminus F \to Y) .$$

Condition (∗) and the contraction property yield

$$\operatorname{ind}(f\colon Y\setminus F \to Y\setminus F) = \operatorname{ind}(f\colon Y\setminus F \to Y),$$

and, finally, the normalization property implies the assertion.

The next result is fundamental for paragraph 1.3.

(1.1.2) COROLLARY. *Let Y be an open subset of a space $X \in \mathcal{F}$. Assume that $f\colon Y \to Y$ is a continuous map which has a compact attractor M and has property (L) in a neighbourhood of M. Let F be an ejective set for f relative to W, and assume that*

(∗) $$f(Y\setminus F) \subset Y\setminus F.$$

Then

$$\operatorname{ind}(f\colon W \to Y) = \Lambda(f\colon Y \to Y) - \Lambda(f\colon Y\setminus F \to Y\setminus F).$$

PROOF. According to (0.1.4), $f\colon Y\setminus F \to Y\setminus F$ has a compact attractor.

(1.1.3) THEOREM. *Let Y be an open subset of a space $X \in \mathcal{F}$. Assume that $f\colon Y \to Y$ is a continuous map which has a compact attractor M and has property (L) in a neighbourhood of M. Let $F \subset Y$ be a closed subset, assume that*

(∗) $$f(Y\setminus F) \subset Y\setminus F,$$

and $f\colon Y\setminus F \to Y\setminus F$ has a compact attractor M'. Furthermore, assume that one of the following conditions is satisfied.

(1) *The inclusion $j\colon Y\setminus F \to Y$ induces an isomorphism $H(j)$ in homology;*

(2) *there exists an open subset U of Y such that $F \subset U \subset \overline{U} \subset Y\setminus M'$ and the inclusion $j\colon Y\setminus U \to Y$ induces an isomorphism $H(j)$ in homology;*

(3) *there exists a neighbourhood V of M' in $Y\setminus F$ and the inclusion $j\colon V \to Y$ induces an isomorphism $H(j)$ in homology. Then*

$$\operatorname{ind}(f\colon W \to Y) = 0$$

for all open subsets W of Y such that

$$F \subset W \quad \text{and} \quad \overline{W} \cap \operatorname{Fix}(f)\setminus F = \emptyset.$$

Moreover, $\Lambda(f\colon Y \to Y) \neq 0$ *implies*

$$\mathrm{Fix}(f)\setminus F \neq \emptyset .$$

PROOF. Note that both $Y\setminus F$ and $Y\setminus U$ are neighbourhoods of M' disjoint from F. Hence it suffices to prove the assertion using (3).

Note that V absorbs the compact subsets of $Y\setminus F$, and hence, in the notation of (0.2.3), we have that $A_V \subset i_* H(V) \subset H(Y\setminus F)$ where $i\colon V \to Y\setminus F$ is the inclusion. Moreover, according to (0.2.3), we have that

$$\Lambda(f_*\colon A_V \to A_V) = \Lambda(f\colon Y\setminus F \to Y\setminus F) .$$

Next, observe that if $a \in H(Y)$, then $j_*^{-1}(a) \in H(V)$, and hence, according to (0.2.3), there exists $n \in N$ such that

$$f_*^n\big(i_*(j_*^{-1}(a))\big) \in A_V \subset i_* H(V) .$$

Now we have a commutative diagram

$$\begin{array}{ccc} V & \xhookrightarrow{\;j\;} & Y \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle f^n} \\ Y\setminus F & & Y \\ {\scriptstyle f^n}\big\downarrow & & \big\| \\ Y\setminus F & \xhookrightarrow[\;l\;]{} & Y \end{array}$$

and thus we obtain

$$f_*^n(a) = l_* f_*^n i_*(j_*^{-1}(a)) \in l_*(A_V) \subset H(Y) .$$

Next, observe that $f_*(A_V) \subset A_V$ and $f_* l_* = l_* f_*$ imply that $f_*(l_*(A_V)) \subset$ $\subset l_*(A_V)$. Applying (0.2.2), we obtain

$$\Lambda(f\colon Y \to Y) = \Lambda\big(f_*\colon l_*(A_V) \to l_*(A_V)\big) .$$

It remains to show that

$$\Lambda\big(f_*\colon l_*(A_V) \to l_*(A_V)\big) = \Lambda(f_*\colon A_V \to A_V) .$$

This follows from (0.2.1) once we have proved that $l_*\colon A_V \to l_*(A_V)$ is an isomorphism. Suppose that l_* is not injective; i.e., there is $a \in A_V$ such that $l_*(a) = 0$ and $a \neq 0$. From the definition of A_V, we have that $a = i_* b$ for some $b \in H(V)$, $b \neq 0$. Now, $l_*(a) = 0$ implies $j_*^{-1} l_* i_*(b) = 0$. However, $j_*^{-1} l_* i_*(b) = b$, and this is a contradiction.

PROBLEM. Does (1.1.3) remain true if we replace the assumption (*) by the assumption that F is ejective?

We can only give a partial answer to this problem. Similar results have been obtained by Nussbaum in [17] and in [5].

(1.1.4) PROPOSITION. *Let P be a closed, convex subset of a Banach space, and let $f: P \to P$ be a continuous map which is condensing on bounded subsets. Assume that $x_0 \in P$ is an ejective fixed point for f relative to W, and assume that $P \setminus \{x_0\}$ is contractible.*

Then

$$\operatorname{ind}(f: W \to P) = 0 .$$

PROOF. Choose $r > 0$ such that $B_r(x_0) \cap P \subset W$.

