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Level-raising and symmetric power functoriality, III

Laurent Clozel and Jack A. Thorne*

December 10, 2015

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1 Introduction

The simplest case of Langlands' functoriality principle asserts the existence of the symmetric powers $\mathrm{Sym}^n \pi$ of a cuspidal representation π of $\mathrm{GL}_2(\mathbb{A}_F)$, where F is a number field. After the publication of Langlands' conjecture [Lan70], Gelbart and Jacquet [GJ78] proved the existence of $\mathrm{Sym}^2 \pi$. After this, progress was slow, eventually leading, through the work of Kim and Shahidi, to the existence of $\mathrm{Sym}^3 \pi$ and $\mathrm{Sym}^4 \pi$ [KS02, Kim03].

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In this series of papers [CT14, CT15], we revisit this problem using recent progress in the deformation theory of modular Galois representations. As a consequence, our methods apply only to π associated to classical modular forms on a totally real number field (in fact, forms of weight at least 2 in each variable, and constant parity). (In this introduction only, we say “ π is classical” for short. In fact, the assumption on parity can be removed; see §7 below.) We assume that π is not of CM type, i.e. not obtained by automorphic induction from an algebraic Hecke character of a CM extension of F . We will say that the n^{th} symmetric power $\text{Sym}^n \pi$ exists if there is a regular algebraic cuspidal automorphic representation Π of $\text{GL}_{n+1}(\mathbb{A}_F)$ such that, for any prime l ,

$$\text{Sym}^n r_l(\pi) \cong r_l(\Pi)$$

where $r_l(\pi) : G_F = \text{Gal}(\overline{F}/F) \rightarrow \text{GL}_2(\overline{\mathbb{Q}}_l)$ is the l -adic representation associated to π (for some embedding of its field of coefficients into $\overline{\mathbb{Q}}_l$) and $r_l(\Pi)$ is similarly defined. Note that this is independent of the apparent choices, by well-known properties of the set of such cuspidal representations under conjugation of the coefficients.

Theorem 1.1 (Theorem 6.1). *1. Assume that F is linearly disjoint from $\mathbb{Q}(\zeta_5)$. Then $\text{Sym}^6 \pi$ exists.*

2. Assume that F is linearly disjoint from $\mathbb{Q}(\zeta_7)$. Then $\text{Sym}^8 \pi$ exists.

Under similar restrictions on F and π , $\text{Sym}^5 \pi$ and $\text{Sym}^7 \pi$ were constructed in [CT15]. As a consequence, one now knows:

Corollary 1.2. *Assume F linearly disjoint from $\mathbb{Q}(\zeta_{35})$, and π classical. Then $\text{Sym}^n \pi$ exists for $n \leq 8$.*

We recall that the “potential” existence of Π , i.e. the existence of some unspecified Galois base change of Π , was obtained by Barnet-Lamb, Gee, and Geraghty [BLGG11], following earlier work of Clozel, Harris, Taylor, and Shepherd-Barron [Tay08, Har09, CHT08, HSBT10]. Therefore the new content of our work is the existence of Π itself, on its proper ground field. In particular this implies:

Corollary 1.3 (Assumptions of Corollary 1.2). *The L -function $L(s, \text{Sym}^n \pi)$ is holomorphic in the whole plane, with the expected functional equation.*

We now give some indications on the proof, concentrating for definiteness on the more difficult case, $n = 8$. Fix $l = 7$, and consider the reduction mod l , \overline{r} , of $r_l(\pi)$. The representation theory of $\text{GL}_2(\overline{\mathbb{F}}_7)$ implies, up to semi-simplification, an isomorphism

$$\text{Sym}^8 \overline{r} \sim (\varphi \overline{r} \otimes \overline{r}) \oplus \overline{\chi}^2 \text{Sym}^4 \overline{r}. \quad (1.1)$$

Here χ is the determinant of r , and φ is the Frobenius endomorphism acting on the coefficients. Since, as we saw, $\varphi \overline{r}$ is automorphic, both summands of (1.1) are; they are irreducible if the image of \overline{r} is sufficiently large.

Keeping to the real field F , we see that $\text{Sym}^8 \overline{r}$ is of the type considered by Arthur in [Art], i.e. the sum of two essentially orthogonal representations of degrees 4, 5. After some arguments involving, in particular, known results about change of weight, we see that this semi-simple residual representation is associated to an automorphic, cohomological representation Σ of the split group Sp_8 over F occurring in the discrete spectrum, an “endoscopic” representation. Our arguments could probably be given in this context, but we would need to transfer Σ to a form of Sp_8 compact at the real primes. This is delicate, and is not completed in [Art, Ch. 9]. We take a different route. We choose a suitable CM extension E of F ; after base change to E we are given two “regular algebraic, essentially conjugate self-dual, cuspidal” representations π_4, π_5 of $\text{GL}_4(\mathbb{A}_E)$ and $\text{GL}_5(\mathbb{A}_E)$. The representation (1.1) becomes a “Schur representation” of unitary type $\overline{R} : G_E \rightarrow \text{GL}_9(\overline{\mathbb{F}}_7)$ in the sense of [CHT08], [Tho15]. One of us (J.T.) has recently proved that deformation theory can be applied in this context, provided that one knows the existence of an initial, automorphic (cuspidal) lifting $R : G_E \rightarrow \text{GL}_9(\overline{\mathbb{Z}}_7)$ corresponding to a representation of $\text{GL}_9(\mathbb{A}_E)$ that is Steinberg at some finite prime. The problem of automorphy of the symmetric power $\text{Sym}^8 r_l(\pi)$ is thus reduced to the problem of the existence of this Steinberg lifting. It is this problem of “level-raising” that occupies us for most of this paper.

This level-raising is done by extending arguments introduced in [CT15]. In §2, we introduce some representations of the odd-dimensional unitary group associated to a ramified quadratic extension of p -adic fields, and study their integral structures. The interplay between these integral structures and the endoscopic decomposition of global spaces of automorphic forms plays a key role in the main level-raising argument. These representations are Iwahori-spherical and are constructed from a certain Langlands parameter using the work of Kazhdan and Lusztig [KL87].

In §3 we construct our initial “endoscopic” representation (with residual representation isomorphic to \overline{R} , as described above) as an automorphic representation of a definite unitary group G . We also study the local theory and identify the representations constructed in §2 as being members of a common L-packet by verifying directly the necessary trace identities (see §3.5).

We should justify the necessity (in our approach) of the complicated interplay of §2 and §3. The L-packets described in Theorem 3.2 are perfectly specified by Mœglin and Mok, by two identities of (twisted) traces. However it is crucial for us to understand the Jacquet modules of their elements. The results of Mok and Mœglin are not sufficient for this purpose.

In §4, we carry out the automorphic part of the level-raising argument, using algebraic modular forms on G . In contrast to the situation of [CT15], although this argument does produce a new automorphic representation that is ‘more ramified’ than the initial one, we can no longer guarantee that this representation is Steinberg locally (the possibilities are enumerated in Proposition 4.1). Moreover, the level-raising argument succeeds only for $l = 5, 7$, whereas the identity corresponding to (1.1), for Sym^{l+1} , holds for an arbitrary prime l . The reason for this failure is given after Lemma 4.4. This is also in contrast to [CT15], where we were able to establish a result for any prime $l > 3$.

We are therefore led in §5 to prove another level-raising result, which uses deformation theory to move from the representation constructed in §4 to one which is indeed Steinberg locally. We note that an argument of Taylor (see [Tay08], [Gee11]) allows one to construct automorphic liftings of an irreducible residual Galois representation with essentially any prescribed local behaviour (for example, with Steinberg type ramification). The key difficulty for us is that we must work with a reducible residual Galois representation (as necessitated e.g. by the equation (1.1)). However, it turns out that the ideas pertaining to the deformation of reducible residual representations introduced in [Tho15] are strong enough to allow one to produce a Steinberg lifting, the key point being that the initial automorphic lift (provided by the work done in §4) does have a place where the local representation is incompatible with the global Galois representation being globally reducible (see especially Lemma 5.3).

Having constructed this automorphic lifting of \overline{R} , Steinberg at some finite place, we can apply the automorphy lifting theorem of [Tho15], much as in our previous paper, to obtain the automorphy of the symmetric power $\mathrm{Sym}^8 r_l(\pi)$, and we carry this out in §6. In fact, we phrase things slightly differently by stating a new automorphy lifting theorem (Theorem 5.7) and applying it to the datum constructed in §4. We hope that this automorphy lifting theorem will have applications elsewhere. Finally, in §7 we show how to reduce the existence of symmetric power liftings of representations of $\mathrm{GL}_2(\mathbb{A}_F)$ which are essentially square-integrable at infinity to the corresponding problem for those which are regular algebraic up to twist (and which is solved in some cases in §6).

We note that as in our earlier paper [CT15], our proofs rely on the work of Mok [Mok15], extending Arthur’s results to unitary groups; and that in turn Mok’s results are conditional on the stabilisation of the twisted trace formula. This has now been announced in preprint form by Mœglin and Waldspurger [MW]. Finally, we again thank Mœglin, here for providing a proof of Proposition 3.7. We also thank all the referees for corrections and criticisms which, we hope, have substantially improved the exposition.

1.1 Notation

We refer to [CT14, §2] for basic notations relating to automorphic representations, including in particular the local Langlands correspondences rec_{F_v} and $\mathrm{rec}_{F_v}^T$ for GL_n ; the definition of RACSDC, RAESDC, RACSD automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$; the notion of an ι -ordinary automorphic representation; the weight λ and infinity type α of a regular algebraic, cuspidal automorphic representation; and the existence of Galois representations $r_\iota(\pi)$, and what it means for a Galois representation ρ to be automorphic.

If F is a field, then we write G_F for its absolute Galois group. If F is a number field and v is a finite place of F , then we will write G_{F_v} for a choice of decomposition group at the place v ; it is endowed with a homomorphism $G_{F_v} \rightarrow G_F$, determined up to G_F -conjugacy. We write $q_v = \#k(v)$ for the size of the residue field at v . If F is a number field and S is a finite set of finite places of F , we write $G_{F,S}$ for the Galois group of the maximal extension of F unramified outside S . If $v \notin S$ is a finite place of F , then we write $\text{Frob}_v \in G_{F,S}$ for a (geometric) Frobenius element, uniquely determined up to $G_{F,S}$ -conjugacy. If l is a prime and $\rho : G_F \rightarrow \text{GL}_n(\overline{\mathbb{Q}}_l)$ is a continuous representation, then after conjugation we can assume that ρ takes values in $\text{GL}_n(\overline{\mathbb{Z}}_l)$. The semi-simplification of the composite $G_F \rightarrow \text{GL}_n(\overline{\mathbb{Z}}_l) \rightarrow \text{GL}_n(\overline{\mathbb{F}}_l)$ is independent of this choice, up to isomorphism, and will be denoted by $\bar{\rho}$.

If G is a reductive group over a non-Archimedean local field F , P is a parabolic subgroup of G , and σ is an admissible representation of P , then we write $\text{n-Ind}_P^G \sigma$ for the normalized (unitary) induction of σ , an admissible representation of G . If N is the unipotent radical of P and π is an admissible representation of G , then we write π_N^{norm} for the normalized Jacquet module of π . We will write St_G for the Steinberg representation of G . If $G = \text{GL}_n(F)$ then we will write $\text{St}_{n,F}$ for the Steinberg representation.

2 Admissible representations of a ramified p -adic unitary group

In this section, which is similar in nature to [CT15, §2], we study the representations of ramified p -adic unitary groups which play a key role in the ‘automorphic part’ of the level-raising arguments in §4. As in [CT15], these arguments will rely on fine properties of the local L-packet containing these representations. The reason for considering *ramified* groups is that only for those can we obtain the crucial Proposition 2.9; for unramified groups this multiplicity-one property fails.

The representations we consider are all Iwahori-spherical. We therefore begin by studying a certain Hecke algebra for the ramified unitary group U_n . We then construct certain representations of this group (see Theorem 2.5) using the theory of Kazhdan–Lusztig [KL87], and study their integral structures (see Proposition 2.6) using the theory of Reeder [Ree00]. Ultimately, the representations we construct will be shown to be the Iwahori-spherical members of a certain L-packet; we state this result as Theorem 2.3, which will however be proved only in §3.5.

2.1 A unitary group Hecke algebra

Let p be an odd prime, F a finite extension of \mathbb{Q}_p , and E a ramified quadratic extension of F . If $t \in E$, we will write t^c or \bar{t} to denote the image of t under the non-trivial element of $\text{Gal}(E/F)$. We write k_F for the residue field of F , and $q = \#k_F$. Fix a uniformizer ϖ of E such that $\varpi^2 \in F$, and let $v : E^\times \rightarrow \mathbb{Z}$ denote the valuation with $v(\varpi) = \frac{1}{2}$. Let $n = 2k + 1 \geq 1$ be an odd integer. Let $I = \{\pm 1, \dots, \pm k\}$, and identify E^n with the set of vectors $(x_1, \dots, x_k, x_0, x_{-k}, \dots, x_{-1})$ with entries $x_i \in E$. There is, up to isomorphism, a unique outer form of GL_n over F which is split by E . One choice is as follows. Define the $n \times n$ matrix

$$J = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & 1 & & \\ & & & & \\ 1 & & \ddots & & \end{pmatrix},$$

and let \mathcal{U}_n denote the smooth affine group scheme over \mathcal{O}_F whose R -points are

$$\mathcal{U}_n(R) = \{g \in \text{GL}_n(R \otimes_{\mathcal{O}_F} \mathcal{O}_E) \mid {}^t g J g^c = J\}.$$

Let U_n denote the F -fiber of \mathcal{U}_n , a reductive group over F . Its derived group is SU_n , the subgroup of matrices with determinant 1, which is simply connected. We will begin by describing the building $\mathcal{B}(U_n, F)$ of the group U_n , following [Tit79]. A maximal F -split torus of U_n is the subgroup $S \subset U_n$ given by the matrices

$$\text{diag}(t_1, \dots, t_k, 1, t_{-k}, \dots, t_{-1}), \quad t_i \in F^\times, \quad t_i t_{-i} = 1.$$

This is also a maximal F -split torus of SU_n . We define cocharacters $e_1, \dots, e_k : \mathbb{G}_m \rightarrow S$ by the formulae

$$e_i(t) = (1, \dots, t, \dots, t^{-1}, \dots, 1),$$

where t sits in the i -entry and t^{-1} in the $(-i)$ -entry; these cocharacters form a basis of the free \mathbb{Z} -module $X_*(S)$. We write a_1, \dots, a_k for the dual basis of $X^*(S)$. If $1 \leq i \leq k$, then we also define $a_{-i} = -a_i \in X^*(S)$.

We write $\Phi = \Phi(U_n, S)$ for the root system of S . If we define $a_{ij} = a_i + a_j \in X^*(S)$ for $i, j \in I$, then we have

$$\Phi = \{a_{ij} \mid i, j \in I, i \neq \pm j\} \cup \{a_i, 2a_i \mid i \in I\}.$$

This is a non-reduced root system of type BC_k . We write W_0 for its Weyl group. Viewing BC_k as the union of its subsystems B_k and C_k , $W_0 \cong \{\pm 1\}^k \rtimes \mathfrak{S}_k$ may be identified with the Weyl group of either one of these subsystems. There is a natural action of W_0 on the set $I \cup \{0\}$. Let $Z = Z_{U_n}(S)$. This is a maximal torus of U_n , consisting of the matrices

$$\text{diag}(t_1, \dots, t_k, t_0, t_{-k}, \dots, t_{-1}), t_i \in E^\times, t_i t_{-i}^c = 1 \text{ for each } i = 0, \dots, k.$$

We let $Z_c \subset Z$ denote the maximal compact subgroup, and set $\Lambda = Z/Z_c \cong \mathbb{Z}^m$, a basis being given by the elements

$$\epsilon_i = \text{diag}(1, \dots, \varpi, \dots, -1/\varpi, \dots, 1), 1 \leq i \leq k,$$

where ϖ occupies the i -entry. The group $X^*(Z) = \text{Hom}_F(Z, \mathbb{G}_m)$ embeds as a finite index subgroup of $X^*(S)$ (via restriction), and there is a natural embedding $\nu : \Lambda \hookrightarrow V = X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ which is characterized by the formula $\chi(\nu(z)) = -v(\chi(z))$ ($z \in Z, \chi \in X^*(Z)$). We let $N = N_{U_n}(Z)$, and $W = N/Z_c$ the extended affine Weyl group of U_n with respect to S ; it is an extension of Λ by $N/Z \cong W_0$.

The standard apartment $A(U_n, S, F)$ consists of the following data (see [Tit79, 1.2]):

- An affine space $A = A(U_n, S, F)$ under the real vector space $V = X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$, equipped with an action of the group $W = N/Z_c$ which extends the translation action of Λ via ν .
- A collection $\Phi_{\text{af}} = \Phi_{\text{af}}(U_n, S, F)$ of affine functions $A \rightarrow \mathbb{R}$ and a mapping $\alpha \mapsto X_\alpha$ which assigns to each $\alpha \in \Phi_{\text{af}}$ a subgroup X_α of U_n . The elements of Φ_{af} are called the affine roots of U_n with respect to S .

These data satisfy some additional conditions that we do not describe here. Having defined the standard apartment, the building $\mathcal{B} = \mathcal{B}(U_n, F)$ is the unique U_n -set up to isomorphism satisfying the following conditions (see [Tit79, 2.1]):

- \mathcal{B} contains A , and is the union of its U_n -translates.
- The group $N \subset U_n$ stabilizes A and its action coincides the given one.
- For each affine root $\alpha \in \Phi_{\text{af}}$, the group X_α fixes the half-apartment $A_\alpha = \alpha^{-1}([0, \infty))$ pointwise.

If $\Omega \subset \mathcal{B}$ is a subset, then we write U_n^Ω for the subgroup fixing Ω pointwise. If $\Omega \subset A$, then U_n^Ω can be described explicitly in terms of the affine root system (see [Tit79, 3.1]).

We now describe the standard apartment for the group U_n . The case of the special unitary group $SU_n \subset U_n$ is treated in detail in [Tit79, 1.15], and is very similar. We can identify $A = V \cong \mathbb{R}^n$, with basis e_1, \dots, e_k . The group N consists of matrices $n(\sigma, a_1, \dots, a_{-1}) = (g_{rs})$, $g_{rs} = \delta_{r\sigma(s)} a_s$ (δ is the Kronecker delta), for all $\sigma \in W_0$ and $a_i \in E$ such that $a_i a_{-i}^c = 1$ for all $i = 0, \dots, k$. The action of N on V is given by the formula

$$n(\sigma, a_1, \dots, a_{-1}) \left(\sum_{i=1}^k x_i e_i \right) = \sum_{i=1}^k y_i e_i,$$

where $y_{\sigma(i)} = x_i + v(a_{-i})$ and we define $y_{-i} = -y_i$. The affine roots are the functions $(i, j \in I)$

$$\{a_{ij} + m \mid i \neq \pm j, m \in \frac{1}{2}\mathbb{Z}\} \cup \{a_i + m \mid m \in \frac{1}{2}\mathbb{Z}\} \cup \{2a_i + \frac{1}{2} + m \mid m \in \mathbb{Z}\}.$$

We recall that the chambers of A are by definition the connected components of the complement of the vanishing hyperplanes of the affine roots $\alpha \in \Phi_{\text{af}}$. A chamber is given in this case by the series of inequalities

$$C : 0 < a_k < a_{k-1} < \cdots < a_1 < \frac{1}{4}.$$

The corresponding root basis is $\{\alpha_0, \dots, \alpha_k\} = \{2a_{-1} + \frac{1}{2}, a_{1,-2}, a_{2,-3}, \dots, a_{k-1,-k}, a_k\}$. Since every element of the quotient $W = N/Z_c$ can be represented by an element of the group $N \cap SU_n$, and SU_n is simply connected, we see that W is generated by the corresponding set $\{s_0, \dots, s_k\}$ of simple reflections [Tit79, 1.13]. The affine reflection $s_0 \in W$ corresponding to the simple affine root $\alpha_0 = 2a_{-1} + \frac{1}{2}$ is, by definition, the unique reflection in $A \cong V$ with vector part $s_{a_{-1}}$ and fixed hyperplane $2a_{-1} + \frac{1}{2} = 0$. For later use, we note the expression (for $\sum_i x_i e_i \in V$):

$$s_0 \left(\sum_{i=1}^k x_i e_i \right) = \sum_{i=2}^k x_i e_i + \left(\frac{1}{2} - x_1 \right) e_1. \quad (2.1)$$

The point 0 of A is a special vertex [Tit79, 1.9], and there is an induced semi-direct product decomposition $W = W_0 \ltimes \Lambda$, where we identify $W_0 = \langle s_1, \dots, s_k \rangle \subset W$. The corresponding group scheme of this special vertex ([Tit79, 3.4]) is canonically identified with \mathcal{U}_n . In particular, the group $K = \mathcal{U}_n(\mathcal{O}_F)$ is a special subgroup of U_n . It follows from [Tit79, 3.1] that the group $\mathfrak{B} = U_n^C$ is the pre-image in K under the canonical reduction map $\mathcal{U}_n(\mathcal{O}_F) \rightarrow \mathcal{U}_n(k_F) \rightarrow \text{O}(J)(k_F)$ of the upper-triangular subgroup of $\text{O}(J)(k_F)$. (Here $\text{O}(J)$ denotes the orthogonal group of the symmetric bilinear form defined by J .) The group \mathfrak{B} contains an Iwahori subgroup with index 2 (pre-image of the upper-triangular subgroup of $\text{SO}_n(k_F) \subset \mathcal{U}_n(k_F) \cong \text{O}_n(k_F)$, see [Tit79, 3.7]).

We define $\mathfrak{P} = U_n^{\mathcal{F}}$, where \mathcal{F} is the subset of A defined by the conditions

$$\mathcal{F} : 0 = a_k < a_{k-1} < \cdots < a_1 < \frac{1}{4}.$$

By [Tit79, 3.1], it is identified with the pre-image in K under the canonical reduction map $\mathcal{U}_n(\mathcal{O}_F) \rightarrow \mathcal{U}_n(k_F) \rightarrow \text{O}(J)(k_F)$ of the matrices in $\text{O}(J)(k_F)$ which are block upper-triangular, with blocks of size $1 + \cdots + 1 + 3 + 1 + \cdots + 1$. We see from the definitions that $[\mathfrak{P} : \mathfrak{B}] = q + 1$ (observe that U_n is residually split).

We let B denote the upper-triangular Borel subgroup of U_n , and P the parabolic subgroup generated by B and the root subgroup corresponding to a_{-k} . We write $B = TN_0$ and $P = MN_P$ for the Levi decompositions of B and P with respect to S ; thus $T = Z$ and the derived subgroup of M is isomorphic to SU_3 .

Lemma 2.1. *Let π be an admissible $\mathbb{C}[U_n]$ -module. Then there are canonical isomorphisms $\pi^{\mathfrak{B}} \cong \pi_{N_0}^{Z_c}$ and $\pi^{\mathfrak{P}} \cong \pi_{N_P}^{M \cap \mathfrak{P}}$, where the subscript indicates unnormalized Jacquet module.*

Proof. The first isomorphism is [Cas80, Proposition 2.4]. The second follows from the first in the same way as in [CT15, Proposition 2.1]. \square

We write $H_{\mathfrak{B}}$ for the convolution algebra of \mathfrak{B} -biinvariant functions $f : U_n \rightarrow \mathbb{Z}$, the Haar measure on G being normalized so that \mathfrak{B} has measure 1. Thus $H_{\mathfrak{B}}$ is a free \mathbb{Z} -module, a basis being given by the double cosets $[\mathfrak{B}g\mathfrak{B}]$ for $g \in G$; $[\mathfrak{B}]$ is the unit element. If R is a ring then we define $H_{\mathfrak{B},R} = H_{\mathfrak{B}} \otimes_{\mathbb{Z}} R$. As the extended affine Weyl group of U_n equals the affine Weyl group, it follows from [Tit79, 3.1.1] that the triple (U_n, \mathfrak{B}, N) is a Tits system (or BN-pair). Consequently, the Hecke algebra $H_{\mathfrak{B}}$ admits the following explicit description [Mat64, Théorème 4].

The affine Weyl group W is generated by the set $\Sigma = \{s_0, \dots, s_k\}$ in the simple affine roots, and the pair (W, Σ) is an affine Coxeter group of type B_k . There is a natural length function $l : W \rightarrow \mathbb{N}$, which takes a Weyl element $w \in W$ to the length of the shortest word in the elements of Σ representing w . For each $w \in W$, the index $q_w = \#BwB/B$ equals $q^{l(w)}$ (see [Tit79, 3.3.1]). The associated braid group B_W is defined as the quotient of the free group in the elements T_w , $w \in W$, by the relations $T_{ww'} = T_w T_{w'}$ if

$l(w'w) = l(w) + l(w')$. The Hecke algebra $H_{\mathfrak{B}}$ is isomorphic to the quotient of the algebra $\mathbb{Z}[B_W]$ by the relations $(T_s - 1)(T_s + q) = 0$, $s \in \Sigma$, the isomorphism being given by the formula $[\mathfrak{B}w\mathfrak{B}] \mapsto T_w$.

