Les Houches lectures on matrix models and topological strings

Marcos Mariño[1]

Department of Physics, CERN
Geneva 23, CH-1211 Switzerland

marcos@mail.cern.ch

Abstract

In these lecture notes for the Les Houches School on Applications of Random Matrices in Physics we give an introduction to the connections between matrix models and topological strings. We first review some basic results of matrix model technology and then we focus on type B topological strings. We present the main results of Dijkgraaf and Vafa describing the spacetime string dynamics on certain Calabi-Yau backgrounds in terms of matrix models, and we emphasize the connection to geometric transitions and to large N gauge/string duality. We also use matrix model technology to analyze large N Chern-Simons theory and the Gopakumar-Vafa transition.

---

[1]Also at Departamento de Matemática, IST, Lisboa, Portugal.

1

Contents

1 Introduction

Topological string theory was introduced by Witten in [70, 72] as a simplified model of string theory which captures topological information of the target space, and it has been intensively studied since then. There are three important lessons that have been learned in the last few years about topological strings:

1) Topological string amplitudes are deeply related to physical amplitudes of type II string theory.

2) The spacetime description of open topological strings in terms of string field theory reduces in some cases to very simple gauge theories.

3) There is an open/closed topological string duality which relates open and closed string backgrounds in a precise way

In these lectures we will analyze a particular class of topological string theories where the gauge theory description in (2) above reduces in fact to a matrix model. This was found by Dijkgraaf and Vafa in a series of beautiful papers [27, 28, 29], where they

also showed that, thanks to the connection to physical strings mentioned in (1), the computation of nonperturbative superpotentials in a wide class of $\mathcal{N} = 1$ gauge theories reduces to perturbative computations in a matrix model. This aspect of the work of Dijkgraaf and Vafa was very much explored and exploited, and rederived in the context of supersymmetric gauge theories without using the connection to topological strings. In these lectures we will focus on the contrary on (2) and (3), emphasizing the string field theory construction and the open/closed string duality. The applications of the results of Dijkgraaf and Vafa to supersymmetric gauge theories have been developed in many papers and reviewed for example in [6], and we will not cover them here. Before presenting the relation between matrix models and topological strings, it is worthwhile to give a detailed conceptual discussion of the general ideas behind (2) and (3) and their connections to large N dualities.

In closed string theory we study maps from a Riemann surface Σ_g to a target manifold X, and the quantities we want to compute are the free energies at genus g, denoted by $F_g(t_i)$. Here, the t_i are geometric data of the target space X, and the free energies are computed as correlation functions of a two-dimensional conformal field theory coupled to gravity. In topological string theory there are two different models, the A and the B model, the target space is a Calabi-Yau manifold (although this condition can be relaxed in the A model), and the parameters t_i are Kähler and complex parameters, respectively. The free energies are assembled together into a generating functional

$$F(g_s, t_i) = \sum_{g=0}^{\infty} g_s^{2g-2} F_g(t_i), \tag{1.1}$$

where g_s is the string coupling constant.

In open string theory we study maps from an open Riemann surface $\Sigma_{g,h}$ to a target X, and we have to provide boundary conditions as well. For example, we can impose Dirichlet conditions by using a submanifold S of X where the open strings have to end. In addition, we can use Chan-Paton factors to introduce a $U(N)$ gauge symmetry. The open string amplitudes are now $F_{g,h}$, and in the cases that will be studied in these lectures the generating functional will have the form

$$F(g_s, N) = \sum_{g=0}^{\infty} \sum_{h=1}^{\infty} F_{g,h} g_s^{2g-2+h} N^h. \tag{1.2}$$

Physically, the introduction of Chan-Paton factors and boundary conditions through a submanifold S of X means that we are wrapping N (topological) D-branes around S. A slightly more general situation arises when there are n submanifolds $S_1, \cdots, S_n$ where the strings can end. In this case, the open string amplitude is of the form $F_{g,h_1,\cdots,h_n}$

3

and the total free energy is now given by

$$F(g_s, N_i) = \sum_{g=0}^{\infty} \sum_{h_1, \cdots, h_n = 1}^{\infty} F_{g, h_1, \cdots, h_n} g_s^{2g-2+h} N_1^{h_1} \cdots N_n^{h_n},\qquad(1.3)$$

where $h = \sum_{i=1}^{n} h_i$. In the case of open strings one can in some situations use string field theory to describe the spacetime dynamics. The open string field theory of Witten [69], which was originally constructed for the open bosonic string theory, can also be applied to topological string theory, and on some particular Calabi-Yau backgrounds the full string field theory of the topological string reduces to a simple $U(N)$ gauge theory, where g_s plays the role of the gauge coupling constant and N is the rank of the gauge group. In particular, the string field reduces in this case to a finite number of gauge fields. As a consequence of this, the open string theory amplitude $F_{g,h}$ can be computed from the gauge theory by doing perturbation theory in the double line notation of 't Hooft [66]. More precisely, $F_{g,h}$ is the contribution of the fatgraphs of genus g and h holes. The idea that fatgraphs of a $U(N)$ gauge theory correspond to open string amplitudes is an old one, and it is very satisfying to find a concrete realization of this idea in the context of a string field theory description of topological strings, albeit for rather simple gauge theories.

The surprising fact that the full string field theory is described by a simple gauge theory is typical of topological string theory, and does not hold for conventional string models. There are two examples where this description has been worked out:

1) The A model on a Calabi-Yau of the form $X = T^*M$, where M is a three-manifold, and there are N topological D-branes wrapping M. In this case, the gauge theory is Chern-Simons theory on M [74].

2) The B model on a Calabi-Yau manifold X which is the small resolution of a singularity characterized by the hyperelliptic curve $y^2 = (W'(x))^2$. If $W'(x)$ has degree n, the small resolution produces n two-spheres, and one can wrap N_i topological D-branes around each two-sphere, with $i = 1, \cdots, n$. In this case Dijkgraaf and Vafa showed that the gauge theory is a multicut matrix model with potential $W(x)$ [27].

In both examples, the open string amplitudes $F_{g,h}$ are just numbers computed by the fatgraphs of the corresponding gauge theories.

The fatgraph expansion of a $U(N)$ gauge theory can be resummed formally by introducing the so called 't Hooft parameter $t = g_s N$. For example, in the case of the free energy, we can rewrite (1.2) in the form (1.1) by defining

$$F_g(t) = \sum_{h=1}^{\infty} F_{g,h} t^h.\qquad(1.4)$$

In other words, starting from an *open* string theory expansion we can obtain a *closed* string theory expansion by resumming the hole expansion as indicated in (1.4). This idea was proposed by 't Hooft [66] and gives a closed string theory interpretation of a gauge theory.

What is the interpretation of the above resummation for the gauge theories that describe the spacetime dynamics of topological open string theories? As was explained in [35] (for the A model example above) and in [15] (for the B model example), there is a *geometric* or *large N* transition that relates the open string Calabi-Yau background X underlying the gauge theory to a closed string Calabi-Yau background X'. The geometric transition typically relates two different ways of smoothing out a singular geometry (the "resolved" geometry and the "deformed" geometry). Moreover, the "master field" that describes the large N limit [68] turns out to encode the target space geometry of the closed string background, and the 't Hooft parameter becomes a geometric parameter of the resulting closed geometry. The idea that an open string background with
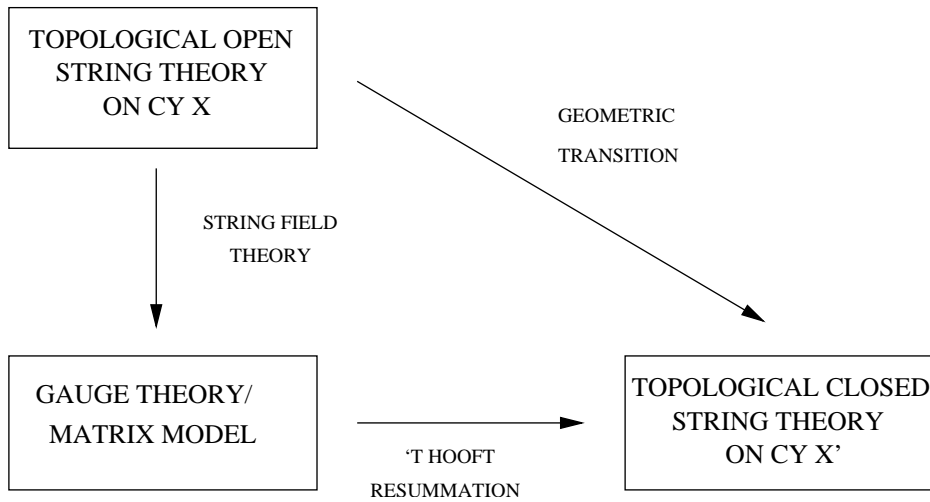

Figure 1: This diagram summarizes the different relations between closed topological strings, open topological strings, and gauge theories.

D-branes is equivalent to a different, geometric closed string background (therefore with no D-branes) appeared originally in the AdS/CFT correspondence [3]. In this correspondence, type IIB theory in flat space in the presence of D-branes is conjectured to be equivalent to type IIB theory in $AdS_5 \times \mathbf{S}^5$ with no D-branes, and where the radius of the $\mathbf{S}^5$ is related to the 't Hooft parameter. The reason this holds is that, at large N, the presence of the D-branes can be traded for a deformation of the background geometry. In other words, we can make the branes disappear if we change

the background geometry at the same time. Therefore, as emphasized by Gopakumar and Vafa in [35], large N dualities relating open and closed strings should be associated to transitions in the geometry. The logical structure of all the connections we have sketched is depicted in Fig. 1.

In these lectures we will mostly focus on the B topological string, the Dijkgraaf-Vafa scenario, and the geometric transition of [15]. For a detailed review of a similar story for the A string, we refer the reader to [54]. The organization of these lectures is as follows. In section 2 we review some basic ingredients of matrix models, including saddle-point techniques and orthogonal polynomials. In section 3 we explain in detail the connection between matrix models and topological strings due to Dijkgraaf and Vafa. We first review the topological B model and its string field theory description, and we show that in the Calabi-Yau background associated to the resolution of a polynomial singularity, the string field theory reduces to a matrix model. We develop some further matrix model technology to understand all the details of this description, and we make the connection with geometric transitions. In section 4 we briefly consider the geometric transition of Gopakumar and Vafa [35] from the point of view of the matrix model description of Chern-Simons theory. This allows us to use matrix model technology to derive some of the results of [35].

2   Matrix models

In this section we develop some aspects and techniques of matrix models which will be needed in the following. There are excellent reviews of this material, such as for example [21, 22].

2.1   Basics of matrix models

Matrix models are the simplest examples of quantum gauge theories, namely, they are quantum gauge theories in zero dimensions. The basic field is a Hermitian $N \times N$ matrix M. We will consider an action for M of the form:

$$\frac{1}{g_s}W(M) = \frac{1}{2g_s}\mathrm{Tr}\,M^2 + \frac{1}{g_s}\sum_{p\geq 3}\frac{g_p}{p}\mathrm{Tr}\,M^p. \tag{2.1}$$

where g_s and g_p are coupling constants. This action has the obvious gauge symmetry

$$M \to UMU^{\dagger}, \tag{2.2}$$

where U is a $U(N)$ matrix. The partition function of the theory is given by

$$Z = \frac{1}{\mathrm{vol}(U(N))}\int dM\, e^{-\frac{1}{g_s}W(M)} \tag{2.3}$$

where the factor $\text{vol}(U(N))$ is the usual volume factor of the gauge group that arises after fixing the gauge. In other words, we are considering here a *gauged* matrix model. The measure in the "path integral" is the Haar measure

$$dM = 2^{\frac{N(N-1)}{2}} \prod_{i=1}^{N} dM_{ii} \prod_{1 \leq i < j \leq N} d\text{Re } M_{ij} d\text{Im } M_{ij}. \tag{2.4}$$

The numerical factor in (2.4) is introduced to obtain a convenient normalization.

A particularly simple example is the *Gaussian matrix model*, defined by the partition function

$$Z_G = \frac{1}{\text{vol}(U(N))} \int dM \, e^{-\frac{1}{2g_s} \text{Tr } M^2}. \tag{2.5}$$

We will denote by

$$\langle f(M) \rangle_G = \frac{\int dM \, f(M) \, e^{-\text{Tr } M^2/2g_s}}{\int dM \, e^{-\text{Tr } M^2/2g_s}} \tag{2.6}$$

the normalized vevs of a gauge-invariant functional $f(M)$ in the Gaussian matrix model. This model is of course exactly solvable, and the vevs (2.6) can be computed systematically as follows. Any gauge-invariant function $f(M)$ can be written as a linear combination of traces of M in arbitrary representations R of $U(N)$. If we represent R by a Young tableau with rows of lengths λ_i, with $\lambda_1 \geq \lambda_2 \geq \cdots$, and with $\ell(R)$ boxes in total, we define the set of $\ell(R)$ integers f_i as follows

$$f_i = \lambda_i + \ell(R) - i, \quad i = 1, \cdots, \ell(R). \tag{2.7}$$

Following [23], we will say that the Young tableau associated to R is even if the number of odd f_i's is the same as the number of even f_i's. Otherwise, we will say that it is odd. If R is even, one has the following result [42, 23]:

$$\langle \text{Tr}_R M \rangle_G = c(R) \dim R, \tag{2.8}$$

where

$$c(R) = (-1)^{\frac{A(A-1)}{2}} \frac{\prod_{f \, \text{odd}} f!! \prod_{f' \, \text{even}} f'!!}{\prod_{f \, \text{odd}, f' \, \text{even}} (f - f')} \tag{2.9}$$

and $A = \ell(R)/2$ (notice that $\ell(R)$ has to be even in order to have a non-vanishing result). Here $\dim R$ is the dimension of the irreducible representation of $SU(N)$ associated to R, and can be computed for example by using the hook formula. On the other hand, if R is odd, the above vev vanishes.

The partition function Z of more general matrix models with action (2.1) can be evaluated by doing perturbation theory around the Gaussian point: one expands the exponential of $\sum_{p \geq 3} (g_p/g_s) \text{Tr} M^p/p$ in (2.3), and computes the partition function as a

power series in the coupling constants g_p. The evaluation of each term of the series involves the computation of vevs like (2.6). Of course, this computation can be interpreted in terms of Feynman diagrams, and as usual the perturbative expansion of the free energy

$$F = \log Z$$

will only involve connected vacuum bubbles.

Since we are dealing with a quantum theory of a field in the adjoint representation we can reexpress the perturbative expansion of F in terms of fatgraphs, by using the double line notation due to 't Hooft [66]. The purpose of the fatgraph expansion is the following: in $U(N)$ gauge theories there is, in addition to the coupling constants appearing in the model (like for example g_s, g_p in (2.1)), a hidden variable, namely N, the rank of the gauge group. The N dependence in the perturbative expansion comes from the group factors associated to Feynman diagrams, but in general a single Feynman diagram gives rise to a polynomial in N involving different powers of N. Therefore, the standard Feynman diagrams, which are good in order to keep track of powers of the coupling constants, are not good in order to keep track of powers of N. If we want to keep track of the N dependence we have to "split" each diagram into different pieces which correspond to a definite power of N. To do that, one writes the Feynman diagrams of the theory as "fatgraphs" or double line graphs, as first indicated by 't Hooft [66]. Let us explain this in some detail, taking the example of the matrix
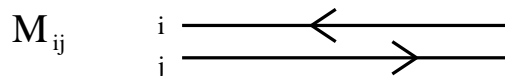
$$\mathbf{M}_{ij}$$

Figure 2: The index structure of the field M_{ij} in the adjoint representation of $U(N)$ is represented through a double line.

model with a cubic potential (*i.e.* $g_p = 0$ in (2.1) for $p > 3$). The fundamental field M_{ij} is in the adjoint representation. Since the adjoint representation of $U(N)$ is the tensor product of the fundamental N and the antifundamental $\overline{N}$, we can look at i (resp. j) as an index of the fundamental (resp. antifundamental) representation. We will represent this double-index structure by a double line notation as shown in Fig. 2. The only thing we have to do now is to rewrite the Feynman rules of the theory by taking into account this double-line notation. For example, the kinetic term of the theory is of the form

$$\frac{1}{g_s} \text{Tr} \, M^2 = \frac{1}{g_s} \sum_{i,j} M_{ij} M_{ji}. \tag{2.10}$$

This means that the propagator of the theory is

$$\langle M_{ij} M_{kl} \rangle = g_s \delta_{il} \delta_{jk} \tag{2.11}$$

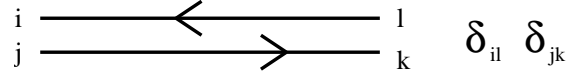and can be represented in the double line notation as in Fig. 3. Next, we consider the

Figure 3: The propagator in the double line notation.

vertices of the theory. For example, the trivalent vertex given by

$$\frac{g_3}{g_s} \mathrm{Tr}\, M^3 = \frac{g_3}{g_s} \sum_{i,j,k} M_{ij}\, M_{jk}\, M_{ki} \tag{2.12}$$

can be represented in the double line notation as in Fig. 4. A vertex of order p can be represented in a similar way by drawing p double lines joined together. Once
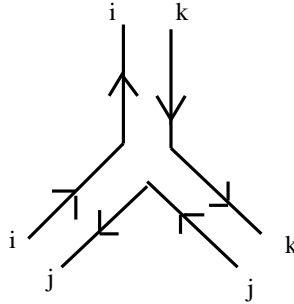
Figure 4: The cubic vertex in the double line notation.

we have rewritten the Feynman rules in the double-line notation, we can construct the corresponding graphs, which look like ribbons and are called ribbon graphs or fatgraphs. It is clear that in general a usual Feynman diagram can give rise to many different fatgraphs. Consider for example the one-loop vacuum diagram $\ominus$, which comes from contracting two cubic vertices. In the double line notation the contraction can be done in two different ways. The first one is illustrated in Fig. 5 and gives a factor

$$\sum_{ijkmnp} \langle M_{ij} M_{mn} \rangle \langle M_{jk} M_{pm} \rangle \langle M_{ki} M_{np} \rangle = g_s^3 N^3. \tag{2.13}$$

Figure 5: Contracting two cubic vertices in the double line notation: the N^3 contribution.

