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Letter to the Editor—On Stochastic Linear Approximation Problems

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ON STOCHASTIC LINEAR APPROXIMATION PROBLEMS

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This paper considers the extension of the linear approximation problem minimize $\|\epsilon\|$ subject to $AX+\epsilon=b$ to the case where the elements of b are independent random variables with known distributions. This extension is accomplished by the use of chance constraints. An analysis of this stochastic problem shows that the problem can be solved by some of the powerful computational methods of approximation theory.

IN MANY applications the problem arises of finding the *best approximate solutions* to a (possibly inconsistent) system of linear equations

$$Ax=b, \tag{1}$$

where A is given $m\times n$ matrix and b is given vector. These solutions are defined as the optimal solutions of the minimization problem:

$$\text{minimize } \|\epsilon\| \quad \text{subject to} \quad Ax+\epsilon=b, \tag{2}$$

where ϵ is the *error* (or *residual*) of the *approximate solution* x, and the norm $\|\ \|$ is some measure of the approximation error. *Minimum* rather than *infimum* is used here and in the following problems because, in each case, minimizing solutions can be shown to exist. For problem (2) this is well known, e.g., by reference 3, p. 20. For problem (3) this can be shown via the equivalent problem (18).

In this paper we consider the extension of the linear approximation problem (2) to the case where the elements b_i, $i=1, \cdots, m$, of the vector b are independent random variables with known distributions $F_i(\ )$, $i=1, \cdots, m$, and where the approximate solution x has to be chosen before b is observed. The stochastic version of (2) considered here is the following chance-constrained problem:

$$\text{minimize } \|\epsilon\| \quad \text{subject to} \quad P\{Ax\geqq b-\epsilon\}\geqq\alpha_1, \quad P\{Ax<b+\epsilon\}\geqq\alpha_2, \quad \epsilon\geqq 0, \tag{3}$$

where $P\{E\}$ denotes the (vector) probability of the event E and the vectors $\alpha_1=(\alpha_{1i})$ and $\alpha_2=(\alpha_{2i})$, $i=1,\cdots,m$, are given lower bounds on the corresponding probability constraints, which are written in detail as

$$P\{\textstyle\sum_{j=1}^{j=n} a_{ij}x_j \geqq b_i-\epsilon_i\}\geqq \alpha_{1i},\qquad P\{\textstyle\sum_{j=1}^{j=n} a_{ij}x_j < b_i+\epsilon_i\}\geqq \alpha_{2i},\qquad i=1,\cdots,m.$$

For the monotonic norm $\|\ \|$ (*see* definition in the next section) and A of full row rank, problem (3) is equivalent to the consistent system of linear equations (20), and the minimal approximation error is given by (19) [under the same assumptions for the linear approximation problem (2), the system (1) is consistent, and thus the approximation error is zero]. For general A, the deterministic equivalent of (3) is a linearly constrained minimization problem [*see* (18) or (21) below] of the type associated with the linear approximation problem (2) with monotonic norm $\|\ \|$.

These results show that the stochastic linear approximation problem (3) is a natural extension of the problem (2), yet is in the domain of applicability of the powerful computational methods of approximation theory (*see*, e. g., reference 3).

PRELIMINARIES AND NOTATIONS

For any vectors $x=(x_i)$, $y=(y_i)$ in R^n we denote by: $|x|$ the vector $(|x_i|)$, $i=1,\cdots,n$; $x\leqq y$ the fact $x_i\leqq y_i$, $i=1,\cdots,n$; $x\leq y$ the fact $x\leqq y$, $x\neq y$; $x<y$ the fact $x_i<y_i$, $i=1,\cdots,n$.