As in [CT15, §2], we now study the algebra $H_{\mathfrak{B}}$ by identifying it with the Iwahori–Hecke algebra of a split reductive group over F . Let Sp_{2k} denote the symplectic group over F in $2k = n - 1$ variables over F , defined by the form

$$J' = \begin{pmatrix} & & & & & 1 \\ & & & & \ddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \ddots & & & & \\ -1 & & & & & \end{pmatrix},$$

and let $S' \subset \mathrm{Sp}_{2k}$ denote the maximal F -split torus consisting of matrices

$$(t_1, \dots, t_k, t_{-k}, \dots, t_{-1}), t_i \in F^\times, t_i t_{-i} = 1 \text{ for each } i = 1, \dots, k.$$

Then S' is also a maximal torus of Sp_{2k} . Let $S'_c \subset S'$ denote the maximal compact subgroup, and let $\Lambda' = S'/S'_c$. Then $\Lambda' \cong \mathbb{Z}^k$, a basis being given by the elements

$$\epsilon'_i = (1, \dots, \varpi^2, \dots, \varpi^{-2}, \dots, 1),$$

where ϖ^2 occupies the i -entry. We identify $\Lambda \cong \Lambda'$ via the isomorphism which sends ϵ_i to ϵ'_i for each $i = 1, \dots, k$.

Let W'_0 denote the Weyl group of S' , and let $N' = N_{\mathrm{Sp}_{2k}}(S')$, $W' = N'/S'_c$. Let \mathfrak{B}' denote the open compact subgroup of Sp_{2k} consisting of matrices with integral entries which are upper-triangular modulo the maximal ideal of \mathcal{O}_F . Then \mathfrak{B}' is an Iwahori subgroup of Sp_{2k} , and we write $H_{\mathfrak{B}'}$ for its associated Hecke algebra. We write $K' \subset \mathrm{Sp}_{2k}$ for the subgroup of matrices with integral entries; it is a hyperspecial maximal compact subgroup of Sp_{2k} , and there is a corresponding semi-direct product decomposition $W' \cong W'_0 \ltimes \Lambda'$.

Proposition 2.2. *There is an isomorphism $H_{\mathfrak{B}} \cong H_{\mathfrak{B}'}$ of Hecke algebras sending $[\mathfrak{B}\epsilon_i\mathfrak{B}]$ to $[\mathfrak{B}'\epsilon'_i\mathfrak{B}']$ for each $i = 1, \dots, k$.*

Proof. Using the identifications $W \cong W_0 \ltimes \Lambda$, $W' \cong W'_0 \ltimes \Lambda'$, we can extend the isomorphisms $\Lambda \cong \Lambda'$, $W_0 \cong W'_0$ to an isomorphism $W \cong W'$. We must check that this isomorphism sends $\Sigma \subset W$ to a set of simple reflections $\Sigma' \subset W'$ corresponding to a choice of basis of affine roots. Indeed, the triple $(\mathrm{Sp}_{2k}, \mathfrak{B}', N')$ is a Tits system (because Sp_{2k} is simply connected). Appealing again to [Mat64], we then see that the isomorphism $f : W \cong W'$ then determines an isomorphism $B_W \cong B_{W'}$ of braid groups which gives, by passage to quotient, an isomorphism $H_{\mathfrak{B}} \cong H_{\mathfrak{B}'}$, which sends $[\mathfrak{B}w\mathfrak{B}]$ to $[\mathfrak{B}'f(w)\mathfrak{B}']$.

To do this check, we note that \mathfrak{B}' acts trivially on a unique chamber C' in the standard apartment $A(\mathrm{Sp}_{2k}, S', F)$. This apartment admits a simple description, because Sp_{2k} is split (see [Tit79, 1.1]), and it is easy to check that the isomorphism $W \cong W'$ sends $\{s_0, \dots, s_k\}$ to the set of simple reflections corresponding to the basis of affine roots corresponding to C' . (The main point is that s_0 is mapped to the simple reflection in $W' - W'_0$, and this follows from the formula $s_0 = s_{a_1} \cdot \epsilon_1$, consequence of (2.1).) \square

We now identify $W_0 = W'_0$ and $\Lambda = \Lambda'$ and introduce the Bernstein presentation of the algebra $H_{\mathfrak{B}, \mathbb{C}} \cong H_{\mathfrak{B}', \mathbb{C}}$, following Lusztig [Lus89, §3]. It is defined in terms of a based root datum $(X, Y, R, \check{R}, \Pi)$. We take $X = \Lambda$, $Y = \mathrm{Hom}(\Lambda, \mathbb{Z})$. We take R to be the set of roots

$$\{\pm\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq k\} \cup \{\pm\epsilon_i \mid 1 \leq i \leq k\},$$

and \check{R} to be the set of coroots

$$\{\pm f_i \pm f_j \mid 1 \leq i < j \leq k\} \cup \{\pm 2f_i \mid 1 \leq i \leq k\},$$

where f_1, \dots, f_k is the basis of Y dual to the basis $\epsilon_1, \dots, \epsilon_k$ of X . We choose the set Π of simple roots to consist of the elements

$$\beta_i = \epsilon_i - \epsilon_{i+1}, 1 \leq i \leq k-1, \text{ and } \beta_k = \epsilon_k.$$

This is the based root datum of the group $\mathrm{SO}_{2k+1}(\mathbb{C})$, the Langlands dual of the simply connected group Sp_{2k} . In terms of this data, Lusztig defines in [Lus89, §3.2] a Hecke algebra H , which is the quotient of the group algebra $\mathbb{C}[v, v^{-1}][B_W]$ by the relations $(T_s + 1)(T_s - v^2)$ ($s \in \Sigma$). It follows from the above discussion that this algebra becomes canonically identified with $H_{\mathfrak{B}, \mathbb{C}}$ after specializing to the value $v = q^{1/2}$. The analysis in [Lus89] therefore applies to our Hecke algebra of interest.

This analysis gives rise to an isomorphism, the Bernstein presentation, of the algebra $H_{\mathfrak{B}, \mathbb{C}}$ with the twisted tensor product

$$H_{\mathfrak{B}, \mathbb{C}} = H_0 \widetilde{\otimes}_{\mathbb{C}} \mathbb{C}[X],$$

where $H_0 \subset H_{\mathfrak{B}, \mathbb{C}}$ is the subalgebra spanned by the elements $T_w, w \in W_0$ (abstractly isomorphic to the group algebra $\mathbb{C}[W_0]$) and $\mathbb{C}[X]$ is the co-ordinate ring of the complex algebraic torus $\mathrm{Hom}(\Lambda, \mathbb{C}^\times)$. The twisted tensor product is the usual tensor product as complex vector spaces, with algebra structure being determined by the following commutation relation. If $\beta \in \Pi$ is a simple root and $s = s_\beta \in W_0$ is the corresponding simple reflection, then $T_s \in H_0$, hence $B_s = T_s - q \in H_0$. If $\theta \in \mathbb{C}[X]$, then the following relation holds:

$$\theta B_s = B_s \theta^s + (\theta^s - \theta) \zeta_\beta,$$

where $\zeta_\beta = (q - e_\beta)/(1 - e_\beta)$ if β is a long root, and $\zeta_\beta = (q - e_\beta)(1 + e_\beta)/(1 - e_{2\beta})$ otherwise. (Here we write $e_\lambda \in \mathbb{C}[X]$ for the basis element corresponding to $\lambda \in \Lambda$, and W_0 acts on $\mathbb{C}[X]$ via its natural right action.)

2.2 Functoriality and base change for the group U_n

We continue with the notation of the previous section. We now recall some elements of the theory of the Langlands parameterization; we will take these ideas up again in §3.1 below, in a more general context. If G is a reductive F -group, we write $\Pi(G)$ for the set of isomorphism classes of irreducible admissible $\mathbb{C}[G]$ -modules, and $\Pi_{\mathrm{temp}}(G)$ for the subset of isomorphism classes of tempered modules. We recall that thanks to the work of Mœglin [Mœg07] and Mok [Mok15] there is a decomposition of $\Pi_{\mathrm{temp}}(U_n)$ as a disjoint union of sets Π_φ , as φ ranges over conjugacy classes of Langlands parameters

$$W_F \times \mathrm{SU}_2(\mathbb{R}) \rightarrow {}^L U_n.$$

Restriction to the subgroup $W_E \times \mathrm{SU}_2(\mathbb{R})$ induces a bijection between the set of these conjugacy classes, and the set of conjugacy classes of parameters

$$W_E \times \mathrm{SU}_2(\mathbb{R}) \rightarrow \mathrm{GL}_n(\mathbb{C})$$

which are conjugate orthogonal; see [Mok15, Lemma 2.2.1]. In particular, there is a map $\Pi_{\mathrm{temp}}(U_n) \rightarrow \Pi_{\mathrm{temp}}(\mathrm{GL}_{n,E})$, with image given by those representations corresponding to conjugate orthogonal parameters. We say that such a representation π of $\mathrm{GL}_{n,E}$ is in the image of the stable base change map. The fibres of the map $\Pi_{\mathrm{temp}}(U_n) \rightarrow \Pi_{\mathrm{temp}}(\mathrm{GL}_{n,E})$ are the L-packets Π_φ .

We are interested in a particular L-packet. Suppose that $n \geq 7$, and let $\Pi_E = \mathrm{St}_{n-4,E} \boxplus \mathrm{St}_{3,E} \boxplus \mathrm{St}_{1,E}$, a tempered representation of $\mathrm{GL}_{n,E}$. Let φ_E denote the corresponding parameter. It extends uniquely to a parameter φ of the previous type. Recall that there is a correspondence $f_E \rightsquigarrow f$ between smooth, compactly supported functions f_E on $U_n(E)$ and f on U_n , associated to stable base change. In the statement of the following theorem, we write $[m_1, \dots, m_k]$ for the character $|\cdot|^{m_1} \otimes |\cdot|^{m_2} \otimes \dots \otimes |\cdot|^{m_k} \otimes 1$ of the group $E^\times \times \dots \times E^\times \times (E^\times)_{\mathbb{N}_{E/F}=1}$, the F -points of the maximal torus $T = Z_{U_n}(S) \subset U_n$. We will call such a character (generally occurring in a Jacquet module) an *exponent*. We write $(?)_{N_0}^{\mathrm{norm}}$ for the normalized Jacquet module with respect to the unipotent radical of the upper-triangular Borel subgroup of U_n .

Theorem 2.3. *1. Suppose that $n = 7$. Then Π_φ contains exactly two elements X, Y , which are uniquely characterized by the following properties:*

- (a) $\langle \text{tr } X + \text{tr } Y, f \rangle = \langle \text{tr } \Pi_E \times I_\theta, f_E \rangle$, where $I_\theta : \Pi_E \cong \Pi_E^c$ is the Whittaker-normalized intertwining operator.
- (b) $\dim X^\mathfrak{B} = \dim X^\mathfrak{B} = 1$ and $\dim Y^\mathfrak{B} = 7$.
- (c) $X_{N_0}^{\text{norm}} = [1, 0, -1]$ and $(Y_{N_0}^{\text{norm}})^{ss} = [1, 0, -1] + 2[1, 0, 1] + 4[1, 1, 0]$.
2. Suppose that $n = 9$. Then Π_φ is a discrete L-packet and contains exactly four elements X, Y, Z, W , which are uniquely characterized by the following properties:
- (a) $\langle \text{tr } X + \text{tr } Y + \text{tr } Z + \text{tr } W, f \rangle = \langle \text{tr } \Pi_E \times I_\theta, f_E \rangle$, where $I_\theta : \Pi_E \cong \Pi_E^c$ is the Whittaker-normalized intertwining operator.
- (b) $\dim X^\mathfrak{B} = \dim X^\mathfrak{B} = 1$, $\dim Y^\mathfrak{B} = 11$, $\dim Z^\mathfrak{B} = 2$, and $\dim W^\mathfrak{B} = 0$.
- (c) We have

$$\begin{aligned} X_{N_0}^{\text{norm}} &= [2, 1, 0, -1] \\ (Y_{N_0}^{\text{norm}})^{ss} &= [2, 1, 0, -1] + 2[2, 1, 0, 1] + 4[2, 1, 1, 0] + [1, 0, 2, 1] + [1, 2, 0, 1] + 2[1, 2, 1, 0] \\ (Z_{N_0}^{\text{norm}})^{ss} &= [1, 0, 2, 1] + [1, 2, 0, 1]. \end{aligned}$$

3. Suppose that $n = 2k + 1 \geq 11$. Then Π_φ is a discrete L-packet and contains exactly four elements X_k, Y_k, Z_k, W_k , which are uniquely characterized by the following (inductively defined) properties:
- (a) $\langle \text{tr } X_k + \text{tr } Y_k + \text{tr } Z_k + \text{tr } W_k, f \rangle = \langle \text{tr } \Pi_E \times I_\theta, f_E \rangle$, where $I_\theta : \Pi_E \cong \Pi_E^c$ is the Whittaker-normalized intertwining operator.
- (b) $\dim X_k^\mathfrak{B} = \dim X_k^\mathfrak{B} = 1$, $\dim Y_k^\mathfrak{B} = \dim Y_{k-1}^\mathfrak{B} + k$, $\dim Z_k^\mathfrak{B} = \dim Z_{k-1}^\mathfrak{B} + (k - 2)$, and $\dim W_k^\mathfrak{B} = 0$.
- (c) The exponents of the Jacquet modules of these representations are as follows.
- i. $X_{k, N_0}^{\text{norm}} = [k - 2, k - 3, \dots, 0, -1]$.
 - ii. $(Y_{k, N_0}^{\text{norm}})^{ss}$ is the sum of the exponents $[1, k - 2, k - 3, \dots, 1]$ with 0 inserted as the i entry for $i = 2, \dots, k - 1$, the exponent $2[1, k - 2, k - 3, \dots, 0]$, together with the exponents of $(Y_{k-1, N_0}^{\text{norm}})^{ss}$ with $k - 2$ inserted as the first entry.
 - iii. $(Z_{k, N_0}^{\text{norm}})^{ss}$ is the sum of the exponents $[1, k - 2, k - 3, \dots, 1]$ with 0 inserted as the i entry for $i = 2, \dots, k - 1$, together with the exponents of $(Z_{k-1, N_0}^{\text{norm}})^{ss}$ with $k - 2$ inserted as the first entry.

The proof of Theorem 2.3 will be given in several stages. Kazhdan and Lusztig have proved [KL87] for certain groups what they call the Deligne–Langlands conjecture, which includes as a special case Theorem 2.4 below. We first apply their construction to obtain Theorem 2.5, which asserts the existence of semi-stable admissible $\mathbb{C}[U_n]$ -modules X, Y and (if $n \geq 9$) Z , defined in terms of the Langlands parameter φ_E , and with Jacquet modules as described by Theorem 2.3 above. It seems reasonable to guess that these modules are the semi-stable modules in the L-packet corresponding to the Langlands parameter. However, this requires proof, and does not follow immediately from the construction of Kazhdan–Lusztig. (Indeed, our L-packets are characterized by an identity of stable traces; since the construction of Kazhdan–Lusztig yields only the representations in the packet with non-zero Iwahori invariants, some further argument must be required.)

In §3 we will describe, following [Mok15], the endoscopic classification of representations of the group U_n , and describe in more detail the L-packets of representations corresponding under this classification to the parameter φ_E . We will then verify in §3.5 that the representations constructed in Theorem 2.5 do indeed appear in this L-packet, and thus deduce Theorem 2.3.

Theorem 2.4. *Let G be a split reductive group over F , with connected center, and let $\mathfrak{I} \subset G$ be an Iwahori subgroup. Then the following two sets are in natural bijection:*

1. The set of representations $\pi \in \Pi_{\text{temp}}(G)$ such that $\pi^\mathfrak{I} \neq 0$.
2. The set of \widehat{G} -conjugacy classes of triples (s, u, ρ) , where:

- (a) $s \in \widehat{G}$ is a semi-simple element contained in a maximal compact subgroup of $\widehat{G}(\mathbb{C})$.
- (b) $u \in \widehat{G}$ is a unipotent element such that $usu^{-1} = u^q$.
- (c) Let \mathcal{B}_u^s be the variety of Borel subgroups $B \subset \widehat{G}$ such that $s \in B$ and $u \in B$. Let $\overline{M}(u, s) = \pi_0((Z_{\widehat{G}}(s) \cap Z_{\widehat{G}}(u))/Z_{\widehat{G}})$. Then ρ is an irreducible representation of $\overline{M}(u, s)$ such that

$$\mathrm{Hom}_{\mathbb{C}[\overline{M}(u, s)]}(\rho, H_*(\mathcal{B}_u^s, \mathbb{C})) \neq 0.$$

Proof. The existence of the bijection is [KL87, Theorem 7.12]; its particular properties in the tempered case are contained in [KL87, Theorem 8.2]. The relevant Hecke-modules are defined in terms of equivariant K-homology, but can be recovered as complex vector spaces as the usual singular homology with complex coefficients, cf. [Ree94, §2]. For a description of these modules directly in terms of (Borel–Moore) homology, see [CG10, Ch. 8]. \square

Theorem 2.5. 1. Suppose that $n = 7$. Then there exist irreducible admissible $\mathbb{C}[U_n]$ -modules X, Y with the following properties.

- (a) $\dim X^{\mathfrak{P}} = \dim X^{\mathfrak{B}} = 1$ and $\dim Y^{\mathfrak{B}} = 7$.
- (b) $X_{N_0}^{\mathrm{norm}} = [1, 0, -1]$ and $(Y_{N_0}^{\mathrm{norm}})^{ss} = [1, 0, -1] + 2[1, 0, 1] + 4[1, 1, 0]$.

2. Suppose that $n = 9$. Then there exist irreducible admissible $\mathbb{C}[U_n]$ -modules X, Y , and Z with the following properties:

- (a) $\dim X^{\mathfrak{P}} = \dim X^{\mathfrak{B}} = 1$, $\dim Y^{\mathfrak{B}} = 11$, and $\dim Z^{\mathfrak{B}} = 2$.
- (b) We have

$$\begin{aligned} X_{N_0}^{\mathrm{norm}} &= [2, 1, 0, -1] \\ (Y_{N_0}^{\mathrm{norm}})^{ss} &= [2, 1, 0, -1] + 2[2, 1, 0, 1] + 4[2, 1, 1, 0] + [1, 0, 2, 1] + [1, 2, 0, 1] + 2[1, 2, 1, 0] \\ (Z_{N_0}^{\mathrm{norm}})^{ss} &= [1, 0, 2, 1] + [1, 2, 0, 1]. \end{aligned}$$

3. Suppose that $n = 2k + 1 \geq 11$. Then there exist irreducible admissible $\mathbb{C}[U_n]$ -modules X_k, Y_k , and Z_k , with the following properties:

- (a) $\dim X_k^{\mathfrak{P}} = \dim X_k^{\mathfrak{B}} = 1$, $\dim Y_k^{\mathfrak{B}} = \dim Y_{k-1}^{\mathfrak{B}} + k$, and $\dim Z_k^{\mathfrak{B}} = \dim Z_{k-1}^{\mathfrak{B}} + (k - 2)$.
- (b) The exponents of the Jacquet modules of these representations are as follows.
 - i. $X_{k, N_0}^{\mathrm{norm}} = [k - 2, k - 3, \dots, 0, -1]$.
 - ii. $(Y_{k, N_0}^{\mathrm{norm}})^{ss}$ is the sum of the exponents $[1, k - 2, k - 3, \dots, 1]$ with 0 inserted as the i entry for $i = 2, \dots, k - 1$, the exponent $2[1, k - 2, k - 3, \dots, 0]$, together with the exponents of $(Y_{k-1, N_0}^{\mathrm{norm}})^{ss}$ with $k - 2$ inserted as the first entry.
 - iii. $(Z_{k, N_0}^{\mathrm{norm}})^{ss}$ is the sum of the exponents $[1, k - 2, k - 3, \dots, 1]$ with 0 inserted as the i entry for $i = 2, \dots, k - 1$, together with the exponents of $(Z_{k-1, N_0}^{\mathrm{norm}})^{ss}$ with $k - 2$ inserted as the first entry.

Proof. All the representations we construct will have non-zero \mathfrak{B} -invariant vectors. Since we have identified (in Proposition 2.2) the Hecke algebra $H_{\mathfrak{B}, \mathbb{C}}$ with the Iwahori–Hecke algebra of a split group $\mathrm{Sp}_{2k}(F)$, we can construct the desired $\mathbb{C}[U_n]$ -modules by constructing representations of $\mathrm{Sp}_{2k}(F)$ with non-zero Iwahori-fixed vectors. We will in fact apply Theorem 2.4 to construct irreducible representations of $\mathrm{PSp}_{2k}(F)$ with irreducible pullback to $\mathrm{Sp}_{2k}(F)$. We fix a maximal torus $T \subset \mathrm{PSp}_{2k}(F)$, and identify the dual group of $\mathrm{Sp}_{2k}(F)$ as $\mathrm{SO}_{2k+1}(\mathbb{C})$. To construct the data (s, u) , note that for each odd integer i there is, up to isomorphism, a unique irreducible representation $V(i)$ of $\mathrm{SL}_2(\mathbb{C})$ of dimension i , which is orthogonal. We fix a choice of symmetric bilinear form on $V(i)$. Given a partition of $n = 2k + 1 = n_1 + \dots + n_r$ into odd integers, there is a homomorphism $\mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{SO}_{2k+1}(\mathbb{C})$, well-defined up to conjugacy in the latter group, giving the

action of $\mathrm{SL}_2(\mathbb{C})$ on $V(n_1) \oplus \cdots \oplus V(n_r)$. This lifts uniquely to a homomorphism $\varphi : \mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{Spin}_{2k+1}(\mathbb{C})$. We associate to the partition the pair (s, u) where

$$s = \varphi \left(\begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix} \right) \text{ and } u = \varphi \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right).$$

We now split into cases, as in the statement of the theorem.

1. The relevant partition is $7 = 3 + 3 + 1$. The variety \mathcal{B}_u^s can be identified with the variety of isotropic flags $F^\bullet = (0 \subset F^1 \subset F^2 \subset F^3)$ in \mathbb{C}^7 such that $s(F^i) \subset F^i$ and $u(F^i) \subset F^i$ for each $i = 1, 2, 3$. The group $\overline{M}(u, s)$ has two elements. Indeed, if we write $S = \mathrm{SL}_2(\mathbb{C})$ and $B_S \subset S$ for its standard upper-triangular Borel subgroup, then for any (algebraic) representation of S on a finite-dimensional \mathbb{C} -vector space V , we have $H^0(S, V) = H^0(B_S, V)$ and $\mathrm{End}_S(V) = \mathrm{End}_{B_S}(V)$. On the other hand, the elements

$$s_0 = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}, u_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

generate a Zariski dense subgroup of B_S . It follows that the groups $M(u, s)$ and $\overline{M}(u, s)$ can be computed in terms of the centralizer of the parameter φ , giving $\overline{M}(u, s) \cong \{\pm 1\}$.

The variety \mathcal{B}_u^s is isomorphic to the disjoint union of varieties X_1, X_2, X_3 on which $\overline{M}(u, s)$ acts. According to [KL87, §7.6], we can associate a character e_α of the maximal torus S' of $\mathrm{Sp}_{2k}(F)$ to each $\overline{M}(u, s)$ -orbit \mathcal{B}^α of connected components of \mathcal{B}_u^s . The semi-simplification of the Jacquet module of the representation associated to a triple (u, s, ρ) is then $\bigoplus_\alpha \dim_{\mathbb{C}} \mathrm{Hom}_{\overline{M}(u, s)}(\rho, H_*(\mathcal{B}^\alpha, \mathbb{C})) \cdot e_\alpha$. The varieties X_i and their associated exponents are as follows:

- (a) X_1 is a union of two points which are interchanged by $\overline{M}(u, s)$. The associated exponent is $[1, 0, -1]$.
- (b) $X_2 \cong \mathbb{P}^1$. The associated exponent is $[1, 0, 1]$.
- (c) $X_3 \cong \mathbb{P}^1 \times \mathbb{P}^1$. The associated exponent is $[1, 1, 0]$.

In general, calculating explicitly the varieties \mathcal{B}_u^s is a difficult problem; see [DCLP88]. We now describe how one can obtain the above description in this case. Let $D \subset \widehat{G} = \mathrm{Spin}_{2k+1}(\mathbb{C})$ denote the image under φ of the diagonal torus in SL_2 , and let $T \subset \widehat{G}$ denote the connected centralizer of the image of φ . Then T is a rank 1 torus, and $D \times T$ is regular, in the sense that its centralizer in \widehat{G} is a maximal torus. In particular, the eigenspaces of $D \times T$ in the standard n -dimensional representation of \widehat{G} are 1-dimensional.