The second one is shown in Fig. 6 and gives a factor

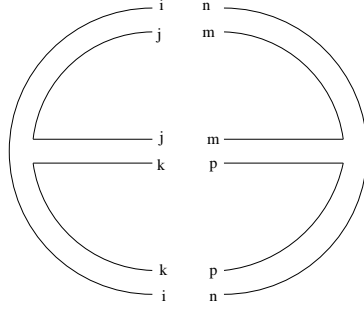$$\sum_{ijkmnp} \langle M_{ij} M_{mn} \rangle \langle M_{jk} M_{np} \rangle \langle M_{ki} M_{pm} \rangle = g_s^3 N. \tag{2.14}$$

In this way we have split the original diagram into two different fatgraphs with a well-defined power of N associated to them. The number of factors of N is simply equal to the number of closed loops in the graph: there are three closed lines in the fatgraph resulting from the contractions in Fig. 5 (see the first graph in Fig. 7), while there is only one in the diagram resulting from Fig. 6. In general, fatgraphs turn out to be
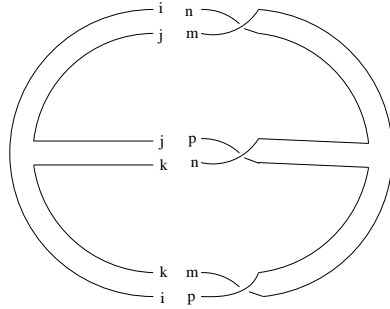
Figure 6: Contracting two cubic vertices in the double line notation: the N contribution.

characterized topologically by the number of propagators or edges E, the number of vertices with p legs V_p, and the number of closed loops h. The total number of vertices is $V = \sum_p V_p$. Each propagator gives a power of g_s, while each interaction vertex with p legs gives a power of g_p/g_s. The fatgraph will then give a factor

$$g_s^{E-V} N^h \prod_p g_p^{V_p}. \tag{2.15}$$

10

The key point now is to regard the fatgraph as a Riemann surface with holes, in which each closed loop represents the boundary of a hole. The genus g of such a surface is determined by the elementary topological relation

$$2g - 2 = E - V - h \tag{2.16}$$

therefore we can write (2.15) as

$$g_s^{2g-2+h} N^h \prod_p g_p^{V_p} = g_s^{2g-2} t^h \prod_p g_p^{V_p} \tag{2.17}$$

where we have introduced the 't Hooft parameter

$$t = N g_s \tag{2.18}$$

The fatgraphs with $g = 0$ are called *planar*, while the ones with $g > 0$ are called *nonplanar*. The graph giving the N^3 contribution in Fig. 5 is planar: it has $E = 3$, $V_3 = 2$ and $h = 3$, therefore $g = 0$, and it is a sphere with three holes. The graph in Fig. 6 is nonplanar: it has $E = 3$, $V_3 = 2$ and $h = 1$, therefore $g = 1$, and represents a torus with one hole (it is easy to see this by drawing the diagram on the surface of a torus).

We can now organize the computation of the different quantities in the matrix model in terms of fatgraphs. For example, the computation of the free energy is given in the usual perturbative expansion by connected vacuum bubbles. When the vacuum bubbles are written in the double line notation, we find that the perturbative expansion of the free energy is given by

$$F = \sum_{g=0}^{\infty} \sum_{h=1}^{\infty} F_{g,h} g_s^{2g-2} t^h, \tag{2.19}$$

where the coefficients $F_{g,h}$ (which depend on the coupling constants of the model g_p) takes into account the symmetry factors of the different fatgraphs. We can now formally define the free energy at genus g, $F_g(t)$, by keeping g fixed and summing over all closed loops h as in (1.4), so that the total free energy can be written as

$$F = \sum_{g=0}^{\infty} F_g(t) g_s^{2g-2}. \tag{2.20}$$

This is the *genus expansion* of the free energy of the matrix model. In (2.20) we have written the diagrammatic series as an expansion in g_s around $g_s = 0$, keeping the 't Hooft parameter $t = g_s N$ fixed. Equivalently, we can regard it as an expansion in $1/N$, keeping t fixed, and then the N dependence appears as N^{2-2g}. Therefore, for t fixed

and N large, the leading contribution comes from planar diagrams with $g = 0$, which go like $\mathcal{O}(N^2)$. The nonplanar diagrams give subleading corrections. Notice that $F_g(t)$, which is the contribution to F to a given order in g_s, is given by an infinite series where we sum over all possible numbers of holes h, weighted by t^h.
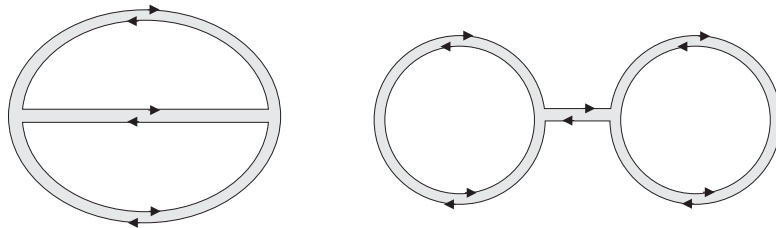
Figure 7: Two planar diagrams in the cubic matrix model.

Example. One can show that

$$\langle (\mathrm{Tr}\, M^3)^2 \rangle_G = g_s^3 (12N^3 + 3N),$$

where the first term corresponds to the two planar diagrams shown in Fig. 7 (contributing $3N^3$ and $9N^3$, respectively), and the second term corresponds to the nonplanar diagram shown in Fig. 6. Therefore, in the cubic matrix model the expansion of the free energy reads, at leading order,

$$F - F_G = \frac{2}{3} g_s g_3^2 N^3 + \frac{1}{6} g_s g_3^2 N + \cdots \tag{2.21}$$

There is an alternative way of writing the matrix model partition function which is very useful. The original matrix model variable has N^2 real parameters, but using the gauge symmetry we can see that, after modding out by gauge transformations, there are only N parameters left. We can for example take advantage of our gauge freedom to diagonalize the matrix M

$$M \to UMU^\dagger = D, \tag{2.22}$$

with $D = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$, impose this as a gauge choice, and use standard Faddeev-Popov techniques in order to compute the gauge-fixed integral (see for example [9]). The gauge fixing (2.22) leads to the delta-function constraint

$$\delta(^U M) = \prod_{i<j} \delta^{(2)}(^U M_{ij}) \tag{2.23}$$

where ${}^U M = UMU^\dagger$. We then introduce

$$\Delta^{-2}(M) = \int dU \, \delta({}^U M). \tag{2.24}$$

It then follows that the integral of any gauge-invariant function $f(M)$ can be written as

$$\int dM \, f(M) = \int dM \, f(M)\Delta^2(M) \int dU \, \delta({}^U M) = \Omega_N \int \prod_{i=1}^{N} d\lambda_i \Delta^2(\lambda) f(\lambda), \quad (2.25)$$

where we have used the gauge invariance of $\Delta(M)$, and

$$\Omega_N = \int dU \tag{2.26}$$

is proportional to the volume of the gauge group $U(N)$, as we will see shortly. We have to evaluate the the factor $\Delta(\lambda)$, which can be obtained from (2.24) by choosing M to be diagonal. If

$$F(M) = 0$$

is the gauge-fixing condition, the standard Faddeev-Popov formula gives

$$\Delta^2(M) = \det\left(\frac{\delta F({}^U M)}{\delta A}\right)_{F=0} \tag{2.27}$$

where we write $U = e^A$, and A is a anti-Hermitian matrix. Since

$$F_{ij}({}^U D) = (UDU^\dagger)_{ij} = A_{ij}(\lambda_i - \lambda_j) + \cdots . \tag{2.28}$$

(2.27) leads immediately to

$$\Delta^2(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j)^2, \tag{2.29}$$

the square of the Vandermonde determinant. Finally, we fix the factor Ω_N as follows. The Gaussian matrix integral can be computed explicitly by using the Haar measure (2.4), and is simply

$$\int dM \, e^{-\frac{1}{2g_s}\operatorname{Tr} M^2} = (2\pi g_s)^{N^2/2}. \tag{2.30}$$

On the other hand, by (2.25) this should equal

$$\Omega_N \int \prod_{i=1}^{N} d\lambda_i \Delta^2(\lambda) e^{-\frac{1}{2g_s}\sum_{i=1}^{N}\lambda_i^2}. \tag{2.31}$$

The integral over eigenvalues can be evaluated in various ways, using for example the Selberg function [55] or the technique of orthogonal polynomials that we describe in the next subsection, and its value is

$$g_s^{N^2/2}(2\pi)^{N/2}G_2(N+2) \tag{2.32}$$

where $G_2(z)$ is the Barnes function, defined by

$$G_2(z+1) = \Gamma(z)G_2(z), \quad G_2(1) = 1. \tag{2.33}$$

Comparing these results, we find that

$$\Omega_N = \frac{(2\pi)^{\frac{N(N-1)}{2}}}{G_2(N+2)}. \tag{2.34}$$

Using now (see for example [60]):

$$\mathrm{vol}(U(N)) = \frac{(2\pi)^{\frac{1}{2}N(N+1)}}{G_2(N+1)}. \tag{2.35}$$

we see that

$$\frac{1}{\mathrm{vol}(U(N))}\int dM\, f(M) = \frac{1}{N!}\frac{1}{(2\pi)^N}\int \prod_{i=1}^{N}d\lambda_i\,\Delta^2(\lambda)f(\lambda). \tag{2.36}$$

The factor $N!$ in the r.h.s. of (2.36) has an obvious interpretation: after fixing the gauge symmetry of the matrix integral by fixing the diagonal gauge, there is still a residual symmetry given by the Weyl symmetry of $U(N)$, which is the symmetric group S_N acting as permutation of the eigenvalues. The "volume" of this discrete gauge group is just its order, $|S_N| = N!$, and since we are considering gauged matrix models we have to divide by it as shown in (2.36). As a particular case of the above formula, it follows that one can write the partition function (2.3) as

$$Z = \frac{1}{N!}\frac{1}{(2\pi)^N}\int \prod_{i=1}^{N}d\lambda_i\,\Delta^2(\lambda)e^{-\frac{1}{2g_s}\sum_{i=1}^{N}W(\lambda_i)}. \tag{2.37}$$

The partition function of the gauged Gaussian matrix model (2.5) is given essentially by the inverse of the volume factor. Its free energy to all orders can be computed by using the asymptotic expansion of the Barnes function

$$\begin{aligned}
\log G_2(N+1) &= \frac{N^2}{2}\log N - \frac{1}{12}\log N - \frac{3}{4}N^2 + \frac{1}{2}N\log 2\pi + \zeta'(-1) \\
&+ \sum_{g=2}^{\infty}\frac{B_{2g}}{2g(2g-2)}N^{2-2g},
\end{aligned} \tag{2.38}$$

14

where B_{2g} are the Bernoulli numbers. Therefore, we find the following expression for the total free energy:

$$
\begin{aligned}
F_G & = \frac{N^2}{2}\left(\log(Ng_s) - \frac{3}{2}\right) - \frac{1}{12}\log N + \zeta'(-1) \\
& + \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)}N^{2-2g}.
\end{aligned}
\tag{2.39}
$$

If we now put $N = t/g_s$, we obtain the following expressions for $F_g(t)$:

$$
\begin{aligned}
F_0(t) & = \frac{1}{2}t^2\left(\log t - \frac{3}{2}\right), \\
F_1(t) & = -\frac{1}{12}\log t, \\
F_g(t) & = \frac{B_{2g}}{2g(2g-2)}t^{2-2g}, \quad g > 1.
\end{aligned}
$$

2.2   Matrix model technology I: saddle-point analysis

The computation of the functions $F_g(t)$ in closed form seems a difficult task, since in perturbation theory they involve summing up an infinite number of fatgraphs (with different numbers of holes h). However, in the classic paper [12] it was shown that, remarkably, $F_0(t)$ can be obtained by solving a Riemann-Hilbert problem. In this section we will review this procedure.

Let us consider a general matrix model with action $W(M)$, and let us write the partition function after reduction to eigenvalues (2.37) as follows:

$$
Z = \frac{1}{N!}\int \prod_{i=1}^{N} \frac{d\lambda_i}{2\pi}e^{N^2 S_{\text{eff}}(\lambda)}
\tag{2.40}
$$

where the effective action is given by

$$
S_{\text{eff}}(\lambda) = -\frac{1}{tN}\sum_{i=1}^{N}W(\lambda_i) + \frac{2}{N^2}\sum_{i<j}\log|\lambda_i - \lambda_j|.
\tag{2.41}
$$

Notice that, since a sum over N eigenvalues is roughly of order N, the effective action is of order $\mathcal{O}(1)$. We can now regard N^2 as a sort of $\hbar^{-1}$ in such a way that, as $N \to \infty$, the integral (2.40) will be dominated by a saddle-point configuration that extremizes the effective action. Varying $S_{\text{eff}}(\lambda)$ w.r.t. the eigenvalue λ_i, we obtain the equation

$$
\frac{1}{2t}W'(\lambda_i) = \frac{1}{N}\sum_{j\neq i}\frac{1}{\lambda_i - \lambda_j}, \quad i = 1, \cdots, N.
\tag{2.42}
$$

The eigenvalue distribution is formally defined for finite N as

$$\rho(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i), \qquad (2.43)$$

where the λ_i solve (2.42). In the large N limit, it is reasonable to expect that this distribution becomes a continuous function with compact support. We will assume that $\rho(\lambda)$ vanishes outside an interval $\mathcal{C}$. This is the so-called *one-cut solution*.

Qualitatively, what is going on is the following. Assume for simplicity that $W(x)$, the potential, has only one minimum x_*. We can regard the eigenvalues as coordinates of a system of N classical particles moving on the real line. The equation (2.42) says that these particles are subject to an effective potential

$$W_{\text{eff}}(\lambda_i) = W(\lambda_i) - \frac{2t}{N} \sum_{j \neq i} \log |\lambda_i - \lambda_j| \qquad (2.44)$$

which involves a logarithmic Coulomb repulsion between eigenvalues. For small 't Hooft parameter, the potential term dominates over the Coulomb repulsion, and the particles tend to be in the minimum x_* of the potential $W'(x_*) = 0$. This means that, for $t = 0$, the interval $\mathcal{C}$ collapses to the point x_*. As t grows, the Coulomb repulsion will force the eigenvalues to be apart from each other and to spread out over an interval $\mathcal{C}$.

We can now write the saddle-point equation in terms of continuum quantities, by using the standard rule

$$\frac{1}{N} \sum_{i=1}^{N} f(\lambda_i) \rightarrow \int_{\mathcal{C}} f(\lambda) \rho(\lambda) d\lambda. \qquad (2.45)$$

Notice that the distribution of eigenvalues $\rho(\lambda)$ satisfies the normalization condition

$$\int_{\mathcal{C}} \rho(\lambda) d\lambda = 1. \qquad (2.46)$$

The equation (2.42) then becomes

$$\frac{1}{2t} W'(\lambda) = \text{P} \int \frac{\rho(\lambda') d\lambda'}{\lambda - \lambda'} \qquad (2.47)$$

where P denotes the principal value of the integral. The above equation is an integral equation that allows one in principle to compute $\rho(\lambda)$, given the potential $W(\lambda)$, as a function of the 't Hooft parameter t and the coupling constants. Once $\rho(\lambda)$ is known,

one can easily compute $F_0(t)$: in the saddle-point approximation, the free energy is given by

$$\frac{1}{N^2}F = S_{\text{eff}}(\rho) + \mathcal{O}(N^{-2}), \tag{2.48}$$

where the effective action in the continuum limit is a functional of ρ:

$$S_{\text{eff}}(\rho) = -\frac{1}{t}\int_{\mathcal{C}} d\lambda\rho(\lambda)W(\lambda) + \int_{\mathcal{C}\times\mathcal{C}} d\lambda d\lambda'\rho(\lambda)\rho(\lambda')\log|\lambda - \lambda'|. \tag{2.49}$$

Therefore, the planar free energy is given by

$$F_0(t) = t^2 S_{\text{eff}}(\rho), \tag{2.50}$$

Since the effective action is evaluated on the distribution of eigenvalues which solves (2.47), one can simplify the expression to

$$F_0(t) = -\frac{t}{2}\int_{\mathcal{C}} d\lambda\rho(\lambda)W(\lambda). \tag{2.51}$$

Similarly, averages in the matrix model can be computed in the planar limit as

$$\frac{1}{N}\langle \text{Tr}\, M^{\ell}\rangle = \int_{\mathcal{C}} d\lambda\, \lambda^{\ell}\rho(\lambda). \tag{2.52}$$

We then see that the planar limit is characterized by a *classical* density of states $\rho(\lambda)$, and the planar piece of quantum averages can be computed as a moment of this density. The fact that the planar approximation to a quantum field theory can be regarded as a classical field configuration was pointed out in [68] (see [20] for a beautiful exposition). This classical configuration is often called the *master field.* In the case of matrix models, the master field configuration is given by the density of eigenvalues $\rho(\lambda)$, and as we will see later it can be encoded in a complex algebraic curve with a deep geometric meaning.

The density of eigenvalues is obtained as a solution to the saddle-point equation (2.47). This equation is a singular integral equation which has been studied in detail in other contexts of physics (see, for example, [57]). The way to solve it is to introduce an auxiliary function called the *resolvent.* The resolvent is defined as a correlator in the matrix model:

$$\omega(p) = \frac{1}{N}\langle \text{Tr}\frac{1}{p - M}\rangle, \tag{2.53}$$

which is in fact a generating functional of the correlation functions (2.52):

$$\omega(p) = \frac{1}{N}\sum_{k=0}^{\infty}\langle \text{Tr} M^k\rangle p^{-k-1} \tag{2.54}$$

Being a generating functional of connected correlators, it admits an expansion of the form [20]:

$$\omega(p) = \sum_{g=0}^{\infty} g_s^{2g} \omega_g(p), \tag{2.55}$$

and the genus zero piece can be written in terms of the eigenvalue density as

$$\omega_0(p) = \int d\lambda \frac{\rho(\lambda)}{p - \lambda} \tag{2.56}$$

The genus zero resolvent (2.56) has three important properties. First of all, as a function of p it is an analytic function on the whole complex plane except on the interval $\mathcal{C}$, since if $\lambda \in \mathcal{C}$ one has a singularity at $\lambda = p$. Second, due to the normalization property of the eigenvalue distribution (2.46), it has the asymptotic behavior

$$\omega_0(p) \sim \frac{1}{p}, \qquad p \to \infty. \tag{2.57}$$

Finally, one can compute the discontinuity of $\omega_0(p)$ as one crosses the interval $\mathcal{C}$. This is just the residue at $\lambda = p$, and one then finds the key equation

$$\rho(\lambda) = -\frac{1}{2\pi i}\big(\omega_0(\lambda + i\epsilon) - \omega_0(\lambda - i\epsilon)\big). \tag{2.58}$$

Therefore, if the resolvent at genus zero is known, the eigenvalue distribution follows from (2.58), and one can compute the planar free energy. On the other hand, by looking again at the resolvent as we approach the discontinuity, we see that the r.h.s. of (2.47) is given by $-(\omega_0(p + i\epsilon) + \omega_0(p - i\epsilon))/2$, and we then find the equation

$$\omega_0(p + i\epsilon) + \omega_0(p - i\epsilon) = -\frac{1}{t}W'(p), \tag{2.59}$$

which determines the resolvent in terms of the potential. In this way we have reduced the original problem of computing $F_0(t)$ to the Riemann-Hilbert problem of computing $\omega_0(\lambda)$. There is in fact a closed expression for the resolvent in terms of a contour integral [56] which is very useful. Let $\mathcal{C}$ be given by the interval $b \leq \lambda \leq a$. Then, one has

$$\omega_0(p) = \frac{1}{2t} \oint_{\mathcal{C}} \frac{dz}{2\pi i} \frac{W'(z)}{p - z} \left(\frac{(p - a)(p - b)}{(z - a)(z - b)} \right)^{\frac{1}{2}}. \tag{2.60}$$

This equation is easily proved by converting (2.59) into a discontinuity equation:

$$\widehat{\omega}_0(p + i\epsilon) - \widehat{\omega}_0(p - i\epsilon) = -\frac{1}{t} \frac{W'(p)}{\sqrt{(p - a)(p - b)}}, \tag{2.61}$$

where $\widehat{\omega}_0(p) = \omega_0(p)/\sqrt{(p-a)(p-b)}$. This equation determines $\omega_0(p)$ to be given by (2.60) up to regular terms, but because of the asymptotics (2.57), these regular terms are absent. The asymptotics of $\omega_0(p)$ also gives two more conditions. By taking $p \to \infty$, one finds that the r.h.s. of (2.60) behaves like $c + d/p + \mathcal{O}(1/p^2)$. Requiring the asymptotic behavior (2.57) imposes $c = 0$ and $d = 1$, and this leads to

$$
\oint_{\mathcal{C}} \frac{dz}{2\pi i} \frac{W'(z)}{\sqrt{(z-a)(z-b)}} = 0,
$$

$$
\oint_{\mathcal{C}} \frac{dz}{2\pi i} \frac{z W'(z)}{\sqrt{(z-a)(z-b)}} = 2t. \tag{2.62}
$$

These equations are enough to determine the endpoints of the cuts, a and b, as functions of the 't Hooft coupling t and the coupling constants of the model.

