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Letter to the Editor—An Alternative Proof of a Conservation Law for the Queue G/G/1

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accompanied by the appropriate polynomial fit between lattice points However, in many cases one does not know beforehand whether the optimal control function will be sufficiently smooth to justify this approach In this case, one has no alternative but to use a fine lattice structure in calculating the optimal control function For problems in this latter category, the procedure in this paper is similar to an adaptive learning process In particular, a coarse lattice structure is first used to learn about the nature of $u^0(\, , k)$, and then this knowledge is applied to the finer lattice structure

Experience on some simple two-dimensional examples in which the optimal control function was known to be piecewise smooth gave reductions on the order of 50 percent with little or no loss in accuracy The extra programming effort to implement the procedure over the standard dynamic programming algorithm is small, and essentially consists of adding an extra iterative loop in the program

ACKNOWLEDGMENT

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REFERENCE

1 S. E. DREYFUS, "Computational Aspects of Dynamic Programming," *Opns Res* 5, 408–415 (1957)

AN ALTERNATIVE PROOF OF A CONSERVATION LAW FOR THE QUEUE *G/G/1*

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(Received January 9 1969)

KLEINROCK first showed that, for a multiclass M/G 1 queue, the expected waiting times for all classes satisfy a simple linear equality constraint that is independent of queue discipline for a large class of disciplines We generalize here the conditions under which this result holds and give a simpler proof

WE STATE and prove a theorem, first given by KLEINROCK,[2] whose original statement was for the queue with Poisson input and general independent service Here we make no similar assumptions about the arrival and service processes other than that the first two moments exist for the service process and the first moment exists for the arrival process Our proof is perhaps slightly simpler than the proof given in reference 2

Consider the queue G/G/1, that is, an arbitrary arrival stream and service process No assumptions about independence are made we require only that

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long-run distributions exist. We assume that (1) the processor is never idle if there is a job available for processing, (2) the processing required by each job and over-all is independent of the discipline used, and (3) there is no preemption. Jobs can be classified into k classes. Define

λ_i = arrival rate of jobs from class i,
$E(P_i^n)$ = nth moment of the processing-time distribution for jobs of class i,
$\rho_i = \lambda_i E(P_i^1)$,
$E(L)$ = expected load in system at a random point in time,
$E(L_q)$ = expected load in the waiting queue at a random point in time,
$E(W_i)$ = expected waiting time for a job of class i.

Observe that L is not affected by the discipline used. L increases by an amount P_i when a job from class i arrives ($i=1, 2, \ldots, k$), otherwise L decreases at unit rate as long as there is load in the system. Thus $E(L)$ is determined strictly by the arrival process and is independent of discipline. Assumption (1) is necessary for this conclusion.

THEOREM. *If* $\sum_{i=1}^{i=k} \rho_i < 1$, *then for all disciplines satisfying the conditions stated above,*

$$\sum_{i=1}^{i=k} \rho_i E(W_i) = E(L) - \sum_{i=1}^{i=k} \lambda_i E(P_i^2)/2$$

Proof. We know $E(L_q) = E(L)$—the expected load in the processor. Given that the processor is busy with a job of class i at some randomly chosen point in time, the expected load in the processor is simply the random modification,[1] $E(P_i^2)/[2\,E(P_i)]$. Assumptions (2) and (3) are necessary for this to be true. Removing the conditioning on the class of job occupying the processor, we can write

$$E(L_q) = E(L) - \sum_{i=1}^{i=k} \lambda_i E(P_i^2)/2$$

The mean arrival rate of load to the queue from class i is $\lambda_i E(P_i)$. The mean time in queue for each unit of load from class i is $E(W_i)$.

If $\sum_{i=1}^{i=k} \rho_i < 1$, then the expected busy-period length is finite, so the load arriving at the queue in an interval of length t approximates the load departing the queue in the same interval as $t \uparrow \infty$, so we can apply LITTLE's theorem,[3] which implies

$$E(L_q) = \sum_{i=1}^{i=k} \lambda_i E(P_i) E(W_i)$$

and the theorem is proved.

The implication of the theorem is that for a nonpreemptive queue the expected waiting times over all classes are restricted by the linear equality constraint given in the theorem. Kleinrock[2] gives this theorem the apt name of "The Conservation Law."

COROLLARY (Kleinrock). *For the case* M/G/1,

$$\sum_{i=1}^{i=k} \rho_i E(W_i) = \rho E(W_F) = [\rho \sum_{j=1}^{j=k} \lambda_j E(P_j^2)]/2[1 - \sum_{i=1}^{i=k} \rho_i]$$

for any discipline satisfying the conditions stated previously, where $\rho = \sum_{i=1}^{i=k} \rho_i$ *and* W_F *is the waiting time for a job if the FIFO discipline were applied to the same system.*

Proof. For the case M/G/1 the expected load in the system is simply the ex-

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pected waiting time if the FIFO discipline were used (*see*, for example, reference 1) so,

$$E(L)=E(W_F)=[\sum_{i=1}^{i=k} \lambda_i E(P_i^2)]/2(1-\sum_{i=1}^{i=k} \rho_i)$$

By applying the theorem and simplifying, the corollary follows

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3 J D C LITTLE, "A Proof of the Formula $L=\lambda W$," *Opns Res* **9**, 383–387 (1961)

A NOTE ON PATTERN SEPARATION

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This note describes a linear programming method of separating two not necessarily disjoint convex polyhedral sets in E^n If the sets are disjoint, a strictly separating hyperplane is generated that maximizes and equalizes the distance between the sets and the hyperplane If the sets intersect a hyperplane is generated that minimizes the maximum error

THE SETS $\{A_i | i=1, 2, \ldots, m\}$ and $\{B_j | j=1, 2, \ldots, k\}$ are collections of points in E^n, their convex hulls being denoted C_a and C_b This note examines linear programming methods of separating C_a and C_b with a hyperplane If some hyperplane separates C_a and C_b strictly, the sets are separable If the sets are not separable, any hyperplane will leave points of C_a or C_b on the wrong side The distance of incorrect points from the hyperplane is called the error distance For example, if $\{x | xc=\gamma\}$ is the hyperplane, points in C_a should satisfy $xc>\gamma$, and $A_ic<\gamma$, then the error for point A_i is $\gamma - A_ic$

MANGASARIAN[1] has designed a linear program with $n+2$ constraints and $k+m$ nonnegative variables for separating C_a and C_b If the sets are disjoint, a strictly separating hyperplane is supplied that fends the sets off equally and maximizes the distance between set and hyperplane The technique detects a nonvoid intersection, but no hyperplane is supplied SMITH[3] has devised a computationally more difficult method for the nonseparable case that minimizes the total error distance If the sets are separable, any separating hyperplane has zero error and is thus optimal, and the program has $n+1$ constraints with $k+m$ upper and lower bounded variables (*see* reference 4) Mangasarian's multi-surface method[2] is designed for the nonseparable case, it constructs a piecewise linear function to separate the

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