Let $\mathcal{B}_u \subset \mathcal{B}$ denote the variety of Borel subgroups containing the unipotent element u . Let $\mathcal{B}_u^D \subset \mathcal{B}_u$ denote the D -fixed subvariety, and $\mathcal{B}_u^{D, T}$ the $D \times T$ -fixed subvariety. Then \mathcal{B}_u^D is smooth and projective, for general reasons; see [DCLP88, §3.6]. On the other hand, we have $\mathcal{B}_u^D = \mathcal{B}_u^s$, hence \mathcal{B}_u^s is smooth and projective.

We now apply the theorem of Bialynicki-Birula (cf. [CG10, Theorem 2.4.3]) to the action of the torus T on the smooth, projective variety \mathcal{B}_u^D . It is easy to show that $\mathcal{B}_u^{D, T}$ is a finite set, containing 8 elements. Indeed, if $F^\bullet \in \mathcal{B}_u^{D, T}$, then each F^i must be a sum of $D \times T$ -eigenspaces. It follows that \mathcal{B}_u^D has a paving by affine spaces X_w , indexed by elements $w \in \mathcal{B}_u^{D, T}$. This immediately implies that $H_*(\mathcal{B}_u^s, \mathbb{C})$ is 8-dimensional, and that the exponents of the Jacquet module of $X + Y$, taken with their multiplicities, are as claimed in the theorem. One can now compute the cells X_w explicitly, together with the action of the group $\overline{M}(u, s)$; this gives the varieties described above.

2. The relevant partition is $9 = 5 + 3 + 1$. The group $\overline{M}(u, s)$ is isomorphic to $\{\pm 1\} \times \{\pm 1\}$. We can choose distinct non-trivial characters ϵ_X, ϵ_Z of this group so that the variety \mathcal{B}_u^s is isomorphic to a union of varieties X_1, \dots, X_6 on which the group $\overline{M}(u, s)$ acts, and having the following properties:
 - (a) X_1 and X_2 are each a union of two points, which are interchanged by the quotient of $\overline{M}(u, s)$ by $\ker \epsilon_Z$. The associated exponents are, respectively, $[1, 0, 2, 1]$ and $[1, 2, 0, 1]$.

- (b) X_3 is isomorphic to \mathbb{P}^1 . The associated exponent is $[1, 2, 1, 0]$.
- (c) X_4 is a union of two points, which are interchanged by the quotient of $\overline{M}(u, s)$ by $\ker \epsilon_X$. The associated exponent is $[2, 1, 0, -1]$.
- (d) X_5 is isomorphic to \mathbb{P}^1 . The associated exponent is $[2, 1, 0, 1]$.
- (e) X_6 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The associated exponent is $[2, 1, 1, 0]$.

This description of the variety \mathcal{B}_u^s can be obtained by applying the inductive procedure of [Ree94, pp. 485–486] and [DCLP88, §3.8].

3. The relevant partition is $n = (n - 4) + 3 + 1$. The group $\overline{M}(u, s)$ is isomorphic to $\{\pm 1\} \times \{\pm 1\}$. We can choose distinct non-trivial characters ϵ_X, ϵ_Z of this group so that the variety \mathcal{B}_u^s is isomorphic to a union of varieties X_1, \dots, X_{k-1} and $\mathcal{B}_u^s(k-1)$ (i.e. the variety associated to the group $\mathrm{PSp}_{2(k-1)}$) and having the following properties.
 - (a) The varieties X_1, \dots, X_{k-2} are each isomorphic to a union of two points, which are interchanged by the quotient of $\overline{M}(u, s)$ by $\ker \epsilon_Z$. The associated exponent of X_i is $[1, k-2, k-3, \dots, 1]$, with an extra 0 inserted as the $i+1$ entry.
 - (b) The variety X_{k-1} is isomorphic to \mathbb{P}^1 . The associated exponent is $[1, k-2, k-3, \dots, 1, 0]$.
 - (c) The action on $\mathcal{B}_u^s(k-1)$ is the one constructed by induction. The associated exponents are those of the variety $\mathcal{B}_u^s(k-1)$, with $k-2$ inserted as the first entry. (Compare the statement of the theorem.)

This description of the variety \mathcal{B}_u^s can be obtained by applying the inductive procedure of [Ree94, pp. 485–486] and [DCLP88, §3.8].

Finally, we have to check that $\dim X^\mathfrak{B} = 1$. By Lemma 2.1, this follows from the computation of $X_{N_0}^{norm}$ and the fact that the representation of $U(3)$ induced from the character $[-1]$ has the trivial representation as its unique submodule. \square

2.3 Integral structures in semi-stable Hecke modules

Let us now suppose that $l = n - 2$ is prime, and fix an isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$. Let $K \subset \overline{\mathbb{Q}}_l$ be a finite extension of \mathbb{Q}_l over which all the representations $\iota^{-1}X, \iota^{-1}Y, \iota^{-1}Z, \iota^{-1}W$ (if the latter two exist) may be defined. (We recall that if V is an irreducible admissible $\overline{\mathbb{Q}}_l[U_n]$ -module, then $V^\mathfrak{B}$ is an irreducible $H_{\mathfrak{B}, \overline{\mathbb{Q}}_l}$ -module; we can therefore find a finite extension K/\mathbb{Q}_l and a $H_{\mathfrak{B}, K}$ -module $M \subset V^\mathfrak{B}$ such that the natural map $M \otimes_K \overline{\mathbb{Q}}_l \rightarrow V^\mathfrak{B}$ is an isomorphism. The procedure of [BH06, 4.3, Proposition] then gives a model of V which is defined over K .) We assume that K contains a square root of q . We write X_K, Y_K etc. for a choice of $K[U_n]$ -modules realizing this descent.

Let \mathcal{O} denote the ring of integers of K , $\lambda \subset \mathcal{O}$ the maximal ideal, $k = \mathcal{O}/\lambda$ the residue field. As explained in the introduction, the proofs of our main theorems are restricted to the cases $l = 5$ and $l = 7$. Therefore, the following computations have been completed only in these cases.

Proposition 2.6. *1. Suppose that $n = 7$, and that q is a primitive root modulo l . Then there exist $H_{\mathfrak{B}, \mathcal{O}}$ -submodules $X_{\mathcal{O}}^\mathfrak{B} \subset X_K^\mathfrak{B}$ and $Y_{\mathcal{O}}^\mathfrak{B} \subset Y_K^\mathfrak{B}$ which are finite \mathcal{O} -modules, and such that the natural maps*

$$X_{\mathcal{O}}^\mathfrak{B} \otimes_{\mathcal{O}} K \rightarrow X_K^\mathfrak{B} \text{ and } Y_{\mathcal{O}}^\mathfrak{B} \otimes_{\mathcal{O}} K \rightarrow Y_K^\mathfrak{B}$$

are isomorphisms, and $X_{\mathcal{O}}^\mathfrak{B} \otimes_{\mathcal{O}} k$ and $Y_{\mathcal{O}}^\mathfrak{B} \otimes_{\mathcal{O}} k$ have no Jordan-Hölder factors as $H_{\mathfrak{B}, k}$ -modules in common.

2. Suppose that $n = 9$, and that q is a primitive root modulo l . Then there exist $H_{\mathfrak{B}, \mathcal{O}}$ -submodules $X_{\mathcal{O}}^\mathfrak{B} \subset X_K^\mathfrak{B}$, $Y_{\mathcal{O}}^\mathfrak{B} \subset Y_K^\mathfrak{B}$, and $Z_{\mathcal{O}}^\mathfrak{B} \subset Z_K^\mathfrak{B}$ which are finite as \mathcal{O} -modules and such that the natural maps

$$X_{\mathcal{O}}^\mathfrak{B} \otimes_{\mathcal{O}} K \rightarrow X_K^\mathfrak{B}, Y_{\mathcal{O}}^\mathfrak{B} \otimes_{\mathcal{O}} K \rightarrow Y_K^\mathfrak{B} \text{ and } Z_{\mathcal{O}}^\mathfrak{B} \otimes_{\mathcal{O}} K \rightarrow Z_K^\mathfrak{B}$$

are isomorphisms, and the sets of Jordan-Hölder factors of the representations $X_{\mathcal{O}}^\mathfrak{B} \otimes_{\mathcal{O}} k$, $Y_{\mathcal{O}}^\mathfrak{B} \otimes_{\mathcal{O}} k$ and $Z_{\mathcal{O}}^\mathfrak{B} \otimes_{\mathcal{O}} k$ have pairwise empty intersection.

Proof. 1. Since $X_K^{\mathfrak{B}}$ is 1-dimensional, we can choose $X_{\mathcal{O}}^{\mathfrak{B}}$ to be any \mathcal{O} -sublattice of $X_K^{\mathfrak{B}}$. To construct $Y_{\mathcal{O}}^{\mathfrak{B}}$, let us first consider the action of $H_{\mathfrak{B}, \mathcal{O}}$ on $M = \mathcal{O}^7$ given by the following matrices:

$$T_{s_1} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$T_{s_2} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ q & -1 & 0 & 0 & 1 & 0 & 0 \\ 2q & 0 & -1 & 0 & 2 & 0 & 0 \\ -2q(q+1) & 2q & q & -1 & 0 & 1 & 0 \\ q(q+1) & 0 & 0 & 0 & q & 0 & 0 \\ -2q(q^2+4q+1) & 2q(q+1) & q(q+1) & 0 & -4q & q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$T_{s_3} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2q & -1 & 1 \\ 0 & 0 & 0 & 0 & 2q(q+1) & 0 & q \end{pmatrix}$$

and

$$\epsilon_1 = \begin{pmatrix} q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1-q^{-1} & q^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-q^{-1} & q^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1-q^{-1}) & q^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q^{-1} \end{pmatrix},$$

$$\epsilon_2 = \begin{pmatrix} q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ -1+q^{-1} & q^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & -1+q^{-1} & q^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(q-1) & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\epsilon_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2(q-1) & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2(q-1) & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(-1+q^{-1}) & q^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q \end{pmatrix}.$$

(These matrices were calculated using the results of Reeder [Ree00], exactly as in the proof of [CT15, Proposition 2.9].) It follows from Reeder's construction that they satisfy the defining relations of the presentation $H_{\mathfrak{B}, \mathbb{C}} = H_0 \widetilde{\otimes}_{\mathbb{C}} \mathbb{C}[X]$, so that M is indeed a $H_{\mathfrak{B}, \mathcal{O}}$ -module. The character $[1, 1, 0]$ of $K[X]$ visibly appears in $M \otimes_{\mathcal{O}} K$ with multiplicity 4; it also has multiplicity 4 in the full representation

induced from this character of T . By Frobenius reciprocity, we must therefore have $M \otimes_{\mathcal{O}} K \cong Y_K^{\mathfrak{B}}$, as both modules are irreducible, and we can take $Y_{\mathcal{O}}^{\mathfrak{B}}$ to be the image of M under such an isomorphism.

To prove the claim about Jordan–Hölder factors, we now show that in fact the module $M \otimes_{\mathcal{O}} k$ is an absolutely irreducible $H_{\mathfrak{B},k}$ -module. This will use the assumption that q is a primitive root modulo 5. Let $V \subset M \otimes_{\mathcal{O}} k$ be a simple $H_{\mathfrak{B},k}$ -submodule; we will show that $V = M \otimes_{\mathcal{O}} k$. We can decompose $V = V_{[1,1,0]} \oplus V_{[1,0,1]} \oplus V_{[1,0,-1]}$ as a sum of generalised $k[X]$ -eigenspaces; if $\mathbf{e}_1, \dots, \mathbf{e}_7$ denotes the standard basis of $k^7 = M \otimes_{\mathcal{O}} k$, then we have

$$V_{[1,1,0]} = V \cap \langle \mathbf{e}_1, \dots, \mathbf{e}_4 \rangle, \quad V_{[1,0,1]} = V \cap \langle \mathbf{e}_5, \mathbf{e}_6 \rangle, \quad \text{and} \quad V_{[1,0,-1]} = V \cap \langle \mathbf{e}_7 \rangle.$$

Since V is non-zero, at least one of these eigenspaces must be non-zero. We treat each possibility in turn. If $V_{[1,0,-1]} \neq 0$, then $\mathbf{e}_7 \in V_{[1,0,-1]}$. The relations

$$\mathbf{e}_6 = T_{s_3} \mathbf{e}_7 - q\mathbf{e}_7, \quad \mathbf{e}_4 = T_{s_2} \mathbf{e}_6 - q\mathbf{e}_6, \quad \mathbf{e}_3 = T_{s_1} \mathbf{e}_4 - q\mathbf{e}_4, \quad \mathbf{e}_1 = T_{s_3} \mathbf{e}_3 - q\mathbf{e}_3$$

show that $\langle \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_6, \mathbf{e}_7 \rangle \subset V$. The relations

$$\mathbf{e}_2 = (1-q^{-1})^{-1}(\epsilon_1 \mathbf{e}_1 - q^{-1} \mathbf{e}_1), \quad \mathbf{e}_5 = q^{-1}(q+1)^{-1}(T_{s_2} \mathbf{e}_1 + \mathbf{e}_1 - q\mathbf{e}_2 - 2q\mathbf{e}_3 + 2q(q+1)\mathbf{e}_4 + 2q(q^2+4q+1)\mathbf{e}_6)$$

then show that $V = M \otimes_{\mathcal{O}} k$.

Now suppose that $V_{[1,0,1]} \neq 0$. Looking at the action of $k[X]$, we see that $\mathbf{e}_6 \in V$, hence (using the above relations) $\langle \mathbf{e}_1, \dots, \mathbf{e}_6 \rangle \subset V$. Then the relation

$$\mathbf{e}_7 = (2q(q+1))^{-1}(T_{s_3} \mathbf{e}_5 + \mathbf{e}_5 - 2q\mathbf{e}_6)$$

shows $V = M \otimes_{\mathcal{O}} k$.

Now suppose finally that $V_{[1,1,0]} \neq 0$. Looking at the action of $k[X]$, we see that $\mathbf{e}_4 \in V$, hence (using the above relations) $\mathbf{e}_3 \in V$, hence $\mathbf{e}_6 = (q(q+1))^{-1}(T_{s_2} \mathbf{e}_3 + \mathbf{e}_3 - q\mathbf{e}_4) \in V$, hence $V_{[1,0,1]} \neq 0$. This shows again $V = M \otimes_{\mathcal{O}} k$.

2. We can again choose $X_{\mathcal{O}}^{\mathfrak{B}}$ to be an arbitrary \mathcal{O} -lattice of $X_K^{\mathfrak{B}}$. We define an action of $H_{\mathfrak{B},\mathcal{O}}$ on $M = \mathcal{O}^{11}$ via the following matrices:

$$T_{s_1} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ q & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2q & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ q(1+q) & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 \\ 0 & 0 & q(1+q) & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & 0 & 2q(1+q) & 0 & 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$T_{s_2} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+q & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{q(1+q+q^2)}{(1+q)^2} & \frac{q^2}{1+q} \end{pmatrix},$$

[illegible]

[illegible]

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$$\epsilon_3 = \begin{pmatrix} \frac{1}{q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1-q}{q} & \frac{1}{q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-q}{q} & \frac{1}{q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(-1+q) & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{q} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{q} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{q^2} \end{pmatrix},$$

$$\epsilon_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2(-1+q) & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2(-1+q) & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{q} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2(1-q)}{q} & \frac{1}{q} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2(-1+q) & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{q} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{q} \end{pmatrix}.$$

Again, by Reeder's results, these matrices satisfy the necessary relations to give M the structure of an $H_{\mathfrak{B}, \mathcal{O}}$ -module. Considering again the exponent $[2, 1, 1, 0]$, which occurs with multiplicity 4, we deduce that $M \otimes_{\mathcal{O}} K \cong Y_K^{\mathfrak{B}}$. We define $Y_{\mathcal{O}}^{\mathfrak{B}}$ to be the image of M under such an isomorphism. We claim that, as in the previous case, $M \otimes_{\mathcal{O}} k$ is an absolutely irreducible $H_{\mathfrak{B}, k}$ -module. To this end, let us write $\mathbf{e}_1, \dots, \mathbf{e}_{11}$ for the standard basis of $M \otimes_{\mathcal{O}} k \cong k^{11}$, and let $V \subset M \otimes_{\mathcal{O}} k$ be a simple $H_{\mathfrak{B}, k}$ -submodule. We have a decomposition

$$V = V_{[2,1,1,0]} \oplus V_{[2,1,0,1]} \oplus V_{[2,1,0,-1]} \oplus V_{[1,2,1,0]} \oplus V_{[1,2,0,1]} \oplus V_{[1,0,2,1]}$$

into generalized $k[X]$ -eigenspaces, where

$$V_{[2,1,1,0]} = V \cap \langle \mathbf{e}_1, \dots, \mathbf{e}_4 \rangle, \quad V_{[2,1,0,1]} = V \cap \langle \mathbf{e}_5, \mathbf{e}_6 \rangle, \quad V_{[2,1,0,-1]} = V \cap \langle \mathbf{e}_7 \rangle,$$

$$V_{[1,2,1,0]} = V \cap \langle \mathbf{e}_8, \mathbf{e}_9 \rangle, \quad V_{[1,2,0,1]} = V \cap \langle \mathbf{e}_{10} \rangle, \quad V_{[1,0,2,1]} = V \cap \langle \mathbf{e}_{11} \rangle.$$

We will show that $V = M \otimes_{\mathcal{O}} k$. Since V is non-zero, one of these generalized eigenspaces is non-zero; we treat each case in turn. If $V_{[2,1,1,0]} \neq 0$, then acting by $k[X]$ shows that $\mathbf{e}_4 \in V$. The relations

$$\mathbf{e}_2 = T_{s_4} \mathbf{e}_4 - q\mathbf{e}_4, \quad \mathbf{e}_3 = T_{s_2} \mathbf{e}_4 - q\mathbf{e}_4, \quad \mathbf{e}_1 = T_{s_4} \mathbf{e}_3 - q\mathbf{e}_3$$

show that $\langle \mathbf{e}_1, \dots, \mathbf{e}_4 \rangle \subset V$. The relations

$$\mathbf{e}_6 = (q(q+1))^{-1}(T_{s_3} \mathbf{e}_3 - q\mathbf{e}_4 + \mathbf{e}_3), \quad \mathbf{e}_5 = (q(q+1))^{-1}(T_{s_3} \mathbf{e}_1 + \mathbf{e}_1 - q\mathbf{e}_2 - 2q\mathbf{e}_3 + 2q(q+1)\mathbf{e}_4 + 2q(1+4q+q^2)\mathbf{e}_6)$$

show that $\langle \mathbf{e}_1, \dots, \mathbf{e}_6 \rangle \subset V$. The relations

$$\mathbf{e}_7 = (2q(q+1))^{-1}(T_{s_4} \mathbf{e}_5 + \mathbf{e}_5 - 2q\mathbf{e}_6), \quad \mathbf{e}_8 = (q(q+1))^{-1}(T_{s_1} \mathbf{e}_1 + \mathbf{e}_1 - q\mathbf{e}_2),$$

$$\mathbf{e}_9 = (q(q+1))^{-1}(T_{s_1} \mathbf{e}_3 + \mathbf{e}_3 - q\mathbf{e}_4), \quad \mathbf{e}_{10} = (2q(q+1))^{-1}(T_{s_1} \mathbf{e}_5 + \mathbf{e}_5 - 2q\mathbf{e}_6)$$

show that $\langle \mathbf{e}_1, \dots, \mathbf{e}_{10} \rangle \subset V$. Finally, the relation

$$\mathbf{e}_{11} = (q+1)^2(q(q^2+q+1))^{-1}(T_{s_2} \mathbf{e}_{10} - (q+1)\mathbf{e}_{10})$$

shows that in fact $V = M \otimes_{\mathcal{O}} k$. (Note that $q^2 + q + 1 \neq 0$ in k , because q is assumed to be a primitive root modulo 7.)

Now suppose instead that $V_{[2,1,0,1]} \neq 0$. Then $\mathbf{e}_6 \in V$, hence $\mathbf{e}_4 = T_{s_3}\mathbf{e}_6 - q\mathbf{e}_6 \in V$. This shows that $V_{[2,1,1,0]} \neq 0$, hence $V = M \otimes_{\mathcal{O}} k$.

Now suppose instead that $V_{[2,1,0,-1]} \neq 0$, hence $\mathbf{e}_7 \in V$. The relation $\mathbf{e}_6 = T_{s_4}\mathbf{e}_7 - q\mathbf{e}_7$ shows that $V_{[2,1,0,1]} \neq 0$, hence $V = M \otimes_{\mathcal{O}} k$.

Now suppose instead that $V_{[1,2,1,0]} \neq 0$. If $v \in V_{[1,2,1,0]}$ is non-zero then $T_{s_1}v - qv \in V_{[2,1,1,0]}$ is non-zero, hence $V = M \otimes_{\mathcal{O}} k$.

Now suppose instead that $V_{[1,2,0,1]} \neq 0$, hence $\mathbf{e}_{10} \in V$. Then the relation $\mathbf{e}_9 = T_{s_3}\mathbf{e}_{10} - q\mathbf{e}_{10}$ shows that $V_{[1,2,1,0]} \neq 0$, hence $V = M \otimes_{\mathcal{O}} k$.

Now suppose that $V_{[1,0,2,1]} \neq 0$, hence $\mathbf{e}_{11} \in V$. The relation $\mathbf{e}_{10} = T_{s_2}\mathbf{e}_{11} - q^2(q+1)^{-1}\mathbf{e}_{11}$ shows that $V_{[1,2,0,1]} \neq 0$, hence $V = M \otimes_{\mathcal{O}} k$. In all cases we therefore have $V = M \otimes_{\mathcal{O}} k$, and this concludes the proof that $M \otimes_{\mathcal{O}} k$ is an absolutely irreducible $H_{\mathfrak{B},k}$ -module.

One can construct an integral structure in $Z_K^{\mathfrak{B}}$ in the same way as for $Y_K^{\mathfrak{B}}$ above; alternatively, it is easy to show using the algebraic modular forms we consider later that one can find a $H_{\mathfrak{B},\mathcal{O}}$ -module N which is finite free as an \mathcal{O} -module and such that there exists an injection $Z_K^{\mathfrak{B}} \hookrightarrow N \otimes_{\mathcal{O}} K$ of $H_{\mathfrak{B},K}$ -modules. One can then define $Z_{\mathcal{O}}^{\mathfrak{B}} = N \cap Z_K^{\mathfrak{B}}$. Since the exponents of the Jacquet module of Z_K are distinct, modulo λ , from the exponent of X_K , the assertion about Jordan-Hölder factors now follows from the assertion that $Y_{\mathcal{O}}^{\mathfrak{B}} \otimes_{\mathcal{O}} k$ is an absolutely irreducible $H_{\mathfrak{B},k}$ -module. This completes the proof of the proposition. \square

Remark 2.7. The matrices described above can be used to define the module $Y_K^{\mathfrak{B}}$ without restriction on the order of q modulo l . We use the hypothesis that q is a primitive root modulo l to ensure that they determine an integral structure $Y_{\mathcal{O}}^{\mathfrak{B}} \subset Y_K^{\mathfrak{B}}$. This observation will be used in the proof of Proposition 2.9 below.

Corollary 2.8. 1. Suppose that $n = 7$, and let M be a $H_{\mathfrak{B},\mathcal{O}}$ -module which is finite flat as an \mathcal{O} -module, and such that $M \otimes_{\mathcal{O}} K \cong (X_K^{\mathfrak{B}})^a \oplus (Y_K^{\mathfrak{B}})^b$ as $H_{\mathfrak{B},K}$ -modules, for some integers $a, b \geq 0$. Let $M_X \subset M$ denote the intersection of M with the $X_K^{\mathfrak{B}}$ -isotypic part of $M \otimes_{\mathcal{O}} K$, and define M_Y similarly.

Suppose that q is a primitive root modulo l . Then the natural map $M_X \oplus M_Y \rightarrow M$ is an isomorphism.

2. Suppose that $n = 9$, and let M be a $H_{\mathfrak{B},\mathcal{O}}$ -module which is finite flat as an \mathcal{O} -module, and such that $M \otimes_{\mathcal{O}} K \cong (X_K^{\mathfrak{B}})^a \oplus (Y_K^{\mathfrak{B}})^b \oplus (Z_K^{\mathfrak{B}})^c$ as $H_{\mathfrak{B},K}$ -modules, for some integers $a, b, c \geq 0$. Let $M_X \subset M$ denote the intersection of M with the $X_K^{\mathfrak{B}}$ -isotypic part of $M \otimes_{\mathcal{O}} K$, and define M_Y, M_Z similarly.

Suppose that q is a primitive root modulo l . Then the natural map $M_X \oplus M_Y \oplus M_Z \rightarrow M$ is an isomorphism.