The norm $\|\ \|$ is called *monotonic* if, for every x, $y\epsilon R^n$,

$$|x|\leqq|y| \quad\text{implies}\quad \|x\|\leqq\|y\|, \tag{5}$$

strictly monotonic if, in addition to (5),

$$|x|\leq|y| \quad\text{implies}\quad \|x\|<\|y\|,$$

absolute if, for every $x\epsilon R^n$,

$$\|x\|=\|(|x|)\| \qquad (=\text{the norm of the vector } |x|). \tag{6}$$

It was shown in reference 1 that (5) and (6) are equivalent, i.e., a norm is absolute if and only if it is monotonic. The best known such norms are the L_p-*norms*

$$\|\epsilon\|=\|\epsilon\|_p=(\textstyle\sum_{i=1}^{i=m}|\epsilon_i|^p)^{1/p},\qquad 1\leqq p,$$

in which case (2) is the L_p-*linear approximation problem*. In particular, for the L_1-*approximation*, $p=1$; for the *least-squares approximation*, $p=2$; and for the *Tchebycheff approximation*, $p=\infty$, i.e., $\|\epsilon\|_\infty=\max\{|\epsilon_i|: i=1,\cdots,m\}$.

In what follows the norm $\|\ \|$ is assumed *monotonic*. The equivalence of (5) and (6) then implies that (2) is equivalent to the minimization problem

$$\text{minimize } \|(|Ax-b|)\|. \tag{7}$$

With the distribution functions F_i, $i=1,\cdots,m$, we associate certain vectors as follows: For any vector $y=(y_i)$ in R^m, $F(y)$ is the vector $[F_i(y_i)]$, $i=1,\cdots,m$.

For any $i=1,\cdots,m$ and $0\leqq\theta\leqq1$, let

$$F_i^{-1}(\theta)=\inf\{\eta: F_i(\eta)\geqq\theta\}\quad\text{and}\quad \hat{F}_i^{-1}(\theta)=\sup\{\eta: F_i(\eta)\leqq\theta\}.$$

The vectors $\alpha_1=(\alpha_{1i})$, $\alpha_2=(\alpha_{2i})$, $i=1,\cdots,m$, given in problem (3) are probability bounds, so necessarily, $0 \leqq \alpha_{ji} \leqq 1$, $j=1, 2$, $i=1, \cdots, m$, or $0 \leqq \alpha_1$, $\alpha_2 \leqq e$, where 0 is the zero vector and e is the vector of ones.

Finally, we introduce the vectors $F^{-1}(\alpha_1)=[F_i^{-1}(\alpha_{1i})]$, $i=1, \cdots, m$, and $\hat{F}^{-1}(e-\alpha_2)=[\hat{F}_i^{-1}(1-\alpha_{2i})]$, $i=1, \cdots, m$.

THE DETERMINISTIC EQUIVALENT

IN THIS SECTION we study the deterministic equivalent of the stochastic approximation problem (3) with the constraints

$$P\{Ax \geqq b-\epsilon\} \geqq \alpha_1, \tag{8}$$

$$P\{Ax < b+\epsilon\} \geqq \alpha_2, \tag{9}$$

$$\epsilon \geqq 0. \tag{10}$$

Using the notations of the previous section, we rewrite (8) as:

$$P\{Ax \geqq b-\epsilon\} = P\{b \leqq Ax+\epsilon\} = F(Ax+\epsilon) \geqq \alpha_1, \tag{8'}$$

which is equivalent to

$$Ax+\epsilon \geqq F^{-1}(\alpha_1). \tag{8''}$$

Similarly, (9) is rewritten as

$$P\{Ax < b+\epsilon\} = P\{b > Ax-\epsilon\} = e-F(Ax-\epsilon) \geqq \alpha_2, \tag{9'}$$

or

$$Ax-\epsilon \leqq \hat{F}^{-1}(e-\alpha_2). \tag{9''}$$

The deterministic equivalent of (3) is therefore

$$\text{minimize } \|\epsilon\| \quad \text{subject to} \tag{11}$$

$$Ax+\epsilon \geqq F^{-1}(\alpha_1), \tag{8''}$$

$$Ax-\epsilon \leqq \hat{F}^{-1}(e-\alpha_2), \tag{9''}$$

$$\epsilon \geqq 0. \tag{10}$$

Subtracting (9″) from (8″), we find that

$$\epsilon \geqq \tfrac{1}{2}\{F^{-1}(\alpha_1)-\hat{F}^{-1}(e-\alpha_2)\}. \tag{12}$$