Define $\varrho: P \to P$ by

$$\varrho(y) = \begin{cases} x_0 + r(y - x_0)\|x_0 - y\|^{-1}, & \text{if } \|x_0 - y\| \geqslant r \\ y, & \text{if } \|x_0 - y\| \leqslant r . \end{cases}$$

Note that if $A \subset P$, then $\varrho(A) \subset \operatorname{co}(\{x_0\} \cup A)$. Thus,

$$\gamma(\varrho(A)) \leqslant \max\{\gamma(\{x_0\}), \gamma(A)\} = \gamma(A) ; \quad \text{i.e., } \gamma(\varrho) \leqslant 1 .$$

This implies $\gamma(f \circ \varrho(A)) \leqslant \gamma(f: \overline{B_r(x_0)} \cap P \to P) \cdot \gamma(A)$ for all $A \subset P$; i.e., $f \circ \varrho: P \to P$ is condensing.

Observe that $f^{-1}(\{x_0\}) \cap B_r(x_0) = \{x_0\}$ whence $f \circ \varrho(P \setminus \{x_0\}) \subset P \setminus \{x_0\}$, since $\varrho(P \setminus \{x_0\}) \subset P \setminus \{x_0\}$. Now we observe that

$$\gamma((f \circ \varrho)^n(P)) \leqslant \left(\gamma(f: \overline{B_r(x_0)} \cap P \to P)\right)^n \cdot 2r ; \quad \text{i.e., } \lim \left(\left(f \circ \varrho(P)\right)^n\right) = 0 .$$

According to (0.3.10), this means that $f \circ \varrho: P \to P$ has a compact attractor. Finally, we can apply (1.1.2) and obtain

$$\begin{aligned} \operatorname{ind}(f: W \to P) &= \operatorname{ind}(f: B_r(x_0) \cap P \to P) = \operatorname{ind}(f \circ \varrho: B_r(x_0) \cap P \to P) \\ &= \Lambda(f \circ \varrho: P \to P) - \Lambda(f \circ \varrho: P \setminus \{x_0\} \to P \setminus \{x_0\}) \\ &= 1 - 1 = 0 \end{aligned}$$

since P and $P \setminus \{x_0\}$ are contractible.

1.2. *Mappings leaving a wedge invariant and the calculation of some Lefschetz numbers.*

Following Schaefer [20], we call a closed, convex subset P of a linear normed space a « wedge » if $x \in P$ implies $t \cdot x \in P$ for $t \geqslant 0$. We call P a « cone » if P is a wedge and $x \in P$, $x \neq 0$, implies that $-x \notin P$. If P is a wedge which has the additional property of a cone for at least one point (i.e., there exists $x_0 \neq 0$, such that $-x_0 \notin P$), then we say P is a « wedge missing a ray ». For $r > 0$, we set

$$B_r = \{x \in P\colon \|x\| < r\} \quad \text{and} \quad S_r = \{x \in P\colon \|x\| = r\}.$$

In the forthcoming paragraphs we shall deal with the following hypotheses.

H_0: There exists $m \in \boldsymbol{N}$ such that

$$f^m(S_r) \subset B_r$$

and, for all $x \in B_r$, there exists $n_x \in \boldsymbol{N}$ such that

$$f^n(x) \in B_r$$

for all $n \geqslant n_x$.

H_∞: There exists $m \in \boldsymbol{N}$ such that

$$f^m(S_r) \subset P \setminus \bar{B}_r$$

and, for all $x \in P \setminus \bar{B}_r$, there exists $n_x \in \boldsymbol{N}$ such that

$$f^n(x) \in P \setminus \bar{B}_r$$

for all $n \geqslant n_x$.

H_0': There exists $m \in \boldsymbol{N}$ such that

$$f^i(S_r) \subset B_r$$

for all $i \geqslant m$, and there exists $n \in \boldsymbol{N}$ such that

$$f^n(B_r) \subset B_r\,.$$

H_∞': There exists $m \in \boldsymbol{N}$ such that

$$f^i(S_r) \subset P \setminus \bar{B}_r$$

for all $i \geq m$, and there exists $n \in N$ such that

$$f^n(P \setminus \bar{B}_r) \subset P \setminus \bar{B}_r .$$

We have the following property which is fundamental for our further considerations (cf. (2.9) of [8]).

(1.2.1) PROPOSITION. *Let P be a wedge missing a ray and let $f: P \to P$ be a continuous map. Assume that one of the conditions H_0, H_∞, H_0', or H_∞' is satisfied. Let*

$$B = \begin{cases} B_r, & \text{in cases } H_0 \text{ and } H_0' \\ P \setminus \bar{B}_r, & \text{in cases } H_\infty \text{ and } H_\infty' . \end{cases}$$

Then $f(W) \subset W$, $f: W \to W$ is a Lefschetz map, and $\Lambda(f: W \to W) = 1$, where

$$W = \bigcup_{i=1}^{\infty} f^{-i}(B) .$$

PROOF. Note that W is open, $f^{-1}(W) \subset W$, and, since $\bar{B} \subset W$, we obtain $f(W) \subset W \cup B \subset W$. Choose $y_0 \in P$ such that $-y_0 \notin P$. We define $\varrho: P \to P$ by

$$\varrho(x) = \begin{cases} x, & \text{if } x \in P \setminus B \\ rx\|x\|^{-1}, & \text{if } x \in \bar{B} \text{ and } B = P \setminus \bar{B}_r \\ r[ry_0 + \|x\|(x - y_0)]\|ry_0 + \|x\|(x - y_0)\|^{-1}, & \text{if } x \in \bar{B} \text{ and } B = B_r . \end{cases}$$

Observe that $\varrho(B) \subset S_r$ and hence $\varrho(W) \subset [(P \setminus B) \cap W] \cup S_r \subset W$. Moreover, $\varrho_{|W} \sim Id_W$ (cf. (2.2) of [8]).

Similar to the proof of (2.9) in [8], one obtains the assertion as a consequence of the following facts:

(1) B is contractible (cf. (2.2) of [8]) and absorbs the compact subsets of W under $f^m \circ \varrho$.