Proof. We prove the second part; see [CT15, Corollary 2.10] for the first part. The map $M_X \oplus M_Y \oplus M_Z \rightarrow M$ becomes an isomorphism after inverting l , so to prove the result it suffices to show that the map $(M_X \oplus M_Y \oplus M_Z) \otimes_{\mathcal{O}} k \rightarrow M \otimes_{\mathcal{O}} k$ is injective. Let L be a simple submodule of the kernel. Being non-zero, the projection of L to at least one of $M_X \otimes_{\mathcal{O}} k$, $M_Y \otimes_{\mathcal{O}} k$, and $M_Z \otimes_{\mathcal{O}} k$ must be non-zero. However, the maps

$$M_X \otimes_{\mathcal{O}} k \rightarrow M \otimes_{\mathcal{O}} k, M_Y \otimes_{\mathcal{O}} k \rightarrow M \otimes_{\mathcal{O}} k, \text{ and } M_Z \otimes_{\mathcal{O}} k \rightarrow M \otimes_{\mathcal{O}} k$$

are injective by definition. It follows that the projection of L to at least two of $M_X \otimes_{\mathcal{O}} k$, $M_Y \otimes_{\mathcal{O}} k$, and $M_Z \otimes_{\mathcal{O}} k$ must be non-zero. However, this implies that these two spaces have a simple submodule in common, contradicting Proposition 2.6. This completes the proof. \square

According to the Bernstein presentation, the algebra $H_{\mathfrak{B},\mathcal{O}}$ has an abelian subalgebras $\mathcal{O}[\Lambda]$. We let $\chi : \mathcal{O}[\Lambda] \rightarrow \mathcal{O}$ denote the character giving the action of $\mathcal{O}[\Lambda]$ on $X_K^{\mathfrak{B}}$, and let $\bar{\chi} : \mathcal{O}[\Lambda] \rightarrow k$ denote its reduction modulo λ . If M is a $\mathcal{O}[\Lambda]$ -module, then we write $M(\bar{\chi})$ for its localization at $\ker \bar{\chi}$. We recall that the algebra $H_{\mathfrak{B},\mathcal{O}}$ has a canonical anti-involution j given on double cosets by $[\mathfrak{B}g\mathfrak{B}] \mapsto [\mathfrak{B}g^{-1}\mathfrak{B}]$. Thus

there is another abelian subalgebra $j\mathcal{O}[\Lambda] \subset H_{\mathfrak{B}, \mathcal{O}}$, with a character $\bar{\chi}j = \bar{\chi} \circ j$, and we write $M(\bar{\chi}j)$ for the analogous localization.

Proposition 2.9. *Suppose that $q \equiv -1 \pmod{l}$. Then:*

1. *Suppose that $n = 7$. Then $X_K^{\mathfrak{B}} = X_K^{\mathfrak{B}}(\bar{\chi}) = X_K^{\mathfrak{B}}(\bar{\chi}j)$ and $\dim_K Y_K^{\mathfrak{B}} \cap Y_K^{\mathfrak{B}}(\bar{\chi}) = \dim_K Y_K^{\mathfrak{B}} \cap Y_K^{\mathfrak{B}}(\bar{\chi}j) = 1$.*
2. *Suppose that $n = 9$. Then $X_K^{\mathfrak{B}} = X_K^{\mathfrak{B}}(\bar{\chi}) = X_K^{\mathfrak{B}}(\bar{\chi}j)$, $\dim_K Y_K^{\mathfrak{B}} \cap Y_K^{\mathfrak{B}}(\bar{\chi}) = \dim_K Y_K^{\mathfrak{B}} \cap Y_K^{\mathfrak{B}}(\bar{\chi}j) = 1$ and $Z_K^{\mathfrak{B}}(\bar{\chi}) = 0$.*

Proof. The assertion that $X_K^{\mathfrak{B}} = X_K^{\mathfrak{B}}(\bar{\chi}) = X_K^{\mathfrak{B}}(\bar{\chi}j)$ follows from the fact that X_K is self-dual (as follows easily from Theorem 3.3) and $X_K^{\mathfrak{B}}$ is 1-dimensional, cf. [CT15, Proposition 2.8]. The assertion in the second case that $Z_K^{\mathfrak{B}}(\bar{\chi}) = 0$ follows from the earlier calculation of Jacquet modules. The assertion that $\dim_K Y_K^{\mathfrak{B}} \cap Y_K^{\mathfrak{B}}(\bar{\chi}) = 1$ follows from the observation that if V is an admissible $\mathbb{C}[U_n]$ -module, $v \in V^{\mathfrak{B}}$ and $T_{s_k}v = qv$, then $v \in V^{\mathfrak{B}}$, together with the above explicit matrices. (Observe that the operator $e_{\mathfrak{P}} = [\mathfrak{P}]/(q+1) = ([\mathfrak{B}] + [\mathfrak{B}s_k\mathfrak{B}])/(q+1)$ is a projector onto the set of \mathfrak{B} -fixed vectors.)

To show that $\dim_K Y_K^{\mathfrak{B}} \cap Y_K^{\mathfrak{B}}(\bar{\chi}j) = 1$, we give another characterization of this subspace. The representation Y_K is self-dual (again, by Theorem 3.3), from which it follows that there exists a perfect duality $\langle \cdot, \cdot \rangle : Y_K^{\mathfrak{B}} \times Y_K^{\mathfrak{B}} \rightarrow K$ satisfying $\langle tx, y \rangle = \langle x, j(t)y \rangle$ for all $t \in H_{\mathfrak{B}, \mathcal{O}}$, $x, y \in Y_K^{\mathfrak{B}}$.

The space $Y_K^{\mathfrak{B}}(\bar{\chi}j)$ can be characterized as the annihilator of the spaces $Y_K^{\mathfrak{B}}(\bar{\eta})$, $\bar{\eta} \neq \bar{\chi}$, under the above duality. The assertion that $Y_K^{\mathfrak{B}} \cap Y_K^{\mathfrak{B}}(\bar{\chi}j)$ is of dimension 1 is therefore equivalent to the assertion that the span of the spaces $Y_K^{\mathfrak{B}}(\bar{\eta})$, $\bar{\eta} \neq \bar{\chi}$, together with the annihilator of $Y_K^{\mathfrak{B}} = e_{\mathfrak{P}}Y_K^{\mathfrak{B}}$, is of codimension 1 in $Y_K^{\mathfrak{B}}$. Since $j(e_{\mathfrak{P}}) = e_{\mathfrak{P}}$, this is itself equivalent to the assertion that the subspace of $Y_K^{\mathfrak{B}}$ spanned by the subspaces $Y_K^{\mathfrak{B}}(\bar{\eta})$, $\bar{\eta} \neq \bar{\chi}$, together with $(1 - e_{\mathfrak{P}})Y_K^{\mathfrak{B}}$, is of codimension 1. This can again be readily checked using the matrices appearing in the proof of Proposition 2.6. \square

3 Unitary groups and transfer

Recall from the introduction (in the special case of Sym^8) the identity (1.1):

$$\text{Sym}^8 \bar{r} \cong (\varphi \bar{r} \otimes \bar{r}) \oplus \bar{\chi}^2 \text{Sym}^4 \bar{r},$$

true for (semi-simplified) representations of $\text{GL}_2(\overline{\mathbb{F}}_7)$, and therefore for a Galois representation \bar{r} of degree 2 (mod 7). We may, for simplicity, assume that \bar{r} has large image, so the two summands are irreducible. If \bar{r} arises from a RAESDC automorphic representation of $\text{GL}_2(\mathbb{A}_F)$, F a totally real field, then the two summands are themselves known to arise from RAESDC representations of $\text{GL}_4(\mathbb{A}_F)$ and $\text{GL}_5(\mathbb{A}_F)$ (these are known to exist, by the results of Ramakrishnan and Kim-Shahidi).

Starting with these representations Π_4, Π_5 , we will construct in this section a “packet” (in the sense of Arthur and Langlands) of representations of a unitary group G in 9 variables over the totally real field F ; G will be compact at the Archimedean places (and quasi-split elsewhere).

If we replace G by its quasi-split form G^* , we are reduced to the construction by Mok of endoscopic L-packets (Theorem 3.5). At the local primes of F ramified in the (CM) extension E defining G , we obtain, for suitable carefully chosen initial data Π_4, Π_5 , the representations described in §2. In order to understand these representations, we complete in this section the proof of Theorem 2.3, and state a more precise result (see Theorem 3.2 and the following paragraph) describing the Arthur–Mok signs of the representations in the local L-packets. This requires delicate local computations (§§3.4 – 3.5).

We are then ready to transfer Mok’s representations to the \mathbb{R} -anistropic group G . This is done in §3.6, using the methods in [CHL11]. The results are stated in Theorem 3.8, Theorem 3.10. It is again crucial that, at the primes ramified in E/F , the local components are controlled by the sign formula.

We should mention that the representation Π_4 directly obtained from our given representation of GL_2 is not directly useful, as $\Pi_4 \boxplus \Pi_5$ does not yield a cohomological representation of G^* , and therefore does not transfer to G . This is easily obviated by a standard weight-changing argument (see [CT15, Proposition 3.10]) which is reviewed in §6.

3.1 Unitary groups

Let E/F be a quadratic extension of local or global fields. We consider a unitary group G in n variables associated to this extension. Thus

$${}^L G = \mathrm{GL}_n(\mathbb{C}) \rtimes W_F$$

where W_F acts through $\mathrm{Gal}(E/F)$, the generator c of $\mathrm{Gal}(E/F)$ acting by

$$g \mapsto \Phi_n {}^t g^{-1} \Phi_n^{-1}, \quad \Phi_n = \begin{pmatrix} & & & -1 \\ & & 1 & \\ & \ddots & & \\ & 1 & & \\ (-1)^n & & & \end{pmatrix}.$$

In our case, $n(=l+2, l \text{ an odd prime})$ will be odd. Over E , G becomes isomorphic to GL_n and there is an L -homomorphism

$${}^L G \longrightarrow {}^L(\mathrm{Res}_{E/F} G_E)$$

of L -groups over F , given by

$$(g, w) \mapsto (g, g, w) \quad (w \in W_F),$$

W_F acting on $(\mathrm{Res}_{E/F} G_E)^\wedge = \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ through $\mathrm{Gal}(E/F)$, c acting by

$$(g, h) \mapsto (\Phi_n {}^t h^{-1} \Phi_n^{-1}, \Phi_n {}^t g^{-1} \Phi_n^{-1}).$$

If F is local, we write $\Phi_{\mathrm{bdd}}(G)$ for the set of equivalence classes of parameters $L_F \rightarrow {}^L G$ with bounded restriction to W_F (cf. [CT15, §2.2]), where

$$L_F = \begin{cases} W_F & F \text{ Archimedean} \\ W_F \times \mathrm{SU}_2(\mathbb{R}) & F \text{ } p\text{-adic.} \end{cases}$$

By composing with ${}^L G \rightarrow {}^L(\mathrm{Res}_{E/F} G)$ we obtain parameters for $\mathrm{GL}_n(E)$. Then, n being odd, we have (see §2.2 and [Mok15, Lemma 2.2.1]):

Lemma 3.1. *This map $\varphi \mapsto \varphi|_{L_E}$ induces a bijection between $\Phi_{\mathrm{bdd}}(G)$ and the set of parameters in $\Phi_{\mathrm{bdd}}(\mathrm{GL}_n(E))$ that are conjugate orthogonal.*

We refer to this map of parameters as the stable base change map. If $\mu : E^\times \rightarrow \mathbb{C}^\times$ is a continuous character such that $\mu|_{F^\times}$ is the character of order two $\varepsilon_{E/F}$ given by class field theory, then the map $\varphi \mapsto \varphi|_{L_E} \otimes \mu$ defines a bijection between $\Phi_{\mathrm{bdd}}(G)$ and the set of parameters in $\Phi_{\mathrm{bdd}}(\mathrm{GL}_n(E))$ that are conjugate symplectic (see [Mok15, Lemma 2.2.1]). In particular, the image of this map is independent of the choice of μ , and we refer to this map on parameters as the unstable base change map.

We now return to the general case (F local or global) and consider the endoscopic group H for G , isomorphic to $U(n-4) \times U(4)$, the unitary groups being quasi-split. The group ${}^L H$ admits the same description, with $\widehat{H} = \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_4(\mathbb{C})$ and the matrices Φ_m, Φ_4 . (We set $m = n-4$.) There is an embedding $\xi : {}^L H \rightarrow {}^L G$ given by the following formulas. We fix a character μ of E^\times (local case), or of $E^\times \backslash \mathbb{A}_E^\times$ (global case) whose restriction to F^\times (resp. \mathbb{A}_F^\times) is the character of order two $\varepsilon_{E/F}$ given by class field theory. Then [Mín11, p. 404] ξ is given by

$$\begin{aligned} (g_1, g_2) \times 1 &\mapsto \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}, \quad (g_1, g_2) \in \widehat{H} \\ 1 \times w &\mapsto \left(\begin{pmatrix} 1_m & \\ & \mu(w)1_4 \end{pmatrix}, w \right), \quad w \in W_E \\ 1 \times w_c &\mapsto \left(\begin{pmatrix} \Phi_m & \\ & \Phi_4 \end{pmatrix} \Phi_n^{-1}, w_c \right) \end{aligned}$$

where $w_c \in W_F - W_E$ is a representative of c .

For some computations it is expedient to replace ξ by the following, conjugate to the previous one by an element of \widehat{G} . Write

$$g_2 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

(blocks of size 2), and consider

$$\begin{aligned} \xi : (g_1, g_2) &\mapsto \begin{pmatrix} A & B \\ C & g_1 & D \end{pmatrix} \\ 1 \times w &\mapsto \left(\begin{pmatrix} \mu(w)1_2 & & \\ & 1_m & \\ & & \mu(w)1_2 \end{pmatrix}, w \right), \quad w \in W_E \\ 1 \times w_c &\mapsto \left(\begin{pmatrix} & \Phi_2 \\ & \Phi_m & \\ \Phi_2 & & \end{pmatrix} \Phi_n^{-1}, w_c \right). \end{aligned} \tag{3.1}$$

3.2 Local L-packets

Assume now that E/F is an extension of p -adic fields. There is a conjugate orthogonal parameter $\varphi_E \in \Phi_{\text{bdd}}(\text{GL}_n(E))$ associated to the induced representation

$$\Pi_E = \text{St}_{m,E} \boxplus \text{St}_{3,E} \boxplus \text{St}_{1,E}.$$

If $l > 5$, thus $m > 3$, this is a discrete parameter in the sense of Mœglin [Mœg07], thus associated to a packet of discrete series for G . If $l = 5$, Π_E is associated to the reducible representation of G :

$$\pi = \text{n-Ind}_P^G(\text{St}_{3,E} \otimes \mathbf{1})$$

where P is the parabolic subgroup of G with Levi subgroup $\text{GL}_3(E) \times U(1)$.

Theorem 3.2 (Mœglin, Mok). *The parameter φ_E determines an L -packet Π_φ of tempered representations of G .*

(i) If $m = 3$, Π_φ has two elements X, Y .

(ii) If $m \geq 5$, Π_φ has four elements X, Y, Z, W which belong to the discrete series of G .

This follows from Arthur's formalism; we refer to [Mok15, Theorem 2.5.1]. Let $\varphi : L_F \rightarrow {}^L G$ be the parameter deduced from φ_E . We must compute

$$\begin{aligned} S_\varphi &= \text{Cent}(\text{Im}\varphi, \widehat{G}), \\ \mathcal{S}_\varphi &= \pi_0(S_\varphi / \{\pm 1\}). \end{aligned}$$

(See [GGP12, §4] for the general calculation.) In case (i) $S_\varphi = \text{O}_2(\mathbb{C}) \times \{\pm 1\}$, $\mathcal{S}_\varphi = \{\pm 1\}$. If we assume φ_E written as $sp(3) \oplus sp(1) \oplus sp(3)$, $sp(r)$ denoting the irreducible representation of degree r of $\text{SU}_2(\mathbb{R})$, the non-trivial element of \mathcal{S}_φ is represented by

$$\begin{pmatrix} & & 1_3 \\ & 1 & \\ 1_3 & & \end{pmatrix}$$

(block scalar matrices).

In case (ii) \mathcal{S}_φ is represented by

$$\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}$$

$(a, b, c = \pm 1) \bmod \{\pm 1\}$, the sizes being $(1, 3, m)$. We recall that there is a non-degenerate pairing of Π_φ with \mathcal{S}_ϕ . In case (i) we set $\varepsilon(X) = -1$ and $\varepsilon(Y) = 1$, ε being the unique non-trivial character. In case (ii) we set, $\varepsilon_1, \varepsilon_2$ being associated with the components a, b :

π	X	Y	Z	W
$\varepsilon_1(\pi)$	1	1	-1	-1
$\varepsilon_2(\pi)$	-1	1	-1	1

For the sign $\varepsilon = \varepsilon_1 \varepsilon_2$, we therefore have $\varepsilon(Y) = \varepsilon(Z) = 1$ and $\varepsilon(X) = \varepsilon(W) = -1$.

We will need information on the Jacquet modules of the components of $\Pi(\varphi_E)$. Recall that an irreducible semi-stable representation is, by definition, a subquotient of an unramified principal series, i.e., a principal series induced from a character of the maximally split torus trivial on its maximal compact subgroup. Let $P_0 = M_0 N_0$ be the Borel subgroup, with Levi subgroup $(E^\times)^k \times U(1)$. (Thus $k = (m+3)/2$.) Let us denote by $e = [e_1, \dots, e_k]$ the character of $(E^\times)^k \times U(1)$

$$(z_1, \dots, z_k, u) \mapsto |z_1|^{e_1} \cdots |z_k|^{e_k} \cdot (z_i \in E^\times, u \in U(1)).$$

For any semi-stable representation π , the normalized Jacquet module $\pi_{N_0}^{norm}$ is, after semi-simplification, a sum of characters (“exponents”) e , with multiplicities.

Theorem 3.3. *Suppose that E/F is ramified, and that the residue characteristic of F is odd. Then:*

- (i) *If $m = 3$, X, Y are semi-stable with exponents $[1, 0, 1]$ and $4[1, 1, 0] + 2[1, 0, 1] + [1, 0, -1]$, respectively.*
- (ii) *If $m = 5$, X, Y, Z are semi-stable with exponents as follows:*

$$\begin{aligned} X &: [2, 1, 0, -1] \\ Y &: [2, 1, 0, -1], 2[2, 1, 0, 1], 4[2, 1, 1, 0], [1, 0, 2, 1], [1, 2, 0, 1], 2[1, 2, 1, 0] \\ Z &: [1, 0, 2, 1], [1, 2, 0, 1]. \end{aligned}$$

- (iii) *For $m > 5$, X, Y, Z are semi-stable and their exponents may be described inductively as follows, k being equal to $\frac{n-1}{2} = \frac{m+3}{2}$. Write X_k, Y_k, Z_k . Then $(X_k)_{N_0}^{norm}$ is 1-dimensional of exponent $[k-2, k-3, \dots, 0, -1]$. The exponents of Y_k and Z_k comprise the exponents $[k-2, e(Y_{k-1})], [k-2, e(Z_{k-1})]$ for all exponents of Y_{k-1}, Z_{k-1} , with the same multiplicities, together with:*

$$\begin{aligned} Y_k &: [1, 0, k-2, \dots, 1], [1, k-2, 0, k-3, \dots, 1], \dots, [1, k-2, k-3, \dots, 0, 1], 2[1, k-2, \dots, 1, 0] \\ Z_k &: [1, 0, \dots, k-2, \dots, 1], [1, k-2, 0, k-3, \dots, 1], \dots, [1, k-2, \dots, 0, 1]. \end{aligned}$$

The proof of Theorem 3.3 will be given in §3.5 below.

3.3 Global L-packets

We now describe the global analogue of this construction, only so far in the quasi-split case. We now assume E/F is a totally imaginary quadratic extension of a totally real field. Choose, for each Archimedean prime v of F , a prime w of E above F so $E_w = \mathbb{C}$. If a is a positive integer, then we will write $U(a)$ for the quasi-split unitary group over F in a variables, split by E .

Lemma 3.4. *There exists a continuous homomorphism $\mu : E^\times \backslash \mathbb{A}_E^\times \rightarrow \mathbb{C}^\times$ such that $\mu|_{\mathbb{A}_F^\times} = \varepsilon_{E/F}$ and, for each Archimedean prime w and $z \in E_w \cong \mathbb{C}$, $\mu(z) = (z/\bar{z})^{1/2}$.*

Proof. Let $E_\infty = \prod_{w|\infty} E_w \cong \prod_w \mathbb{C}$, and let $U \subset \mathbb{A}_E^{\infty, \times}$ be an open subgroup satisfying the following conditions:

- We have $U \cap E^\times \subset \mathcal{O}_F^{\times, +}$ (the subgroup of totally positive units of F).

- We have $\varepsilon_{E/F}|_{U \cap \mathbb{A}_F^{\infty, \times}} = 1$.
- The group U is stable under the action of $c \in \text{Gal}(E/F)$.

Then $E^\times \cdot \mathbb{A}_F^\times \cdot (U \times E_\infty^\times)$ is an open subgroup of \mathbb{A}_E^\times , and $E^\times \cap (\mathbb{A}_F^\times \cdot (U \times E_\infty^\times)) \subset F^\times$. Indeed, if $e \in E^\times$ satisfies $e = \alpha \cdot (u, (z_w)_{w|\infty})$ for $\alpha \in \mathbb{A}_F^\times$, $u \in U$, and $z_w \in E_w^\times$, then $f = e/e^c = (u/u^c, (z_w/\bar{z}_w)_{w|\infty})$ lies in $E^\times \cap U \subset \mathcal{O}_F^{\times, +}$. We thus have $f = f^c = f^{-1}$, hence $f^2 = 1$, hence $f = 1$ (as f is totally positive).

We can therefore define a homomorphism $\mu : E^\times \cdot \mathbb{A}_F^\times \cdot (U \times E_\infty^\times) \rightarrow \mathbb{C}^\times$ by the formula $e \cdot \alpha \cdot (u, (z_w)_{w|\infty}) \mapsto \varepsilon_{E/F}(\alpha) \prod_w (z_w/\bar{z}_w)^{1/2}$. This character is continuous because it is continuous on restriction to the open subgroup $U \times E_\infty^\times$. The lemma is completed on choosing any extension of μ to a homomorphism $\mathbb{A}_E^\times \rightarrow \mathbb{C}^\times$. \square

We will consider endoscopic representations π of $G(\mathbb{A})$ associated to conjugate self-dual cuspidal representations Π_m, Π_4 of $\text{GL}_m(\mathbb{A}_E), \text{GL}_4(\mathbb{A}_E)$. We assume first that

$$\Pi_m^\vee \cong \Pi_m^c, \quad (3.2)$$

and that for each Archimedean prime w , the Langlands parameter of Π_m is given by

$$z \mapsto \text{diag}((z/\bar{z})^{\frac{m-1}{2}}, (z/\bar{z})^{\frac{m-3}{2}}, \dots, (z/\bar{z})^{\frac{1-m}{2}}). \quad (3.3)$$

Note that this parameter comes, by *stable* base change, from a parameter for $U(m)$, at least at the Archimedean primes (cf. Lemma 3.1).

For the other representation we assume that

$$\Pi_4^\vee \cong \Pi_4^c, \quad (3.4)$$

and that the Langlands parameter is given, at each Archimedean prime w , by

$$z \mapsto \text{diag}((z/\bar{z})^{\frac{n-2}{2}}, (z/\bar{z})^{\frac{n-4}{2}}, (z/\bar{z})^{\frac{2-n}{2}}, (z/\bar{z})^{-\frac{n}{2}}). \quad (3.5)$$

Again, the representation Π_4 originates from stable base change since $n-2$ is odd. The representation $\Pi_4 \otimes \mu$ is then still conjugate self-dual, and originates from unstable base change. Its Archimedean parameters are

$$z \mapsto \text{diag}((z/\bar{z})^{\frac{n-1}{2}}, (z/\bar{z})^{\frac{n-3}{2}}, (z/\bar{z})^{\frac{3-n}{2}}, (z/\bar{z})^{\frac{1-n}{2}}). \quad (3.6)$$

We can consider the datum $(\Pi_m, \Pi_4 \otimes \mu)$ as an Arthur datum ψ in the sense of [Art], [Mok15], for the unitary group G . It defines a global group \mathcal{S}_ψ [Mok15, Definition 2.4.8], isomorphic to $\{\pm 1\}$ seen as before as the quotient of the diagonal-scalar matrices (a, b) ($a, b = \pm 1$) of size $(m, 4)$ by $Z(\widehat{G})^{W_F} = \{\pm 1\}$. We write $s \in \mathcal{S}_\psi$ for the non-trivial element.