The above expressions are in fact valid for very general potentials (we will apply them to logarithmic potentials in section 4), but when $W(z)$ is a polynomial, one can find a very convenient expression for the resolvent: if we deform the contour in (2.60) we pick up a pole at $z = p$, and another one at infinity, and we get

$$
\omega_0(p) = \frac{1}{2t} W'(p) - \frac{1}{2t} \sqrt{(p-a)(p-b)} M(p), \tag{2.63}
$$

where

$$
M(p) = \oint_0 \frac{dz}{2\pi i} \frac{W'(1/z)}{1-pz} \frac{1}{\sqrt{(1-az)(1-bz)}}. \tag{2.64}
$$

Here, the contour is around $z = 0$. These formulae, together with the expressions (2.62) for the endpoints of the cut, completely solve the one-matrix model with one cut in the planar limit, for polynomial potentials.

Another way to find the resolvent is to start with (2.42), multiply it by $1/(\lambda_i - p)$, and sum over i. One finds, in the limit of large N,

$$
(\omega_0(p))^2 - \frac{1}{t} W'(p) \omega_0(p) + \frac{1}{4t^2} R(p) = 0, \tag{2.65}
$$

where

$$
R(p) = 4t \int d\lambda \, \rho(\lambda) \frac{W'(p) - W'(\lambda)}{p - \lambda}. \tag{2.66}
$$

Notice that (2.65) is a quadratic equation for $\omega_0(p)$ and has the solution

$$
\omega_0(p) = \frac{1}{2t} \left(W'(p) - \sqrt{(W'(p))^2 - R(p)} \right), \tag{2.67}
$$

which is of course equivalent to (2.63).

A useful way to encode the solution to the matrix model is to define

$$y(p) = W'(p) - 2t\,\omega_0(p). \tag{2.68}$$

Notice that the force on an eigenvalue is given by

$$f(p) = -W'_{\text{eff}}(p) = -\frac{1}{2}(y(p + i\epsilon) + y(p - i\epsilon)). \tag{2.69}$$

In terms of $y(p)$, the quadratic equation (2.65) determining the resolvent can be written as

$$y^2 = W'(p)^2 - R(p). \tag{2.70}$$

This is nothing but the equation of a hyperelliptic curve given by a certain deformation (measured by $R(p)$) of the equation $y^2 = W'(p)^2$ typical of singularity theory. We will see in the next section that this result has a beautiful interpretation in terms of topological string theory on certain Calabi-Yau manifolds.

Example. *The Gaussian matrix model.* Let us now apply this technology to the simplest case, the Gaussian model with $W(M) = M^2/2$. Let us first look for the position of the endpoints from (2.62). Deforming the contour to infinity and changing $z \to 1/z$, we find that the first equation in (2.62) becomes

$$\oint_0 \frac{dz}{2\pi i} \frac{1}{z^2} \frac{1}{\sqrt{(1 - az)(1 - bz)}} = 0, \tag{2.71}$$

where the contour is now around $z = 0$. Therefore $a + b = 0$, in accord with the symmetry of the potential. Taking this into account, the second equation becomes:

$$\oint_0 \frac{dz}{2\pi i} \frac{1}{z^3} \frac{1}{\sqrt{1 - a^2 z^2}} = 2t, \tag{2.72}$$

and gives

$$a = 2\sqrt{t}. \tag{2.73}$$

We see that the interval $\mathcal{C} = [-a, a] = [-2\sqrt{t}, 2\sqrt{t}]$ opens as the 't Hooft parameter grows up, and as $t \to 0$ it collapses to the minimum of the potential at the origin, as expected. We immediately find from (2.63)

$$\omega_0(p) = \frac{1}{2t}\left(p - \sqrt{p^2 - 4t}\right), \tag{2.74}$$

and from the discontinuity equation we derive the density of eigenvalues

$$\rho(\lambda) = \frac{1}{2\pi t}\sqrt{4t - \lambda^2}. \tag{2.75}$$

20

The graph of this function is a semicircle of radius $2\sqrt{t}$, and the above eigenvalue distribution is the famous Wigner-Dyson semicircle law. Notice also that the equation (2.70) is in this case

$$y^2 = p^2 - 4t. \tag{2.76}$$

This is the equation for a curve of genus zero, which resolves the singularity $y^2 = p^2$. We then see that the opening of the cut as we turn on the 't Hooft parameter can be interpreted as a deformation of a geometric singularity. This will be later interpreted in section 3.5 from the point of view of topological string theory on Calabi-Yau manifolds.

Exercise. *Resolvent for the cubic matrix model*. Consider the cubic matrix model with potential $W(M) = M^2/2 + g_3 M^3/3$. Derive an expression for the endpoints of the one-cut solution as a function of t, g_3, and find the resolvent and the planar free energy. The solution is worked out in [12].

Although we will not need it in this review, there are well-developed techniques to obtain the higher genus $F_g(t)$ as systematic corrections to the saddle-point result $F_0(t)$ [5, 32]. Interestingly enough, these corrections can be computed in terms of integrals of differentials defined on the hyperelliptic curve (2.70).

We have so far considered the so-called one cut solution to the one-matrix model. This is not, however, the most general solution, and we now will consider the multicut solution in the saddle-point approximation. Recall from our previous discussion that the cut appearing in the one-matrix model was centered around a minimum of the potential. If the potential has many minima, one can have a solution with various cuts, centered around the different minima. The most general solution has then s cuts (where s is lower or equal than the number of minima n), and the support of the eigenvalue distribution is a disjoint union of s intervals

$$\mathcal{C} = \cup_{i=1}^{s} \mathcal{C}_i, \tag{2.77}$$

where

$$\mathcal{C}_i = [x_{2i}, x_{2i-1}] \tag{2.78}$$

and $x_{2s} < \cdots < x_1$. The equation (2.67) still gives the solution for the resolvent, and it is easy to see that the way to have multiple cuts is to require $\omega_0(p)$ to have $2s$ branch points corresponding to the roots of the polynomial $W'(z)^2 - R(z)$. Therefore we have

$$\omega_0(p) = \frac{1}{2t}W'(p) - \frac{1}{2t}\sqrt{\prod_{k=1}^{2s}(p - x_k)M(p)}, \tag{2.79}$$

which can be solved in a compact way by

$$\omega_0(p) = \frac{1}{2t} \oint_{\mathcal{C}} \frac{dz}{2\pi i} \frac{W'(z)}{p - z} \left(\prod_{k=1}^{2s} \frac{p - x_k}{z - x_k}\right)^{\frac{1}{2}}. \tag{2.80}$$

In order to satisfy the asymptotics (2.57) the following conditions must hold:

$$\delta_{\ell s} = \frac{1}{2t} \oint_{\mathcal{C}} \frac{dz}{2\pi i} \frac{z^\ell W'(z)}{\prod_{k=1}^{2s} (z - x_k)^{\frac{1}{2}}}, \qquad \ell = 0, 1, \cdots, s. \tag{2.81}$$

In contrast to the one-cut case, these are only $s+1$ conditions for the $2s$ variables x_k representing the endpoints of the cut. For $s > 1$, there are not enough conditions to determine the solution of the model, and we need extra input to determine the positions of the endpoints x_k. Usually, the extra condition which is imposed is that the different cuts are at equipotential lines (see for example [11, 4]). It is easy to see that in general the effective potential is constant on each cut,

$$W_{\text{eff}}(p) = \Gamma_i, \qquad p \in \mathcal{C}_i, \tag{2.82}$$

but the values of Γ_i will be in general different for the different cuts. This means that there can be eigenvalue tunneling from one cut to the other. The way to guarantee equilibrium is to choose the endpoints of the cuts in such a way that $\Gamma_i = \Gamma$ for all $i = 1, \cdots, s$. This gives the $s - 1$ conditions:

$$W_{\text{eff}}(x_{2i+1}) = W_{\text{eff}}(x_{2i}), \qquad i = 1, \cdots, s - 1, \tag{2.83}$$

which, together with the $s+1$ conditions (2.81) provide $2s$ constraints which allow one to find the positions of the $2s$ endpoints x_i. We can also write the equation (2.83) as

$$\int_{x_{2i+1}}^{x_{2i}} dz\, M(z) \prod_{k=1}^{2s} (z - x_k)^{\frac{1}{2}} = 0. \tag{2.84}$$

In the context of the matrix models describing topological strings, the multicut solution is determined by a different set of conditions and will be described in section 3.4.

2.3   Matrix model technology II: orthogonal polynomials

Another useful technique to solve matrix models involves orthogonal polynomials. This technique was developed in [8, 9] (which we follow quite closely), and provides explicit expressions for $F_g(t)$ at least for low genus. This technique turns out to be particularly useful in the study of the so-called double-scaling limit of matrix models [13]. We will use this technique to study Chern-Simons matrix models, in section 4, therefore this subsection can be skipped by the reader who is only interested in the conventional matrix models involved in the Dijkgraaf-Vafa approach.

The starting point of the technique of orthogonal polynomials is the eigenvalue representation of the partition function

$$Z = \frac{1}{N!} \int \prod_{i=1}^{N} \frac{d\lambda_i}{2\pi} \Delta^2(\lambda) e^{-\frac{1}{g_s} \sum_{i=1}^{N} W(\lambda_i)}, \tag{2.85}$$

22

where $W(\lambda)$ is an arbitrary potential. If we regard

$$d\mu = e^{-\frac{1}{g_s}W(\lambda)}\frac{d\lambda}{2\pi} \tag{2.86}$$

as a measure in $\mathbf{R}$, one can introduce *orthogonal polynomials* $p_n(\lambda)$ defined by

$$\int d\mu \, p_n(\lambda)p_m(\lambda) = h_n\delta_{nm}, \quad n \geq 0, \tag{2.87}$$

where $p_n(\lambda)$ are normalized by requiring the behavior $p_n(\lambda) = \lambda^n + \cdots$. One can now compute Z by noting that

$$\Delta(\lambda) = \det p_{j-1}(\lambda_i). \tag{2.88}$$

By expanding the determinant as

$$\sum_{\sigma \in S_N} (-1)^{\epsilon(\sigma)} \prod_k p_{\sigma(k)-1}(\lambda_k) \tag{2.89}$$

where the sum is over permutations σ of N indices and $\epsilon(\sigma)$ is the signature of the permutation, we find

$$Z = \prod_{i=0}^{N-1} h_i = h_0^N \prod_{i=1}^{N} r_i^{N-i}, \tag{2.90}$$

where we have introduced the coefficients

$$r_k = \frac{h_k}{h_{k-1}}, \quad k \geq 1. \tag{2.91}$$

One of the most important properties of orthogonal polynomials is that they satisfy recursion relations of the form

$$(\lambda + s_n)p_n(\lambda) = p_{n+1}(\lambda) + r_n p_{n-1}(\lambda). \tag{2.92}$$

It is easy to see that the coefficients r_n involved in this relation are indeed given by (2.91). This follows from the equality

$$h_{n+1} = \int d\mu \, p_{n+1}(\lambda)\lambda p_n(\lambda), \tag{2.93}$$

together with the use of the recursion relation for $\lambda p_{n+1}(\lambda)$. For even potentials, $s_n = 0$.

As an example of this technique, we can consider again the simple case of the Gaussian matrix model. The orthogonal polynomials of the Gaussian model are well-known: they are essentially the Hermite polynomials $H_n(x)$, which are defined by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \tag{2.94}$$

More precisely, one has

$$p_n(x) = \left(\frac{g_s}{2}\right)^{n/2} H_n(x/\sqrt{2g_s}), \tag{2.95}$$

and one can then check that

$$h_n^G = \left(\frac{g_s}{2\pi}\right)^{\frac{1}{2}} n! g_s^n, \quad r_n^G = n\, g_s. \tag{2.96}$$

Using now (2.90) we can confirm the result (2.33) that we stated before.

It is clear that a detailed knowledge of the orthogonal polynomials allows the computation of the partition function of the matrix model. It is also easy to see that the computation of correlation functions also reduces to an evaluation in terms of the coefficients in the recursion relation. To understand this point, it is useful to introduce the *orthonormal* polynomials

$$\mathcal{P}_n(\lambda) = \frac{1}{\sqrt{h_n}} p_n(\lambda), \tag{2.97}$$

which satisfy the recursion relation

$$\lambda \mathcal{P}_n(\lambda) = -s_n \mathcal{P}_n(\lambda) + \sqrt{r_{n+1}} \mathcal{P}_{n+1}(\lambda) + \sqrt{r_n} \mathcal{P}_{n-1}(\lambda). \tag{2.98}$$

Let us now consider the normalized vev $\langle \operatorname{Tr} M^\ell \rangle$, which in terms of eigenvalues is given by the integral

$$\langle \operatorname{Tr} M^\ell \rangle = \frac{1}{N!Z} \int \prod_{i=1}^{N} e^{-\frac{1}{g_s} W(\lambda_i)} \frac{d\lambda_i}{2\pi} \Delta^2(\lambda) \left(\sum_{i=1}^{N} \lambda_i^\ell \right). \tag{2.99}$$

By using (2.88) it is easy to see that this equals

$$\sum_{j=0}^{N-1} \int d\mu \lambda^\ell \mathcal{P}_j^2(\lambda). \tag{2.100}$$

This integral can be computed in terms of the coefficients in (2.97). For example, for $\ell = 2$ we find

$$\langle \operatorname{Tr} M^2 \rangle = \sum_{j=0}^{N-1} (s_j^2 + r_{j+1} + r_j), \tag{2.101}$$

where we put $r_0 = 0$. A convenient way to encode this result is by introducing the Jacobi matrix

$$\mathcal{J} = \begin{pmatrix} 0 & r_1^{1/2} & 0 & 0 & \cdots \\ r_1^{1/2} & 0 & r_2^{1/2} & 0 & \cdots \\ 0 & r_2^{1/2} & 0 & r_3^{1/2} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{2.102}$$

24

as well as the diagonal matrix

$$\mathcal{S} = \begin{pmatrix} s_0 & 0 & 0 & 0 & \cdots \\ 0 & s_1 & 0 & 0 & \cdots \\ 0 & 0 & s_2 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \tag{2.103}$$

It then follows that

$$\langle \mathrm{Tr}\, M^\ell \rangle = \mathrm{Tr}\, (\mathcal{J} - \mathcal{S})^\ell. \tag{2.104}$$

The results we have presented so far give the exact answer for the correlators and the partition function, at all orders in $1/N$. As we have seen, we are particularly interested in computing the functions $F_g(t)$ which are obtained by resumming the perturbative expansion at fixed genus. As shown in [8, 9], one can in fact use the orthogonal polynomials to provide closed expressions for $F_g(t)$ in the one-cut case. We will now explain how to do this in some detail.

The object we want to compute is

$$\mathcal{F} = F - F_G = \log Z - \log Z_G. \tag{2.105}$$

If we write the usual series $\mathcal{F} = \sum_{g \geq 0} \mathcal{F}_g g_s^{2g-2}$, we have

$$g_s^2 \mathcal{F} = \frac{t^2}{N^2} (\log Z - \log Z_G) = \frac{t^2}{N} \log \frac{h_0}{h_0^G} + \frac{t^2}{N} \sum_{k=1}^{N} (1 - \frac{k}{N}) \log \frac{r_k(N)}{k g_s}. \tag{2.106}$$

The planar contribution to the free energy $\mathcal{F}_0(t)$ is obtained from (2.106) by taking $N \to \infty$. In this limit, the variable

$$\xi = \frac{k}{N}$$

becomes a continuous variable, $0 \leq \xi \leq 1$, in such a way that

$$\frac{1}{N} \sum_{k=1}^{N} f(k/N) \to \int_0^1 d\xi f(\xi)$$

as N goes to infinity. Let us assume as well that $r_k(N)$ has the following asymptotic expansion as $N \to \infty$:

$$r_k(N) = \sum_{s=0}^{\infty} N^{-2s} R_{2s}(\xi). \tag{2.107}$$

We then find

$$\mathcal{F}_0(t) = -\frac{1}{2} t^2 \log t + t^2 \int_0^1 d\xi (1 - \xi) \log \frac{R_0(\xi)}{\xi}. \tag{2.108}$$

25

This provides a closed expression for the planar free energy in terms of the large N limit of the recursion coefficients r_k.

It is interesting to see how to recover the density of states $\rho(\lambda)$ in the saddle-point approximation from orthogonal polynomials. Let us first try to evaluate (2.104) in the planar approximation, following [9]. A simple argument based on the recursion relations indicates that, at large N,

$$(\mathcal{J}^\ell)_{nn} \sim \frac{\ell!}{(\ell/2)!^2} r_n^{\ell/2}. \tag{2.109}$$

Using now the integral representation

$$\frac{\ell!}{(\ell/2)!^2} = \int_{-1}^1 \frac{dy}{\pi} \frac{(2y)^\ell}{\sqrt{1-y^2}},$$

we find

$$\frac{1}{N}\langle \mathrm{Tr}\, M^\ell \rangle = \int_0^1 d\xi \int_{-1}^1 \frac{dy}{\pi} \frac{1}{\sqrt{1-y^2}} (2y R_0^{1/2}(\xi) - s(\xi))^\ell,$$

where we have denoted by $s(\xi)$ the limit as $N \to \infty$ of the recursion coefficients $s_k(N)$ which appear in (2.92). Since the above average can be also computed by (2.52), by comparing we find

$$\rho(\lambda) = \int_0^1 d\xi \int_{-1}^1 \frac{dy}{\pi} \frac{1}{\sqrt{1-y^2}} \delta\big(\lambda - (2y R_0^{1/2}(\xi) - s(\xi))\big),$$

or, more explicitly,

$$\rho(\lambda) = \int_0^1 \frac{d\xi}{\pi} \frac{\theta[4R_0(\xi) - (\lambda + s(\xi))^2]}{\sqrt{4R_0(\xi) - (\lambda + s(\xi))^2}}. \tag{2.110}$$

Here, θ is the step function. It also follows from this equation that $\rho(\lambda)$ is supported on the interval $[b(t), a(t)]$, where

$$b(t) = -2\sqrt{R_0(1)} - s(1), \quad a(t) = 2\sqrt{R_0(1)} - s(1). \tag{2.111}$$

Example. In the Gaussian matrix model $R_0(\xi) = t\xi$, and $s(\xi) = 0$. We then find that the density of eigenvalues is supported in the interval $[-2\sqrt{t}, 2\sqrt{t}]$ and it is given by

$$\rho(\lambda) = \frac{1}{\pi} \int_0^1 d\xi \frac{\theta[4\xi t - \lambda^2]}{\sqrt{4\xi t - \lambda^2}} = \frac{1}{2\pi t}\sqrt{4t - \lambda^2}$$

which reproduces of course Wigner's semicircle law.