(2) If $i: B \to W$ denotes the inclusion, then $i_* H(B)$ is acyclic, f_*-invariant, and $\hat{f}_* := f_*|_{i_* H(B)} = Id$. Thus, $\Lambda(\hat{f}_*) = 1$.

(3) Let $\tilde{f}_*$ denote the quotient homomorphism of $f_*: H(W) \to H(W)$ and $\hat{f}_*$: Then $\tilde{f}_*$ is weakly nilpotent. This is a consequence of (1). Hence $\Lambda(\tilde{f}_*) = 0$.

(4) Since $\hat{f}_*$ and $\tilde{f}_*$ are Leray endomorphisms, one concludes that $f_*: H(W) \to H(W)$ is a Leray endomorphism and

$$\Lambda(f_*) = \Lambda(\hat{f}_*) + \Lambda(\tilde{f}_*) = 1 + 0 = 1 .$$

The next proposition is only designed for section 1.4. It can be obtained by going through the proof of (1.2.1) with obvious modifications.

(1.2.2) PROPOSITION. *Let P be a wedge missing a ray, $X \subset P$ an open subset, and $f: X \to X$ a continuous map. Assume that, with the notation of* (1.2.1), *$B \subset X$ and that the assumptions of* (1.2.1) *are satisfied. Then $f(W) \subset W$, $f: W \to W$ is a Lefschetz map, and*

$$\Lambda(f: W \to W) = 1 .$$

1.3. *Condensing mappings of wedges.*

The following result for cones is due to Nussbaum [17].

(1.3.1) PROPOSITION. *Let P be a wedge missing a ray in a Banach space, and let $f: P \to P$ be a continuous map which is condensing on bounded subsets. If $0 \in P$ is an ejective fixed point for f relative to U, then*

$$\operatorname{ind}(f: U \to P) = 0 .$$

PROOF. According to (1.1.4), it suffices to show that $P \setminus \{0\}$ is contractible. To see this, choose $r > 0$ and observe that the radial retraction $\varrho: P \setminus \{0\} \to S_r$ defined by

$$\varrho(x) = rx\|x\|^{-1}$$

is homotopic to $Id_{P\setminus\{0\}}$. Since P is missing a ray we find $y_0 \in P$ such that $-y_0 \notin P$. Define $h: S_r \times [0, 1] \to S_r$ by

$$h(x, t) = r(ty_0 + (1-t)x)\|ty_0 + (1-t)x\|^{-1} ;$$

this homotopy is well defined by the choice of y_0.

Nussbaum's proof substantially uses the fact that, if P is a cone, then $0 \in P$ is an extremal point.

The following results for mappings which are compact on bounded sets are due to G. Fournier and H. O. Peitgen [8].

(1.3.2) PROPOSITION. *Let P be a wedge missing a ray in a Banach space, and let $f: P \to P$ be a continuous map which is condensing on bounded subsets. Assume that H_∞ or H'_∞ is satisfied.*

Then

$$\operatorname{ind}(f: B_r \to P) = 0 .$$

PROOF. We shall use the notation of proposition (1.2.1). Note that we have $\varrho(A) \subset \operatorname{co}\{\{0\} \cup A\}$ for all $A \subset P$ whereby $\gamma(\varrho) \leqslant 1$; i.e., $f \circ \varrho: P \to P$ is condensing. Since

$$\gamma\big((f\circ\varrho)^n(P)\big) \leqslant \gamma\big((f\circ\varrho)^{n-1}(f(\bar{B}_r)\big) \leqslant 2r\big(\gamma(f\colon \bar{B}_r \to P)\big)^n ,$$

we have that $\lim\limits_{n\to\infty} \gamma\big((f\circ\varrho)^n(P)\big) = 0$ and, according to (0.3.10), this implies that $f\circ\varrho: P \to P$ has a compact attractor.

Note that $S_r \subset W$, thus $\varrho(W) \subset W$ and $f\circ\varrho(W) \subset W$. Moreover, $F = P \setminus W$ is closed and ejective for f relative to B_r. Since $f_{|\bar{B}_r} = f\circ\varrho_{|\bar{B}_r}$, F is also ejective for $f\circ\varrho$. Now we can apply (1.1.2) and obtain

$$\operatorname{ind}(f\colon B_r \to P) = \operatorname{ind}(f\circ\varrho\colon B_r \to P) = \Lambda(f\circ\varrho\colon P \to P) - \Lambda(f\circ\varrho\colon W \to W) .$$

The first of these Lefschetz numbers is 1 because P is contractible. The second, however, is also 1 since $f\circ\varrho_{|W} \sim f_{|W}$, and thus we can apply (1.2.1).

(1.3.3) PROPOSITION. *Let P be a wedge missing a ray in a Banach space, and let $f: P \to P$ be a continuous map which has property (L) for each compact, f-invariant subset and which is eventually condensing on bounded subsets. Assume that either H_0 is satisfied and there exists $n_0 \in N$ such that*

$$(*) \qquad B_r \subset \bigcup_{i=1}^{n_0} f^{-i}(B_r)$$

or H_0' is satisfied.

Then, if

$$W = \bigcup_{i=1}^{\infty} f^{-i}(B_r) ,$$

we have that W is f-invariant, $f\colon W \to W$ has a compact attractor, and

$$\operatorname{ind}(f\colon B_r \to P) = 1 .$$

PROOF. Note that either H_0 together with condition $(*)$ or H_0' implies that there exists $m_0 \in N$ such that

$$B_r \cup S_r \subset \bigcup_{i=1}^{m_0} f^{-i}(B_r) \subset \bigcup_{i=1}^{m_0} f^{-i}(\bar{B}_r) = F .$$

F is closed in P and, since $f^{-1}(W) \subset W$ and $\bar{B}_r \subset W$, we have that $F \subset W$.

