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Letter to the Editor—A Conjecture Concerning the Smallest Bound on the Iterations in Linear Programming

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A CONJECTURE CONCERNING THE SMALLEST BOUND ON THE ITERATIONS IN LINEAR PROGRAMMING

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IN SOLVING linear programming problems accurate estimates of the number of iterations needed to reach the optimum are important to have It has been mentioned in the literature that computing experience indicates this number of iterations to be of the order of twice the number of constraints We have been informed by two computing groups that a number of large linear programming problems have been left unsolved because, after many hours of machine operation, it was not known how much longer the process would continue

Here through heuristic arguments based on what appears to be a reasonable conjecture we give an upper bound to the number of iterations for algorithms which change one vector at a time, such as the simplex process

A linear programming problem, as defined in matrix notation, requires a vector $x \geqq 0$ (nonnegativity constraints) be found that satisfies the constraints $Ax \leqq b$, and maximizes the linear function cx Here $x=(x_1, \quad, x_n)$, $A=[a_{ij}]$ $(i=1, \quad, m, j=1, \quad, n)$, $b=(b_1, \quad, b_m)$ and $c=(c_1, \quad, c_n)$ is the cost vector With the original (the primal) problem is associated the dual problem $yA \geqq c$, $y \geqq 0$, $by=$ minimum, where $y=(y_1, \quad, y_m)$ A duality theorem asserts that if either the primal or the dual has a solution, then the values of the objective functions of both problems at the optimum are the same It is a relatively easy matter to obtain the solution vector of one problem from that of the other

In the original problem one usually has $m<n$ Thus all the vertices of the region of solutions lie on the coordinate planes This follows from the fact that, generally, in n-dimensions, n hyperplanes each of dimension $(n-1)$ intersect at a point The dual problem defines a polytope in m-dimensional space In this case not all vertices need lie on the coordinate planes

Geometrically, the inequality constraints of the problem define a convex set (the feasible region) whose boundary is a polytope in n-dimensions, and the objective function defines a hyperplane that is translated in a parallel direction towards that point of the convex region which yields the optimum (e g , if minimizing this point yields the shortest distance of the objective hyperplane from the origin) It is intuitively obvious, at least in 3 dimensions, that the optimum is on the boundary and is usually a vertex of the polyhedron However, WEYL has proven this fact for the n-dimensional case If all vertices were easily obtainable, then one could evaluate the objective function at each vertex until the optimum is attained From this it is clear that there should be a natural interest in the number of vertices of polytopes There is, of course, the stronger interest in estimating the number of steps required to solve a linear programming problem by various procedures and particularly by the simplex process Because specific criteria for choosing new vectors, etc , are used, the simplex process does not require the use of all the vertices It simply follows a network path towards the optimum

To each hyperplane of the constraint set corresponds a vector that may be used to form a basis. A feasible basis corresponds to a vertex of the constraint polyhedron. This vertex is the intersection of hyperplanes from which the basis vectors are taken. Note that the number of planes defining the vertex may exceed the dimension and hence one chooses from among them a number that corresponds to the dimension. Each change of basis is effected by dropping one vector and selecting a new one. Geometrically, this corresponds to moving from one vertex of the polyhedron to an adjacent vertex, which is the intersection of planes that include all but one of those planes that intersect in the previous vertex. If we associate with each vector a point in the plane and join that point to all other associated points, we have all possible ways of changing vectors, having started with a given basis. Some of these changes may not be feasible since not all hyperplanes define vertices of the constraint polyhedron. In all there are $\binom{m+n}{2}$ lines joining these associated points and therefore there are at most that many changes of bases. However, there is a question as to whether it may not be possible that a connecting line may be repeated, i.e., that one vector may replace the same vector it previously replaced in another basis. We assume that this cannot happen, i.e., two given vectors may be exchanged no more than once (with one of them replacing the other) in all choices of bases. It would appear that there is no theoretical reason why algorithms may not exist that satisfy this conjecture. For example, there is computing evidence supporting it for the simplex algorithm under nondegeneracy and without cycling. Our search among experts using the simplex process to solve linear programs has not produced counter examples. It is believed that practical experience that shows that the number of iterations of the simplex process of the order of twice the number of constraints increases the plausibility of this conjecture since our subsequent argument produces an upper bound close to that estimated from computing experience.

If our assumption is accepted we have an upper bound of $\binom{m+n}{2}$ possible changes of bases. However, this number may be reduced because of the fact that one does not return to an old basis. Thus, for example, the simplex process improves the value of the objective function or leaves it the same from one change of basis to the next, i.e., from one vertex of the polyhedron to that which replaces it in a change of basis.

The argument now proceeds as follows. In m dimensions (the simplex process selects a basis of m vectors for an n variable, m essential constraints problem) there are at least m edges meeting at a vertex of a polyhedron. With a change of basis the objective hyperplane transits across one of these edges to a new adjacent vertex, since there is only one edge joining any two vertices of a polyhedron. However, the objective hyperplane, in its move towards the optimum vertex, will never return to the vertex it has just left, and hence it will never transit across any of the remaining $(m-1)$ or more edges leading out of the previous vertex to other vertices. These vertices may be reached through other edges connecting from other vertices but never again from the given vertex. This implies, in the change of basis interpretation, that, whatever the total number of changes of bases is, i.e., $\binom{m+n}{2}$, it must be reduced by a factor of at least $\frac{1}{2}m$, i e , out of every m possible changes, one is chosen, and the $\frac{1}{2}$ arises from the fact that a change of basis involves two vertices and a direction connecting them is counted twice, i e , once at each vertex Thus we must consider a reduction of only $\frac{1}{2}m$ in the number of changes of bases Thus, finally we have

$$\binom{m+n}{2} \Big/ \tfrac{1}{2} m = \frac{2}{m}\binom{m+n}{2}$$

as an upper bound to the number of iterations This dominates $2(m+n)$ the quantity observed in practice Incidently, we have seen examples in which the iterations of the simplex method exceed $2(m+n)$ but not the above bound

If a result is desired for n-dimensions, i e , if one uses the dual-simplex process, then the following is the upper bound to the number of feasible bases

$$\frac{2}{n}\binom{m+n}{2},$$

which is dominated by

$$2\,(m+n)$$

since $n>m$ Note that the binomial coefficient in the foregoing results includes nonfeasible changes of bases

It is possible by means of parametric programming to transform all the regions contained in the intersections of the $(n-1)$-dimensional hyperplanes of the problem to a convex region This may be described by the blowing up of spikes projecting from the feasible region A spike is defined by no more than $(m+n-1)$ planes and hence it is basically another polytope, at least one of whose $(n-1)$-dimensional faces coincides with an $(n-1)$-dimensional face of the convex polytope of feasible solutions Then the linear programming problem can be solved with respect to the new region With this modification, $(2/n)\binom{m+n}{2}$ remains the upper bound to the total number of iterations of the dual simplex method and $(2/m)\binom{m+n}{2}$ for the simplex method itself In other words the estimate also applies to problems in which there are many extraneous (nonfeasible) intersections, which by parametrization may be made to define vertices of feasible regions for new problems

REMARK The use of artificial bases (unit vectors) with very high costs, enables circumventing degeneracy, i e , the nonexistence of the appropriate dimensional bases By starting at a distant point in the positive orthant with an artificial basis, the objective function reaches the feasible region from (generally) a best direction, i e , with a 'smallest' number of iterations