As shown in [8, 9], orthogonal polynomials can be used as well to obtain the higher genus free energies $\mathcal{F}_g$. The key ingredient to do that is simply the Euler-MacLaurin formula, which reads

$$\frac{1}{N}\sum_{k=1}^{N} f\left(\frac{k}{N}\right) = \int_0^1 f(\xi)d\xi + \frac{1}{2N}[f(1) - f(0)] + \sum_{p=1}^{\infty} \frac{1}{N^{2p}}\frac{B_{2p}}{(2p)!}[f^{(2p-1)}(1) - f^{(2p-1)}(0)],$$

(2.112)

and should be regarded as an asymptotic expansion for N large which gives a way to compute systematically $1/N$ corrections. We can then use it to calculate (2.106) at all orders in $1/N$, where

$$f(k/N) = \left(1 - \frac{k}{N}\right)\log\frac{Nr_k(N)}{k},$$

(2.113)

and we use the fact that r_k has an expansion of the form (2.107). In this way, we find for example that

$$\mathcal{F}_1(t) = t^2 \int_0^1 d\xi(1-\xi)\frac{R_2(\xi)}{R_0(\xi)} + \frac{t^2}{12}\frac{d}{d\xi}\left[(1-\xi)\log\frac{R_0(\xi)}{\xi}\right]_0^1,$$

and so on. We will use this formulation in section 4 to compute $\mathcal{F}_g(t)$ in the matrix model that describes Chern-Simons theory on $\mathbf{S}^3$.

It is clear from the above analysis that matrix models can be solved with the method of orthogonal polynomials, in the sense that once we know the precise form of the coefficients in the recursion relation we can compute all quantities in an $1/N$ expansion. Since the recursion relation is only known exactly in a few cases, we need methods to determine its coefficients for general potentials $W(M)$. In the case of *polynomial* potentials, of the form

$$W(M) = \sum_{p\geq 0}\frac{g_p}{p}\text{Tr}\,M^p,$$

there are well-known techniques to obtain explicit results [9], see [21, 22] for reviews. We start by rewriting the recursion relation (2.92) as

$$\lambda\,p_n(\lambda) = \sum_{m=0}^{n+1} B_{nm}p_m,$$

where B is a matrix. The identities

$$r_n \int d\lambda e^{-\frac{1}{g_s}W(\lambda)}W'(\lambda)p_n(\lambda)p_{n-1}(\lambda) = nh_n g_s,$$

$$\int d\lambda\frac{d}{d\lambda}(p_n e^{-\frac{1}{g_s}W(\lambda)}p_n) = 0$$

(2.114)

lead to the matrix equations

$$
\begin{aligned}
(W'(B))_{nn-1} &= ng_s, \\
(W'(B))_{nn} &= 0.
\end{aligned}
\tag{2.115}
$$

These equations are enough to determine the recursion coefficients. Consider for example a quartic potential

$$
W(\lambda) = \frac{g_2}{2}\lambda^2 + \frac{g_4}{4}\lambda^4.
$$

Since this potential is even, it is easy to see that the first equation in (2.115) is automatically satisfied, while the second equation leads to

$$
r_n\{g_2 + g_4(r_n + r_{n-1} + r_{n+1})\} = ng_s
$$

which at large N reads

$$
R_0(g_2 + 3g_4 R_0) = \xi t.
$$

In general, for an even potential of the form

$$
W(\lambda) = \sum_{p\geq 0} \frac{g_{2p+2}}{2p+2}\lambda^{2p+2}
\tag{2.116}
$$

one finds

$$
\xi t = \sum_{p\geq 0} g_{2p+2}\binom{2p+1}{p}R_0^{p+1}(\xi),
\tag{2.117}
$$

which determines R_0 as a function of ξ. The above equation is sometimes called – especially in the context of double-scaled matrix models– the *string equation*, and by setting $\xi = 1$ we find an explicit equation for the endpoints of the cut in the density of eigenvalues as a function of the coupling constants and t.

Exercise. Verify, using saddle-point techniques, that the string equation correctly determines the endpoints of the cut. Compute $R_0(\xi)$ for the quartic and the cubic matrix model, and use it to obtain $\mathcal{F}_0(t)$ (for the quartic potential, the solution is worked out in detail in [9]).

3   Type B topological strings and matrix models

3.1   The topological B model

The topological B model was introduced in [49, 73] and can be constructed by twisting the $\mathcal{N} = 2$ superconformal sigma model in two dimensions. There are in fact two different twists, called the A and the B twist in [49, 73], and in these lectures we will

focus on the second one. A detailed review of topological sigma models and topological strings can be found in [39].

The topological B model is a theory of maps from a Riemann surface Σ_g to a Calabi-Yau manifold X of complex dimension d. The Calabi-Yau condition arises in order to cancel an anomaly that appears after twisting (see for example Chapter 3 of [52] for a detailed analysis of this issue). Indices for the real tangent bundle of X will be denoted by $i = 1, \cdots, 2d$, while holomorphic and antiholomorphic indices will be denoted respectively by $I, \overline{I} = 1, \cdots, d$. The holomorphic tangent bundle will be simply denoted by TX, while the antiholomorphic tangent bundle will be denoted by $\overline{TX}$. One of the most important properties of Calabi-Yau manifolds (which can actually be taken as their defining feature) is that they have a holomorphic, nonvanishing section Ω of the canonical bundle $K_X = \Omega^{3,0}(X)$. Since the section is nowhere vanishing, the canonical line bundle is trivial and $c_1(K_X) = 0$. We will always consider examples with complex dimension $d = 3$.

The field content of the topological B model is the following. First, since it is a nonlinear sigma model, we have a map $x : \Sigma_g \to X$, which is a scalar, commuting field. Besides the field x, we have two sets of Grassmann fields $\eta^{\overline{I}}, \theta^{\overline{I}} \in x^*(\overline{TX})$, which are scalars on Σ_g, and a Grassmannian one-form on Σ_g, ρ_α^I, with values in $x^*(TX)$. We also have commuting auxiliary fields $F^I, F^{\overline{I}}$ (we will follow here the off-shell formulation of [49, 50]). The action for the theory is:

$$
\begin{aligned}
\mathcal{L} \;=\; & t \int_{\Sigma_g} d^2z \Big[G_{I\overline{J}} \big(\partial_z x^I \partial_{\bar z} x^{\overline{J}} + \partial_{\bar z} x^I \partial_z x^{\overline{J}} \big) - \rho_z^I \big(G_{I\overline{J}} D_{\bar z} \eta^{\overline{J}} + D_{\bar z} \theta_I \big) \\
& - \rho_{\bar z}^I \big(G_{I\overline{J}} D_z \eta^{\overline{J}} - D_z \theta_I \big) - R^I{}_{\overline{J}\overline{L}K} \eta^{\overline{L}} \rho_z^J \rho_{\bar z}^K \theta_I - G_{I\overline{J}} F^I F^{\overline{J}} \Big],
\end{aligned}
\tag{3.1}
$$

In this action, we have picked local coordinates $z, \bar z$ on Σ_g, and d^2z is the measure $-idz \wedge d\bar z$. t is a parameter that plays the role of $1/\hbar$, the field θ_I is given by $\theta_I = G_{I\overline{J}} \theta^{\overline{J}}$, and the covariant derivative D_α acts on sections ψ^i of the tangent bundle as

$$
D_\alpha \psi^i = \partial_\alpha \psi^i + \partial_\alpha x^j \Gamma^i_{jk} \psi^k.
\tag{3.2}
$$

The theory also has a BRST, or topological, charge $\mathcal{Q}$ which acts on the fields according to

$$
\begin{array}{llll}
[\mathcal{Q}, x^I] &=& 0, & \qquad [\mathcal{Q}, x^{\overline{I}}] &=& \eta^{\overline{I}}, \\
\{\mathcal{Q}, \eta^{\overline{I}}\} &=& 0, & \qquad \{\mathcal{Q}, \theta_I\} &=& G_{I\overline{J}} F^{\overline{J}}, \\
\{\mathcal{Q}, \rho_z^I\} &=& \partial_z x^I, & \qquad [\mathcal{Q}, F^I] &=& D_z \rho_{\bar z}^I - D_{\bar z} \rho_z^I + R^I{}_{\overline{J}\overline{L}K} \eta^{\overline{L}} \rho_z^J \rho_{\bar z}^K, \\
\{\mathcal{Q}, \rho_{\bar z}^I\} &=& \partial_{\bar z} x^I, & \qquad [\mathcal{Q}, F^{\overline{I}}] &=& -\Gamma^{\overline{I}}_{\overline{J}K} \eta^{\overline{J}} F^{\overline{K}},
\end{array}
$$

The action of $\mathcal{Q}$ explicitly depends on the splitting between holomorphic and antiholomorphic coordinates on X, in other words, it depends explicitly on the choice of

29

complex structure on X. It is easy to show that $\mathcal{Q}^2 = 0$, and that the action of the model is $\mathcal{Q}$-exact:

$$\mathcal{L} = \{\mathcal{Q}, V\} \tag{3.3}$$

where V (sometimes called the gauge fermion) is given by

$$V = t \int_{\Sigma_g} d^2 z \left[G_{I\bar{J}} \left(\rho_z^I \partial_{\bar{z}} x^{\bar{J}} + \rho_{\bar{z}}^I \partial_z x^{\bar{J}} \right) - F^I \theta_I \right]. \tag{3.4}$$

Finally, we also have a $U(1)$ ghost number symmetry, in which x, η, θ and ρ have ghost numbers 0, 1, 1, and -1, respectively. The Grassmannian charge $\mathcal{Q}$ then has ghost number 1. Notice that, if we interpret $\eta^{\bar{I}}$ as a basis for antiholomorphic differential forms on X, the action of $\mathcal{Q}$ on x^I, $x^{\bar{I}}$ may be interpreted as the Dolbeault antiholomorphic differential $\bar{\partial}$.

It follows from (3.3) that the energy-momentum tensor of this theory is given by

$$T_{\alpha\beta} = \{\mathcal{Q}, b_{\alpha\beta}\}, \tag{3.5}$$

where $b_{\alpha\beta} = \delta V / \delta g^{\alpha\beta}$ and has ghost number -1. The fact that the energy-momentum tensor is $\mathcal{Q}$-exact means that the theory is *topological*, in the sense that the partition function does not depend on the background two-dimensional metric. This is easily proved: the partition function is given by

$$Z = \int \mathcal{D}\phi \, e^{-\mathcal{L}}, \tag{3.6}$$

where ϕ denotes the set of fields of the theory, and we compute it in the background of a two-dimensional metric $g_{\alpha\beta}$ on the Riemann surface. Since $T_{\alpha\beta} = \delta\mathcal{L}/\delta g^{\alpha\beta}$, we find that

$$\frac{\delta Z}{\delta g^{\alpha\beta}} = -\langle \{\mathcal{Q}, b_{\alpha\beta}\} \rangle, \tag{3.7}$$

where the bracket denotes an unnormalized vacuum expectation value. Since $\mathcal{Q}$ is a symmetry of the theory, the above vacuum expectation value vanishes, and we find that Z is metric-independent, at least formally.

The $\mathcal{Q}$-exactness of the action itself also has an important consequence: the same argument that we used above implies that the partition function of the theory is independent of t. Now, since t plays the role of $1/\hbar$, the limit of t large corresponds to the semiclassical approximation. Since the theory does not depend on t, the semiclassical approximation is *exact*. The classical configurations for the above action are *constant* maps $x : \Sigma_g \to X$. Therefore, it follows that path integrals of the above theory reduce to integrals over X [73].

What are the operators to consider in this theory? Since the most interesting aspect of this model is the independence w.r.t. to the two-dimensional metric, we want to look for operators whose correlation functions satisfy this condition. It is easy to see that the operators in the cohomology of $\mathcal{Q}$ do the job: topological invariance requires them to be $\mathcal{Q}$-closed, and on the other hand they cannot be $\mathcal{Q}$-exact, since otherwise their correlation functions would vanish. One can also check that the $\mathcal{Q}$-cohomology is given by operators of the form

$$\mathcal{O}_\phi = \phi^{J_1\cdots J_q}_{\bar{I}_1\cdots\bar{I}_p}\eta^{\bar{I}_1}\cdots\eta^{\bar{I}_p}\theta_{J_1}\cdots\theta_{J_q}, \tag{3.8}$$

where

$$\phi = \phi^{J_1\cdots J_q}_{\bar{I}_1\cdots\bar{I}_p}dx^{\bar{I}_1}\wedge\cdots\wedge dx^{\bar{I}_p}\frac{\partial}{\partial x^{J_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{J_q}} \tag{3.9}$$

is an element of $H^p_{\bar{\partial}}(X, \wedge^q TX)$. Therefore, the $\mathcal{Q}$-cohomology is in one-to-one correspondence with the twisted Dolbeault cohomology of the target manifold X. We can then consider correlation functions of the form

$$\langle\prod_a \mathcal{O}_{\phi_a}\rangle. \tag{3.10}$$

This correlation function vanishes unless the following selection rule is satisfied

$$\sum_a p_a = \sum_a q_a = d(1-g), \tag{3.11}$$

where g is the genus of the Riemann surface. This selection rule comes from a $U(1)_L \times U(1)_R$ anomalous global current. Due to the arguments presented above, this correlation function can be computed in the semiclassical limit, where the path integral reduces to an integration over the target X. The product of operators in (3.10) corresponds to a form in $H^d_{\bar{\partial}}(X, \wedge^d TX)$. To integrate such a form over X we crucially need the Calabi-Yau condition. This arises as follows. In a Calabi-Yau manifold we have an invertible map

$$\begin{aligned}\Omega^{0,p}(\wedge^q TX) &\longrightarrow \Omega^{d-q,p}(X)\\ \phi^{I_1\cdots I_q}_{\bar{J}_1\cdots\bar{J}_p} &\mapsto \Omega_{I_1\cdots I_q I_{q+1}\cdots I_d}\phi^{I_1\cdots I_q}_{\bar{J}_1\cdots\bar{J}_p}\end{aligned} \tag{3.12}$$

where the $(d,0)$-form Ω is used to contract the indices. Since Ω is holomorphic, this descends to the $\bar{\partial}$-cohomology. It then follows that an element in $H^d_{\bar{\partial}}(X, \wedge^d TX)$ maps to an element in $H^d_{\bar{\partial}}(X)$. After further multiplication by Ω, one can then integrate a (d,d)-form over X. This is the prescription to compute correlation functions like (3.10). A simple and important example of this procedure is the case of a Calabi-Yau threefold, $d = 3$, and operators associated to forms in $H^1_{\bar{\partial}}(X, TX)$, or by using

31

(3.12), to forms in $H^{2,1}_{\bar{\partial}}(X)$. These operators are important since they correspond to infinitesimal deformations of the complex structure of X. The selection rule (3.11) says that we have to integrate three of these operators, and the correlation function reads in this case

$$\langle \mathcal{O}_{\phi_1} \mathcal{O}_{\phi_2} \mathcal{O}_{\phi_3} \rangle = \int_X (\phi_1)^{I_1}_{\bar{J}_1} (\phi_2)^{I_2}_{\bar{J}_2} (\phi_3)^{I_3}_{\bar{J}_3} \, \Omega_{I_1 I_2 I_3} dz^{\bar{J}_1} dz^{\bar{J}_2} dz^{\bar{J}_3} \wedge \Omega. \qquad (3.13)$$

It turns out that the full information of the correlators (3.13) at genus zero can be encoded in a single function called the *prepotential*. We will quickly review here some of the basic results of special geometry and the theory of the prepotential for the topological B model, and we refer the reader to [17, 39] for more details. The correlation functions in the B model, like for example (3.13), depend on a choice of complex structure, as we have already emphasized. The different complex structures form a moduli space $\mathcal{M}$ of dimension $h^{2,1}$. A convenient parametrization of $\mathcal{M}$ is the following. Choose first a symplectic basis for $H_3(X)$, denoted by (A_a, B^a), with $a = 0, 1, \cdots, h^{2,1}$, and such that $A_a \cap B^b = \delta^b_a$. We then define the *periods* of the Calabi-Yau manifold as

$$z_a = \int_{A_a} \Omega, \qquad \mathcal{F}^a = \int_{B^a} \Omega, \quad a = 0, \cdots, h^{2,1}. \qquad (3.14)$$

Of course, the symplectic group $\mathrm{Sp}(2h^{2,1} + 2, \mathbf{R})$ acts on the vector $(z^a, \mathcal{F}_a)$. A basic result of the theory of deformation of complex structures says that the z^a are (locally) complex projective coordinates for $\mathcal{M}$. Inhomogeneous coordinates can be introduced in a local patch where one of the projective coordinates, say z_0, is different from zero, and taking

$$t_a = \frac{z_a}{z_0}, \qquad a = 1, \cdots, h^{2,1}. \qquad (3.15)$$

The coordinates z_a are called *special projective coordinates*, and since they parametrize $\mathcal{M}$ we deduce that the other set of periods must depend on them, *i.e.* $\mathcal{F}^a = \mathcal{F}^a(z)$. Using the periods (3.14) we can define the *prepotential* $\mathcal{F}(z)$ by the equation

$$\mathcal{F} = \frac{1}{2} \sum_{a=0}^{h^{2,1}} z_a \mathcal{F}^a. \qquad (3.16)$$

The prepotential satisfies

$$\mathcal{F}^a(z) = \frac{\partial \mathcal{F}}{\partial z_a} \qquad (3.17)$$

and turns out to be a homogeneous function of degree two in the z_a. Therefore, one can rescale it in order to obtain a function of the inhomogeneous coordinates t_a:

$$F_0(t_a) = \frac{1}{z_0^2} \mathcal{F}(z_a). \qquad (3.18)$$

The fact that the coordinates z_a are projective is related to the freedom in normalizing the three-form Ω. In order to obtain expressions in terms of the inhomogeneous coordinates t_a, we simply have to rescale $\Omega \to \frac{1}{z_0}\Omega$, and the periods $(z_a, \mathcal{F}^a)$ become

$$(1, t_a, 2F_0 - \sum_{a=1}^{h^{2,1}} t_a \frac{\partial F_0}{\partial t_a}, \frac{\partial F_0}{\partial t_a}). \tag{3.19}$$

One of the key results in special geometry is that the correlation functions (3.13) can be computed in terms of the prepotential $F_0(t)$. Given a deformation of the complex structure parametrized by t_a, the corresponding tangent vector $\partial/\partial t_a$ is associated to a differential form of type $(2,1)$. This form leads to an operator $\mathcal{O}_a$, and the three-point functions involving these operators turn out to be given by

$$\langle \mathcal{O}_a \mathcal{O}_b \mathcal{O}_c \rangle = \frac{\partial^3 F_0}{\partial t_a \partial t_b \partial t_c}. \tag{3.20}$$

The prepotential $F_0(t)$ encodes the relevant information about the B model on the sphere, and it has an important physical meaning, since it gives the four-dimensional supergravity prepotential of type IIB string theory compactified on X (and determines the leading part of the vector multiplet effective action).