Moreover, F is f-invariant because

$$f(F) \subset \bigcup_{i=1}^{m_0-1} f^{-i}(\bar{B}_r) \cup B_r \subset F .$$

Now we use F to show that $f\colon W \to W$ has a compact attractor M: we observe that $\bar{B}_r \subset \bigcup_{i=1}^{m_0} f^{-i}(\bar{B}_r)$ implies that

$$f^n(\bar{B}_r) \subset \bigcup_{i=1}^{m_0} f^i(\bar{B}_r)$$

for all $n \in N$; in fact, if $x \in \bar{B}_r$, then there are $i_1, \ldots, i_s \in N$ such that $1 \leqslant i_j \leqslant m_0$, $n - (i_1 + \ldots + i_s) \leqslant m_0$, and

$$f^{i_j} \circ f^{i_{j-1}} \circ \ldots \circ f^{i_1}(x) \in B_r$$

for $j \in \{1, \ldots, s\}$.

Since f is eventually condensing on bounded subsets of P, we have that $f^n(\bar{B}_r)$ is bounded for each $n \in N$, and therefore we can find $R > 0$ such that

$$\bigcup_{i=1}^{m_0} f^i(\bar{B}_r) \subset B_R .$$

Now choose $k \in N$ such that $\gamma(f^k\colon B_R \to P) < 1$. Let $n > m_0 + k$ and let $p_i, q_i \in N$ be chosen such that $n - i = p_i \cdot k + q_i$, $i \in \{1, \ldots, m_0\}$, $p_i \geqslant 1$, and $0 \leqslant q_i < k$. We have

$$\begin{aligned}\gamma(f^n(F)) &\leqslant \max_{i=1,\ldots,m_0} \gamma(f^{n-i}(\bar{B}_r)) = \max_{i=1,\ldots,m_0} \gamma(f^{q_i} \circ f^{p_i \cdot k}(\bar{B}_r)) \\ &\leqslant \max_{i=1,\ldots,m_0} \gamma(f^{q_i}\colon \bar{B}_R \to P) \cdot (\gamma(f^k\colon \bar{B}_R \to P))^{p_i} \cdot 2r ;\end{aligned}$$

hence $\lim_{n\to\infty} \gamma(\overline{f^n(F)}) = 0$. According to (0.3.6), we obtain that

$$M = \bigcap_{i=0}^{\infty} \overline{f^i(F)}$$

is compact, nonempty, and, for all open neighbourhoods U of M, there exists $n_U \in N$ such that

$$\overline{f^i(F)} \subset U$$

whenever $i \geqslant n_U$. Since F is f-invariant, and since $M \subset F \subset W$, we have that M is f-invariant.

It remains to show that M attracts the compact subsets of W. Let $K \subset W$ be a compact subset. There exists $n_K \in \boldsymbol{N}$ such that

$$K \subset \bigcup_{i=1}^{n_K} f^{-i}(B_r) .$$

Hence $f^{n_K}(K) \subset B_r \cup \ldots \cup f^{n_K}(B_r) \subset F \cup \ldots \cup f^{n_K}(F) \subset F$; that is,

$$f^i(K) \subset \overline{f^{i-n_K}(F)} \subset U , \quad \text{if } i \geqslant n_U + n_K .$$

Now, f has property (L) in a neighbourhood of M; thus (0.5.4) implies that $f\colon W \to W$ belongs to the fixed point index class and

$$\operatorname{ind}(f\colon W \to W) = \Lambda(f\colon W \to W) = 1 .$$

Finally, the excision and contraction properties together with (1.2.1) imply

$$\operatorname{ind}(f\colon B_r \to P) = \operatorname{ind}(f\colon B_r \to W) = \operatorname{ind}(f\colon W \to W) = 1 .$$

PROBLEM. Does (1.3.3) remain true without assuming condition $(*)$?

We can only give a partial answer here (see also 1.5.2).

(1.3.4) PROPOSITION. *Let P be a wedge missing a ray in a Banach space, and let $f\colon P \to P$ be a continuous map which has property (L) for each compact, f-invariant subset and which is eventually compact on bounded subsets. Assume that H_0 is satisfied. Then if*

$$W = \bigcup_{i=1}^{\infty} f^{-i}(B_r) ,$$

we have that W is f-invariant, $f\colon W \to W$ has a compact attractor, and

$$\operatorname{ind}(f\colon B_r \to P) = 1 .$$

PROOF. Choosing $k \in \boldsymbol{N}$ such that $\overline{f^k(B_r)}$ is compact, consider

$$V = B_r \cap f^{-k}(B_r) \quad \text{and} \quad K = f^k(V) \subset \bar{B}_r \subset W .$$

We have that $\bar{K}$ is compact in W. From H_0 and the definition of W, we have that, for each $x \in W$, there exists $n_x \in \boldsymbol{N}$ such that $f^n(x) \in B_r$ whenever $n \geqslant n_x$. This means $f^{n_x}(x) \in B_r \cap f^{-k}(B_r) = V$, and therefore we have that

$$W \subset \bigcup_{i=1}^{\infty} f^{-i}(V) .$$

Since $\bar{K}$ is compact, we find $n \in \boldsymbol{N}$ such that

$$\bar{K} \subset \bigcup_{i=1}^{n} f^{-i}(V) \subset \bigcup_{i=1}^{n} f^{-k-i}(\bar{K}) .$$

Set $M = \bigcup_{i=0}^{n+k-1} f^i(\bar{K})$. Then M is f-invariant since

$$f(M) \subset \bigcup_{i=1}^{n+k} f^i(\bar{K}) \subset M \cup f^{n+k}(\bar{K}) \subset M \cup \bigcup_{i=0}^{n-1} f^i(\bar{K}) \subset M .$$

Moreover, M absorbs the compact subsets of W. To see this, let L be a compact subset of W. Then

$$L \subset \bigcup_{i=0}^{j} f^{-i}(V) \quad \text{for some } j \in \boldsymbol{N}$$

and hence

$$\begin{aligned} f^{j+k+l}(L) &\subset \bigcup_{i=0}^{j} f^{i+k+l}(V) \\ &\subset \bigcup_{i=0}^{j} f^{i+l}(\bar{K}) \\ &\subset \bigcup_{i=0}^{j} f^{i+l}(M) \subset M , \end{aligned}$$

for all $l \in \boldsymbol{N}$.