For each place v of F , ψ determines a local (tempered) parameter $\varphi_v : W_F \times \text{SU}_2(\mathbb{R}) \rightarrow {}^L G$. If $\pi_v \in \Pi_{\varphi_v}$ the pairing $\langle s, \pi_v \rangle$ is well-defined. We now have:

Theorem 3.5 (Mok). *Assume given, for each v , a representation $\pi_v \in \Pi_{\varphi_v}$, with π_v almost everywhere unramified. Then the representation $\pi = \otimes_v \pi_v$ occurs in $L_{\text{cusp}}^2(G(F) \backslash G(\mathbb{A}))$ if and only if $\prod_v \langle s, \pi_v \rangle = 1$. In this case, it occurs with multiplicity one.*

This is essentially [Mok15, Theorem 2.5.2], taking into account the fact that the sign ε_ψ is 1 since the parameter ψ is tempered. We have added the fact that π is cuspidal in our case: this follows from Mok's proof, implying that π_v is associated by unramified base change, at almost all primes, with the induced representation $\Pi_m \boxplus \Pi_4 \otimes \mu$. In particular it is tempered; however, a residual representation is tempered at no place [Clo93, Proposition 4.10]. In the next sections, we will prove the analogue of this theorem when G is compact at the Archimedean primes.

3.4 Transfer factors

In this section we prove lemmas needed to transfer the representation $\Pi_m \boxplus (\Pi_4 \otimes \mu)$ of §3.3 to a compact unitary group. We will also obtain the local information necessary to prove Theorem 3.3.

Since n is odd, there exists a unitary group G , associated to a Hermitian form on E^n , such that $G(F_v)$ is compact for each Archimedean prime and quasi-split at finite primes. We now denote by G^* the quasi-split inner form of G . The other elliptic endoscopic groups are of the form $U(a) \times U(b)$ ($n = a + b$); we are essentially concerned with $H = U(m) \times U(4)$. We follow the arguments of [CT15, §3.4–3.8] to which we will refer when convenient.

We consider a decomposed, smooth, K_∞ -finite function $f = \otimes_v f_v$ on $G(\mathbb{A}_F)$. There are associated functions f^H on $H(\mathbb{A}_F)$ where H is any endoscopic group. At finite primes, $f_v \rightsquigarrow f_v^H$ is given by the Whittaker normalization of transfer factors. (Note that since we are considering unitary groups of odd absolute rank, there is a unique equivalence class of Whittaker data.) As in [CT15, §3.5] one checks that they are simply the Langlands–Shelstad transfer factors. At the Archimedean primes we have to compare Kottwitz’s transfer factor used in [Clo11] and the Langlands–Shelstad factor Δ_0 . At each real prime we find, as in [CT15, §3.5], for our particular H :

$$\Delta_K^{G^*}(\gamma, \delta) = \pm \Delta_0(\gamma, \delta)$$

($\gamma \in H(F_v)$, $\delta \in G(F_v)$), the sign depending only on the real groups which are the same at all Archimedean places, and

$$\Delta_K^{G^*}(\gamma, \gamma) = \Delta_K^G(\gamma, \gamma).$$

If $[F : \mathbb{Q}]$ is even, we conclude that we can use Kottwitz’s factors at infinity and the Langlands–Shelstad factors at finite primes.

We must next understand the spectral transfer factors, first at the real primes [CT15, §3.6]. Since we are using Kottwitz’s transfer factors we can use the analysis in [Clo11]. Let $\Theta_{\varphi_v, H}$ be the (stable) character of the L–packet for $H(\mathbb{R})$ associated to the representation of $H(\mathbb{C}) = \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_4(\mathbb{C})$ that is the local component of $(\Pi_m, \Pi_4 \otimes \mu)$ at a place $w|v$. Let \mathbb{C} denote the trivial representation of $G(\mathbb{R})$. Then, the transfer $f_v \rightsquigarrow f_v^H$ being defined by Kottwitz’s factor,

$$\langle \Theta_{\varphi_v, H}, f_v^H \rangle = \det(w) \langle \Theta_{\mathbb{C}}, f_v \rangle$$

where w is described in [Clo11, Theorem 3.4]. (This is due to Kottwitz and Shelstad.) In our case $\det(w) = 1$. Also note that the cocycle $a_{w\sigma}$ there is equal to 1 since $G(F_v)$ is compact.

We now come to the geometric transfer factors at the p -adic places. They are the Langlands–Shelstad factors, but we will need explicit formulas for them on (associated) maximally split tori in $G = G(F_v)$ and $H = H(F_v)$. We could use the formulas in [Wal10], but we hope this exposition will clarify the relation with our parameters. We have $T_G \cong (E^\times)^k \times U(1)$, and

$$T_H \cong ((E^\times)^{k-2} \times U(1)) \times (E^\times)^2 \subset U(m) \times U(4).$$

Elements of T_G , T_H can be conjugated by the obvious isomorphisms in $\mathrm{GL}_n(E)$, in particular replacing some entries z_i by z_i^c . We write $\gamma = t_H \in T_H$, $\delta = t \in T_G$. Note that in G , a regular stable conjugacy class is parameterized, modulo conjugacy, by

$$\ker(H^1(F, T_G) \longrightarrow H^1(F, G)).$$

Since $\det : G \rightarrow U(1)$ induces an isomorphism in cohomology, we see that stable conjugacy coincides with conjugacy. The endoscopic identity then reduces to

$$O_{t_H}(f^H) = \Delta_0(t_H, t) O_t(f)$$

for associated elements (t_H, t) . Write

$$\Delta_0(t_H, t) = \Delta_I \Delta_{II} \Delta_{III,1} \Delta_{III,2} \Delta_{IV}.$$

Here $\Delta_{IV} = D_G(t)/D_H(t_H)$, where $D_G(t) = |\det[(\text{Ad}(t) - 1); \text{Lie}(G)/\text{Lie}(T_G)]|^{1/2}$ is a geometric factor that will occur naturally later and D_H is the analogue for H . If $T_{sc} = T \cap SU(n)$, the previous remark implies easily that $H^1(F, T_{sc}) = \{1\}$; thus $\Delta_I, \Delta_{III,1}$ are trivial (see [CT15, §3.7] for the relevant references to Langlands–Shelstad). The factor Δ_{II} [LS87, p.243] requires the choice of data $a_\alpha \in E^\times$ and χ_α , a character of F^\times or E^\times , depending on whether α is fixed or not by $\text{Gal}(E/F)$, for the (absolute) roots of (G, T_0) . We can take all data equal to 1, giving $\Delta_{II} = 1$.

The amusing factor is $\Delta_{III,2}$, which depends on the embedding of ${}^L H$ in ${}^L G$, and in particular on the choice of the character μ . This is described in [LS87, p. 246]. If G is a quasi-split unitary group and (T, B) is the usual datum of a Borel subgroup and its maximal torus (over F), the construction of the L-group yields naturally a map $\xi_T : {}^L T \rightarrow {}^L G$, $(t, w) \mapsto (t, w)$ ($t \in \widehat{T}$, $w \in W_F$). (This is associated to trivial χ -data, in the terminology of [LS87].) If G has rank r , we take as usual $\widehat{T} = (\mathbb{C}^\times)^r \subset \text{GL}_r(\mathbb{C})$. This applies to G and H . Consider now the embedding $\xi : {}^L H \rightarrow {}^L G$. We have two tori T_H, T and the map $\xi \circ \xi_{T_H} : \widehat{T}_H \rightarrow \widehat{T} \subset \widehat{G}$ is W_F -equivariant. It follows that $\xi \circ \xi_{T_H}(t, w) = a(w)\xi_T(t, w)$ where $a(w)$ is a 1-cocycle of W_F for the action of W_F on \widehat{T} .

In order to compute $a(w)$, it is convenient to consider the conjugate embedding given by (3.1). Let $\sigma \in W_F$ be a representative of $W_F - W_E$. The cocycle $a(w)$ is the product of a trivial (middle) matrix of size m and an outer matrix $g_2 \in \text{GL}_4(\mathbb{C})$, which we identify with $a(w)$. This outer component is equal to

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

We find, the other component being trivial:

$$\begin{aligned} a(w) : W_F &\longrightarrow (\mathbb{C}^\times)^4 \subset \text{GL}_4(\mathbb{C}) \\ \sigma &\longmapsto (1, 1, -1, -1) \\ w &\longmapsto \mu(w), \quad w \in W_E. \end{aligned}$$

The corresponding component of $T_H = T$ is isomorphic to $(E^\times)^2$. Thus $a(w)$ defines componentwise two equal characters of E^\times , with L-group

$$\mathbb{C}^\times \times \mathbb{C}^\times, \quad \sigma(z, w) = (w^{-1}, z^{-1})$$

associated to the cocycle

$$\begin{aligned} \alpha : w &\longmapsto (\mu(w), \mu(w)) \quad (w \in W_E) \\ \sigma &\longmapsto ((1, -1), \sigma). \end{aligned}$$

By Langlands' description of Shapiro's lemma for L-groups [Lan89], this is associated to the character $\mu(z)$ of $z \in E^\times$. We have shown:

Lemma 3.6. *Assume $t \in T_G$ and $t_H \in T_H \subset U(m) \times U(4)$ are associated, with $t_H = (t', z_1, z_2)$, $(z_1, z_2) \in E^\times \times E^\times \subset U(4)$. Then $\Delta(t_H, t) = \mu(z_1)\mu(z_2)\Delta_{IV}(t_H, t)$.*

Note that since $\mu(\bar{z}^{-1}) = \mu(z)$, this does not depend on the choice of the representative (z_1, z_2) for the second component.

3.5 Proof of Theorem 2.3

Now that the local transfer factors are available to us, we can compare the elements of the local L-packet of Mœglin–Mok (Theorem 3.2) and the representations appearing in Theorem 2.5, thus proving Theorem 2.3 and Theorem 3.3. We recall that E/F is assumed to be ramified, and F of odd residue characteristic.

We first assume $m \geq 5$ and consider the semi-stable modules X, Y, Z of Theorem 2.5, which we now denote by X', Y', Z' . With φ_E as in §3.2, let $\varphi : W_F \rightarrow {}^L G$ be the associated parameter. The computation of \mathcal{S}_φ shows that φ factors through $\xi : {}^L H \rightarrow {}^L G$ and therefore defines a parameter φ_H for H , which by

composition yields the parameter associated to $\text{St}_{m,E} \otimes (\text{St}_{1,E} \boxplus \text{St}_{3,E})$ for $H(E) = \text{GL}_m(E) \times \text{GL}_4(E)$. However, ξ contains, in the component of ${}^L H$ associated to $U(4)$, the character μ , so $\varphi_H|_{L_E}$ is, on this component,

$$w \mapsto \mu^{-1}(w)(sp(1) \oplus sp(3)),$$

$sp(i)$ being the irreducible representation of degree i of $\text{SU}_2(\mathbb{R})$.

Using Mœglin's results [Mœg07] we find that the parameter $sp(1) \oplus sp(3)$ for $U(4)$, originating from unstable base change, is associated to two discrete series representations τ_4^+, τ_4^- of $U(4)$. We want to understand their Jacquet modules (for the Borel subgroup.) The full representation of $GL(4, E)$, $St(3)_E \boxplus St(1)_E$, has a Jacquet module of length 4, with exponents

$$[1, 0, -1, 0], [0, 1, 0, -1], [1, 0, 0, -1](\times 2).$$

(We have used the natural notation for exponents on $GL(4, E)$, with maximal split torus $(E^\times)^4$.) Thus $St(3)_E \boxplus St(1)_E$ is a submodule of the principal series induced from $[1, 0, 0, -1]$, and it is its only irreducible submodule. This follows for instance from the realisation of the Zelevinsky involution from an involution of the Iwahori–Hecke algebra, since this representation has a unique quotient and the involution sends it to its dual; see [Pro98].

Let us denote, for the end of this proof, by θ the Galois automorphism of $GL(4, E)$ defined by our unitary group. Then θ acts naturally on the Jacquet module, the only stable exponent being $[1, 0, 0, -1]$. As in [Mœg07], the twisted character of $St(3)_E \boxplus St(1)_E$ on the maximally split torus is associated to the character of $\tau_4^+ \oplus \tau_4^-$ on same (for $U(4)$). However, the representation induced from $[1, 0, 0, -1]$ is induced from the θ -stable parabolic of type $(1, 2, 1)$, from a θ -stable representation. A standard computation then gives its twisted character, which does not vanish on the maximally split torus, and contains the exponent $[1, 0]$. Thus θ has non-zero trace on the subspace of the Jacquet module associated to $[1, 0, 0, -1]$. This implies that the Jacquet module of $\tau_4^+ \oplus \tau_4^-$ is equal to $2[1, 0]$. Twisting by μ^{-1} , we see that the stable packet associated to the component of degree 4 of φ_H is therefore composed of two representations π_4^+, π_4^- and has Jacquet module

$$2 \mu^{-1}(z_1 z_2) \otimes [1, 0]$$

with obvious notation.¹ Of course, the representation of $U(m) \subset H$ is the Steinberg representation.

Consider now the tempered modules X', Y' of Theorem 2.5, with a distinguished exponent $e_0 = [k - 2, k - 3, \dots, 0, -1]$. Each defines, by stable base change, a representation Π of $\text{GL}_n(E)$ – perhaps not the same Π . Arguing as in [CT15, Lemma 4.3] we find that Π is a semi-stable representation of $\text{GL}_n(E)$ (i.e, one having Iwahori-fixed vectors) originating from stable base change. Thus the parameter of Π is of the form

$$\bigoplus_i sp(n_i) \otimes \chi_i$$

with $\chi_i(z^c) = \chi_i(z^{-1})$, χ_i unramified, and the usual multiplicity 1 condition – no factor of the form $(sp(n) \otimes \chi) \oplus (sp(n) \otimes \chi(z^{-c}))$ – since X', Y' are in the discrete series as is immediately seen from their exponents. Comparing Jacquet modules (twisted for Π) we see first that the χ_i are trivial – thus n_i odd since the parameter is conjugate orthogonal (§2.2).

Moreover the θ -stable exponent for $\text{GL}_n(E)$ associated to e_0 ,

$$[k - 2, \dots, 0, -1, 0, 1, 0, \dots, 2 - k]$$

must occur in $\Pi_{N_0(E)}^{norm}$. Since the Jacquet module of Π is a sum of \mathfrak{S}_n -conjugates of the total exponents determined from the segments $[\frac{n_i-1}{2}, \dots, \frac{1-n_i}{2}]$, this implies $\{n_i\} = \{m, 3, 1\}$. Consequently X', Y' are contained in the L-packet described in Theorem 3.2. We now use the character identities associated to base change. Let f_E be a smooth, compactly supported function on $G(E)$, f associated to f_E on $G = G(F)$, and f^H similarly on H . For $\pi \in \Pi_\varphi$, let $\varepsilon(\pi)$ be the sign associated to H .

The stable identity

$$\sum_\pi \langle \text{trace } \pi, f \rangle = \text{trace}(\Pi_E(f_E) I_\theta),$$

¹We thank, once more, Mœglin for this argument.

where π runs over Π_φ and I_θ is the Whittaker-normalized intertwining operator associated to base change, tells us little because the twisted trace of I_θ on the Jacquet module of Π_E involves indeterminate signs on the eigenspaces. (The part of the identity associated to the exponent e_0 has already been exploited.) Consider now the identity

$$\sum_{\pi} \varepsilon(\pi) \langle \text{trace } \pi, f \rangle = \langle \text{trace } \pi_H, f^H \rangle \quad (3.7)$$

where we write, for simplicity, π_H for the L-packet consisting of $St_m \otimes \pi_4^+$ and $St_m \otimes \pi_4^-$; St_m is the Steinberg representation of the factor $U(m)$; and the Jacquet module of $\pi_4^+ \oplus \pi_4^-$ is equal to $2\mu^{-1}(z_1 z_2) \otimes [1, 0]$.

We imitate the method of [CT15, §3.7]. For functions f whose orbital integrals are supported on the regular part of $T = T_G$ (notation of §3.4) we have

$$\langle \text{trace } \pi, f \rangle = \int_{T^+} \Theta(\pi_{N_0}^{norm})(t) D_G(t) O_t(f) dt. \quad (3.8)$$

Similarly

$$\langle \text{trace } \pi_H, f^H \rangle = \int_{T_H^+} \Theta(\pi_{H, N_H}^{norm}) D_H(t_H) O_{t_H}(f^H) dt_H.$$

Recall that $\Delta^{IV}(t_H, t) = D_G(t)/D_H(t_H)$. The transfer factor $\Delta(t_H, t)$ contains the character $\mu(z_1 z_2)$, whose inverse occurs in π_H . Writing $e_H = ([k-2, k-1, \dots, 1], [1, 0])$ for the exponent of (each component of) π_H corrected by this transfer factor, we now have, with $g(t_H, t) = \Delta(t_H, t) O_{t_H}(f^H)$:

$$\langle \text{trace } \pi_H, f^H \rangle = 2 \int_{T_H^+} e_H(t_H) g(t_H) dt_H.$$

We recall that $t_H \in T_H^+$ is associated to several elements of T_G^+ . Neglecting the factor $U(1)$ of both tori (on which our data can be taken equal to 1), we can pretend that T_G^+ is parameterized by $(z_1, \dots, z_k) : |z_1| < |z_2| < \dots < |z_k| < 1$ ($z_i \in E^\times$) and T_H^+ by

$$(z_1, \dots, z_{k-2}; z_{k-1}, z_k) : |z_1| < \dots < |z_{k-2}|, |z_{k-1}| < |z_k| < 1.$$

Thus an element of T_H^+ is obtained from an element $t \in T_G^+$ by $w \in \mathfrak{S}_k$, w increasing on the two obvious intervals. We see that the alternating sum of exponents given by the left-hand side of (3.7) is equal to $2 \sum_w w([k-2, \dots, 1, 1, 0])$; here w runs over the specified elements.

Now assume $m = 5$, thus $k = 4$. The associated sum is easily seen to be equal to

$$jY' - jX' + jZ'$$

(j = normalized Jacquet module), cf. Theorem 2.3. Since the Jacquet modules of X' , Y' , Z' are clearly independent in the Grothendieck group of T , we deduce that $Y = Y'$, $X = X'$, X and Y being as in Theorem 3.3, with $\varepsilon(Y) = 1$, $\varepsilon(X) = -1$, and that the Jacquet module of $\pm(Z - W)$ is $j(Z') = [1, 0, 2, 1] + [1, 2, 0, 1]$. However, the expression of $j(Z')$ implies that

$$Z' \hookrightarrow \text{n-Ind}(\text{St}_{2,E} | \cdot |^{1/2} \otimes \text{St}_{U(5)}), \quad (3.9)$$

the induction being unitary, from the Levi subgroup $\text{GL}_2(E) \times U(5)$. The induced representation has a unique irreducible submodule. (It is an induced representation, the parameter being in the opposite chamber of that giving a unique Langlands *quotient*). Since (3.9) is now true for Z or W , we see that one of them is Z' : it is the representation Z described in Theorem 3.3. Furthermore $\varepsilon(Z) = 1$.

For $m > 5$, we can separate the set of increasing elements of \mathfrak{S}_k in two sets, those fixing 1 and those sending 1 to a larger index (thus $k-2$ to a smaller power). The inductive description of $j(X_k)$, $j(Y_k)$, $j(Z_k)$ in Theorem 3.3 is then immediately seen to be compatible with the signs $\varepsilon(X) = -1$, $\varepsilon(Y) = 1$; and the identification of Z given by the theorem is obtained by the same argument, implying $\varepsilon(Z) = 1$. Since ε is a character of the L-packet, we must have $\varepsilon(W) = -1$ (including for $m = 5$); the computation of Jacquet modules shows in fact that W is not semi-stable.

Finally, consider $m = 3$. Thus the L-packet of Mœglin, Mok has two elements. The exponent $[1, 0, 1]$ in Theorem 3.3 again shows, by the argument given at the beginning of this paragraph, that both X' and Y' belong to the L-packet.

Alternately, it would be easy to show that X' and Y' (described in Theorem 2.5) are the two summands of $\text{n-Ind}(\text{St}_{3,E} \boxtimes \mathbf{1})$ from the $\text{GL}_3(E) \times U(1)$ Levi subgroup of $U(7)$. The endoscopic identity implies $\varepsilon(Y) = 1$, $\varepsilon(X) = -1$ since the transfer to H of $\varepsilon(Y) + \varepsilon(X)$ is a positive sum of representations.

We now complete the identification of X, Y, \dots by the corresponding characters of \mathcal{S}_ϕ given after Theorem 3.2. The reader will easily see that this is not necessary for the proof of our main results, only $\varepsilon = \varepsilon_1 \varepsilon_2$ being relevant; we will therefore be brief. We need another endoscopic identity. Let us denote, for this paragraph, by H the endoscopic group of type $(1, n-1)$, thus associated to the character ε_1 in §3.2. There is an endoscopic embedding $\xi : {}^L H \rightarrow {}^L G$ with ${}^L H = \widehat{H} \rtimes W_F$, $\widehat{H} = \text{GL}_1(\mathbb{C}) \times \text{GL}_{n-1}(\mathbb{C})$, given by formulas similar to those in 3.1. Here $w \in W_E$ is sent to the diagonal matrix $(1, \mu(w)1_{n-1})$. Our parameter φ (associated to φ_E by stable base change) is equal to $\xi \circ \varphi_H$, where $\varphi_H|_{\text{SU}_2(\mathbb{R})}$ is conjugate to

$$s \mapsto (1, \text{Sym}^2 s \oplus \text{Sym}^{m-1} s)$$

and

$$\varphi_H|_{W_E} : w \mapsto (1, \mu(w)^{-1}1_{n-1}).$$

In particular the second component of φ_H defines an unstable discrete L-packet for $U(n-1)$ with two elements π_1, π_2 [Mœg07]. The extra identity, cf. (3.7), is

$$\sum_{\pi} \varepsilon_1(\pi) \langle \text{trace } \pi, f \rangle = \langle \text{trace } \pi_H, f^H \rangle$$

where $\pi_H = \mathbf{1} \otimes (\pi_1 \oplus \pi_2)$.

As in §3.4 we consider orbital integrals on T_G, T_H . The transfer factor is computed in the same fashion. Note that H , and indeed its factor $U(n-1)$, has the same rank as G . Assume $m = 5$, $k = 4$. As before we are reduced to a computation on T_G^+ and T_H^+ , whose descriptions coincide in this case. Moreover the character μ occurring in the endoscopic functoriality is compensated by the transfer factor, as before. By the comparison, the module $\sum_{\pi} \varepsilon_1(\pi) j_{\pi}$ ($\pi = X, Y, Z, W$) is effective (i.e., a positive sum). This implies by Theorem 2.5 $\varepsilon_1(Y) = 1$, compatibly with the fact that Y is in fact generic. Thus $\varepsilon_1(X)$ or $\varepsilon(Z) = -1$. The description of ε_1 is complete if we show that $\varepsilon_1(X) = 1$, implying that $\varepsilon_1(Z) = \varepsilon_1(W) = -1$. Note that

$$j(X) + j(Y) - j(Z)$$

is indeed effective. If $\varepsilon_1(X) = -1$, the exponent $[2, 1, 0, -1]$ disappears in the Jacquet module of π_H . The proof is therefore completed by

Proposition 3.7 (Mœglin). *The Jacquet module of π_H contains, with multiplicity 2, the character*

$$\mu(z)^{-1}[2, 1, 0, -1]$$

of T_H . More generally, assume $n = a + b$ with a, b odd. If π is the L-packet constructed from the datum $(\text{St}_{a,E} \boxplus \text{St}_{b,E})$, then the full Jacquet module contains

$$\left[\frac{b-1}{2}, \frac{b-3}{2}, \dots, \frac{1-a}{2} \right] \quad (a \leq b).$$

Proof. This follows from the arguments of [Mœg07], just as in [CT15, Appendix A]. The (induced) representation associated to the datum $\text{St}_{3,E} \boxplus \text{St}_{3,E}$ (we neglect the constant twist by μ^{-1} in the next arguments) has a Jacquet module of exponent $[1, 0, -1]$ with multiplicity 2. It splits into two summands, each having a Jacquet module of dimension 1 with this exponent. For $\text{St}_{5,E} \boxplus \text{St}_{3,E}$, we now use Mœglin's hereditary formula for Jacquet modules; with her notation, we have:

$$\text{Jac}_{|\cdot|^2}^{\tilde{G}}(\text{St}_{5,E} \boxplus \text{St}_{3,E}) = \text{St}_{3,E} \boxplus \text{St}_{3,E}.$$

This means here, by the identity between character for $U(8)$ and twisted character, that the exponents of the full L-packet π starting with 2 are those of $\text{St}_{3,E} \boxplus \text{St}_{3,E}$ with the same multiplicity, whence the result; the argument in the general case is the same, starting with $\text{St}_{a,E} \boxplus \text{St}_{b,E}$. \square

Finally, a similar argument applies for $m > 5$, using the inductive description of the modules.

3.6 Compact transfer: proofs

Returning to the global situation, we are now able to obtain the analogue of Theorem 3.5 for our \mathbb{R} -anisotropic group G . We will also need the similar theorem when the datum is a cuspidal representation of $G(\mathbb{A}_E)$. For clarity we assume first that $l > 5$ (thus $n = l + 2 > 7$).