In order to obtain interesting quantities at higher genus one has to couple the topological B model to two-dimensional gravity, using the fact that the structure of the twisted theory is very close to that of the bosonic string [30, 74, 7]. In the bosonic string, there is a nilpotent BRST operator, $\mathcal{Q}_{\text{BRST}}$, and the energy-momentum tensor turns out to be a $\mathcal{Q}_{\text{BRST}}$-commutator: $T(z) = \{\mathcal{Q}_{\text{BRST}}, b(z)\}$. In addition, there is a ghost number with anomaly $3\chi(\Sigma_g) = 6 - 6g$, in such a way that $\mathcal{Q}_{\text{BRST}}$ and $b(z)$ have ghost number 1 and -1, respectively. This is precisely the same structure that we found in (3.5), and the composite field $b_{\alpha\beta}$ plays the role of an antighost. Therefore, one can just follow the prescription of coupling to gravity for the bosonic string and define a genus $g \geq 1$ free energy as follows:

$$F_g = \int_{\overline{M}_g} \langle \prod_{k=1}^{6g-6} (b, \mu_k) \rangle, \tag{3.21}$$

where

$$(b, \mu_k) = \int_{\Sigma_g} d^2 z (b_{zz}(\mu_k)_{\bar{z}}^{\ z} + b_{\bar{z}\bar{z}}(\overline{\mu}_k)_{z}^{\ \bar{z}}), \tag{3.22}$$

and μ_k are the usual Beltrami differentials. The vacuum expectation value in (3.21) refers to the path integral over the fields of the topological B model, and gives a differential form on the moduli space of Riemann surfaces of genus g, $\overline{M}_g$, which is

33

then integrated over. The free energies F_g of the B model coupled to gravity for $g \geq 1$ are also related to variation of complex structures. A target space description of this theory, called Kodaira-Spencer theory of gravity, was found in [7], and can be used to determine recursively the F_g in terms of special geometry data.

3.2 The open type B model and its string field theory description

The topological B model can be formulated as well for open strings, *i.e.*, when the worldsheet is an open Riemann surface with boundaries $\Sigma_{g,h}$ [74, 59]. In order to construct the open string version we need boundary conditions (b.c.) for the fields. It turns out that the appropriate b.c. for the B model are Dirichlet along holomorphic cycles of X, S, and Neumann in the remaining directions. Moreover, one can add Chan-Paton factors to the model, and this is implemented by considering a $U(N)$ holomorphic bundle over the holomorphic cycle S. The resulting theory can then be interpreted as a topological B model in the presence of N topological D-branes wrapping S. Since we will be interested in finding a spacetime description of the open topological B model, we can consider the case in which the branes fill spacetime (the original case considered in [74]) and deduce the spacetime action for lower dimensional branes by dimensional reduction. In the spacetime filling case, when $S = X$, the boundary conditions for the fields are $\theta = 0$ along $\partial \Sigma_{g,h}$ and that the pullback to $\partial \Sigma_{g,h}$ of $*\rho$ vanishes (where $*$ is the Hodge operator).

The open topological B model can also be coupled to gravity following the same procedure that is used in the closed case, and one obtains in this way the open type B topological string propagating along the Calabi-Yau manifold X. We are now interested in providing a description of this model when the N branes are spacetime filling. As shown by Witten in [74], the most efficient way to do that is to use the cubic string field theory introduced in [69].

In bosonic open string field theory we consider the worldsheet of the string to be an infinite strip parameterized by a spatial coordinate $0 \leq \sigma \leq \pi$ and a time coordinate $-\infty < \tau < \infty$, and we pick the flat metric $ds^2 = d\sigma^2 + d\tau^2$. We then consider maps $x : I \to X$, with $I = [0, \pi]$ and X the target of the string. The string field is a functional of open string configurations $\Psi[x(\sigma)]$, of ghost number one (the string functional depends as well on the ghost fields, but we do not indicate this dependence explicitly). In [69], Witten defines two operations on the space of string functionals. The first one is the *integration*, which is defined formally by folding the string around

its midpoint and gluing the two halves:

$$\int \Psi = \int \mathcal{D}x(\sigma) \prod_{0 \leq \sigma \leq \pi/2} \delta[x(\sigma) - x(\pi - \sigma)]\Psi[x(\sigma)]. \tag{3.23}$$

The integration has ghost number -3, which is the ghost number of the vacuum. This corresponds to the usual fact that in open string theory on the disk one has to soak up three zero modes. One also defines an associative, noncommutative *star product* $\star$ of string functionals through the following equation:

$$\int \Psi_1 \star \cdots \star \Psi_N = \int \prod_{i=1}^{N} \mathcal{D}x_i(\sigma) \prod_{i=1}^{N} \prod_{0 \leq \sigma \leq \pi/2} \delta[x_i(\sigma) - x_{i+1}(\pi - \sigma)]\Psi_i[x_i(\sigma)], \tag{3.24}$$

where $x_{N+1} \equiv x_1$. The star product simply glues the strings together by folding them around their midpoints, and gluing the first half of one with the second half of the following (see for example the review [65] for more details), and it doesn't change the ghost number. In terms of these geometric operations, the string field action is given by

$$S = \frac{1}{g_s} \int \left(\frac{1}{2} \Psi \star Q_{\text{BRST}} \Psi + \frac{1}{3} \Psi \star \Psi \star \Psi \right) \tag{3.25}$$

where g_s is the string coupling constant. Notice that the integrand has ghost number 3, while the integration has ghost number -3, so that the action (3.25) has zero ghost number. If we add Chan-Paton factors, the string field is promoted to a $U(N)$ matrix of string fields, and the integration in (3.25) includes a trace Tr. The action (3.25) has all the information about the spacetime dynamics of open bosonic strings, with or without D-branes. In particular, one can derive the Born-Infeld action describing the dynamics of D-branes from the above action [64].

We will not need all the technology of string field theory in order to understand open topological strings. The only piece of relevant information is the following: the string functional is a function of the zero mode of the string (which corresponds to the position of the string midpoint), and of the higher oscillators. If we decouple all the oscillators, the string functional becomes an ordinary function of spacetime, the $\star$ product becomes the usual product of functions, and the integral is the usual integration of functions. The decoupling of the oscillators is in fact the point-like limit of string theory. As we will see, this is the relevant limit for topological open type B strings on X.

We can now exploit again the analogy between open topological strings and the open bosonic string that we used to define the coupling of the topological B model to gravity (*i.e.*, that both have a nilpotent BRST operator and an energy-momentum tensor that

is $\mathcal{Q}_{\mathrm{BRST}}$-exact). Since both theories have a similar structure, the spacetime dynamics of open topological type B strings is governed as well by (3.25), where $\mathcal{Q}_{\mathrm{BRST}}$ is given in this case by the topological charge defined in (3.3), and where the star product and the integration operation are as in the bosonic string. The construction of the cubic string field theory also requires the existence of a ghost number symmetry, which is also present in the topological sigma model in the form of a $U(1)_R$ symmetry, as we discussed in 3.1. It is convenient to consider the $U(1)_R$ charge of the superconformal algebra in the Ramond sector, which is shifted by $-d/2$ with respect to the assignment presented in 3.1 (here, d is the dimension of the target). When $d = 3$ this corresponds to the normalization used in [69], in which the ghost vacuum of the bc system is assigned the ghost number $-1/2$.

In order to provide the string field theory description of open topological type B strings on X, we have to determine the precise content of the string field, the $\star$ algebra and the integration of string functionals for this particular model. As in the conventional string field theory of the bosonic string, we have to consider the Hamiltonian description of topological open strings. We then take Σ to be an infinite strip and consider maps $x : I \to X$, with $I = [0, \pi]$. The Hilbert space is made up out of functionals $\Psi[x(\sigma), \cdots]$, where x is a map from the interval as we have just described, and the $\cdots$ refer to the Grassmann fields (which play here the rôle of ghost fields). Notice that, since $\rho_{z,\bar{z}}^I$ are canonically conjugate to η, θ, we can choose our functional to depend only on η, θ. It is easy to see that the Hamiltonian has the form

$$ H = \int_0^\pi d\sigma \left(tG_{ij} \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma} + \cdots \right). \tag{3.26} $$

We then see that string functionals with $dx^i/d\sigma \neq 0$ cannot contribute: as we saw in the previous subsection, the physics is t-independent, therefore we can take $t \to \infty$. In this limit the functional gets infinitely massive and decouples from the spectrum, unless $dx^i/d\sigma = 0$. Therefore, the map $x : I \to X$ has to be constant and in particular it must be a point in X. A similar analysis holds for the Grassmann fields as well. Since $\theta = 0$ at the boundary, it follows that string functionals are functions of the commuting zero modes x^i and $\eta^{\bar{I}}$, and can be written as

$$ \Psi = A^{(0)}(x) + \sum_{p \geq 1} \eta^{\bar{I}_1} \cdots \eta^{\bar{I}_p} A^{(p)}_{\bar{I}_1 \cdots \bar{I}_p}(x). \tag{3.27} $$

These functionals can be interpreted as a sum of $(0, p)$-forms on X. If we have N D-branes wrapping X, these forms will be valued in $\mathrm{End}(E)$ (where E is a holomorphic $U(N)$ bundle). The $\mathcal{Q}$ symmetry acts as on these functionals as the Dolbeault operator $\bar{\partial}$ with values in $\mathrm{End}(E)$. Notice that a differential form of degree p will have ghost number p.

We are now ready to write the string field action for topological open type B strings on X with N spacetime filling branes. We have seen that the relevant string functionals are of the form (3.27). Since in string field theory the string field has ghost number one, we must have

$$\Psi = \eta^T A_{\overline{I}}(x), \tag{3.28}$$

where $A_{\overline{I}}(x)$ is a $(0,1)$-form taking values in the endomorphisms of some holomorphic vector bundle E. In other words, the string field is just the $(0,1)$ piece of a gauge connection on E. Since the string field only depends on commuting and anticommuting zero modes, the star product becomes the wedge products of forms in $\Omega^{(0,p)}(\mathrm{End}(E))$, and the integration of string functionals becomes ordinary integration of forms on X wedged by Ω. We then have the following dictionary:

$$\Psi \to A, \qquad \mathcal{Q}_{\mathrm{BRST}} \to \overline{\partial}$$

$$\star \to \wedge, \qquad \int \to \int_X \Omega \wedge\,. \tag{3.29}$$

The string field action (3.25) is then given by

$$S = \frac{1}{2g_s} \int_X \Omega \wedge \mathrm{Tr}\left(A \wedge \overline{\partial} A + \frac{2}{3} A \wedge A \wedge A \right). \tag{3.30}$$

This is the so-called *holomorphic Chern-Simons action*. It is a rather peculiar quantum field theory in six dimensions, but as we will see, when we consider D-branes of lower dimension, we will be able to obtain from (3.30) more conventional theories by dimensional reduction.

3.3 Topological strings and matrix models

We have seen that the spacetime description of the open B model with spacetime filling branes reduces to a six-dimensional theory (3.30). We will see now that, in some circumstances, this theory simplifies drastically and reduces to a matrix model.

In order to simplify the spacetime description one should study simple Calabi-Yau manifolds. The simplest example of a local Calabi-Yau threefold is a Riemann surface together with an appropriate bundle over it. The motivation for considering this kind of models is the following. Consider a Riemann surface Σ_g holomorphically embedded inside a Calabi-Yau threefold X, and let us consider the holomorphic tangent bundle of X restricted to Σ_g. We then have

$$TX|_{\Sigma_g} = T\Sigma_g \oplus N_{\Sigma_g} \tag{3.31}$$

where N_{Σ_g} is a holomorphic rank two complex vector bundle over Σ_g, called the normal bundle of Σ_g, and the CY condition $c_1(X) = 0$ gives

$$c_1(N_{\Sigma_g}) = 2g - 2. \tag{3.32}$$

The Calabi-Yau X "near Σ_g" looks precisely like the total space of the bundle

$$N \to \Sigma_g \tag{3.33}$$

where N is regarded here as a bundle over Σ_g satisfying (3.32). The space (3.33) is an example of a *local* Calabi-Yau threefold, and it is noncompact.

When $g = 0$ and $\Sigma_g = \mathbb{P}^1$ it is possible to be more precise about the bundle N. A theorem due to Grothendieck says that any holomorphic bundle over $\mathbb{P}^1$ splits into a direct sum of line bundles (for a proof, see for example [36], pp. 516-7). Line bundles over $\mathbb{P}^1$ are all of the form $\mathcal{O}(n)$, where $n \in \mathbf{Z}$. The bundle $\mathcal{O}(n)$ can be easily described in terms of two charts on $\mathbb{P}^1$: the north pole chart, with coordinates z, Φ for the base and the fiber, respectively, and the south pole chart, with coordinates z', Φ'. The change of coordinates is given by

$$z' = 1/z, \quad \Phi' = z^{-n}\Phi. \tag{3.34}$$

We also have that $c_1(\mathcal{O}(n)) = n$. We then find that local Calabi-Yau manifolds that are made out of a two-sphere together with a bundle over it are all of the form

$$\mathcal{O}(-a) \oplus \mathcal{O}(a - 2) \to \mathbb{P}^1, \tag{3.35}$$

since the degree of the bundles have to sum up to -2 due to (3.32).

Let us now consider the string field theory of type B open topological strings on the Calabi-Yau manifold (3.35). We will consider a situation where we have Dirichlet boundary conditions associated to $\mathbb{P}^1$, in other words, there are N topological D-branes wrapping $\mathbb{P}^1$. Since the normal directions to the D-brane worldvolume are noncompact, the spacetime description can be obtained by considering the dimensional reduction of the original string field theory action (3.30). As usual in D-brane physics, the gauge potential A splits into a gauge potential on the worldvolume of the brane and Higgs fields describing the motion along the noncompact, transverse directions. In a nontrivial geometric situation like the one here, the Higgs fields are sections of the normal bundle. We then get three different fields:

$$A, \quad \Phi_0, \quad \Phi_1, \tag{3.36}$$

where A is a $U(N)$ $(0,1)$ gauge potential on $\mathbb{P}^1$, Φ_0 is a section of $\mathcal{O}(-a)$, and Φ_1 is a section of $\mathcal{O}(a - 2)$. Both fields, Φ_0 and Φ_1, take values in the adjoint representation

of $U(N)$. It is easy to see that the action (3.30) becomes

$$S = \frac{1}{g_s} \int_{\mathbf{P}^1} \mathrm{Tr}\left(\Phi_0 \overline{D}_A \Phi_1\right), \tag{3.37}$$

where $\overline{D}_A = \overline{\partial} + [A, \cdot]$ is the antiholomorphic covariant derivate. Notice that this theory is essentially a gauged $\beta\gamma$ system, since Φ_0, Φ_1 are quasiprimary conformal fields of dimensions $a/2$, $1 - a/2$, respectively.

We will now consider a more complicated geometry. We start with the Calabi-Yau manifold (3.35) with $a = 0$, *i.e.*

$$\mathcal{O}(0) \oplus \mathcal{O}(-2) \to \mathbf{P}^1. \tag{3.38}$$

In this case, Φ_0 is a scalar field on $\mathbf{P}^1$, while Φ_1 is a $(1,0)$ form (since $K_{\mathbf{P}^1} = \mathcal{O}(-2)$). If we cover $\mathbf{P}^1$ with two patches with local coordinates z, z' related by $z' = 1/z$, the fields in the two different patches, Φ_0, Φ_1, and Φ_0', Φ_1' will be related by

$$\Phi_0' = \Phi_0, \quad \Phi_1' = z^2 \Phi_1. \tag{3.39}$$

We can regard this geometry as a *family* of $\mathbf{P}^1$s located at $\Phi_1 = 0$ (the zero section of the nontrivial line bundle $\mathcal{O}(-2)$) parametrized by $\Phi_0 = \Phi_0' = x \in \mathbf{C}$. The idea is to obtain a geometry where we get n isolated $\mathbf{P}^1$s at fixed positions of x. To do that, we introduce an arbitrary polynomial of degree $n + 1$ on Φ_0, $W(\Phi_0)$, and we modify the gluing rules above as follows [15]:

$$z' = 1/z, \quad \Phi_0' = \Phi_0, \quad \Phi_1' = z^2 \Phi_1 + W'(\Phi_0)z. \tag{3.40}$$

Before, the $\mathbf{P}^1$ was in a family parameterized by $\Phi_0 \in \mathbf{C}$. Now, we see that there are n isolated $\mathbf{P}^1$s located at fixed positions of Φ_0 given by $W'(\Phi_0) = 0$, since this is the only way to have $\Phi_1 = \Phi_1' = 0$.

The geometry obtained by imposing the gluing rules (3.40) can be interpreted in yet another way. Call $\Phi_0 = x$ and define the coordinates

$$u = 2\Phi_1', \quad v = 2\Phi_1, \quad y = i(2z'\Phi_1' - W'(x)). \tag{3.41}$$

The last equation in (3.40) can now be written as

$$uv + y^2 + W'(x)^2 = 0. \tag{3.42}$$

This is a singular geometry, since there are singularities along the line $u = v = y = 0$ for every x_* such that $W'(x_*) = 0$. For example, if $W'(x) = x$, (3.42) becomes, after writing $u, v \to u - iv, u + iv$

$$u^2 + v^2 + x^2 + y^2 = 0. \tag{3.43}$$

This Calabi-Yau manifold is called the *conifold*, and it is singular at the origin. For arbitrary polynomials $W(x)$, the equation (3.42) describes more general, singular Calabi-Yau manifolds. Notice that locally, around the singular points $u = v = y = 0$, $x = x_*$, the geometry described by (3.42) looks like a conifold (whenever $W''(x_*) = 0$). The manifold described by (3.40) is obtained after blowing up the singularities in (3.42), *i.e.* we modify the geometry by "inflating" a two-sphere $\mathbb{P}^1$ at each singularity. This process is called resolution of singularities in algebraic geometry, and for this reason we will call the manifold specified by (3.40) the *resolved manifold* X_{res}.

We can now consider the dynamics of open type B topological strings on X_{res}. We will consider a situation in which we have in total N D-branes in such a way that N_i D-branes are wrapped around the i-th $\mathbb{P}^1$, with $i = 1, \cdots, n$. As before, we have three fields in the adjoint representation of $U(N)$, Φ_0, Φ_1 and the gauge connection A. The action describing the dynamics of the D-branes turns out to be given by

$$S = \frac{1}{g_s} \int_{\mathbb{P}^1} \mathrm{Tr} \left(\Phi_1 \overline{D}_A \Phi_0 + \omega W(\Phi_0) \right) \tag{3.44}$$

where ω is a Kähler form on $\mathbb{P}^1$ with unit volume. This action was derived in [43, 27]. A quick way to see that the modification of the gluing rules due to adding the polynomial $W'(\Phi_0)$ leads to the extra term in (3.44) is to use standard techniques in CFT [27]. The fields Φ_0, Φ_1 are canonically conjugate and on the conformal plane they satisfy the OPE

$$\Phi_0(z)\Phi_1(w) \sim \frac{g_s}{z - w}. \tag{3.45}$$

Let us now regard the geometry described in (3.40) as two disks (or conformal planes) glued through a cylinder. Since we are in the cylinder, we can absorb the factors of z in the last equation of (3.40). The operator that implements the transformation of Φ is

$$U = \exp \frac{1}{g_s} \oint \mathrm{Tr}\, W(\Phi_0(z))\, dz, \tag{3.46}$$

since from (3.45) it is easy to obtain

$$\Phi_1' = U\Phi_1 U^{-1}. \tag{3.47}$$

We can also write

$$U = \exp \frac{1}{g_s} \int_{\mathbb{P}^1} \mathrm{Tr}\, W(\Phi_0(z))\omega \tag{3.48}$$

where ω is localized to a band around the equator of $\mathbb{P}^1$ (as we will see immediately, the details of ω are unimportant, as long as it integrates to 1 on the two-sphere).