For the final part of the proof, see the last part of the proof for (1.3.3).

Combining the results of this section, we obtain a fixed point principle that can be regarded as an asymptotic version of the principle due to M. A. Krasnosel'skiĭ [12] which has come to be known as the principle for mappings « expanding » or « compressing » a cone.

(1.3.5) THEOREM. *Let P be a wedge missing a ray in a Banach space, and let $f: P \to P$ be a continuous map which is condensing on bounded subsets. Let $r = r_1 > 0$. Assume that one of the following conditions is satisfied:*

(1) H_0';

(2) H_0 *and f is eventually compact on bounded subsets*;

(3) H_0 *and* $B_{r_1} \subset \bigcup_{i=1}^{n_0} f^{-i}(B_{r_1})$ *for some* n_0.

Let $r = r_2 > 0$. *Assume that one of the following conditions is satisfied.*

(4) H_∞;

(5) H'_∞;

(6) $0 \in P$ *is an ejective fixed point for* f *relative to* B_{r_2} *and* $r_2 < r_1$.

Then

$$\operatorname{ind}(f\colon U \to P) = \begin{cases} +1\,, & \text{if } r_1 > r_2 \\ -1\,, & \text{if } r_1 < r_2\,. \end{cases}$$

Thus, f *has a fixed point in*

$$U = \{x \in P\colon \min\{r_1, r_2\} < \|x\| < \max\{r_1, r_2\}\}\,.$$

PROOF. This is an immediate consequence of the additivity property for the fixed point index and (1.3.1)-(1.3.4).

1.4. *Eventually condensing mappings of wedges.*

Propositions (1.3.1) and (1.3.2) do not seem to generalize to mappings which are condensing on bounded subsets and which have property (L) for each compact, invariant subset. However, we still can obtain a fixed point principle.

(1.4.1) THEOREM. *Let* P *be a wedge missing a ray in a Banach space, and let* $f\colon P \to P$ *be a continuous map which has property* (L) *for each compact,* f*-invariant subset and which is eventually condensing on bounded subsets. Let* $r = r_1 > 0$, *and assume that*

(1) H'_0 *is satisfied; or*

(2) H_0 *is satisfied and* $B_{r_1} \subset \bigcup_{i=1}^{n_0} f^{-i}(B_{r_1})$ *for some* $n_0 \in \mathbf{N}$.

Let $r = r_2 > 0$, *and assume that*

(3) H_∞ *is satisfied; or*

(4) H'_∞ *is satisfied; or*

(5) $0 \in P$ *is an ejective fixed point of* f *relative to* B_{r_2} *and* $f(P \setminus \{0\}) \subset$ $\subset P \setminus \{0\}$.

Then, if $r_1 > r_2$, *we have that*

$$\operatorname{ind}(f\colon U \to P) = 1\,.$$

Thus, f has a fixed point in

$$U = \{x \in P : r_2 < \|x\| < r_1\}.$$

PROOF. According to (1.3.3), if $W = \bigcup_{i=1}^{\infty} f^{-i}(B_{r_1})$, then W is f-invariant, $f\colon W \to W$ has a compact attractor, and $\operatorname{ind}(f\colon B_{r_1} \to P) = 1$.

Case where $0 \in P$ *is ejective*: $f(W \setminus \{0\}) \subset f(P \setminus \{0\}) \cap W \subset W \setminus \{0\}$ and, moreover, $F = \{0\}$ is an ejective fixed point of $f\colon W \to W$.

Other cases: Note that $B_{r_2} \subset B_{r_1} \subset W$. Hence, if $W' = \bigcup_{i=1}^{\infty} f^{-i}(P \setminus \bar{B}_{r_2})$, then $F = P \setminus W' \subset B_{r_2} \setminus W' \subset W \setminus W'$ and $f(W') \subset W'$. Thus, we have that F is an ejective set for $f\colon W \to W$ and $f(W \setminus F) \subset W \setminus F$.

According to (1.1.2), we have in all cases that

$$\operatorname{ind}(f\colon B_{r_2} \to P) = \operatorname{ind}(f\colon B_{r_2} \to W) = \Lambda(f\colon W \to W) - \Lambda(f\colon W \setminus F \to W \setminus F).$$

Now, (1.2.1) implies that $\Lambda(f\colon W \to W) = 1$. Since

$$W \setminus F = \bigcup_{i=1}^{\infty} f^{-i}(W \setminus \bar{B}_{r_2}),$$

we can apply (1.2.2) to the mapping $f\colon W \to W \supset B_{r_2}$ and obtain $\Lambda(f\colon W \setminus F \to W \setminus F) = 1$. Thus, $\operatorname{ind}(f\colon B_{r_2} \to P) = 1 - 1 = 0$, and finally we obtain the assertion by using the additivity property of the index:

$$\operatorname{ind}(f\colon U \to P) = \operatorname{ind}(f\colon B_{r_1} \to P) - \operatorname{ind}(f\colon B_{r_2} \to P) = 1 - 0 = 1.$$

PROBLEM. Do (1.3.1), (1.3.2) and (1.3.5) generalize to mappings which have property (L) for each compact, invariant subset and which are eventually condensing on bounded subsets?

1.5. *Special wedges.*

In this section, we shall restrict attention to «special wedges»; that is, a wedge P for which there exists $y_0 \in P$ such that $\|x + y_0\| \geq \|x\|$ for all $x \in P$. Notice that if $\lambda > 0$, for all $x \in P$,

$$\|x + \lambda y_0\| = \lambda \|x \cdot \lambda^{-1} + y_0\| \geq \lambda \|x\lambda^{-1}\| = \|x\|.$$

The following lemma is our main reason for considering this type of wedge.