We fix a CM extension E/F as before, and places v_0, \dots, v_s of F which are ramified in E and of odd residue characteristic. We write w_0, \dots, w_s for the places of E dividing v_0, \dots, v_s . We assume $[F : \mathbb{Q}]$ even. Assume given σ_1, σ_2 , two conjugate self-dual, cuspidal representations of $\text{GL}_4(\mathbb{A}_E)$ and $\text{GL}_m(\mathbb{A}_E)$ respectively. We will assume that at each Archimedean prime v , the Langlands parameter of $\sigma_1 \boxplus \sigma_2$ is

$$z \mapsto \text{diag} \left((z/\bar{z})^{\frac{n-1}{2}}, (z/\bar{z})^{\frac{n-3}{2}}, \dots, (z/\bar{z})^{\frac{1-n}{2}} \right), \quad (3.10)$$

cf. §3.3. (Thus σ_2 is RACSDC, σ_1 is a twist of such by μ .) There is then an element $w_v \in \mathfrak{S}_n$ conjugating the Langlands parameter $\varphi_{\sigma_1} \oplus \varphi_{\sigma_2}$ to the Langlands parameter associated to (3.10), as in §3.4 and [Clo11, p.362]. We assume

$$\prod_{v|\infty} \det(w_v) = 1. \quad (3.11)$$

Note that this does not depend on the prime of E dividing v . We assume in addition that all other primes of ramification of σ_1, σ_2 are above split primes of F . If w is a place of E ramified over the place v of F , recall that we have associated to the representation

$$\Pi_{E_w} = \text{St}_{m,E_w} \boxplus \text{St}_{3,E_w} \boxplus \text{St}_{1,E_w}$$

of $\text{GL}_n(E_w)$ the L-packet Π_{φ_v} and a map $\varepsilon_v : \Pi_{\varphi_v} \longrightarrow \{\pm 1\}$, with

$$\varepsilon_v(X_v) = \varepsilon_v(W_v) = -1, \quad \varepsilon_v(Y_v) = \varepsilon_v(Z_v) = 1.$$

Theorem 3.8. *1. Suppose that σ_1 and σ_2 are conjugate self-dual, cuspidal representations of the groups $\text{GL}_4(\mathbb{A}_E)$ and $\text{GL}_m(\mathbb{A}_E)$, respectively, which satisfy the following conditions:*

(a) $\sigma_1 \boxplus \sigma_2$ is regular algebraic of infinity type

$$((l+1)/2, (l-1)/2, \dots, (-1-l)/2)$$

at every infinite place, and (with notation as above) we have $\prod_{v|\infty} \det(w_v) = 1$.

(b) There exist places v_0, \dots, v_s of F , ramified in E and of odd residue characteristic, such that for each $i = 0, \dots, s$, $\sigma_{1,w_i} \cong \text{St}_{3,E_{w_i}} \boxplus \text{St}_{1,E_{w_i}}$, and $\sigma_{2,w_i} \cong \text{St}_{m,E_{w_i}}$.

(c) Any other place of E at which $\sigma_1 \boxplus \sigma_2$ is ramified is split over F . Any place $v \neq v_0, \dots, v_s$ of F is unramified in E .

Then there exist exactly $4^{s+1}/2$ automorphic representations σ of $G(\mathbb{A}_F)$ with stable base change $\pi = \sigma_1 \boxplus \sigma_2$. They each appear with multiplicity one, and are in bijection with the tuples $(\sigma_{v_i})_{i=0}^s \in \prod_{i=0}^s \Pi_{\varphi_v}$ satisfying $\prod_{i=0}^s \varepsilon_{v_i}(\sigma_{v_i}) = 1$.

2. Let π be a RACSDC automorphic representation of $\text{GL}_n(\mathbb{A}_E)$ satisfying the following hypotheses:

(a) π is regular algebraic of infinity type

$$((l+1)/2, (l-1)/2, \dots, (-1-l)/2)$$

at every infinite place.

- (b) There exist places v_0, \dots, v_s of F , ramified in E and of odd residue characteristic, such that for each $i = 0, \dots, s$, $\pi_{w_i} \cong \text{St}_{m, E_{w_i}} \boxplus \text{St}_{3, E_{w_i}} \boxplus \text{St}_{1, w_i}$.
- (c) Any other place of E at which π is ramified is split over F . Any place $v \neq v_0, \dots, v_s$ of F is unramified in E .

Then there are exactly 4^{s+1} automorphic representations σ of $G(\mathbb{A}_F)$ with stable base change π , each appearing with multiplicity one. They are naturally in bijection with tuples $(\sigma_{v_i})_{i=0}^s \in \prod_{i=0}^s \Pi_{\varphi_{v_i}}$.

We note that over any finite place of F different from v_0, \dots, v_s E/F is unramified or split, and there is only one element in the local L-packet for $G(F_v)$ associated to $\sigma_1 \boxplus \sigma_2$ or π . (In the inert case, note that there is only, up to conjugacy, one hyperspecial group for $G(F_v)$.)

Remark 3.9. The condition $\prod_{v|\infty} \det(w_v) = 1$ in the first part of the theorem is implied by the following two assumptions:

1. For every place v of F dividing l , v splits in E and $[F_v : \mathbb{Q}_l]$ is even.
2. There exists an isomorphism $\iota : \overline{\mathbb{Q}_l} \cong \mathbb{C}$ such that $\sigma_1 \boxplus \sigma_2$ is ι -ordinary.

Indeed, if $\tau : E \hookrightarrow \mathbb{C}$ is an embedding inducing the place v of F , then $\det(w_v)$ depends (by [CT14, Lemma 2.6]) only on the place of F dividing l induced by the embedding $\iota^{-1}\tau : E \hookrightarrow \overline{\mathbb{Q}_l}$. Since we are assuming $[F_v : \mathbb{Q}_l]$ to be even, the signs in the product $\prod_{v|\infty} \det(w_v)$ must cancel, so the condition $\prod_{v|\infty} \det(w_v) = 1$ holds. We will assume these extra conditions in the applications in §4 below. The same remarks apply to Theorem 3.10 below.

This can also be seen in terms of Shelstad's formalism for endoscopy, cf. [She82, She10, She], since $\prod_{v|\infty} \det w_v$ equals $\prod_{v|\infty} \chi(g_v s g_v^{-1})$, where:

- $g_v \in \widehat{G}$ conjugates $\xi \circ \varphi_{H,v}$ to a Langlands parameter whose restriction to \mathbb{C}^\times is diagonal and dominant for the upper-triangular Borel, so that the centralizer of $\text{Ad}(g_v) \circ \xi \circ \varphi_{H,v}$ is the subgroup $\text{diag}(\pm 1, \dots, \pm 1)$ of \widehat{G} having 2^n elements.
- χ is the character $((-1)^{(n-1)/2}, (-1)^{(n+1)/2}, (-1)^{(n-1)/2}, \dots)$ (alternating signs) of this group (χ is trivial on $Z_{\widehat{G}}^{\text{Gal}(\overline{F}_v/F_v)} = \{\pm 1\}$).

Compare [Mok15, §7.1], in particular Theorem 7.1.1, and the reference to Shelstad's work there.

The proof of Theorem 3.8 will rely on Theorem 3.5. Consider first the quasi-split group G^* . By Arthur's stabilization of the trace formula [Art03] (we use Arthur's formulation in [Art05]) we have, for a function $f_* = \bigotimes_v f_{*,v}$ on $G^*(\mathbb{A}_F)$:

$$I_{disc}^{G^*}(f_*) = \sum_{\mathcal{E}} \iota(G^*, \mathcal{E}) ST_{disc}^{\mathcal{E}}(f_*^{\mathcal{E}}) \quad (3.12)$$

where \mathcal{E} runs over the equivalence classes of elliptic endoscopic data, including H . We refer to Arthur for the precise definitions; here $I_{disc}^{G^*}$ denotes the discrete part of the trace formula for G^* , while $ST_{disc}^{\mathcal{E}}$ is a stabilised form of this expression, for the endoscopic group associated to \mathcal{E} .

Similarly, for f a function on $G(\mathbb{A}_F)$:

$$T_{disc}^G(f) = I_{disc}^G(f) = \sum_{\mathcal{E}} \iota(G, \mathcal{E}) ST_{disc}^{\mathcal{E}}(f^{\mathcal{E}}). \quad (3.13)$$

An argument already introduced in [CHL11] implies that the only relevant data, when we consider representations with base change π , are (G^*, H) . The argument is the same as in [CHL11, Lemma 4.2]. By the main result in [Lab11], each term in $ST_{disc}^{\mathcal{E}}(f^{\mathcal{E}})$ in (3.13) is associated by base change – say, at almost all primes – to ‘discrete’, cohomological automorphic representations of the group $H'(\mathbb{A}_E) = GL_a(\mathbb{A}_E) \times GL_b(\mathbb{A}_E)$ stable by the Galois automorphism associated to the unitary group H' deduced from \mathcal{E} . The meaning of ‘discrete’

here is precisely explained in [CHL11, p. 486]: they are discrete parameters for descent to the unitary group. For each block GL_a or GL_b , they are Langlands sums of cuspidal representations, or more generally of Speh representations. Since our representation π is a Langlands sum of two cuspidal representations of GL_4 and GL_m , we see first that the ‘discrete’ representations must be sums of cuspidal representations, and then, by the disjunction results of Jacquet–Shalika [JS81b, JS81a] that the only possibilities for H' are as indicated.

The functions f and f_* coincide at finite primes. In (3.13), f^{G^*} is associated to f , and we take it of course to be equal to f_* . At the Archimedean primes, f_v can be taken to be the constant function equal to 1. Assume $\pi_\infty = \otimes \pi_v$ is a discrete series representation of $G^*(F_\infty)$, in the L–packet associated to the trivial representation. Let $f_{*,\infty}$ be a pseudo-coefficient of π_∞ : thus $\text{trace } \pi_\infty(f_{*,\infty}) = 1$. It is well-known that $f_{*,\infty}$ is associated to the constant function 1 on $G(F_\infty)$. By our condition on the Weyl element w , we have, $\Pi(H)$ being the L–packet of discrete series for H associated to $\sigma_1\mu^{-1} \boxplus \sigma_2$:

$$\langle \text{trace } \Pi(H), f_\infty^H \rangle = \langle \text{trace } \mathbb{C}, f_\infty \rangle = 1.$$

By the endoscopic identity for (H, G^*) at the Archimedean prime,

$$\langle \text{trace } \Pi(H), f_{*,\infty}^H \rangle = \sum_{\pi} \langle s, \pi \rangle \langle \text{trace } \pi, f_{*,\infty} \rangle,$$

where π runs over the L–packet of discrete series representations of $G^*(F_\infty)$ associated to the trivial representation of $G(F_\infty)$. By assumption this is equal to $\langle s, \pi_\infty \rangle$ for our chosen representation π_∞ .

Consider now another L–packet $\Pi_1(H)$ transferring to this same L–packet (cf. [Clo11]). Then the corresponding automorphic representations of H give rise to Galois representations having different Hodge–Tate weights [Gue11, Theorem 2.3]². We now introduce the family ψ of Hecke eigenvalues, at (almost all) unramified primes, corresponding to π . By unramified functoriality, it defines such a family for all the endoscopic groups. They are associated to the corresponding Galois representations. By the Chebotarev density theorem applied to these representations, we see that their contributions to the traces vanish once we consider the parts of the trace formulas associated to ψ . We denote these terms by a lower index ψ .

Assuming π_∞ chosen so that $\langle s, \pi_\infty \rangle = 1$ we see that f_∞^H can be chosen equal to $f_{*,\infty}^H$. We now have

$$I_{disc}^{G^*}(f_*)_\psi = T_{disc}^G(f)_\psi.$$

Since $\langle s, \pi_\infty \rangle = 1$, Mok’s theorem (here Theorem 3.5) implies that there are $4^{s+1}/2$ representations of $G^*(\mathbb{A}_F)$ in the discrete or cuspidal spectrum with Archimedean component π_∞ and with stable base change π . Recall that $I_{disc}^{G^*}$ contains, in addition to the trace $T_{disc}^{G^*}$ in the discrete spectrum, terms associated to Eisenstein series whose parameters are fixed by certain Weyl groups elements, see [Art05, (21.19)]. However the family of Hecke eigenvalues for G^* , at the unramified primes, determined by $\sigma_1 \boxplus \sigma_2$, is easily seen to be distinct from that of any Eisenstein representation. Thus we actually get the identity

$$T_{disc}^{G^*}(f_*)_\psi = T_{disc}^G(f)_\psi.$$

Since the groups coincide at the finite primes, and f_v is the constant function equal to 1 at the Archimedean primes, the identity of traces implies the result.

The proof of the second part of Theorem 3.8 is similar. In this case, after separation of Hecke eigenvalues, only the terms of (3.12) and (3.13) given by G^* have to be considered. We take for $f_{*,\infty}$ any pseudo-coefficient of the L–packet. The global datum being stable, the pairing $\langle s, \pi \rangle$ is, at all primes, the trivial character. The result follows.

We note that the same arguments imply that the (strong) stable base change of an automorphic representation σ of $G(\mathbb{A}_F)$ exists: it is a representation $\pi = \boxplus_i \pi_i$ of $GL_n(\mathbb{A}_E)$ with local components at all primes predicted by Langlands functoriality, well-defined if the local components of σ are tempered. Indeed recall that there is an n -dimensional semi-simple l -adic representation associated to σ [Gue11, §2]. We will only be interested in the case where this representation is irreducible or with irreducible factors $(4, m)$. In this case it is easy to see that σ is associated to such a representation π with either π cuspidal or two cuspidal

²Guerberoff assumes that E/F is unramified at finite primes. It is easy to see that this assumption is irrelevant here.

summands π_i of sizes $(4, m)$. Consider again the formulas (3.12), (3.13). If π is associated to an irreducible Galois representation, standard arguments [CHL11] show that both sums reduce to the stable trace when we consider the part relative to the given Hecke eigenvalues almost everywhere. Now the previous arguments imply that these sums are equal, and this implies that strong, stable base change for G follows from the analogous result for G^* proved by Mok: see [Mok15, Theorem 2.5.2].

Assume the semi-simplification is of type $(4, m)$. Then only the terms relative to G^* and H have to be compared, and we are, as above, reduced to the term relative to H . We must compare the Archimedean functions $f_\infty^H, f_{*,\infty}^H$. The function f_∞^H is defined by

$$\langle \text{trace } \Pi(H), f_\infty^H \rangle = \det(w)$$

for any L-packet $\Pi(H)$ for H such that $\xi \circ \varphi_H = \varphi_G$, φ_G being associated to the trivial representation. Similarly $f_{*,\infty}^H$ is defined by

$$\langle \text{trace } \pi(H), f_{*,\infty}^H \rangle = \det(w) \langle a_{w\tau}, s \rangle,$$

cf. [Clo11, p. 366]. Here $\tau \in \Omega/\Omega_{\mathbb{R}} = \mathfrak{S}_n/(\mathfrak{S}_r \times \mathfrak{S}_{r+1})$ ($n = 2r + 1$) parameterizes a discrete series representation for G^* in the L-packet of the trivial representation, and $f_{*,\infty}^H$ is, as before, a pseudo-coefficient of this discrete series. Fix such a σ occurring in $T_{disc}^G(f)$, and therefore a sequence of Hecke eigenvalues at the unramified primes, and assume the corresponding H -term is non-zero. By base change (cf. [Lab11, Theorem 5.1], which has already been used through Guerberoff's result quoted above) there is a corresponding representation π of $H(\mathbb{A}_E)$, uniquely defined. This determines the infinity type of π , and therefore the L-packet $\Pi(H)$, hence w .

Recall that $w \in \Omega \mapsto a_w$ is a 1-cocycle of Ω with values in $H^1(\mathbb{R}, T)$ where T is the diagonal anisotropic torus in G^* (we refer to [Clo11] for the relevant choices, implicit throughout this paper as in [CT15]). In particular $a_w = 1$ if $w = 1$. If we choose τ such that $\langle a_{w\tau}, s \rangle = 1$, we deduce that after separation of the family ψ of Hecke eigenvalues corresponding to σ ,

$$ST_{disc}^H(f_\infty^H)_\psi = ST_{disc}(f^H)_\psi.$$

Finally we have, for this choice of $f_\infty, f_{*,\infty}$:

$$T_{disc}(f)_\psi = T_{disc}(f_*)_\psi$$

Note that the f_v for finite primes, outside the set specifying ψ , are arbitrary. Mok's results, specifically [Mok15, Theorem 2.5.2] now applied to base change to $G(E)$ rather than descent, imply the existence of a stable base change π of σ - a RACSD representation of $G(\mathbb{A}_E)$ - with local components in the correct L-packets at all finite primes; at infinite primes π is cohomological and its infinitesimal character is associated to the trivial representation of $G(F_\infty)$. Thus π is, at all primes, the stable base change of σ .

Consider now the case where $l = 5$, hence $n = 7$. The assumptions and data are the same, except that the L-packet associated to $\text{St}_{3,E_w} \boxplus \text{St}_{3,E_w} \boxplus \text{St}_{1,E_w}$ has now two elements X_v, Y_v (Theorem 3.3). We define a local sign by the formula $\varepsilon_v(X_v) = -1, \varepsilon_v(Y_v) = 1$. Again, the data determine elements $w_v \in \mathfrak{S}_7$, and we assume $\prod_{v|\infty} \det(w_v) = 1$. Exactly the same argument now yields:

Theorem 3.10. *1. Suppose that σ_1, σ_2 are conjugate self-dual, cuspidal representations of $\text{GL}_4(\mathbb{A}_E)$, $\text{GL}_3(\mathbb{A}_E)$, respectively satisfying the following hypotheses:*

(a) $\sigma_1 \boxplus \sigma_2$ is regular algebraic of infinity type

$$(3, 2, \dots, -3)$$

at every infinite place, and (with notation as above) we have $\prod_{v|\infty} \det(w_v) = 1$.

(b) There exist places v_0, \dots, v_s of F , ramified in E and of odd residue characteristic, such that for each $i = 0, \dots, s$, $\sigma_{1,w_i} \cong \text{St}_{3,E_{w_i}} \boxplus \text{St}_{1,E_{w_i}}$, and $\sigma_{2,w_i} \cong \text{St}_{3,E_{w_i}}$. (Here we write w_i for the place of E above v_i .)

(c) Any other place of E at which $\sigma_1 \boxplus \sigma_2$ is ramified is split over F . Any place $v \neq v_0, \dots, v_s$ of F is unramified in E .

Then there exist exactly 2^s automorphic representations σ of $G(\mathbb{A}_F)$ with stable base change $\pi = \sigma_1 \boxplus \sigma_2$. They each appear with multiplicity one, and are in bijection with the tuples $(\sigma_{v_i})_{i=0}^s \in \prod_{i=0}^s \Pi_{\varphi_{v_i}}$ satisfying $\prod_{i=0}^s \varepsilon_{v_i}(\sigma_{v_i}) = 1$.

2. Let π be a RACSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ satisfying the following hypotheses:

(a) π is regular algebraic, of infinity type

$$(3, 2, \dots, -3)$$

at every infinite place.

(b) There exist places v_0, \dots, v_s of F , ramified in E and of odd residue characteristic, such that for each $i = 0, \dots, s$, $\pi_{w_i} \cong \mathrm{St}_{3, E_{w_i}} \boxplus \mathrm{St}_{3, E_{w_i}} \boxplus \mathrm{St}_{1, E_{w_i}}$.

(c) Any other place of E at which π is ramified is split over F . Any place $v \neq v_0, \dots, v_s$ of F is unramified in E .

Then there are exactly 2^{s+1} automorphic representations σ of $G(\mathbb{A}_F)$ with stable base change π , each appearing with multiplicity one. They are naturally in bijection with tuples $(\sigma_{v_i})_{i=0}^s \in \prod_{i=0}^s \Pi_{\varphi_{v_i}}$.

Finally, the same arguments as in the previous case show that stable base change obtains for representations of $G(\mathbb{A}_F)$.

4 Level-raising and algebraic modular forms

This section is devoted to the key level-raising argument. In this introduction we still sketch it in the case of Sym^8 and $l = 7$.

Assume, at one ramified place v (for E/F), that our given representation of $\mathrm{GL}_2(\mathbb{A}_F)$ (for which we hope to establish the existence of the symmetric power lifting) is Steinberg. Then the local component at v of the representation of $G(\mathbb{A}_F)$ constructed in §3 will base change to $\mathrm{St}_{3, E_w} \boxplus \mathrm{St}_{1, E_w} \boxplus \mathrm{St}_{5, E_w}$, and therefore lie in the L-packet of representations which is described by Theorem 2.3.

The flexibility needed in §6 forces us, in fact, to consider a large number v_0, \dots, v_s of such ramified primes. We also assume $q_{v_0} \equiv -1 \pmod{l}$. We can define two level subgroups U_0, U_1 of $G(\mathbb{A}_F^\infty)$; at the places $v_i \neq v_0$ their local factor is the group \mathfrak{B} considered in §2, which contains an Iwahori subgroup with index 2. At v_0 , U_1 is again this group and U_0 is the parahoric subgroup \mathfrak{P} such that $\dim X^{\mathfrak{P}} = 1$ for the distinguished module X of §2.

The global L-packet constructed in §3 determines a character of an unramified Hecke algebra that, considered mod l , defines a maximal ideal \mathfrak{m} . Let K/\mathbb{Q}_l be a sufficiently large extension. For $A = K, \mathcal{O}, k$ (integers and residue field of K), we have the (finite) spaces of automorphic forms on G , localized, $S(U_i, A)_{\mathfrak{m}}$ ($i = 0, 1$). There is a natural injection $S(U_0) \rightarrow S(U_1)$.

The first main result of this section, Proposition 4.1, controls the Jordan decomposition of a local component Σ_v ($v = v_0, \dots, v_s$) occurring in characteristic 0, so essentially in $S(U_i, \mathcal{O})_{\mathfrak{m}}$. After base change to E , it can obviously be of type $(1, 3, 5)$; the only other possibilities are $(7, 1, 1)$ and (9) (the last corresponding to a twist of Steinberg). This follows from the restrictions on the degeneracy mod l of Jordan matrices, and the existence of the Galois representations associated to primes of the Hecke algebra above \mathfrak{m} .

The second main result — the level-raising result — consists in showing that (at least one of) these second Jordan decompositions do occur. In other terms, we must rule out the case where all (base changed) representations associated to \mathfrak{m} are of type $(1, 3, 5)$. These are exactly the local representations of the groups $G(F_v)$ studied in §§2 – 3. The argument is based on the following facts:

1. There is a perfect pairing $S(U_1, A)_{\mathfrak{m}} \times S(U_1, A)_{\mathfrak{m}^\vee} \rightarrow A$ for $A = \mathcal{O}$ or $A = k$. (Here \mathfrak{m}^\vee is the maximal ideal associated to the contragredient representation.)

2. For $A = k$, this pairing vanishes on $S(U_0, k)_{\mathfrak{m}} \subset S(U_1, k)_{\mathfrak{m}}$. This is a direct consequence of our assumption $q_{v_0} \equiv -1 \pmod{l}$.

Recall that the module X has a distinguished exponent $e = [2, 1, 0, -1]$, which occurs in $X_{N_0}^{\text{norm}} \cong X^{\mathfrak{B}}$, this space being one-dimensional. We can refine the localization of $S(U_i, k)$ at \mathfrak{m} by using the action of Bernstein's maximal abelian subalgebra of the Iwahori–Hecke algebra, the characters of which are exactly the exponents. Now, using this decomposition for the semi-stable modules X, Y, Z of §2, we are able to conclude that the isotropic subspace $S(U_0, k)$ (localized at \mathfrak{m} and e , or rather $\bar{e} = e \pmod{l}$), is too large, by using Mok's multiplicity formula. We refer the reader to the end of §4 for the precise argument.

It is worth comparing the work we do here with a theorem of the second-named author [Tho14], which (loosely speaking) allows one to replace a local Weil–Deligne representation (r, N) , where N has two Jordan blocks, with a Weil–Deligne representation where N has a single Jordan block (and is therefore regular nilpotent). Unfortunately the methods of [Tho14] require that the residue characteristic l satisfy some stringent conditions (in particular, it should be banal for the local group at the prime where level-raising occurs). Since the strategy of this paper requires us to work with relatively small primes l , these methods can not be applied here.

We also note that the argument sketched above is not alone sufficient to obtain a representation which is Steinberg locally, in contrast to the analogous argument in [CT15]; although we show that one of the possibilities (7, 1, 1) and (9) does occur, we are not able to decide which, and we need the latter to occur in order to be able to apply the automorphy lifting theorem proved in [Tho15]. We must therefore take up again the problem of level-raising in the next §5, where we use Galois-theoretic techniques to obtain the necessary improvements to the results of this section.

4.1 Set-up

Let $l \geq 5$ be a prime, and choose an isomorphism $\iota : \overline{\mathbb{Q}_l} \cong \mathbb{C}$. Let E be an imaginary CM field with maximal totally real subfield F , and satisfying the following conditions:

1. Each place $v|l$ of F is split in E , and the local degree $[F_v : \mathbb{Q}_l]$ is even.
2. There exist places v_0, \dots, v_s of F , ramified in E and of odd residue characteristic, such that $q_{v_0} \equiv -1 \pmod{l}$ and each q_{v_i} , $i = 1, \dots, s$, is a primitive root modulo l . Every other place of F is unramified in E . We write w_0, \dots, w_s for the places of E dividing v_0, \dots, v_s . (We assume $s \geq 1$.)