Writing

$$\epsilon^0 = \tfrac{1}{2}\{F^{-1}(\alpha_1)-\hat{F}^{-1}(e-\alpha_2)\}, \tag{13}$$

we see from (12) that, for $\epsilon^0 \geqq 0$, the constraint (10) is redundant, since it is satisfied whenever (8″) and (9″) are. From (13) and the definitions of $F^{-1}(\alpha_1)$ and $\hat{F}^{-1}(e-\alpha_2)$, it is clear that $\epsilon^0 \geqq 0$ if $\alpha_1 > e-\alpha_2$, that is,

$$\alpha_{1i}+\alpha_{2i} > 1, \qquad i=1, \cdots, m. \tag{14}$$

Recalling from (4) that α_{1i}, α_{2i}, $i=1, \cdots, m$, are somewhat like confidence levels (hence normally close to 1) it is clear that (14) will be satisfied in all meaningful applications.

Combining (12) and (13) gives now

$$\epsilon = \epsilon^0 + \delta, \qquad \delta \geqq 0, \tag{15}$$

which, when substituted in (8″) and (9″), results in the two-sided constraint

$$-\delta \leqq Ax - \delta^0 \leqq \delta, \tag{16}$$

where

$$\delta^0 = \tfrac{1}{2}\{F^{-1}(\alpha_1) + \hat{F}^{-1}(e-\alpha_2)\} \quad \text{and} \quad \delta \geqq 0. \tag{17}$$

Thus we have proved:

THEOREM. *Let $\alpha_1 > e - \alpha_2$ and let b_i, $i = 1, \cdots, m$, be independent random variables. Then the stochastic linear approximation problem* (3) *is equivalent to the minimization problem.*

$$\text{minimize } \|\epsilon^0 + \delta\| \quad \text{subject to} \quad -\delta \leqq Ax - \delta^0 \leqq \delta \quad \text{and} \quad \delta \geqq 0, \tag{18}$$

where ϵ^0 and δ^0 are given by (13) *and* (17), *respectively.*

As an easy consequence of this theorem we have:

COROLLARY. *Let the norm* $\|\ \|$ *be monotonic. Then:* (i) *The value*

$$\|\epsilon^0\| = \tfrac{1}{2}\|F^{-1}(\alpha_1) - \hat{F}^{-1}(e-\alpha_2)\| \tag{19}$$

is a lower bound for the optimal value of problem (3). (ii) *If the system*

$$Ax = (\delta^0 = \tfrac{1}{2}\{F^{-1}(\alpha_1) + \hat{F}^{-1}(e-\alpha_2)\}) \tag{20}$$

is consistent, then its solutions are optimal solutions of (3) *with optimal approximation error ϵ^0 given by* (13) *and optimal value equal to* (19).

Finally, we observe that the conditions $\epsilon^0 \geqq 0$ and $\delta \geqq 0$ imply that $\|\epsilon^0 + \delta\| = \|(\epsilon^0 + |\delta|)\|$. Thus (18), and hence (3), is equivalent to the minimization problem

$$\text{minimize } \|(\epsilon^0 + |Ax - \delta^0|)\|. \tag{21}$$

Comparing (21) with (7), we note that the former has (19) for a lower bound. Hence if $\|\ \|$ is monotonic and A is of rank m [i.e., (1) is solvable for all b] so that (7) is error free, the optimal error in (21) will still be ϵ^0, which is dependent only on the confidence levels α_1, α_2. Every component of ϵ^0 is a monotonic nondecreasing function of the corresponding components of α_1, α_2. Raising the confidence levels thus increases the error.

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