With its aid, it will be possible to eliminate one of the crucial assumptions which were necessary in 1.3.

(1.5.1) LEMMA. *Let P be a special wedge in a linear normed space, and let $R>0$. For all $\varepsilon>0$, there exists a retraction $\varrho\colon \bar{B}_R\to S_R$ such that $\varrho_{|S_R}=Id_{S_R}$ and $\gamma(\varrho)\leqslant 1+\varepsilon$.*

PROOF. Choose $y_0\in P$ such that $\|y_0\|=1$ and $\|x+y_0\|\geqslant\|x\|$ for all $x\in P$. Now, for all $n\in\boldsymbol{N}$, define a map $h_n\colon\bar{B}_R\to P$ by

$$h_n(x)=\|x\|R^{-1}x+(1-\|x\|R^{-1})\,2nRy_0\,.$$

Observe that, for all $A\subset\bar{B}_R$, $h_n(A)\subset\operatorname{co}(A\cup\{2nRy_0\})$, and therefore $\gamma(h_n(A))\leqslant\gamma(A)$; i.e., $\gamma(h_n\colon\bar{B}_R\to P)\leqslant 1$. Furthermore, we have the estimates

$$\begin{aligned}\|h_n(x)\|&=\big\|\,\|x\|R^{-1}x+(1-\|x\|R^{-1})2nRy_0\big\|\\&\geqslant 2nR|1-\|x\|R^{-1}|\,\|y_0\|-\|x\|R^{-1}\|x\|\\&\geqslant 2nR|1-\|x\|R^{-1}|-R\geqslant R\end{aligned}$$

provided that $\|x\|\leqslant(n-1)n^{-1}R$ and since P is a special wedge

$$\|h_n(x)\|\geqslant\|x\|R^{-1}\|x\|\geqslant((n-1)n^{-1})^2\cdot R$$

provided $\|x\|\geqslant(n-1)n^{-1}R$. This implies that

$$h_n(\bar{B}_R)\subset P\diagdown B_r$$

where $r=((n-1)n^{-1})^2R$.

Next, we define $\varrho_n\colon P\diagdown B_r\to S_r$ by

$$\varrho_n(x)=r\|x\|^{-1}x\,.$$

Then $\varrho_n(A)\subset\operatorname{co}(A\cup\{0\})$ for all $A\subset P\diagdown B_r$, and hence $\gamma(\varrho_n(A))\leqslant\gamma(A)$; i.e., $\gamma(\varrho_n\colon P\diagdown B_r\to P)\leqslant 1$.

Finally, define $\pi_n\colon S_r\to S_R$ by

$$\pi_n(x)=Rr^{-1}x\,.$$

Then

$$\begin{aligned}\gamma(\pi_n)\leqslant R\cdot r^{-1}=(n(n-1)^{-1})^2&=1+2(n-1)^{-1}+(n-1)^{-2}\\&\leqslant 1+3(n-1)^{-1}\qquad\text{if } n\geqslant 2\,.\end{aligned}$$

To obtain $\varrho\colon \bar{B}_R \to S_R$, setting

$$\varrho = \pi_n \circ \varrho_n \circ h_n\,, \quad \text{we have}$$

$\gamma(\varrho) \leqslant \gamma(\pi_n)\cdot\gamma(\varrho_n)\cdot\gamma(h_n) \leqslant 1 + 3(n-1)^{-1}$.

(1.5.2) PROPOSITION. *Let P be a special wedge in a Banach space, and let $f\colon P \to P$ be a continuous map which is condensing on bounded subsets. Assume that H_0 is satisfied.*
Then

$$\operatorname{ind}(f\colon B_r \to P) = 1\,.$$

PROOF. Set $W = \bigcup_{i=1}^{\infty} f^{-i}(B_r)$. Then $W \supset \bar{B}_r$ is open in P, $f(W) \subset W$, $f^{-1}(W) \subset W$, and, according to (1.2.1), we have that $\Lambda(f\colon W \to W) = 1$. Now, if $\varrho\colon \bar{B}_r \to S_r$ is any retraction, define $\pi\colon W \to W$ by

$$\pi(x) = \begin{cases} \varrho(x)\,, & \text{if } x \in \bar{B}_r \\ x\,, & \text{if } x \in W \setminus \bar{B}_r\,. \end{cases}$$

Since $\bar{B}_r$ is convex, we have that $\pi_{|\bar{B}_r} \sim Id_{\bar{B}_r}$ and $\pi \sim Id_W$. Set $g := \pi \circ f\colon W \to W$, and observe that $g \sim f$; hence $\Lambda(g\colon W \to W) = \Lambda(f\colon W \to W) = 1$.

Now, observe that $W \subset \bigcup_{i=1}^{\infty} g^{-i}(S_r)$.

In fact, if $x \in W$, then there is $n \in \boldsymbol{N}$ such that $f^n(x) \in B_r$. Suppose that n is minimal with this property; i.e., $f^i(x) \notin B_r$, $1 \leqslant i < n$. Then $g^n(x) = \pi \circ f^n(x) \in S_r$.