Let $n = l + 2$, $m = l - 2$. In this case there exists a unitary group G in n variables over F , determined uniquely up to isomorphism by the following conditions:

1. The group G is split by E .
2. $G(F \otimes_{\mathbb{Q}} \mathbb{R})$ is compact.
3. G is quasi-split at each finite place.

We suppose given cuspidal, conjugate self-dual automorphic representations π_1 and π_2 of $\text{GL}_m(\mathbb{A}_E)$ and $\text{GL}_4(\mathbb{A}_E)$, respectively, that satisfy the following conditions:

1. For each $i = 0, \dots, s$, we have $\pi_{1, w_i} \cong \text{St}_{m, E_{w_i}}$ and $\pi_{2, w_i} \cong \text{St}_{3, E_{w_i}} \boxplus \text{St}_{1, E_{w_i}}$. If $w \neq w_0, \dots, w_s$ is any place of E such that $\pi_{1, w}$ or $\pi_{2, w}$ is ramified, then w is split over F .
2. $\pi = \pi_1 \boxplus \pi_2$ is regular algebraic and ι -ordinary, of infinity type $\mathbf{a} = (a_{\tau})_{\tau \in \text{Hom}(E, \mathbb{C})}$:

$$a_{\tau} = ((n-1)/2, (n-3)/2, \dots, (1-n)/2).$$

The residual representation $\overline{r_{\iota}(\pi)}$ is a direct sum of two absolutely irreducible constituents $\bar{\rho}_1$ and $\bar{\rho}_2$. We have $\dim \bar{\rho}_1 = m$, $\dim \bar{\rho}_2 = 4$. (We remind the reader that $r_{\iota}(\pi)$ is the Galois representation associated to the regular algebraic automorphic representation π ; see §1.1 for our conventions, which are those of [CT14].)

3. For each $i = 0, \dots, s$, let $t_{w_i} \in G_{E_{w_i}}$ denote a generator of tame inertia. Then the element $\bar{\rho}_1(t_{w_i}) \in \mathrm{GL}_m(\bar{\mathbb{F}}_l)$ is a regular unipotent element (having a single Jordan block). The element $\bar{\rho}_2(t_{w_i}) \in \mathrm{GL}_4(\bar{\mathbb{F}}_l)$ has Jordan decomposition corresponding to the partition $4 = 3 + 1$.

In this case there exists, by the discussion in §3.6, an automorphic representation σ of $G(\mathbb{A}_F)$ with stable base change π . We may suppose that, in the notation of §3.6, we have $\sigma_{v_0} \cong X_{v_0}$ and σ_{v_i} is semi-stable for each $i = 1, \dots, s$; for example, we can choose $\sigma_{v_1} = X_{v_1}$ and $\sigma_{v_i} = Y_{v_i}$ for each $i \geq 2$. In conjunction with Theorem 2.3, this ensures that we can choose a subgroup U_0 satisfying the condition 5 below.

Let $K \subset \bar{\mathbb{Q}}_l$ be a finite extension of \mathbb{Q}_l with ring of integers \mathcal{O} , maximal ideal λ , and residue field k . If $U = \prod_v U_v \subset G(\mathbb{A}_F^\infty)$ is an open compact subgroup, then we can define for any \mathcal{O} -algebra A the space $S(U, A)$ of algebraic modular forms with level U , weight zero, and coefficients in A , as the set of functions $G(F) \backslash G(\mathbb{A}_F^\infty) / U \rightarrow A$. We choose as in [CT15, §3.3] the following auxiliary data:

1. An integral model for G over \mathcal{O}_F .
2. If w is a place of E split over the place v of F , an isomorphism $\iota_w : G(F_v) \cong \mathrm{GL}_n(E_w)$.
3. For each place $v|l$ of F , a place \tilde{v} of E above it, and a uniformizer $\varpi_{\tilde{v}}$ of $\mathcal{O}_{E_{\tilde{v}}}$.

We fix an identification between each group $G_{F_{v_i}}$ and the quasi-split group U_n of §2. Having made this choice, the open compact subgroups $\mathfrak{B}_{v_i} \subset \mathfrak{P}_{v_i} \subset G(F_{v_i})$ are defined. We now choose an open compact subgroup $U_0 = \prod_v U_{0,v} \subset G(\mathbb{A}_F^\infty)$ satisfying the following properties:

1. $U_{0,v_0} = \mathfrak{P}_{v_0}$.
2. If $i = 1, \dots, s$ then $U_{0,v_i} = \mathfrak{B}_{v_i}$.
3. If v is a place of F inert in E , then U_v is a hyperspecial maximal compact subgroup of $G(F_v)$.
4. There exists $c > 0$ such that $U_{0,v} = \iota_{\tilde{v}}^{-1} \mathrm{Iw}_c(\tilde{v})$ for each place $v|l$, the subgroup $\mathrm{Iw}_{\tilde{v}}(c)$ being as in [CT15, §3.3].
5. $\sigma^{U_0} \neq 0$ and U_0 is sufficiently small (i.e. some $U_{0,v}$ contains no element of finite order).

We define $U_1 = \prod_v U_{1,v}$ by setting $U_{1,v_0} = \mathfrak{B}_{v_0}$ and $U_{1,v} = U_{0,v}$ if $v \neq v_0$. Then U_0 contains U_1 with index $q_{v_0} + 1$.

The Hecke operators U_v^j at places $v|l$, as defined in [CT15, §3.3], act on $S(U_0, \mathcal{O})$ and $S(U_1, \mathcal{O})$. If $U = U_0$ or U_1 we define the Hecke algebra $\mathbb{T}(U, \mathcal{O}) \subset \mathrm{End}_{\mathcal{O}}(S(U, \mathcal{O}))$ as the commutative \mathcal{O} -subalgebra generated by the unramified Hecke operators at places v of F split in E such that U_v is a maximal compact subgroup, together with the operators U_v^j at places $v|l$. The operators U_v^j depend on the choice of uniformizer $\varpi_{\tilde{v}}$, but the algebra $\mathbb{T}(U, \mathcal{O})$ does not.

If Σ is an automorphic representation of G such that Σ_∞ is the trivial representation, then there is a canonical embedding $\iota^{-1} \Sigma^{\infty, U} \hookrightarrow S(U, \mathcal{O}) \otimes_{\mathcal{O}} \bar{\mathbb{Q}}_l = S(U, \bar{\mathbb{Q}}_l)$. The subspace $\iota^{-1} \Sigma^{\infty, U, \mathrm{ord}} \subset \iota^{-1} \Sigma^{\infty, U}$ is defined, as in [CT15, §3.3]. If $N \subset S(U, \mathcal{O})$ is an \mathcal{O} -submodule such that

$$N \cap \iota^{-1} \Sigma^{\infty, U, \mathrm{ord}} \neq 0$$

inside $S(U, \bar{\mathbb{Q}}_l)$, then we say that Σ contributes to N . In particular, if Σ contributes to $S(U, \mathcal{O})$, then $\Sigma^{\infty, U} \neq 0$ and Σ is ι -ordinary.

After possibly enlarging K , we can assume that representation σ contributes to $S(U_0, \mathcal{O})$, in the above sense, and therefore defines a homomorphism $\mathbb{T}(U_0, \mathcal{O}) \rightarrow \mathcal{O}$, and we write \mathfrak{m} for the kernel of the composite homomorphism $\mathbb{T}(U_1, \mathcal{O}) \rightarrow \mathbb{T}(U_0, \mathcal{O}) \rightarrow \mathcal{O} \rightarrow k$. The dual representation σ^\vee also contributes to $S(U_0, \mathcal{O})$, and we write $\mathfrak{m}^\vee \subset \mathbb{T}(U_1, \mathcal{O})$ for the corresponding maximal ideal.

Proposition 4.1. *Suppose that Σ is an automorphic representation of $G(\mathbb{A}_F)$ which contributes to $S(U_1, \mathcal{O})_{\mathfrak{m}}$. Let Π denote the stable base change of Σ to $\mathrm{GL}_n(\mathbb{A}_E)$, a RACSD automorphic representation. Then Π is ι -ordinary, and one of the following holds:*

1. For each $i = 0, \dots, s$, we have $\Pi_{w_i} \cong \text{St}_{m, E_{w_i}} \boxplus \text{St}_{3, E_{w_i}} \boxplus \text{St}_{1, E_{w_i}}$.

2. There exists $i \in \{0, \dots, s\}$ such that $r_\iota(\Pi)(t_{w_i})$ has Jordan decomposition corresponding either to the partition $n = (n-2) + 1 + 1$ or to the trivial partition $n = n$. Moreover, Π is cuspidal.

Proof. Let us choose some $i \in \{0, \dots, s\}$ and now write $w = w_i$ to simplify notation. Then Π_w is semi-stable, just as in [CT15, Lemma 4.3]. [In fact, the proof of [CT15, Lemma 4.3] should be corrected as follows, starting on line 9 of the proof (notations as in loc. cit.).]

Now fix $v = v_i$, $w = w_i$ for some $i = 0, \dots, s$. Since $\sigma_0 = \otimes_{i=0}^s \sigma_{0, v_i}$ is semi-stable, we can take for $j \neq i$ the function $f_{v_j} = [\mathfrak{B}_{v_i}]$ (the characteristic function of \mathfrak{B}_{v_i}). Letting f_v vary, we obtain an identity

$$\sum_{\sigma} c(\sigma) \langle \text{tr } \sigma, f_v \rangle = \langle \text{tr } \pi_w \times I_c, f_{E_w} \rangle,$$

with σ running over the representations of $G(F_v)$, $c(\sigma)$ a positive integer (a multiplicity), and $c(\sigma_0) > 0$.

The rest of the argument is correct. Note however that the trace $\langle \text{tr } \pi_{N_0(E)}^{\text{norm}}, (f_{E_w})^{P_0} \rangle$ should be a twisted trace. We also point out that it should follow from Mok's results, completed by Mœglin [Mœg07], that semi-stable representations are always sent to semi-stable representations by base change.] Since it is in the image of the stable base change map, there exists by [Mok15, Lemma 2.2.1] a partition $n = n_1 + \dots + n_r$, the even parts occurring twice ("parity condition"), and quadratic unramified characters $\psi_1, \dots, \psi_r : E_w^\times \rightarrow \mathbb{C}^\times$ such that $\Pi_w \cong \text{St}_{n_1, E_w}(\psi_1) \boxplus \dots \boxplus \text{St}_{n_r, E_w}(\psi_r)$.

We have $r_\iota(\Pi) \cong \bar{\rho}_1 \oplus \bar{\rho}_2$, by the Chebotarev density theorem. We consider the natural (closure) ordering on conjugacy classes of nilpotent $n \times n$ matrices (see e.g. [Tay08, §2]). If a nilpotent $N \in M_n(\mathcal{O})$ specializes to $\bar{N} \in M_n(k)$, we have

$$\dim_K \ker(N^j) \leq \dim_k \ker(\bar{N}^j).$$

We can choose, after possibly enlarging K , a continuous homomorphism $r : G_E \rightarrow \text{GL}_n(\mathcal{O})$, together with an isomorphism $r \otimes_{\mathcal{O}} \mathbb{Q}_\ell \cong r_\iota(\Pi)$, such that the reduction $r \bmod \lambda$ is semi-simple. Let $N = r(t_w) - 1$. The above inequality shows that $(n_1, \dots, n_r) \geq (m, 3, 1)$ in the partial ordering of partitions corresponding to that of nilpotent conjugacy classes. We see that $r \leq 3$; the parity condition implies that $r = 1$ or $r = 3$. We find that the possible partitions of length 3 are $(m, 3, 1)$, $(m+1, 2, 1)$ and $(m+2, 1, 1)$, the middle one being impossible by the parity condition. We must show that in the case $(m, 3, 1)$, the characters ψ_1, ψ_2 and ψ_3 are all trivial.

In this case, the natural map $(\ker N^i) \otimes_{\mathcal{O}} k \rightarrow \ker(N^i \otimes_{\mathcal{O}} k)$ is an isomorphism for each $i \geq 0$. Let $\phi_w \in G_{E_w}$ denote a lift of Frobenius; then ϕ_w acts on $\ker N^i$. Comparing these eigenvalues for the representations \bar{r} and $\bar{\rho}_1 \oplus \bar{\rho}_2$, we see that we must have $\iota^{-1}\psi_1 \equiv \iota^{-1}\psi_2 \equiv \iota^{-1}\psi_3 \equiv 1 \bmod \lambda$. Since these are quadratic characters, they must in fact be trivial.

Finally, Π must be cuspidal in case 2 since $\bar{\rho}_1, \bar{\rho}_2$ are absolutely irreducible. If Π was not cuspidal, the associated semi-simple Galois representation $r_\iota(\Pi)$ would be reducible with degrees $(m, 4)$ and this is incompatible with the Jordan decomposition. \square

4.2 Raising the level

We now come to the main theorem of this section.

Theorem 4.2. *Suppose that $l = 5$ or $l = 7$. Then there exists an automorphic representation Σ of $G(\mathbb{A}_F)$ falling into case 2 of Proposition 4.1.*

The remainder of this section is devoted to the proof of this result. We assume from now on that $l = 7$, the case $l = 5$ being similar (and simpler, given what we have done so far). We also now assume, for contradiction, the following:

(\star) There does not exist an automorphic representation Σ of $G(\mathbb{A}_F)$ falling into case 2 of Proposition 4.1.

In this case, we can partition the representations Σ contributing to the space $S(U_1, \mathcal{O})_{\mathfrak{m}}$ into two types, according to Theorem 3.8 and the discussion following it:

1. The base change Π of Σ is not cuspidal. In this case, there are exactly $4^{s+1}/2$ automorphic representations of $G(\mathbb{A}_F)$ with base change Π , each appearing with multiplicity one.
2. The base change Π of Σ is cuspidal. In this case, there are exactly 4^{s+1} automorphic representations of $G(\mathbb{A}_F)$ with base change Π , each appearing with multiplicity one.

Choose for each $i = 1, \dots, s$ an Iwahori-spherical element R_i of the L-packet $\{X_{v_i}, Y_{v_i}, Z_{v_i}, W_{v_i}\}$ of representations of $G(F_{v_i})$ with base change $\text{St}_{m, E_{w_i}} \boxplus \text{St}_{3, E_{w_i}} \boxplus \text{St}_{1, E_{w_i}}$, so that $\prod_{i=1}^s \varepsilon_{v_i}(R_i) = -1$. Let \mathcal{M} denote the intersection, as $i = 1, \dots, s$, of the $R_i^{\mathfrak{B}_{v_i}}$ -isotypic parts of $S(U_1, \mathcal{O})_{\mathfrak{m}}$. By Theorem 3.8 and Corollary 2.8, \mathcal{M} is non-zero and is a direct summand $H_{\mathfrak{B}_{v_0}, \mathcal{O}}$ -submodule of $S(U_1, \mathcal{O})_{\mathfrak{m}}$.

We define a perfect pairing $\langle \cdot, \cdot \rangle : S(U_1, \mathcal{O}) \times S(U_1, \mathcal{O}) \rightarrow \mathcal{O}$ by the formula

$$\langle f, g \rangle = \sum_{x \in G(F) \backslash G(\mathbb{A}_F^\infty) / U_1} f(x)g(x).$$

Since the group U_1 is sufficiently small, the groups $G(F) \cap gU_1g^{-1}$ ($g \in G(\mathbb{A}_F^\infty)$), a priori finite, are trivial. A calculation which is essentially the same as the one in [Tay89, p. 271] then shows that this pairing satisfies the formula $\langle af, g \rangle = \langle f, j(a)g \rangle$ for all $a \in H_{\mathfrak{B}_{v_0}, \mathcal{O}}$ and $f, g \in S(U_1, \mathcal{O})$, $j : H_{\mathfrak{B}_{v_0}, \mathcal{O}} \rightarrow H_{\mathfrak{B}_{v_0}, \mathcal{O}}$ being the canonical anti-involution.

Proposition 4.3. *1. The above pairing induces perfect pairings*

$$\langle \cdot, \cdot \rangle_{\mathfrak{m}} : S(U_1, \mathcal{O})_{\mathfrak{m}} \times S(U_1, \mathcal{O})_{\mathfrak{m}^\vee} \rightarrow \mathcal{O}.$$

and

$$\langle \cdot, \cdot \rangle_{\mathfrak{m}} : S(U_1, k)_{\mathfrak{m}} \times S(U_1, k)_{\mathfrak{m}^\vee} \rightarrow k.$$

2. The above pairing vanishes on restriction to the subspace $S(U_0, k)_{\mathfrak{m}} \times S(U_0, k)_{\mathfrak{m}^\vee}$.
3. The above pairing induces a perfect pairing $\mathcal{M} \times \mathcal{N} \rightarrow \mathcal{O}$. The induced perfect duality between $\mathcal{M} \otimes_{\mathcal{O}} k \times \mathcal{N} \otimes_{\mathcal{O}} k$ vanishes on restriction to the subspace $\mathcal{M}^{U_0} \otimes_{\mathcal{O}} k \times \mathcal{N}^{U_0} \otimes_{\mathcal{O}} k$.

Proof. The proof is the same as that of [CT15, Proposition 4.5]. □

Lemma 4.4. *With assumptions as above, there are isomorphisms of $H_{\mathfrak{B}_{v_0}, \mathbb{C}}$ -modules:*

$$\mathcal{M} \otimes_{\mathcal{O}, \iota} \mathbb{C} \cong (X_{v_0}^{\mathfrak{B}_{v_0}})^a \oplus (Y_{v_0}^{\mathfrak{B}_{v_0}})^b \oplus (Z_{v_0}^{\mathfrak{B}_{v_0}})^c \cong \mathcal{N} \otimes_{\mathcal{O}, \iota} \mathbb{C},$$

where $a > b = c \geq 0$.

Proof. Since the spaces \mathcal{M} and \mathcal{N} are dual, it is enough to prove the assertion for \mathcal{M} . By Theorem 3.8, there exist finite sets S^{cusp} , S^{end} of automorphic representations Π of $\text{GL}_n(\mathbb{A}_E)$, finite-dimensional and non-zero \mathbb{C} -vector spaces V_Π for each $\Pi \in S^{\text{cusp}} \cup S^{\text{end}}$, and a decomposition of $H_{\mathfrak{B}_{v_0}, \mathbb{C}}$ -modules:

$$\mathcal{M} \otimes_{\mathcal{O}, \iota} \mathbb{C} \cong \left[\bigoplus_{\Pi \in S^{\text{cusp}}} V_\Pi \otimes_{\mathbb{C}} (X_{v_0}^{\mathfrak{B}_{v_0}} \oplus Y_{v_0}^{\mathfrak{B}_{v_0}} \oplus Z_{v_0}^{\mathfrak{B}_{v_0}}) \right] \oplus \left[\bigoplus_{\Pi \in S^{\text{end}}} V_\Pi \otimes_{\mathbb{C}} X_{v_0}^{\mathfrak{B}_{v_0}} \right].$$

Moreover, S^{end} is non-empty (apply Theorem 3.8 to our initial datum $\pi_1 \boxplus \pi_2$). It follows that we can take $a = \sum_{\Pi \in S^{\text{cusp}}} \dim V_\Pi + \sum_{\Pi \in S^{\text{end}}} \dim V_\Pi$, and $b = c = \sum_{\Pi \in S^{\text{cusp}}} \dim V_\Pi$. □

We recall that $H_{\mathfrak{B}_{v_0}, \mathcal{O}}$ has an abelian subalgebra $\mathcal{O}[\Lambda]$. Let $\chi : \mathcal{O}[\Lambda] \rightarrow \mathcal{O}$ be the character giving the action of $\mathcal{O}[\Lambda]$ on $X_{v_0}^{\mathfrak{B}_{v_0}} \otimes_{\mathbb{C}, \iota^{-1}} \overline{\mathbb{Q}}_l$, and let $\bar{\chi} : \mathcal{O}[\Lambda] \rightarrow k$ be its reduction modulo l . There is an induced perfect duality $\mathcal{M}(\bar{\chi}) \times \mathcal{N}(j\bar{\chi}) \rightarrow \mathcal{O}$, where we write, as in §2.3, $(?) (\bar{\chi})$ for the localization at $\ker \bar{\chi} \subset \mathcal{O}[\Lambda]$.

Let us write $\mathcal{M}_1 \subset \mathcal{M}$ and $\mathcal{N}_1 \subset \mathcal{N}$ for the \mathfrak{P}_{v_0} -fixed vectors inside the spaces $\mathcal{M}(\bar{\chi})$ and $\mathcal{N}(j\bar{\chi})$, respectively. These are direct summands as \mathcal{O} -modules, of rank equal to $a + b$, by Proposition 2.9. On the other hand, Theorem 2.5 shows that X_{v_0} has one exponent congruent mod l to $[2, 1, 0, -1]$, Y_{v_0} has three, and Z_{v_0} none, since $q \equiv -1 \pmod{l}$. Thus $\mathcal{M}(\bar{\chi})$ and $\mathcal{N}(j\bar{\chi})$ are of rank $a + 3b$.

Remark 4.5. The reader can check that this argument fails for $l > 7$. In particular, the space $Z_{v_0}(\bar{\chi})$ is no longer trivial.

Since the perfect duality between $\mathcal{M}(\bar{\chi}) \otimes_{\mathcal{O}} k$ and $\mathcal{N}(j\bar{\chi}) \otimes_{\mathcal{O}} k$ vanishes on restriction to the subspace $\mathcal{M}_1 \otimes_{\mathcal{O}} k \times \mathcal{N}_1 \otimes_{\mathcal{O}} k$, we must have $2(a + b) \leq a + 3b$, hence $a \leq b$. This contradicts the assertion of Lemma 4.4 that $a > b$. This contradiction implies that our assumption (\star) above must be incorrect, and completes the proof of Theorem 4.2.

5 Level-raising and Galois theory

In this section, which is essentially self-contained, we prove a level-raising result, using Galois-theoretic arguments. The need for this result is explained at the beginning of §4. We start with a residual representation $\bar{\rho} = \bar{\rho}_1 \oplus \bar{\rho}_2$ which is a sum of two irreducible pieces, say of dimensions n_1, n_2 . We accomplish level-raising when there is an automorphic lift of $\bar{\rho}$ which locally has a Weil–Deligne representation (r, N) such that N has a Jordan block of dimension strictly greater than $\sup(n_1, n_2)$. Any such lift is automatically irreducible. This tension between lifts which are reducible and lifts which have prescribed ramification locally allows us to exploit the geometric techniques developed in [Tho15] to pass to another automorphic lift which has regular nilpotent ramification locally at the given place.

In this way we prove Theorem 5.1. In order to streamline its later application, we then state and prove Theorem 5.7, which essentially combines Theorem 5.1 with the main result of [Tho15]. This result will then be applied, together with the work done in §4, to deduce the automorphy of symmetric power Galois representations in §6.

5.1 A level-raising theorem

Let E be an imaginary CM field with maximal totally real subfield F . Fix a prime l and an isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$. Fix an integer $n \geq 2$ and a RACSDC automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_E)$ satisfying the following hypotheses:

1. $\overline{r_\iota(\Pi)} = \bar{\rho}_1 \oplus \bar{\rho}_2$, where $\dim \bar{\rho}_i = n_i$ and the $\bar{\rho}_i$ are pairwise non-isomorphic, absolutely irreducible, and each satisfy $\bar{\rho}_i^c \cong \bar{\rho}_i^\vee \epsilon^{1-n}$. We assume that $n_1 > n_2$ and that $r_\iota(\Pi)$ is primitive, i.e. not induced from a proper extension of E .
2. There exists a place w_0 of E such that Π_{w_0} has an Iwahori-fixed vector and, writing $\mathrm{rec}_{E_{w_0}}^T(\Pi_{w_0}) = (r, N)$, N has a Jordan block of length strictly greater than n_1 . We write v_0 for the place of F below w_0 .

The purpose of this section is to prove the following theorem.

Theorem 5.1. *With notation as above, suppose that the following further hypotheses are in effect:*

1. Π is ι -ordinary.
2. $l > 3$ and l is coprime to each of n_1, n_2 and n . The product $n_1 n_2$ is even.
3. Each $\bar{\rho}_i|_{G_{E(\zeta_l)}}$ is adequate, and $E(\zeta_l)$ is not contained in the fixed field of $\mathrm{ad} \overline{r_\iota(\Pi)}$.

4. There exists a CM extension E_0/E linearly disjoint from the extension of $E(\zeta_l)$ cut out by $\bar{\rho}$, and RAECSDC automorphic representations (π_1, χ_1) and (π_2, χ_2) of $\mathrm{GL}_{n_1}(\mathbb{A}_{E_0})$ and $\mathrm{GL}_{n_2}(\mathbb{A}_{E_0})$, respectively, such that π_1, π_2 are ι -ordinary, and $r_\iota(\pi_i) \cong \bar{\rho}_i|_{G_{E_0}}$ for $i = 1, 2$.