One easy check of the above action is that the equations of motion lead to the geometric picture of D-branes wrapping n holomorphic $\mathbb{P}^1$s in the geometry. The gauge connection is just a Lagrange multiplier enforcing the condition

$$[\Phi_0, \Phi_1] = 0, \tag{3.49}$$

therefore we can diagonalize Φ_0 and Φ_1 simultaneously. The equation of motion for Φ_0 is simply

$$\overline{\partial}\Phi_0 = 0, \tag{3.50}$$

and since we are on $\mathbb{P}^1$, we have that Φ_0 is a constant, diagonal matrix. Finally, the equation of motion for Φ_1 is

$$\overline{\partial}\Phi_1 = W'(\Phi_0)\omega, \tag{3.51}$$

and for nonsingular Φ_1 configurations both sides of the equation must vanish simultaneously, as we can see by integrating both sides of the equation over $\mathbb{P}^1$. Therefore, $\Phi_1 = 0$ and the constant eigenvalues of Φ_0 satisfy

$$W'(\Phi_0) = 0 \tag{3.52}$$

i.e. they must be located at the critical points of $W(x)$. In general, we will have N_i eigenvalues of Φ_0 at the i-th critical point, $i = 1, \cdots, n$, and this is precisely the D-brane configuration we are considering.

What happens in the quantum theory? In order to analyze it, we will use the approach developed in [10] for the analysis of two-dimensional gauge theories[2]. First of all, we choose the maximally Abelian gauge for Φ_0, *i.e.* we write

$$\Phi_0 = \Phi_0^{\mathbf{k}} + \Phi_0^{\mathbf{t}}, \tag{3.53}$$

where $\Phi_0^{\mathbf{t}}$ is the projection on the Cartan subalgebra $\mathbf{t}$, and $\Phi_0^{\mathbf{k}}$ is the projection on the complementary part $\mathbf{k}$. The maximally Abelian gauge is defined by the condition

$$\Phi_0^{\mathbf{k}} = 0 \tag{3.54}$$

which means that the nondiagonal entries of Φ_0 are gauge-fixed to be zero. This is in fact the same gauge that we used before to write the matrix model in the eigenvalue basis. After fixing the gauge the usual Faddeev-Popov techniques lead to a ghost functional determinant given by

$$\frac{1}{N!}\mathrm{Det}_{\mathbf{k}}(\mathrm{ad}(\Phi_0^{\mathbf{t}}))_{\Omega^0(\mathbf{P}^1)} \tag{3.55}$$

---

[2]I'm grateful to George Thompson for very useful remarks on this derivation.

where the subscript $\mathbf{k}$ means that the operator $\Phi_0^{\mathbf{t}}$ acts on the space $\mathbf{k}$, and the normalization factor $1/N!$ is the inverse of the order of the residual symmetry group, namely the Weyl group which permutes the N entries of Φ_0^t. The integrand of (3.44) reads, after gauge fixing,

$$\mathrm{Tr}\left(\Phi_1^{\mathbf{t}}\overline{\partial}\Phi_0^{\mathbf{t}} + W(\Phi_0^{\mathbf{t}})\right) + 2\sum_\alpha A^\alpha \Phi_1^{-\alpha}\alpha(\Phi_0^{\mathbf{t}}), \tag{3.56}$$

where α are roots, E_α is a basis of $\mathbf{k}$, and we have expanded $\Phi_1^{\mathbf{k}} = \sum_\alpha \Phi_1^\alpha E_\alpha$ as well as $A^{\mathbf{k}}$. We can now integrate out the A^α to obtain

$$\frac{1}{\mathrm{Det}_{\mathbf{k}}(\mathrm{ad}(\Phi_0^{\mathbf{t}}))_{\Omega^{1,0}(\mathbb{P}^1)}} \prod_{\alpha>0} \delta(\Phi_1^\alpha). \tag{3.57}$$

Here we have used the functional generalization of the standard formula $\delta(ax) = |a|^{-1}\delta(x)$. We can now trivially integrate over $\Phi_1^{\mathbf{k}}$. The inverse determinant in (3.57) combines with (3.55) to produce

$$\frac{\mathrm{Det}_{\mathbf{k}}(\mathrm{ad}(\Phi_0^{\mathbf{t}}))_{H^0(\mathbb{P}^1)}}{\mathrm{Det}_{\mathbf{k}}(\mathrm{ad}(\Phi_0^{\mathbf{t}}))_{H^{1,0}(\mathbb{P}^1)}} \tag{3.58}$$

where (as usual) nonzero modes cancel (since they are paired by ∂) and one ends with the determinants evaluated at the cohomologies. Similarly, integrating out $\Phi_1^{\mathbf{t}}$ in (3.56) leads to $\overline{\partial}\Phi_0^{\mathbf{t}} = 0$, therefore $\Phi_0^{\mathbf{t}}$ must be constant. The quotient of determinants is easy to evaluate in this case, and one finds

$$\left[\prod_{i<j}(\lambda_i - \lambda_j)^2\right]^{h^0(\mathbb{P}^1)-h^{1,0}(\mathbb{P}^1)}, \tag{3.59}$$

where λ_i are the constant eigenvalues of $\Phi_0^{\mathbf{t}}$. Since $h^0(\mathbb{P}^1) = 1$, $h^{1,0}(\mathbb{P}^1) = 0$, we just get the square of the Vandermonde determinant and the partition function reads:

$$Z = \frac{1}{N!} \int \prod_{i=1}^N d\lambda_i \prod_{i<j}(\lambda_i - \lambda_j)^2 \, e^{-\frac{1}{g_s}\sum_{i=1}^N W(\lambda_i)}. \tag{3.60}$$

In principle, as explained in [10], one has to include a sum over nontrivial topological sectors of the Abelian gauge field $A^{\mathbf{t}}$ in order to implement the gauge fixing (3.54) correctly. Fortunately, in this case the gauge-fixed action does not depend on $A^{\mathbf{t}}$, and the inclusion of topological sectors is irrelevant. The expression (3.60) is (up to a factor $(2\pi)^N$) the gauge-fixed version of the matrix model

$$Z = \frac{1}{\mathrm{vol}(U(N))} \int \mathcal{D}\Phi \, e^{-\frac{1}{g_s}\mathrm{Tr}\,W(\Phi)} \tag{3.61}$$

We have then derived a surprising result due to Dijkgraaf and Vafa [27]: the string field theory action for open topological B strings on the Calabi-Yau manifold described by (3.40) is a matrix model with potential $W(\Phi)$.

3.4   Open string amplitudes and multicut solutions

The total free energy $F(N_i, g_s)$ of topological B strings on the Calabi-Yau (3.40) in the background of $N = \sum_i N_i$ branes wrapped around n $\mathbb{P}^1$'s is of the form (1.3), and as we have just seen it is given by the free energy of the matrix model (3.61). In particular, the coefficients $F_{g,h_1,\cdots,h_n}$ can be computed perturbatively in the matrix model. We have to be careful however to specify the classical vacua around which we are doing perturbation theory. Remember from the analysis of the matrix model that the classical solution which describes the brane configuration is characterized by having N_i eigenvalues of the matrix located at the i-th critical point of the potential $W(x)$. In the saddle-point approximation, this means that we have to consider a *multicut* solution, with eigenvalues "condensed" around *all* the extrema of the potential. Therefore, in contrast to the multicut solution discussed in 2.2, we have that (1) all critical points of $W(x)$ have to be considered, and not only the minima, and (2) the number of eigenvalues in each cut is not determined dynamically as in (2.83), but it is rather fixed to be N_i in the i-th cut. In other words, the integral of the density of eigenvalues $\rho(\lambda)$ along each cut equals a *fixed* filling fraction $\nu_i = N_i/N$:

$$\int_{x_{2i}}^{x_{2i-1}} d\lambda\, \rho(\lambda) = \nu_i, \qquad (3.62)$$

where $N = \sum_{i=1}^{n} N_i$ is the total number of eigenvalues. Let us introduce the partial 't Hooft couplings

$$t_i = g_s N_i = t\nu_i. \qquad (3.63)$$

Taking into account (2.58) and (2.68), we can write (3.62) as

$$t_i = \frac{1}{4\pi i} \oint_{A_i} y(\lambda) d\lambda, \quad i = 1, \cdots, n, \qquad (3.64)$$

where A_i is the closed cycle of the hyperelliptic curve (2.70) which surrounds the cut $\mathcal{C}_i$. Assuming for simplicity that all the t_i are different from zero, and taking into account that $\sum_i t_i = t$, we see that (3.64) gives $n - 1$ independent conditions, where n is the number of critical points of $W(x)$. These conditions, together with (2.81), determine the positions of the endpoints x_i as functions of the t_i and the coupling constants in $W(x)$. It is clear that the solution obtained in this way is not an equilibrium solution of the matrix model, since cuts can be centered around local maxima and different cuts will have different values of the effective potential. This is not surprising, since we are not considering the matrix model as a quantum mechanical system *per se*, but as an effective description of the original brane system. The different choices of filling fractions correspond to different choices of classical vacua for the brane system.

A subtle issue concerning the above matrix model is the following. The matrix field Φ in (3.61) comes from the B model field Φ_0, which is a holomorphic field. Therefore, the matrix integral (3.60) should be understood as a contour integral, and in order to define the theory a choice of contour should be made. This can be done in perturbation theory, by choosing for example a contour that leads to the usual results for Gaussian integration, and therefore at this level the matrix model is not different from the usual Hermitian matrix model [27, 75]. In some cases, however, regarding (3.61) as a holomorphic matrix model can be clarifying, see [51] for an exhaustive discussion.

The above description of the multicut solution refers to the saddle-point approximation. What is the meaning of the multicut solutions from the point of view of perturbation theory? To address this issue, let us consider for simplicity the case of the cubic potential:

$$\frac{1}{g_s} W(\Phi) = \frac{1}{2g_s} \text{Tr}\, \Phi^2 + \frac{1}{3} \frac{\beta}{g_s} \text{Tr}\, \Phi^3. \tag{3.65}$$

This potential has two critical points, $a_1 = 0$ and $a_2 = -1/\beta$. The most general multicut solution will have two cuts. There will N_1 eigenvalues sitting at $\Phi = 0$, and N_2 eigenvalues sitting at $\Phi = -1/\beta$. The partition function Z of the matrix model is:

$$Z = \frac{1}{N!} \int \prod_{i=1}^{N} \frac{d\lambda_i}{2\pi} \Delta^2(\lambda) e^{-\frac{1}{2g_s} \sum_i \lambda_i^2 - \frac{\beta}{3g_s} \sum_i \lambda_i^3}, \tag{3.66}$$

where $\Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j)$ is the Vandermonde determinant. We can now expand the integrand around the vacuum with $\lambda_i = 0$ for $i = 1, \ldots, N_1$ and $\lambda_i = -\frac{1}{\beta}$ for $i = N_1 + 1, \ldots, N$. Denoting the fluctuations by μ_i and ν_j, the Vandermonde determinant becomes

$$\Delta^2(\lambda) = \prod_{1 \le i_1 < i_2 \le N_1} (\mu_{i_1} - \mu_{i_2})^2 \prod_{1 \le j_1 < j_2 \le N_2} (\nu_{j_1} - \nu_{j_2})^2 \prod_{\substack{1 \le i \le N_1 \\ 1 \le j \le N_2}} \left(\mu_i - \nu_j + \frac{1}{\beta}\right)^2. \tag{3.67}$$

We also expand the potential around this vacuum and get

$$W = \sum_{i=1}^{N_1} \left(\frac{1}{2g_s} \mu_i^2 + \frac{\beta}{3g_s} \mu_i^3\right) - \sum_{i=1}^{N_2} \left(\frac{m}{2g_s} \nu_i^2 - \frac{\beta}{3g_s} \nu_i^3\right) + \frac{1}{6\beta^2 g_s} N_2. \tag{3.68}$$

Notice that the propagator of the fluctuations around $-1/\beta$ has the 'wrong' sign, since we are expanding around a local maximum. The interaction between the two sets of eigenvalues, which is given by the last factor in (3.67), can be exponentiated and included in the action. This generates an interaction term between the two eigenvalue bands

$$W_{\text{int}} = 2N_1 N_2 \log \beta + 2 \sum_{k=1}^{\infty} \frac{1}{k} \beta^k \sum_{i,j} \sum_{p=0}^{k} (-1)^p \binom{k}{p} \mu_i^p \nu_j^{k-p}. \tag{3.69}$$

By rewriting the partition function in terms of matrices instead of their eigenvalues, we can represent this model as an effective two-matrix model, involving an $N_1 \times N_1$ matrix Φ_1, and an $N_2 \times N_2$ matrix Φ_2:

$$Z = \frac{1}{\text{Vol}(U(N_1)) \times \text{Vol}(U(N_2))} \int D\Phi_1 D\Phi_2 e^{-W_1(\Phi_1)-W_2(\Phi_2)-W(\Phi_1,\Phi_2)}, \qquad (3.70)$$

where

$$
\begin{aligned}
W_1(\Phi_1) &= +\text{Tr}\left(\frac{1}{2g_s}\Phi_1^2 + \frac{\beta}{3g_s}\Phi_1^3\right), \\
W_2(\Phi_2) &= -\text{Tr}\left(\frac{1}{2g_s}\Phi_2^2 - \frac{\beta}{3g_s}\Phi_2^3\right), \\
W_{\text{int}}(\Phi_1, \Phi_2) &= 2\sum_{k=1}^{\infty} \frac{\beta^k}{k} \sum_{p=0}^{k}(-1)^p \binom{k}{p} \text{Tr}\,\Phi_1^p\,\text{Tr}\,\Phi_2^{k-p} \\
&\quad + N_2 W(a_2) + N_1 W(a_1) - 2N_1 N_2 \ln \beta. \qquad (3.71)
\end{aligned}
$$

Here, $\text{Tr}\,\Phi_1^0 = N_1$, $\text{Tr}\,\Phi_2^0 = N_2$, $W(a_1) = 0$ and $W(a_2) = 1/(6g_s\beta^2)$. Although the kinetic term for Φ_2 has the 'wrong' sign, we can still make sense of the model in perturbation theory by using formal Gaussian integration, and this can in fact be justified in the framework of holomorphic matrix models [51]. Therefore, the two-cut solution of the cubic matrix model can be formally represented in terms of an effective two-matrix model. It is now straightforward to compute the free energy $F_{\text{pert}} = \log\big(Z(\beta)/Z(\beta=0)\big)$ in perturbation theory. It can be expanded as

$$F_{\text{pert}} = -N_1 W(a_1) - N_2 W(a_2) - 2N_1 N_2 \ln \beta + \sum_{h=1}^{\infty}\sum_{g\geq 0}(g_s\beta^2)^{2g-2+h} F_{g,h}(N_1, N_2) \quad (3.72)$$

where $F_{g,h}$ is a homogeneous polynomial in N_1 and N_2 of degree h. One finds, up to fourth order in the coupling constant β, the following result [45]:

$$
\begin{aligned}
F_{\text{pert}} = & -N_1 W(a_1) - N_2 W(a_2) - 2N_1 N_2 \ln \beta \\
+ \ & g_s\beta^2 \left[\left(\frac{2}{3}N_1^3 - 5N_1^2 N_2 + 5N_1 N_2^2 - \frac{2}{3}N_2^3\right) + \frac{1}{6}(N_1 - N_2)\right] \\
+ \ & g_s^2\beta^4 \left[\left(\frac{8}{3}N_1^4 - \frac{91}{3}N_1^3 N_2 + 59N_1^2 N_2^2 - \frac{91}{3}N_1 N_2^3 + \frac{8}{3}N_2^4\right) + \left(\frac{7}{3}N_1^2 - \frac{31}{3}N_1 N_2 + \frac{7}{3}N_2^2\right)\right] \\
+ \ & \cdots
\end{aligned}
$$

$$(3.73)$$

From this explicit perturbative computation one can read off the first few coefficients F_{g,h_1,h_2}. Of course, this procedure can be generalized, and the n-cut solution can

be represented by an effective n matrix model with interactions among the different matrices that come from the expansion of the Vandermonde determinant. These interactions can be also incorporated in terms of ghost fields, as explained in [24]. This makes possible to compute corrections to the saddle-point approximation in perturbation theory. One can also use the multicut solution to the loop equations [4, 47] with minor modifications to compute the genus one correction in closed form [45, 26, 18].

3.5 Master field and geometric transition

We have seen that the open topological string amplitudes on the Calabi-Yau manifold X_{res} are computed by a multicut matrix model whose planar solution (or, equivalently, its master field configuration) is given by a hyperelliptic curve

$$y^2 = W'(x)^2 - R(x). \tag{3.74}$$

Moreover, we also saw in (3.64) that the partial 't Hooft couplings can be understood as integrals around the A_i cycles of this curve, with $i = 1, \cdots, n$. Let us now compute the variation of the free energy $F_0(t_i)$ when we vary t_i. The variation w.r.t. t_i (keeping the t_j, $j \neq i$, fixed) can be obtained by computing the variation in the free energy as we move one eigenvalue from the cut $\mathcal{C}_i$ to infinity [27]. This variation is given by (minus) the integral of the force exerted on an eigenvalue, as we move it from the endpoint of the cut to infinity. The path from the endpoint of $\mathcal{C}_i$ to infinity, which does not intersect the other cuts $\mathcal{C}_j$, will be denoted by B_i. Taking into account (2.69), and the fact that $y(p)$ has no discontinuities outside the cuts $\mathcal{C}_j$, we find

$$\frac{\partial F_0}{\partial t_i} = \int_{B_i} y(x) dx. \tag{3.75}$$

Usually this integral is divergent, but can be easily regularized by taking B_i to run up to a cutoff point $x = \Lambda$, and subtracting the divergent pieces as the cutoff Λ goes to infinity. For example, for the Gaussian matrix model one has

$$\frac{\partial F_0}{\partial t} = \int_{2\sqrt{t}}^{\Lambda} dx \sqrt{x^2 - 4t} = t(\log t - 1) - 2t \log \Lambda + \frac{1}{2}\Lambda^2 + \mathcal{O}(1/\Lambda^2). \tag{3.76}$$

Therefore, the regularized integral gives $t(\log t - 1)$, which is indeed the right result. It is now clear that (3.64) and (3.75) look very much like the relations (3.14) that define the periods (therefore the prepotential) in special geometry. What is the interpretation of the appearance of special geometry?