Next, since there exists $m \in \boldsymbol{N}$ such that $f^m(S_r) \subset B_r$, we have that

$$S_r \subset \bigcup_{i=1}^{m} g^{-i}(S_r) = F \subset W\,.$$

Moreover, one obtains from $f^m(S_r) \subset B_r$ that

$$g^n(S_r) = (\pi \circ f)^n(S_r) \subset \bigcup_{i=0}^{m} f^i(S_r)\,.$$

Since f is condensing on bounded sets, we have that $f^i(\bar{B}_r)$ is bounded, and therefore we can find $R > 0$ such that

$$\bigcup_{i=0}^{m} f^i(\bar{B}_r) \subset B_R\,.$$

Now, choose $\varepsilon > 0$ such that

$$k = (1+\varepsilon)\cdot\gamma(f\colon B_R \to P) < 1 .$$

According to (1.5.1), we may assume that $\gamma(\varrho) \leqslant 1+\varepsilon$, and therefore $\gamma(\pi) \leqslant \max\{\gamma(\varrho), \gamma(Id_W)\} \leqslant 1+\varepsilon$. Since $S_r \subset F$, one obtains that F is f-invariant. We compute $\gamma(g^n(F))$ for $n > m$:

$$\gamma(g^n(F)) \leqslant \max_{i=1,\dots,m} \gamma(g^{n-i}(S_r)) \leqslant \gamma(g\colon B_R \to P)^{n-m}\cdot 2r \leqslant k^{n-m}\cdot 2r .$$

Hence, $\lim_{n\to\infty} \gamma(\overline{g^n(F)}) = 0$, and, from (0.3.6), we obtain that $M = \bigcap_{i=0}^{\infty} \overline{g^i(F)} \subset F \subset W$ is compact, nonempty, and, for all neighbourhoods U of M, there exists $n_U \in N$ such that $\overline{g^i(F)} \subset U$ whenever $i \geqslant n_U$. Moreover, we have that M is g-invariant since $g(F) \subset F$.

Furthermore, M attracts the compact subsets of W under g. To show this, let K be a compact subset of W and let U be an open neighbourhood of M. Since $K \subset \bigcup_{i=1}^{\infty} f^{-i}(B_r)$, we find $n_K \in N$ such that $K \subset \bigcup_{i=1}^{n_K} f^{-i}(B_r)$; i.e., $K \subset \bigcup_{i=1}^{n_K} g^{-i}(S_r)$. Now, $g^{n_K}(K) \subset S_r \cup \bigcup_{i=1}^{n_K-1} g^i(S_r) \subset F \cup \bigcup_{i=1}^{n_K-1} g^i(F) \subset F$. Hence, $g^i(K) \subset g^{i-n_K}(F) \subset U$, whenever $i \geqslant n_K + n_U$.

Now, the normalization property for the index implies that

$$\operatorname{ind}(g\colon W \to W) = \Lambda(g\colon W \to W) = 1 ,$$

and, since $g(x) \neq x$ for all $x \in W \setminus B_r$, we have that $\operatorname{ind}(g\colon B_r \to W) = \operatorname{ind}(g\colon W \to W)$. Thus, to obtain the assertion, we have to show that $\operatorname{ind}(g\colon B_r \to W) = \operatorname{ind}(f\colon B_r \to P)$. First, $\operatorname{ind}(f\colon B_r \to P) = \operatorname{ind}(f\colon B_r \to W)$ by (0.5.3). Then, consider the homotopy

$$h\colon \bar{B}_r \times [0,1] \to P$$

defined by $h(x,t) = tf(x) + (1-t)g(x)$. We have that $\gamma\big(h(A\times[0,1])\big) \leqslant \gamma\big(\operatorname{co}(f(A)\cup g(A))\big) \leqslant k\gamma(A)$ for all $A \subset \bar{B}_r$ and that $h(x,t) \neq x$ for all $(x,t) \in S_r \times [0,1]$. Therefore, from (0.5.5), it follows that

$$\operatorname{ind}(g\colon B_r \to W) = \operatorname{ind}(f\colon B_r \to W) .$$

(1.5.3) THEOREM. *Let P be a special wedge in a Banach space, and let $f\colon P \to P$ be a continuous map which is condensing on bounded subsets. Let $r = r_1$ and assume that H_0 or H_0' is satisfied. Let $r = r_2$ and either assume that H_∞ or H_∞' is satisfied or assume that $0 \in P$ is an ejective fixed point of f relative to B_{r_2} and $r_2 < r_1$.*

Then

$$\operatorname{ind}(f\colon U \to P) = \begin{cases} +1, & \text{if } r_1 > r_2 \\ -1, & \text{if } r_1 < r_2, \end{cases}$$

and f has a fixed point in

$$U = \{x \in P\colon \min\{r_1, r_2\} < \|x\| < \max\{r_1, r_2\}\}.$$

PROOF. This is immediate from the additivity property of the fixed point index, (1.3.1)-(1.3.5), and (1.5.2).

1.6. *Continua of solutions for mappings leaving a wedge invariant.*

Let P be a wedge missing a ray in a Banach space, and let $a, b \in \boldsymbol{R}$, $a < b$. In this paragraph, we consider mappings $F\colon P \times (a, b) \to P$ and wish to establish conditions under which the nonlinear eigenvalue problem

$$F(x, \lambda) = x$$

admits a continuum (i.e., a closed, connected set) of solutions. Our result here is a generalization of Peitgen [19] where the case of mappings which are compact on bounded sets is treated. Since the structure of the proof is taken from [19], we only outline the main steps here and refer to [19].

First, we fix some notation.

Let $r, R\colon (a, b) \to \boldsymbol{R}_+ \setminus \{0\}$ be continuous maps such that $r(\lambda) < R(\lambda)$ for all λ or $r(\lambda) > R(\lambda)$ for all λ. If $\lambda_0, \lambda_1 \in (a, b)$ are fixed elements then we set

$$\begin{aligned} S_0 &= \{x \in P\colon \|x\| = r(\lambda_0)\}, \qquad B_0 = \{x \in P\colon \|x\| < r(\lambda_0)\}, \\ S_1 &= \{x \in P\colon \|x\| = R(\lambda_1)\}, \qquad B_1 = \{x \in P\colon \|x\| < R(\lambda_1)\}, \\ Z &= P \times (a, b), \\ Z_r^R &= \{(x, \lambda) \in Z\colon \min\{r(\lambda), R(\lambda)\} \leqslant \|x\| \leqslant \max\{r(\lambda), R(\lambda)\}\}, \end{aligned}$$

and

$$U_r^R = \{(x, \lambda) \in Z\colon \min\{r(\lambda), R(\lambda)\} < \|x\| < \max\{r(\lambda), R(\lambda)\}\}.$$

If $\Omega \subset Z$, then $\Omega(\lambda) = \{x \in P\colon (x, \lambda) \in \Omega\}$ is the section over λ. Observe that U_r^R is open in Z.