Recall that our starting point was a Calabi-Yau geometry obtained as a blowup of the singularity given in (3.42). However, there is another way of smoothing out singularities

in algebraic geometry, which is by *deforming* them rather than by resolving then. For example, the conifold singularity given in (3.43) can be smoothed out by deforming the geometry to

$$x^2 + y^2 + u^2 + v^2 = \mu. \tag{3.77}$$

This is the so called *deformed conifold*. Geometrically, turning on μ corresponds to inflating a three-sphere in the geometry, since the real section of the conifold is indeed an $\mathbf{S}^3$. As $\mu \to 0$, the three-sphere collapses to zero size, so we can interpret the singularity as arising from a collapsing three-cycle in the geometry. In the more general singularity (3.42), the generic deformation requires turning on a generic polynomial of degree $n - 1$ $R(x)$, and we get the Calabi-Yau manifold

$$u^2 + v^2 + y^2 + W'(x)^2 = R(x). \tag{3.78}$$

We will call this geometry the *deformed manifold* X_{def}. The deformation by $R(x)$ introduces in fact n three-spheres in the geometry, one for each singularity (recall that each of the singular points in (3.42) is locally like the conifold). The noncompact Calabi-Yau manifold (3.78) has a holomorphic three-form:

$$\Omega = \frac{1}{2\pi} \frac{dxdydu}{v} \tag{3.79}$$

The three-spheres created by the deformation can be regarded as two-spheres fibered over an interval in the complex x-plane. To see this, let us consider for simplicity the case of the deformed conifold (3.77), with μ real. This geometry contains a three-sphere which is given by the restriction of (3.77) to real values of the variables. If we now consider a fixed, real value of x in the interval $-\sqrt{\mu} < x < \sqrt{\mu}$, we get of course a two-sphere of radius $\sqrt{\mu - x^2}$. The sphere collapses at the endpoints of the interval, $x = \pm\sqrt{\mu}$, and the total geometry of the two-sphere together with the interval $[-\sqrt{\mu}, \sqrt{\mu}]$ is a three-sphere. In the more general case, the curve $W'(x)^2 - R(x)$ has n cuts with endpoints x_{2i}, x_{2i-1}, $i = 1, \cdots, n$, and the n three-spheres are $\mathbf{S}^2$ fibrations over these cuts.

Let us now consider *closed* type B topological strings propagating on X_{def}. As we saw in 3.1, the genus zero theory is determined by the periods of the three-form Ω given in (3.79). We then choose a symplectic basis of three-cycles $\widehat{A}_i$, $\widehat{B}^j$, with $\widehat{A}_i \cap \widehat{B}^j = \delta_i^j$. Here, the $\widehat{A}_i$ cycles are the n three-spheres, and they project to cycles A_i surrounding the cut $\mathcal{C}_i = [x_{2i}, x_{2i-1}]$ in the x-plane. The $\widehat{B}_i$ cycles are dual cycles which project in the x plane to the B_i paths [15]. The periods of Ω are then given by

$$t_i = \frac{1}{4\pi} \oint_{\widehat{A}_i} \Omega, \quad \frac{\partial F_0}{\partial t_i} = \int_{\widehat{B}^i} \Omega. \tag{3.80}$$

It is easy to see that these periods reduce to the periods (3.63) and (3.75) on the hyperelliptic curve (3.74), respectively. Let us consider again the case of the deformed conifold (3.77), which is simpler since there is only one three-sphere. Let us compute the A-period over this three-sphere, which is an $\mathbf{S}^2$ fibration over the cut $[-\sqrt{\mu}, \sqrt{\mu}]$, by first doing the integral over $\mathbf{S}^2$, and then doing the integral over the cut. Since $v = \sqrt{\mu - x^2 - \rho^2}$, where $\rho^2 = y^2 + u^2$, the integral of Ω over $\mathbf{S}^2$ is simply

$$\frac{1}{2\pi} \int_{\mathbf{S}^2} \frac{dy\,dz}{\sqrt{\mu - x^2 - \rho^2}} = \sqrt{\mu - x^2}. \tag{3.81}$$

Therefore, the A-period becomes

$$t = \frac{1}{2\pi} \int_{-\sqrt{\mu}}^{\sqrt{\mu}} y(x)\,dx, \tag{3.82}$$

where y is now given by $y^2 + x^2 = \mu$. This is nothing but the A-period (3.63) (up to a redefinition $y \to -iy$). The general case is very similar, and one finally obtains that the special geometry (3.80) of the deformed Calabi-Yau geometry (3.78) is equivalent to the planar solution of the matrix model, given by the hyperelliptic curve (3.74) and the equations for the partial 't Hooft couplings (3.64) and the planar free energy (3.75).

The physical interpretation of this result is that there is an equivalence between an *open* topological string theory on the manifold X_{res}, with N D-branes wrapping the n spheres obtained by blowup, and a *closed* topological string theory on the manifold X_{def}, where the N D-branes have disappeared. Moreover, the 't Hooft couplings t_i in the open string theory become geometric periods in the closed string theory. Since the open topological strings on X_{res} are described by a matrix model, the fact that the planar solution reproduces very precisely the deformed geometry is important evidence for this interpretation. This duality relating an open and a closed string theory is an example of a *geometric*, or *large N*, transition. Notice that, as a consequence of this duality, the 't Hooft resummation of the matrix model corresponds to a closed string theory propagating on X_{def}. The master field controlling the planar limit (which is encoded in the planar resolvent, or equivalently in the quantity $y(\lambda)$) leads to an algebraic equation that describes very precisely the *target* of the closed string theory dual. The large N transition between these two geometries was proposed in [15]. The fact that the open string side can be described by a matrix model was discovered in [27].

3.6 Extensions and applications

The results derived above can be extended to more complicated Calabi-Yau backgrounds with branes [28, 29]. For example, one can consider ADE type geometries with

branes wrapping two-spheres [16, 14], and the string field theory description reduces to the ADE matrix models considered in [46]. In the one-matrix model described before, the master field is given by a hyperelliptic curve $F(x, y) = 0$ which is then regarded as the Calabi-Yau manifold

$$uv + F(x, y) = 0 \qquad (3.83)$$

in disguise. In some of the examples considered in [28, 29], however, the master field is no longer described by a hyperelliptic curve, but involves a more complicated geometry. This geometry is the Calabi-Yau closed string background that is obtained by geometric transition from the open string background with branes. A detailed study of the more complicated master field geometries that arise in multimatrix models can be found in [33].

Another consequence of the result of Dijkgraaf and Vafa, together with the geometric transition of [15], is that the Kodaira-Spencer theory of gravity [7] on the noncompact Calabi-Yau manifold (3.78) is equivalent to the 't Hooft resummation of the matrix model with potential $W(x)$. For the simple example of the cubic potential, this was explicitly checked at genus one in [45]. The formalism developed in [32] seems to be very appropriate to establish this equivalence in detail.

As we mentioned in the introduction, the main application of the results of Dijkgraaf and Vafa has been the computation of effective superpotentials in supersymmetric gauge theories by using matrix model techniques. This is based on the fact [7, 29] that the resummation $F_0(t)$ of the open string amplitudes is deeply related to the superpotential of the gauge theory which can be obtained from string backgrounds with branes. We refer the reader to [6, 62] for an exposition of these results.

4 Type A topological strings, Chern-Simons theory and matrix models

The conceptual structure of what we have seen in the B model is the following: first one shows, by using string field theory, that the target space description of open topological B strings reduces to a matrix model in certain backgrounds. Then one solves the model in the planar limit, and a geometry emerges which is interpreted as a closed string dual to the original open string theory. Both geometries are related by a large N transition. The first transition of this type was discovered in the context of topological A strings by Gopakumar and Vafa [35]. What we will do here is to rederive their result by using the language and technology of matrix models. The key ingredient is the fact pointed out in [53] that the partition function of Chern-Simons theory can be written in terms of a somewhat exotic matrix model. We will only focus on the matrix model aspects

of this correspondence. A detailed review of Chern-Simons theory and the geometric transition for the A model can be found in [54].

4.1 Solving the Chern-Simons matrix model

The Chern-Simons action with gauge group G on a generic three-manifold M is defined by

$$S = \frac{k}{4\pi} \int_M \text{Tr}\Big(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\Big) \tag{4.1}$$

Here, k is the coupling constant, and A is a G-gauge connection on the trivial bundle over M. We will consider Chern-Simons theory with gauge group $G = U(N)$. As noticed in [71], since the action (4.1) does not involve the metric, the resulting quantum theory is topological, at least formally. In particular, the partition function

$$Z(M) = \int [\mathcal{D}A] e^{iS} \tag{4.2}$$

should define a topological invariant of M. A detailed analysis shows that this is in fact the case, with an extra subtlety related to a choice of framing of the three-manifold.

The partition function of Chern-Simons theory can be computed in a variety of ways. In [71] it was shown that in fact the theory is exactly solvable by using nonperturbative methods and the relation to the Wess-Zumino-Witten (WZW) model. In particular, the partition function of the $U(N)$ theory on the three-sphere $\mathbf{S}^3$ is given by

$$Z(\mathbf{S}^3) = \frac{1}{(k+N)^{N/2}} \sum_{w \in \mathcal{W}} \epsilon(w) \exp\Big(-\frac{2\pi i}{k+N} \rho \cdot w(\rho)\Big), \tag{4.3}$$

where the sum over w is a sum over the elements of the Weyl group $\mathcal{W}$ of $U(N)$, $\epsilon(w)$ is the signature of w, and ρ is the Weyl vector of $SU(N)$. By using Weyl's denominator formula,

$$\sum_{w \in \mathcal{W}} \epsilon(w) e^{w(\rho) \cdot u} = \prod_{\alpha > 0} 2 \sinh \frac{\alpha \cdot u}{2}, \tag{4.4}$$

where α are positive roots, one finds

$$Z(\mathbf{S}^3) = \frac{1}{(k+N)^{N/2}} \prod_{\alpha > 0} 2 \sinh\Big(\frac{(\alpha \cdot \rho)}{2} g_s\Big) \tag{4.5}$$

where

$$g_s = \frac{2\pi i}{k+N}. \tag{4.6}$$

It was found by Witten that open topological type A strings on $T^* \mathbf{S}^3$ (which is nothing but the deformed conifold geometry (3.77)) in the presence of N D-branes wrapping

$\mathbf{S}^3$ are in fact described by $U(N)$ Chern-Simons theory on $\mathbf{S}^3$ [74]. This is the type A model analog to the fact that open type B strings on the geometry described by (3.40) are captured by a matrix model, and in both cases this is shown by using open string field theory. The free energy of Chern-Simons theory on $\mathbf{S}^3$ has an expansion of the form (1.2), with g_s given in (4.6), and the coefficients $F_{g,h}$, which can be computed by standard perturbation theory, have the interpretation of open string amplitudes on $T^*\mathbf{S}^3$.

The analogy between the A story and the B story can be taken even further, since it turns out that the partition function of Chern-Simons on $\mathbf{S}^3$, as well as on many other three-manifolds, can be represented as a matrix integral [53]. In the case of $\mathbf{S}^3$ most of the physical information in $Z(\mathbf{S}^3)$ can be obtained by other means, but for other three-manifolds like lens spaces and Seifert spaces, the matrix model representation is crucial in order to extract the coefficients $F_{g,h}$ [53]. The Chern-Simons matrix model on $\mathbf{S}^3$ gives however a particularly clean way to derive the resummed free energies $F_g(t)$ and the geometry of the master field, and we will devote the rest of these lectures to presenting this analysis.

In the case of $\mathbf{S}^3$ the easiest way to derive the matrix model representation of the Chern-Simons partition function is through direct computation. Consider the following integral:

$$Z_{CS} = \frac{e^{-\frac{g_s}{12}N(N^2-1)}}{N!} \int \prod_{i=1}^{N} \frac{d\beta_i}{2\pi} \, e^{-\sum_i \beta_i^2/2g_s} \prod_{i<j} \left(2\sinh\frac{\beta_i-\beta_j}{2}\right)^2. \qquad (4.7)$$

It can easily be seen that this reproduces the partition function of $U(N)$ Chern-Simons theory on $\mathbf{S}^3$, given in (4.5), and the derivation is left as an exercise.

Exercise. Use the Weyl formula (4.4) to write (4.7) as a Gaussian integral, and show that it reproduces (4.3).

The measure factor in (4.7)

$$\prod_{i<j} \left(2\sinh\frac{\beta_i-\beta_j}{2}\right)^2 \qquad (4.8)$$

is not the standard Vandermonde determinant, although it reduces to it for small separations among the eigenvalues. In fact, for very small g_s, the Gaussian potential in (4.7) will be very narrow, forcing the eigenvalues to be close to each other, and one can expand the sinh in (4.8) in power series. At leading order we find the usual Gaussian matrix model, while the corrections to it can be evaluated systematically by computing correlators in the Gaussian theory. In this way one obtains the perturbative expansion of Chern-Simons theory, see [53] for details.

Here we will take a slightly different route in order to analyze the model. First of all, we want to write the above integral as a standard matrix integral with the usual Vandermonde discriminant. This can be achieved with the change of variables [67]

$$\exp(\beta_i + t) = \lambda_i, \tag{4.9}$$

where $t = Ng_s$, as usual. It is easy to see that the above integral becomes, up to a factor $\exp(-N^3 g_s/2)$,

$$Z_{SW} = \frac{1}{N!} \int \prod_{i=1}^{N} \frac{d\lambda_i}{2\pi} \Delta^2(\lambda) \exp\left(-\sum_{i=1}^{N} (\log \lambda_i)^2/2g_s\right), \tag{4.10}$$

therefore we are considering the matrix model

$$Z_{SW} = \frac{1}{\text{vol}(U(N))} \int dM \, e^{-\frac{1}{2g_s} \text{Tr}\,(\log M)^2}. \tag{4.11}$$

We will call this model the *Stieltjes-Wigert matrix model*, hence the subscript in (4.10) and (4.11). This is because it can be exactly solved with the so-called Stieltjes-Wigert polynomials, as we will explain in a moment.

Matrix integrals with logarithmic potentials are somewhat exotic, but have appeared before in connection with the Penner model [61], with the $c = 1$ string at the self-dual radius [25, 41], and with the $\mathbb{P}^1$ model [31]. We want to analyze now the saddle-point approximation to the matrix integral (4.7), or equivalently to (4.10). Since the model in (4.10) has the standard Vandermonde, we can use the techniques of section 2.2. Although the formulae there were obtained for a polynomial potential, some of them generalize to arbitrary polynomials. In particular, to obtain the resolvent $\omega_0(p)$ we can use the formula (2.60) with

$$W'(z) = \frac{\log z}{z}. \tag{4.12}$$

Notice that this potential has a minimum at $z = 1$. We then expect a one-cut solution where the endpoints of the interval $a(t)$, $b(t)$ will satisfy $a(0) = b(0) = 1$. In order to compute the integral (2.60) we deform the integration contour. In the case of polynomial potentials, we picked a residue at $z = p$ and at infinity. Here, since the logarithm has a branch cut, we cannot push the contour to infinity. Instead, we deform the contour as indicated in Fig. 8: we pick the pole at $z = p$, and then we surround the cut of the logarithm along the negative real axis and the singularity at $z = 0$ with a small circle C_ϵ of radius ϵ. This kind of situation is typical of the solution of matrix models with the character expansion [44]. The resulting integrals are:

$$\frac{1}{2t}\left\{-\int_{-\infty}^{-\epsilon} \frac{dz}{z(z-p)\sqrt{(z-a)(z-b)}} + \oint_{C_\epsilon} \frac{dz \log z}{z(z-p)\sqrt{(z-a)(z-b)}}\right\}. \tag{4.13}$$
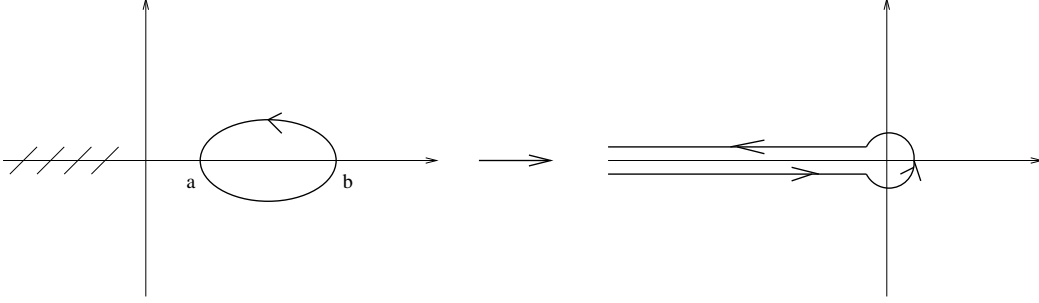
Figure 8: This shows the deformation of the contour needed to compute the planar resolvent of the Chern-Simons matrix integral. We pick a residue at $z = p$, and we have to encircle the singularity at the origin as well as the branch cut of the logarithm, which on the left hand side is represented by the dashed lines.

Both are singular as $\epsilon \to 0$, but singularities cancel, and after some computations one finds for the resolvent:

$$\omega_0(p) = -\frac{1}{2tp} \log \left[\frac{(\sqrt{a}\sqrt{p-b} - \sqrt{b}\sqrt{p-a})^2}{(\sqrt{p-a} - \sqrt{p-b})^2 p^2} \right] + \frac{\sqrt{(p-a)(p-b)}}{2tp\sqrt{ab}} \log \left[\frac{4ab}{2\sqrt{ab} + a + b} \right].$$
(4.14)

In order to satisfy the asymptotics (2.57) the second term must vanish, and the first one must go like $1/p$. This implies

$$
\begin{aligned}
4ab &= 2\sqrt{ab} + a + b, \\
\sqrt{a} + \sqrt{b} &= 2e^t,
\end{aligned}
$$
(4.15)

and from here we obtain the positions of the endpoints of the cut a, b as a function of the 't Hooft parameter:

$$
\begin{aligned}
a(t) &= 2e^{2t} - e^t + 2e^{\frac{3t}{2}}\sqrt{e^t - 1}, \\
b(t) &= 2e^{2t} - e^t - 2e^{\frac{3t}{2}}\sqrt{e^t - 1}.
\end{aligned}
$$
(4.16)

Notice that, for $t = 0$, $a(0) = b(0) = 1$, as expected. The final expression for the resolvent is then:

$$\omega_0(p) = -\frac{1}{tp} \log \left[\frac{1 + e^{-t}p + \sqrt{(1 + e^{-t}p)^2 - 4p}}{2p} \right],$$
(4.17)

and from here we can easily find the density of eigenvalues

$$\rho(\lambda) = \frac{1}{\pi t\lambda} \tan^{-1} \left[\frac{\sqrt{4\lambda - (1 + e^{-t}\lambda)^2}}{1 + e^{-t}\lambda} \right].$$
(4.18)

If we now define

$$u(p) = t(1 - p\omega_0(p)) + \pi i \qquad (4.19)$$

we see that it solves the equation

$$e^u + e^v + e^{v-u+t} + 1 = 0 \qquad (4.20)$$

where we put $p = e^{t-v}$. This was found in [2] by a similar analysis. The equation (4.20) is the analog of (3.74) in the case of polynomial matrix models, and can be regarded as an algebraic equation describing a noncompact Riemann surface. In fact, (4.20) is nothing but the *mirror* of the resolved conifold geometry (see for example [40, 1]), and t is the Kähler parameter of the geometry. This is of course in agreement with the result of [35], who argued that the 't Hooft resummation of Chern-Simons theory leads to a closed string theory propagating on the resolved conifold. As in the B model that we analyzed before, the master field of the matrix model encodes the information about the target geometry of the closed string description, and provides evidence for the geometric transition relating $T^*\mathbf{S}^3$ and the resolved conifold geometry.