(1.6.1) THEOREM. *Let $F\colon Z \to P$ be a continuous map which is condensing, and assume that $F(x, \lambda) \neq x$ for all $(x, \lambda) \in Z_r^R \setminus U_r^R$.*

Let $\lambda_0, \lambda_1 \in (a, b)$ *be fixed elements. Let* $r = r(\lambda_0)$, *and assume that* $F(\cdot, \lambda_0): P \to P$ satisfies

— *condition* H_0'; *or*

— *condition* H_0 *and is eventually compact on bounded subsets*; *or*

— *condition* H_0 *and there is* $n_0 \in N$ *such that*

$$B_r \subset \bigcup_{i=1}^{n_0} F(\cdot, \lambda_0)^{-i}(B_r)\,; \qquad or$$

— *condition* H_0 *and* P *is a special wedge.*

Let $r = R(\lambda_1)$, *and assume that* $F(\cdot, \lambda_1): P \to P$ *satisfies*

— *condition* H_∞; *or*

— *condition* H_∞'; *or*

— $0 \in P$ *is an ejective fixed point of* $F(\cdot, \lambda_1)$ *relative to* B_r *and* $R(\lambda) < r(\lambda)$.

Then

(i) *for any* $\lambda \in (a, b)$,

$$\operatorname{ind}(F(\cdot, \lambda): U_r^R(\lambda) \to P) = \begin{cases} -1, & \text{if } r(\lambda) < R(\lambda) \\ +1, & \text{if } r(\lambda) > R(\lambda)\,; \end{cases}$$

(ii) *for any* $\varepsilon > 0$, *there exists a continuum*

$$\mathcal{C}_\varepsilon \subset \mathcal{S} = \{(x, \lambda) \in Z: F(x, \lambda) = x,\ x \neq 0\}$$

such that $(x, \lambda) \in \mathcal{C}_\varepsilon$ *implies* $x \in U_r^R(\lambda)$ *and* $\lambda \in [a + \varepsilon, b - \varepsilon]$;

(iii) *the projection of* $\mathcal{C}_\varepsilon$ *onto the* λ*-axis fills the entire interval* $[a + \varepsilon, b - \varepsilon]$.

PROOF. Let us assume that $r(\lambda) < R(\lambda)$ for all $\lambda \in (a, b)$; the case $r(\lambda) > R(\lambda)$ is proved similarly.

(1) Since $F(x, \lambda) \neq x$ for all $(x, \lambda) \in Z_r^R \setminus U_r^R$, the generalized homotopy property (cf. [14], p. 245) together with the index computations of the previous paragraphs give

$$1 = \operatorname{ind}(F(\cdot, \lambda_0): B_{r(\lambda_0)} \to P) = \operatorname{ind}(F(\cdot, \lambda): B_{r(\lambda)} \to P)$$

and

$$0 = \mathrm{ind}(F(\cdot, \lambda_1): B_{R(\lambda_1)} \to P) = \mathrm{ind}(F(\cdot, \lambda): B_{R(\lambda)} \to P) ;$$

thus, we obtain from the additivity property that

$$\mathrm{ind}(F(\cdot, \lambda): U_r^R(\lambda) \to P) = 0 - 1 = -1 \quad \text{for all } \lambda \in (a, b) .$$

(2) Choose $\varepsilon > 0$. Let K_ε be the set

$$K_\varepsilon = \{(x, \lambda) \in U_r^R: F(x, \lambda) = x, a + \varepsilon \leqslant \lambda \leqslant b - \varepsilon\} .$$

Then K_ε is compact (cf. [14], p. 245) and (1) implies that $K_\varepsilon \neq \emptyset$. Now assume that there is no continuum joining $K_\varepsilon(a + \varepsilon) \neq \emptyset$ with $K_\varepsilon(b - \varepsilon) \neq \emptyset$. Then, by a lemma of Whyburn ([21], Chap. 1, Theorem 9.3), K_ε decomposes into two disjoint closed subsets K_1, K_2 such that

$$K_1 \cup K_2 = K_\varepsilon , \quad K_\varepsilon(a + \varepsilon) \times \{a + \varepsilon\} \subset K_1 \text{ and } K_\varepsilon(b - \varepsilon) \times \{b - \varepsilon\} \subset K_2 .$$

Choose Ω open in U_r^R such that $K_1 \subset \Omega$ and $K_2 \cap \bar{\Omega} = \emptyset$. Observe that the generalized homotopy property (cf. [14]) and the excision property imply ($\lambda \geqslant a + \varepsilon$)

$$\mathrm{ind}(F(\cdot, \lambda): \Omega(\lambda) \to P) = \mathrm{ind}(F(\cdot, a + \varepsilon): \Omega(a + \varepsilon) \to P)$$
$$= \mathrm{ind}(F(\cdot, a + \varepsilon): U_r^R(a + \varepsilon) \to P) = -1 .$$

However, since $\Omega(b - \varepsilon) \cap K_\varepsilon(b - \varepsilon) = \emptyset$, we have $\mathrm{ind}(F(\cdot, b - \varepsilon): \Omega(b - \varepsilon) \to$ $\to P) = 0$, and this is a contradiction.

It is obvious from the above proof that (1.6.1) can be generalized to mappings considered in the previous paragraphs after the generalized homotopy property has been extended appropriately. This, however, is omitted here for reasons of length.

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