As we mentioned before, the matrix model (4.11) can be solved exactly with a set of orthogonal polynomials called the Stieltjes-Wigert polynomials. The fact that the Chern-Simons matrix model is essentially equivalent to the Stieltjes-Wigert matrix model was pointed out by Tierz in [67]. The Stieltjes-Wigert polynomials are defined as follows [63]:

$$p_n(x) = (-1)^n q^{n^2 + \frac{n}{2}} \sum_{\nu=0}^{n} \begin{bmatrix} n \\ \nu \end{bmatrix} q^{\frac{\nu(\nu-n)}{2} - \nu^2} (-q^{-\frac{1}{2}} x)^{\nu} \qquad (4.21)$$

and satisfy the orthogonality condition (2.87) with

$$d\mu(x) = e^{-\frac{1}{2g_s}(\log x)^2} \frac{dx}{2\pi} \qquad (4.22)$$

and

$$h_n = q^{\frac{3}{4}n(n+1) + \frac{1}{2}} [n]! \left(\frac{g_s}{2\pi}\right)^{\frac{1}{2}},$$

where

$$q = e^{g_s}. \qquad (4.23)$$

In the above equations,

$$[n] = q^{\frac{n}{2}} - q^{-\frac{n}{2}}, \quad \begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]![n-m]!}. \qquad (4.24)$$

The recursion coefficients appearing in (2.92) are in this case

$$r_n = q^{3n}(q^n - 1), \qquad s_n = -q^{\frac{1}{2}+n}(q^{n+1} + q^n - 1).$$

54

The Stieltjes-Wigert ensemble can be regarded as a q-deformation (in the sense of quantum group theory) of the usual Gaussian ensemble. For example, as $g_s \to 0$ one has that $[n] \to ng_s$, therefore

$$h_n \to h_n^G, \tag{4.25}$$

where h_n^G is given in (2.96). Also, one can easily check that the normalized vev of $\mathrm{Tr}_R M$ in this ensemble is given by

$$\langle \mathrm{Tr}_R M \rangle_{SW} = e^{\frac{3t\ell(R)}{2}} q^{\frac{\kappa_R}{2}} \dim_q R, \tag{4.26}$$

where $\ell(R)$ is the number of boxes of R, κ_R is a quantity defined by

$$\kappa_R = \ell(R) + \sum_i \lambda_i(\lambda_i - 2i) \tag{4.27}$$

in terms of lengths of rows λ_i in R, and $\dim_q R$ is the quantum dimension of the representation R

$$\dim_q R = \prod_{\alpha > 0} \frac{[\alpha \cdot (\Lambda + \rho)]}{[\alpha \cdot \rho]} \tag{4.28}$$

where Λ is the highest weight associated to R. As $g_s \to 0$, the vev (4.26) becomes just $\dim R$, the classical dimension of R, which is essentially the vev in the Gaussian ensemble (2.8).

Notice that, for this set of orthogonal polynomials, the expansion (2.107) is very simple since

$$
\begin{aligned}
R_0(\xi) &= e^{4t\xi}(1 - e^{-t\xi}), \quad R_{2s}(\xi) = 0, \quad s > 0, \\
s(\xi) &= e^{t\xi}(1 - 2e^{t\xi}).
\end{aligned}
\tag{4.29}
$$

As we pointed out in section 2.3, $R_0(\xi)$ and $s(\xi)$ can be used to determine the endpoints of the cut in the resolvent through (2.111). It is easy to see that (4.29) indeed lead to (4.16), and that by using (2.110) one obtains (4.18). In fact, it is well-known that the expression (4.18) is the density of zeroes of the Stieltjes-Wigert polynomials [48, 19].

We can now use the technology developed in section 2.3 to compute $\mathcal{F}_g(t)$. Since

$$F_{CS} = F_{SW} - \frac{7}{12}t^3 + \frac{1}{12}t, \tag{4.30}$$

the formula (2.108) gives

$$F_0^{CS}(t) = \frac{t^3}{12} - \frac{\pi^2 t}{6} - \mathrm{Li}_3(e^{-t}) + \zeta(3), \tag{4.31}$$

where the polylogarithm of index j is defined by:

$$\text{Li}_j(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^j}. \tag{4.32}$$

The above result is in precise agreement with the result in [35] obtained by resumming the perturbative series. With some extra work we can also compute $F_g^{CS}(t)$, for all $g > 0$, starting from (2.112). We just have to compute $f^{(p)}(1) - f^{(p)}(0)$, for p odd, where

$$f(\xi) = (1 - \xi)\phi(\xi, t), \quad \phi(\xi, t) = \log \frac{1 - e^{-t\xi}}{\xi} + 4t\xi.$$

It is easy to see that

$$\phi^{(p)}(\xi, t) = (-1)^{p+1} \Big\{ \text{Li}_{1-p}(e^{-t\xi})t^p - \frac{(p-1)!}{\xi^p} \Big\},$$

and by using the expansion

$$\frac{1}{1 - e^{-t}} = \frac{1}{t} + \sum_{k=0}^{\infty} (-1)^{k+1} B_{k+1} \frac{t^k}{(k+1)!}$$

one gets

$$\phi^{(p)}(0, t) = \frac{(-1)^p B_p}{p} t^p.$$

Putting everything together, we find for $g > 1$

$$\mathcal{F}_g(t) = \frac{B_{2g} B_{2g-2}}{2g(2g-2)(2g-2)!} + \frac{B_{2g}}{2g(2g-2)!}\text{Li}_{3-2g}(e^{-t}) - \frac{B_{2g}}{2g(2g-2)}t^{2-2g}.$$

Since the last piece is the free energy at genus g of the Gaussian model, we conclude that the Chern-Simons free energy at genus g is given by

$$F_g^{CS}(t) = \frac{B_{2g} B_{2g-2}}{2g(2g-2)(2g-2)!} + \frac{B_{2g}}{2g(2g-2)!}\text{Li}_{3-2g}(e^{-t}) \tag{4.33}$$

which agrees with the resummation of [35] and also with the genus g closed string amplitude of type A topological strings on the resolved conifold (see [54] for more details).

4.2 Extensions

We have seen that the matrix model reformulation of Chern-Simons theory provides an efficient way to obtain the master field geometry and to resum the perturbative

expansion. The result (4.33) can be derived as well from the perturbation series [35, 34], but the existence of a matrix model description of Chern-Simons theory turns out to be useful in other situations as well. For example, one can easily write a matrix integral for Chern-Simons theory for other gauge groups [53], and the corresponding models have been analyzed in [37]. Moreover, the matrix representation of Chern-Simons partition functions can be extended to lens spaces and Seifert spaces, and provides a useful way to study perturbative expansions around nontrivial flat connections. The matrix models that describe these expansions have been studied in perturbation theory in [53, 2] and the saddle-point approximation to lens space matrix models has been studied in [38]. There are as well multimatrix models describing A topological strings on some noncompact Calabi-Yau geometries [2] that can be studied by using saddle-point techniques [76], and it is possible as well to formulate the Chern-Simons partition function on $\mathbf{S}^3$ in terms of a unitary model [58]. However, all these matrix models are usually much harder to analyze than conventional ones, and more work is needed to understand their large N properties.

Acknowledgments

References

[1] M. Aganagic, A. Klemm and C. Vafa, "Disk instantons, mirror symmetry and the duality web," Z. Naturforsch. A **57**, 1 (2002) [arXiv:hep-th/0105045].

[2] M. Aganagic, A. Klemm, M. Mariño and C. Vafa, "Matrix model as a mirror of Chern-Simons theory," JHEP **0402**, 010 (2004) [arXiv:hep-th/0211098].

[3] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N field theories, string theory and gravity," Phys. Rept. **323**, 183 (2000) [arXiv:hep-th/9905111].

[4] G. Akemann, "Higher genus correlators for the Hermitian matrix model with multiple cuts," Nucl. Phys. B **482**, 403 (1996) [arXiv:hep-th/9606004].

[5] J. Ambjorn, L. Chekhov, C. F. Kristjansen and Y. Makeenko, "Matrix model calculations beyond the spherical limit," Nucl. Phys. B **404**, 127 (1993) [Erratum-ibid. B **449**, 681 (1995)] [arXiv:hep-th/9302014].

[6] R. Argurio, G. Ferretti and R. Heise, "An introduction to supersymmetric gauge theories and matrix models," Int. J. Mod. Phys. A **19**, 2015 (2004) [arXiv:hep-th/0311066].

[7] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, "Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes," Commun. Math. Phys. **165**, 311 (1994) [arXiv:hep-th/9309140].

[8] D. Bessis, "A New Method In The Combinatorics Of The Topological Expansion," Commun. Math. Phys. **69**, 147 (1979).

[9] D. Bessis, C. Itzykson and J. B. Zuber, "Quantum Field Theory Techniques In Graphical Enumeration," Adv. Appl. Math. **1**, 109 (1980).

[10] M. Blau and G. Thompson, "Lectures on 2-d gauge theories: Topological aspects and path integral techniques," arXiv:hep-th/9310144.

[11] G. Bonnet, F. David and B. Eynard, "Breakdown of universality in multi-cut matrix models," J. Phys. A **33**, 6739 (2000) [arXiv:cond-mat/0003324].

[12] E. Brézin, C. Itzykson, G. Parisi and J. B. Zuber, "Planar Diagrams," Commun. Math. Phys. **59**, 35 (1978).

[13] E. Brézin and V. A. Kazakov, "Exactly Solvable Field Theories Of Closed Strings," Phys. Lett. B **236**, 144 (1990) . M. R. Douglas and S. H. Shenker, "Strings In Less Than One-Dimension," Nucl. Phys. B **335**, 635 (1990). D. J. Gross and A. A. Migdal, "Nonperturbative Two-Dimensional Quantum Gravity," Phys. Rev. Lett. **64**, 127 (1990).

[14] F. Cachazo, B. Fiol, K. A. Intriligator, S. Katz and C. Vafa, "A geometric unification of dualities," Nucl. Phys. B **628**, 3 (2002) [arXiv:hep-th/0110028].

[15] F. Cachazo, K. A. Intriligator and C. Vafa, "A large N duality via a geometric transition," Nucl. Phys. B **603**, 3 (2001) [arXiv:hep-th/0103067].

[16] F. Cachazo, S. Katz and C. Vafa, "Geometric transitions and N = 1 quiver theories," arXiv:hep-th/0108120.

[17] P. Candelas and X. C. De La Ossa, "Moduli Space Of Calabi-Yau Manifolds," Nucl. Phys. B **355**, 455 (1991).

[18] L. Chekhov, "Genus one correlation to multi-cut matrix model solutions," arXiv:hep-th/0401089.

[19] Y. Chen and N. Lawrence, "Density of zeros of some orthogonal polynomials," Methods Appl. Anal. **5**, No.4, 367 (1998).

[20] S. Coleman, "1/N," in *Aspects of symmetry*, Cambridge University Press, 1988.

[21] P. Di Francesco, "Matrix model combinatorics: Applications to folding and coloring," arXiv:math-ph/9911002; "2D quantum gravity, matrix models and graph combinatorics," arXiv:math-ph/0406013.

[22] P. Di Francesco, P. Ginsparg and J. Zinn-Justin, "2-D Gravity and random matrices," Phys. Rept. **254**, 1 (1995) [arXiv:hep-th/9306153].

[23] P. Di Francesco and C. Itzykson, "A Generating function for fatgraphs," Annales Poincare Phys. Theor. **59**, 117 (1993) [arXiv:hep-th/9212108].

[24] R. Dijkgraaf, S. Gukov, V. A. Kazakov and C. Vafa, "Perturbative analysis of gauged matrix models," Phys. Rev. D **68**, 045007 (2003) [arXiv:hep-th/0210238].

[25] R. Dijkgraaf, G. W. Moore and R. Plesser, "The Partition function of 2-D string theory," Nucl. Phys. B **394**, 356 (1993) [arXiv:hep-th/9208031].

[26] R. Dijkgraaf, A. Sinkovics and M. Temurhan, "Matrix models and gravitational corrections," Adv. Theor. Math. Phys. **7**, 1155 (2004) [arXiv:hep-th/0211241].

[27] R. Dijkgraaf and C. Vafa, "Matrix models, topological strings, and supersymmetric gauge theories," Nucl. Phys. B **644**, 3 (2002) [arXiv:hep-th/0206255].

[28] R. Dijkgraaf and C. Vafa, "On geometry and matrix models," Nucl. Phys. B **644**, 21 (2002) [arXiv:hep-th/0207106].

[29] R. Dijkgraaf and C. Vafa, "A perturbative window into non-perturbative physics," arXiv:hep-th/0208048.

[30] R. Dijkgraaf, E. Verlinde and H. Verlinde, "Notes on topological string theory and two-dimensional topological gravity," in Trieste Spring School 1990.

[31] T. Eguchi and S. K. Yang, "The Topological $\mathbb{P}^1$ model and the large N matrix integral," Mod. Phys. Lett. A **9**, 2893 (1994) [arXiv:hep-th/9407134]. T. Eguchi, K. Hori and S. K. Yang, "Topological sigma models and large N matrix integral," Int. J. Mod. Phys. A **10**, 4203 (1995) [arXiv:hep-th/9503017].

[32] B. Eynard, "Topological expansion for the 1-hermitian matrix model correlation functions," arXiv:hep-th/0407261.

[33] F. Ferrari, "Planar diagrams and Calabi-Yau spaces," Adv. Theor. Math. Phys. **7**, 619 (2004) [arXiv:hep-th/0309151].

[34] R. Gopakumar and C. Vafa, "M-theory and topological strings. I," arXiv:hep-th/9809187.

[35] R. Gopakumar and C. Vafa, "On the gauge theory/geometry correspondence," Adv. Theor. Math. Phys. **3**, 1415 (1999) [arXiv:hep-th/9811131].

[36] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley, 1977.

[37] N. Halmagyi and V. Yasnov, "Chern-Simons matrix models and unoriented strings," JHEP **0402**, 002 (2004) [arXiv:hep-th/0305134].

[38] N. Halmagyi and V. Yasnov, "The spectral curve of the lens space matrix model," arXiv:hep-th/0311117. N. Halmagyi, T. Okuda and V. Yasnov, "Large N duality, lens spaces and the Chern-Simons matrix model," JHEP **0404**, 014 (2004) [arXiv:hep-th/0312145].

[39] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow, *Mirror symmetry*, American Mathematical Society, 2003.

[40] K. Hori and C. Vafa, "Mirror symmetry," arXiv:hep-th/0002222.

[41] C. Imbimbo and S. Mukhi, "The Topological matrix model of c = 1 string," Nucl. Phys. B **449**, 553 (1995) [arXiv:hep-th/9505127].

[42] C. Itzykson and J. B. Zuber, "Matrix Integration And Combinatorics Of Modular Groups," Commun. Math. Phys. **134**, 197 (1990).

[43] S. Kachru, S. Katz, A. E. Lawrence and J. McGreevy, "Open string instantons and superpotentials," Phys. Rev. D **62**, 026001 (2000) [arXiv:hep-th/9912151].

[44] V. A. Kazakov, M. Staudacher and T. Wynter, "Character expansion methods for matrix models of dually weighted graphs," Commun. Math. Phys. **177**, 451 (1996) [arXiv:hep-th/9502132].

[45] A. Klemm, M. Mariño and S. Theisen, "Gravitational corrections in supersymmetric gauge theory and matrix models," JHEP **0303**, 051 (2003) [arXiv:hep-th/0211216].

[46] I. K. Kostov, "Gauge invariant matrix model for the A-D-E closed strings," Phys. Lett. B **297**, 74 (1992) [arXiv:hep-th/9208053]. S. Kharchev, A. Marshakov, A. Mironov, A. Morozov and S. Pakuliak, "Conformal matrix models as an alternative to conventional multimatrix models," Nucl. Phys. B **404**, 717 (1993) [arXiv:hep-th/9208044].

[47] I. K. Kostov, "Conformal field theory techniques in random matrix models," arXiv:hep-th/9907060.

[48] A.B.J. Kuijlaars and W. Van Assche, "The asymptotic zero distribution of orthogonal polynomials with varying recurrence coefficients," J. Approximation Theory **99**, No.1, 167-197 (1999).

[49] J. M. F. Labastida and P. M. Llatas, "Topological matter in two-dimensions," Nucl. Phys. B **379**, 220 (1992) [arXiv:hep-th/9112051].

[50] J. M. F. Labastida and M. Mariño, "Type B topological matter, Kodaira-Spencer theory, and mirror symmetry," Phys. Lett. B **333**, 386 (1994) [arXiv:hep-th/9405151].

[51] C. I. Lazaroiu, "Holomorphic matrix models," JHEP **0305**, 044 (2003) [arXiv:hep-th/0303008].

[52] M. Mariño, "The geometry of supersymmetric gauge theories in four dimensions," arXiv:hep-th/9701128.

[53] M. Mariño, "Chern-Simons theory, matrix integrals, and perturbative three-manifold invariants," arXiv:hep-th/0207096.

[54] M. Mariño, "Chern-Simons theory and topological strings," arXiv:hep-th/0406005.

[55] M.L. Mehta, *Random matrices*, 2nd edition, Academic Press, 1991.

[56] A. Migdal, "Loop equations and $1/N$ expansion," Phys. Rept. **102**, 199 (1983).

[57] N.I. Muskhelishvili, *Singular integral equations*, Noordhoff, 1953.

[58] T. Okuda, "Derivation of Calabi-Yau Crystals from Chern-Simons Gauge Theory," arXiv:hep-th/0409270.

[59] H. Ooguri, Y. Oz and Z. Yin, "D-branes on Calabi-Yau spaces and their mirrors," Nucl. Phys. B **477**, 407 (1996) [arXiv:hep-th/9606112].

[60] H. Ooguri and C. Vafa, "Worldsheet derivation of a large N duality," Nucl. Phys. B **641**, 3 (2002) [arXiv:hep-th/0205297].

[61] R.C. Penner, "Perturbative series and the moduli space of Riemann surfaces," J. Differ. Geom. **27**, No.1, 35-53 (1988).

[62] P. B. Rønne, "On the Dijkgraaf-Vafa conjecture," arXiv:hep-th/0408103.

[63] G. Szegö, *Orthogonal polynomials*, American Mathematical Society, 1959.

[64] W. Taylor, "D-brane effective field theory from string field theory," Nucl. Phys. B **585**, 171 (2000) [arXiv:hep-th/0001201].

[65] W. Taylor and B. Zwiebach, "D-branes, tachyons, and string field theory," arXiv:hep-th/0311017.

[66] G. 't Hooft, "A planar diagram theory for strong interactions," Nucl. Phys. B **72** (1974) 461.

[67] M. Tierz, "Soft matrix models and Chern-Simons partition functions," Mod. Phys. Lett. A **19**, 1365 (2004) [arXiv:hep-th/0212128].

[68] E. Witten, "The $1/N$ Expansion In Atomic And Particle Physics," in G. 't Hooft *et al.* (eds.), *Recent developments in gauge theories*, Plenum Press, 1980.

[69] E. Witten, "Noncommutative Geometry And String Field Theory," Nucl. Phys. B **268**, 253 (1986).

[70] E. Witten, "Topological Sigma Models," Commun. Math. Phys. **118**, 411 (1988).

[71] E. Witten, "Quantum Field Theory And The Jones Polynomial," Commun. Math. Phys. **121**, 351 (1989).

[72] E. Witten, "On The Structure Of The Topological Phase Of Two-Dimensional Gravity," Nucl. Phys. B **340**, 281 (1990).

[73] E. Witten, "Mirror manifolds and topological field theory," arXiv:hep-th/9112056.

[74] E. Witten, "Chern-Simons gauge theory as a string theory," Prog. Math. **133**, 637 (1995) [arXiv:hep-th/9207094].

[75] E. Witten, "Perturbative gauge theory as a string theory in twistor space," arXiv:hep-th/0312171.

[76] V. Yasnov, "Chern-Simons matrix model for local conifold," arXiv:hep-th/0409136.