Then there exists a soluble CM extension E_1 of E , linearly disjoint from the extension of $E(\zeta_l)$ cut out by $\overline{r_\iota(\Pi)}$, and a RACSDC automorphic representation Π_1 of $\mathrm{GL}_n(\mathbb{A}_{E_1})$ satisfying the following conditions:

1. Π_1 is ι -ordinary.
2. $\overline{r_\iota(\Pi_1)} \cong \overline{r_\iota(\Pi)}|_{G_{E_1}}$.
3. There exists a place w_1 of E_1 dividing w_0 such that Π_{1,w_1} is an unramified twist of the Steinberg representation.

The proof will make heavy use of the techniques of the paper [Tho15], to which we make frequent references. After making a preliminary soluble base change, we can assume that the following further hypotheses hold:

1. There exists a regular algebraic, conjugate self-dual and ι -ordinary automorphic representation Π' of $\mathrm{GL}_n(\mathbb{A}_E)$ which is everywhere unramified, and satisfies $\overline{r_\iota(\Pi')} \cong \overline{r_\iota(\Pi)}$. (Π' can be constructed using an easy generalization of [CHT08, Lemma 4.4.1].)
2. Every place of F dividing l or above which Π is ramified splits in E . For each place $v|l$ of F , the local degree $[F_v : \mathbb{Q}_l]$ is even.
3. The extension E/F is everywhere unramified, and Π is everywhere unipotently ramified.
4. If w is a place of E dividing l or at which Π is ramified, then $\overline{r_\iota(\Pi)}|_{G_{E_w}}$ is trivial.
5. For each place $w|l$ of E , we have $[E_w : \mathbb{Q}_l] > n(n+1)/2 + 1$. For each place $w \nmid l$ of E at which Π is ramified, the highest power of l dividing $q_w - 1$ is strictly greater than n .

In this case $[F : \mathbb{Q}]$ is even and there exists a unitary group G in n variables over F satisfying the following properties:

1. G is split by E , and is quasi-split at every finite place.
2. $G(F_\infty)$ is compact.

Choose a finite set V_0 of finite places of E with the following properties:

1. If M/E is a Galois extension of E contained in the extension of $E(\zeta_l)$ cut out by $\overline{r_\iota(\Pi)}$ and $\mathrm{Gal}(M/E)$ is simple, then there exists a place of V_0 which does not split in M .
2. V_0 contains no place of E which divides l or above which Π is ramified.

We write S_l for the set of places of F above l . Let S_a denote a finite, non-empty set of places of F , split in E and coprime to V_0 , such that if w is a place of E above an element of S_a , then Π_w is unramified, $\mathrm{ad} r_\iota(\Pi)(\mathrm{Frob}_w)$ is trivial and $q_w \not\equiv 1 \pmod{l}$. (Such places exist because of the assumption on $\mathrm{ad} r_\iota(\Pi)$.) Let T denote the set of places of F below a place at which Π is ramified, together with $S_l \cup S_a$. Thus by definition, T is coprime to V_0 and every place of T splits in E . We choose for each place $v \in T$ a place \tilde{v} of E dividing it, and write \tilde{T} for the set of these places. We define $\tilde{S}_l = \{\tilde{v} \mid v \in S_l\}$ and $\tilde{S}_a = \{\tilde{v} \mid v \in S_a\}$.

We say that a soluble CM extension L/E is *good* if every place of $V_0 \cup \tilde{S}_a$ is split in L . Given such an extension, we write L^+ for the maximal totally real subfield of L and $S_{l,L}$, $S_{a,L}$ and T_L for the sets of places of L^+ dividing S_l , S_a and T , respectively. If $v \in T_L$ then there is a unique place \tilde{v} of L dividing v and extending $\tilde{v}|_F$. If $S \subset T_L$ is any set of places then we define $\tilde{S} = \{\tilde{v} \mid v \in S\}$, as above.

We fix an integral model of the group G over \mathcal{O}_F exactly as in [CT15, §3.3], as well as identifications $\iota_{\tilde{v}} : G(F_v) \cong \mathrm{GL}_n(E_{\tilde{v}})$ for each $v \in T$. After base extension to \mathcal{O}_L , L a good extension, we also obtain identifications $\iota_{\tilde{v}} : G(L_v^+) \cong \mathrm{GL}_n(L_{\tilde{v}})$ for any $v \in T_L$.

We fix a finite extension $K \subset \overline{\mathbb{Q}_l}$ of \mathbb{Q}_l . We write $\mathcal{O} \subset K$ for the ring of integers, $\lambda \subset \mathcal{O}$ for the maximal ideal, and $k = \mathcal{O}/\lambda$ for the residue field. If $v \in T_L$, we will consider the following local deformation problems, in the sense of [Tho15, Definition 3.6]:

1. If $v \in S_{l,L}$, then there is the ordinary lifting ring R_v^Δ defined in [Tho15, §3.3.2], following Geraghty [Ger].
2. If $v \in S_{a,L}$ then we consider the unrestricted local deformation problem, represented by the ring R_v^\square . With assumptions as above, R_v^\square is formally smooth over \mathcal{O} , the universal lifting being unramified.
3. If $v \nmid l$, $q_v \equiv 1 \pmod l$, and $\chi_v = (\chi_{v,1}, \dots, \chi_{v,n})$ is a choice of n -tuple of characters $\chi_{v,i} : \mathcal{O}_{L_{\bar{v}}}^\times \rightarrow \mathcal{O}^\times$ which are trivial after reduction modulo λ , then there is the ring $R_v^{\chi_v}$ which classifies liftings of $\overline{r_l(\Pi)}|_{G_{L_{\bar{v}}}}$ whose restriction to inertia has characteristic polynomial $\prod_{i=1}^n (X - \chi_{v,i})$, as defined in [Tho15, §3.3.3]. In the special case that the characters $\chi_{v,i}$ are all trivial, we write $R_v^{\chi_v} = R_v^1$.
4. If $v \nmid l$, and $q_v \equiv 1 \pmod l$, then there is the ring R_v^{St} classifying unipotently ramified lifts for which a Frobenius lift has eigenvalues $\alpha, q_v \alpha, \dots, q_v^{n-1} \alpha$ for some α , as defined in [Tho15, §3.3.4].
5. If $v \nmid l$ and $q_v \equiv 1 \pmod l$, then we define a new local deformation problem as follows. Let us write \mathbf{m} for a choice of partition $n = m_1 + \dots + m_k$, $m_1 \geq m_2 \geq \dots \geq m_k$. The local deformation ring $R_v^{\mathbf{m}}$ is the maximal \mathcal{O} -flat reduced quotient of R_v^1 classifying lifts for which a lift of $\text{Frob}_{\bar{v}}^{-1}$ has characteristic polynomial in the scheme $\text{Pol}_n(\mathbf{m}, q_v)$ of [Tay08, §2]. Note that if \mathbf{m} is the partition $n = n$, then $R_v^{\mathbf{m}} = R_v^{\text{St}}$ by definition. (We can specify a local deformation problem by specifying a local deformation ring, thanks to [BLGHT11, Lemma 3.2].)

Lemma 5.2. *Let $x \in \text{Spec } R_v^{\mathbf{m}}[1/l]$ be a closed point induced by a homomorphism $\rho : G_{L_{\bar{v}}} \rightarrow \text{GL}_n(\mathcal{O})$ such that $\rho \otimes_{\mathcal{O}} \overline{\mathbb{Q}_l}$ is pure, in the sense of [TY07, Lemma 1.4]. Then $\text{Spec } R_v^1[1/l]$ is formally smooth over K at x , and there is a unique minimal prime Q_v of R_v^1 in the kernel of the induced homomorphism $R_v^1 \rightarrow \mathcal{O}$. Moreover, Q_v contains the kernel of the homomorphism $R_v^1 \rightarrow R_v^{\mathbf{m}}$.*

Proof. We first note that the underlying reduced ring of R_v^1 is \mathcal{O} -flat and equidimensional of dimension $1+n^2$, cf. [Tay08, Proposition 3.1] and [Tho12, Lemma 3.15]. On the other hand, writing R_v^\square for the unrestricted lifting ring of which R_v^1 is a quotient, one knows (cf. [BLGGT14, Lemma 1.3.2]) that $\text{Spec } R_v^\square[1/l]$ is equidimensional of dimension n^2 , and formally smooth over K at those closed points corresponding to pure representations. It follows that there is a unique irreducible component of $\text{Spec } R_v^1[1/l]$ passing through each closed point x as in the statement of the lemma, which is formally smooth over K at x .

Let us write $Q_v \subset R_v^1$ for the corresponding minimal prime; it remains to show that Q_v contains the kernel of the homomorphism $R_v^1 \rightarrow R_v^{\mathbf{m}}$. This follows from the proof of [Tho12, Lemma 3.15]. Indeed, in the notation at the bottom of [Tho12, p. 865], x defines a K -point of $\mathcal{M}(\mathbf{m})$ (in fact, of $\mathcal{M}(\mathbf{m})^0$) which is contained in no other irreducible component of \mathcal{M} . The inclusion $\mathcal{M}(\mathbf{m}) \subset \mathcal{M}(\mathbf{m})'$ now implies the result. \square

Recall the disconnected group \mathcal{G}_n over \mathbb{Z} classifying representations of unitary type considered in [CHT08, §2.1], [Tho15, §3.1]. There exists an extension of $\overline{r_l(\Pi)}$ to a homomorphism $\bar{r} : G_F \rightarrow \mathcal{G}_n(k)$, which is necessarily Schur in the sense of [Tho15, Definition 3.2] and satisfies $\nu \circ \bar{r}(c) = -1$. After possibly enlarging K , we can also choose a lift of \bar{r} to a homomorphism $r : G_F \rightarrow \mathcal{G}_n(\mathcal{O})$ such that $r|_{G_E \otimes_{\mathcal{O}} \overline{\mathbb{Q}_l}} \cong \overline{r_l(\Pi)}$. Suppose given a good extension L/E , a finite set R of places of L^+ dividing $T \setminus (S_l \cup S_a)$, and for each $v \in R$ a local lifting ring R_v of one of the types 3–5 defined above. In terms of the pair $\mathcal{D} = (L, \{R_v\}_{v \in R})$ we define:

1. The Iwasawa algebra $\Lambda_L = \widehat{\otimes}_{v \in S_{l,L}} \Lambda_v$, where $\Lambda_v = \mathcal{O}[(I_{L_{\bar{v}}}(l))^{ab}]^n$.
2. A global deformation problem in the sense of [Tho15, §3.2], $\mathcal{S}(\{R_v\}_{v \in R}) =$

$$\left(L/L^+, R \cup S_{l,L} \cup S_{a,L}, \tilde{R} \cup \tilde{S}_{l,L} \cup \tilde{S}_{a,L}, \Lambda_L, \bar{r}, \epsilon^{1-n} \delta_{L/L^+}^n, \{R_v^\Delta\}_{v \in S_{l,L}} \cup \{R_v^\square\}_{v \in S_{a,L}} \cup \{R_v\}_{v \in R} \right).$$

3. An open compact subgroup $U = \prod_v U_v \subset G(\mathbb{A}_{L^+}^{l,\infty})$ defined as follows:

- (a) For v inert in L , $U_v \subset G(L_v^+)$ is a hyperspecial maximal compact subgroup.
- (b) For $v \notin R \cup S_{l,L} \cup S_{a,L}$ split in L , $U_v = G(\mathcal{O}_{L_v^+})$.
- (c) For $v \in S_{a,L}$, $U_v = \iota_{\tilde{v}}^{-1} \ker(\mathrm{GL}_n(\mathcal{O}_{L_{\tilde{v}}}) \rightarrow \mathrm{GL}_n(k(\tilde{v})))$.
- (d) For $v \in R$, $U_v = \iota_{\tilde{v}}^{-1} \mathrm{Iw}(\tilde{v})$.

Then for any open compact subgroup $U_l \subset G(L_l^+)$, $U \cdot U_l$ is sufficiently small, as the set S_a is non-empty.

4. A space of modular forms $S_{\mathcal{D}}$, which is the space denoted $S_{\chi}(U(\mathfrak{l}^{\infty}), K/\mathcal{O})^{\vee}$ in [Tho15, Definition 4.2]. The datum $\chi = \{\chi_v\}_{v \in R}$ here is taken to consist of the corresponding character χ_v for those local deformation problems R_v such that $R_v = R_v^{\chi_v}$, and to be the trivial character otherwise. This is a finite free Λ_L -module (as the level is sufficiently small, cf. [Tho15, Proposition 4.3]).
5. A Hecke algebra $\mathbb{T}_{\mathcal{D}}$, which is the localization of the algebra denoted $\mathbb{T}_{\chi}^{T_L}(U(\mathfrak{l}^{\infty}), \mathcal{O})$ in [Tho15, Definition 4.2] (with the same choice of χ as in the previous point) at the maximal ideal corresponding to a descent of the representation Π' to G with non-zero U^l -fixed vectors. (This descent exists, by [CT14, Proposition 2.9], and the maximal ideal is independent of this choice.)

We refer to a pair $(L, \{R_v\}_{v \in R})$ satisfying the above conditions as a deformation datum. Associated to a choice of deformation datum is the global deformation ring $R_{\mathcal{D}} = R_{S(\{R_v\}_{v \in R})}^{\mathrm{univ}}$ (notation as in [Tho15, Proposition 3.8]) and its subring the universal pseudodeformation ring $P_{\mathcal{D}} = P_{S(\{R_v\}_{v \in R})}$ (notation as in [Tho15, Definition 3.27]). The inclusion $P_{\mathcal{D}} \subset R_{\mathcal{D}}$ is a finite homomorphism of Noetherian Λ_L -algebras ([Tho15, Proposition 3.29]). If for each $v \in R \setminus (S_{l,L} \cup S_{a,L})$, $R_v = R_v^{\chi_v}$ for some choice of χ_v , then there is a surjective homomorphism $P_{\mathcal{D}} \rightarrow \mathbb{T}_{\mathcal{D}}$ of Λ_L -algebras. (The proof of this assertion follows similar lines to that of [Tho15, Proposition 4.12].) In this case we write $I_{\mathcal{D}}$ for the kernel of this algebra homomorphism, and $J_{\mathcal{D}} = I_{\mathcal{D}} R_{\mathcal{D}}$. If \mathcal{D} is any deformation datum, then we write $R_{\mathcal{D}}^{\mathrm{red}}$ for the quotient of $R_{\mathcal{D}}$ which classifies reducible deformations ([Tho15, Definition 3.31]). Whenever it is defined, $\mathbb{T}_{\mathcal{D}}$ is a finite torsion-free Λ_L -algebra ([Tho15, Proposition 4.3]). Whenever $R_v = R_v^{\chi_v}$ or R_v^{St} for each $v \in R$, every minimal prime of $R_{\mathcal{D}}$ has dimension at least $\dim \Lambda_L = 1 + n[L^+ : \mathbb{Q}]$ ([Tho15, Proposition 3.9]).

Lemma 5.3. *Fix a partition \mathbf{m} with $m_1 > n_1 = \dim \overline{\rho}_1$. Let $\mathcal{D} = (L, \{R_v\}_{v \in R_0} \cup \{R_v^{\mathbf{m}}\}_{v \in R_1})$ be a deformation datum. Let Δ denote the Galois group of the maximal abelian l -extension of L unramified outside l , and let Δ_0 denote the Galois group of the maximal abelian l -extension of L unramified outside l in which every prime of $\tilde{R}_1 \cup \tilde{R}_1^c$ splits completely. Let*

$$d = \dim_{\mathbb{Q}_l} \ker(\Delta \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \rightarrow \Delta_0 \otimes_{\mathbb{Z}_l} \mathbb{Q}_l)^{c=-1}.$$

Then $\dim R_{\mathcal{D}}^{\mathrm{red}} \leq 1 + n[L^+ : \mathbb{Q}] - d$, and $R_{\mathcal{D}}^{\mathrm{red}}$ is a finite Λ_L -algebra.

Proof. It suffices to show that for each minimal prime $Q \subset R_{\mathcal{D}}^{\mathrm{red}}$, we have $\dim R_{\mathcal{D}}^{\mathrm{red}}/Q \leq 1 + n[L^+ : \mathbb{Q}] - d$ and $R_{\mathcal{D}}^{\mathrm{red}}/Q$ is a finite Λ_L -algebra. Let us now fix such a prime and write $C = R_{\mathcal{D}}^{\mathrm{red}}/Q$. We may write \bar{r} as a sum of homomorphisms $\bar{r}_i : G_F \rightarrow \mathcal{G}_{n_i}(k)$, as in [Tho15, §3.5]. For $i = 1, 2$, we define $\Lambda_{i,L} = \otimes_{v \in S_{l,L}} \Lambda_{i,v}$ and $\Lambda_{i,v} = \mathcal{O}[[I_{L_{\tilde{v}}}(l)^{n_i}]]$. We introduce the auxiliary deformation problems

$$\begin{aligned} \mathcal{S}_1 &= \left(L/L^+, S_{l,L} \cup S_{a,L} \cup R_0 \cup R_1, \tilde{S}_{l,L} \cup \tilde{S}_{a,L} \cup \tilde{R}_0 \cup \tilde{R}_1, \right. \\ &\quad \left. \Lambda_{1,L}, \bar{r}_1, \epsilon^{1-n} \delta_{L/L^+}^n, \{R_v^{\Delta}\}_{v \in S_{l,L}} \cup \{R_v^{\square}\}_{v \in S_a} \cup \{R_v^1\}_{v \in R_0 \cup R_1} \right), \\ \mathcal{S}_2 &= \left(L/L^+, S_{l,L} \cup S_{a,L} \cup R_0 \cup R_1, \tilde{S}_{l,L} \cup \tilde{S}_{a,L} \cup \tilde{R}_0 \cup \tilde{R}_1, \right. \\ &\quad \left. \Lambda_{2,L}, \bar{r}_2, \epsilon^{1-n} \delta_{L/L^+}^n, \{R_v^{\Delta}\}_{v \in S_{l,L}} \cup \{R_v^{\square}\}_{v \in S_a} \cup \{R_v^1\}_{v \in R_0 \cup R_1} \right). \end{aligned}$$

(We have abused notation slightly here by writing R_v^{Δ} , R_v^{\square} , R_v^1 also for the local lifting rings corresponding to the local representations $\bar{r}_1|_{G_{L_{\tilde{v}}}}$ and $\bar{r}_2|_{G_{L_{\tilde{v}}}}$ of dimensions n_1 and n_2 , respectively.) Arguing as in the proof of [Tho15, Theorem 7.1], one can show the following:

1. Let $r_{\mathcal{D}}^{\text{red}} \bmod Q = r_1 \oplus r_2 : G_{L^+} \rightarrow \mathcal{G}_n(C)$ denote the reduction mod Q of the universal deformation. Then for each $v \in S_{l,L}$ there exists an isomorphism $\Lambda_{1,L} \widehat{\otimes}_{\mathcal{O}} \Lambda_{2,L} \cong \Lambda_L$ such that, with the induced structures on C of $\Lambda_{1,L}$ - and $\Lambda_{2,L}$ -algebra, r_1 is a deformation of type \mathcal{S}_1 and r_2 is a deformation of type \mathcal{S}_2 .
2. For each $i = 1, 2$, the universal deformation ring $R_{\mathcal{S}_i}^{\text{univ}}$ is finite over $\Lambda_{i,L}$ of dimension $1 + n_i[L^+ : \mathbb{Q}]$.
3. The induced map $R_{\mathcal{S}_1}^{\text{univ}} \widehat{\otimes}_{\mathcal{O}} R_{\mathcal{S}_2}^{\text{univ}} \rightarrow C$ is surjective.

(This is where we use the assumption in Theorem 5.1 that each of $\bar{\rho}_1, \bar{\rho}_2$ are potentially ordinarily automorphic.) It follows that C is a finite Λ_L -algebra. We now bound the dimension of C . For each $i = 1, 2$, let $\psi_i : G_L \rightarrow \mathcal{O}^\times$ be the Teichmüller lift of $\det \bar{r}_i|_{G_L}$. Write $R_{\mathcal{S}_i, \psi_i}^{\text{univ}}$ for the quotient of $R_{\mathcal{S}_i}^{\text{univ}}$ classifying deformations with determinant ψ_i . According to [Tho15, Lemma 3.36], there are isomorphisms $R_{\mathcal{S}_i}^{\text{univ}} \cong R_{\mathcal{S}_i, \psi_i}^{\text{univ}} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[[\Delta/(c+1)]]$, and hence

$$R_{\mathcal{S}_1}^{\text{univ}} \widehat{\otimes}_{\mathcal{O}} R_{\mathcal{S}_2}^{\text{univ}} / \lambda \cong R_{\mathcal{S}_1, \psi_1}^{\text{univ}} / \lambda \widehat{\otimes}_k R_{\mathcal{S}_2, \psi_2}^{\text{univ}} / \lambda \widehat{\otimes}_k k[[\Delta/(c+1)]] \widehat{\otimes}_k k[[\Delta/(c+1)]].$$

Let

$$\Psi_1, \Psi_2 : \Delta/(c+1) \rightarrow (k[[\Delta/(c+1)]] \widehat{\otimes}_k k[[\Delta/(c+1)]])^\times$$

denote the universal characters valued in this ring. With the above identification, the universal deformation has the form $(r_{\mathcal{S}_1, \psi_1}^{\text{univ}} \otimes \Psi_1) \oplus (r_{\mathcal{S}_2, \psi_2}^{\text{univ}} \otimes \Psi_2)$. (Recall that n_1 and n_2 are coprime to l .) Let $A = R_{\mathcal{S}_1, \psi_1}^{\text{univ}} \widehat{\otimes}_{\mathcal{O}} R_{\mathcal{S}_2, \psi_2}^{\text{univ}}$, with maximal ideal \mathfrak{m}_A . We have (by [Mat89, Theorem 15.1]) $\dim C \leq \dim A + \dim C/\mathfrak{m}_A C$, and $C/\mathfrak{m}_A C$ is a quotient of $B = k[[\Delta/(c+1)]] \widehat{\otimes}_k k[[\Delta/(c+1)]]$. In fact, we have $\dim C/\mathfrak{m}_A C \leq \dim B/J$, where B/J is the quotient of B classifying pairs Ψ_1, Ψ_2 such that $[(\bar{r}_1 \otimes \Psi_1) \oplus (\bar{r}_2 \otimes \Psi_2)]|_{G_{L_{\bar{v}}}}$ is of type $R_v^{\mathfrak{m}}$ for each $v \in R_1$. We now show that $\dim B/J \leq \dim B - d$, which will complete the proof.

Let $P \subset B$ be a prime such that $[(\bar{r}_1 \otimes \Psi_1) \oplus (\bar{r}_2 \otimes \Psi_2)]|_{G_{L_{\bar{v}}}} \bmod P$ is of type $R_v^{\mathfrak{m}}$, for each $v \in R_1$. Let ϕ_v be a Frobenius lift at \bar{v} . Then $[(\bar{r}_1 \otimes \Psi_1) \oplus (\bar{r}_2 \otimes \Psi_2)](\phi_v)$ has an eigenvalue of multiplicity $m_1 > n_1 > n_2$; since its eigenvalues are visibly either distinct of multiplicities $n_1 + n_2 = n$ or equal, we must be in the latter case, and so $\Psi_1(\phi_v) = \Psi_2(\phi_v)$. Since P was arbitrary, we see that $\Psi_1(\text{Frob}_{\bar{v}}) = \Psi_2(\text{Frob}_{\bar{v}})$ modulo the nilradical of J , for each $v \in R_1$. The result now follows from the definition of d . \square

Lemma 5.4. *Let $\mathcal{D} = (L, \{R_v^{\chi_v}\}_{v \in R_0} \cup \{R_v^{\mathfrak{m}}\}_{v \in R_1})$ be a deformation datum, where \mathfrak{m} is the partition of n corresponding to the Jordan decomposition of N , $\text{rec}_{E_{w_0}}^T(\Pi_{w_0}) = (r, N)$, and R_1 consists of places of L^+ dividing v_0 . Let $\mathcal{D}_0 = (L, \{R_v^{\chi_v}\}_{v \in R_0} \cup \{R_v^1\}_{v \in R_1})$. (Thus there is a surjection $R_{\mathcal{D}_0} \rightarrow R_{\mathcal{D}}$ of Noetherian Λ_L -algebras.) Then $\dim R_{\mathcal{D}}/J_{\mathcal{D}_0} R_{\mathcal{D}} = \dim \Lambda_L$.*

Proof. We use the properties of the Hecke algebra described in the paragraph preceding Lemma 5.3. The ring $R_{\mathcal{D}_0}/J_{\mathcal{D}_0} R_{\mathcal{D}_0}$ is a finite $\mathbb{T}_{\mathcal{D}_0}$ -algebra, hence a finite Λ_L -algebra. It follows that $\dim R_{\mathcal{D}}/J_{\mathcal{D}_0} R_{\mathcal{D}} \leq \dim \Lambda_L$. To prove the lemma, we must therefore show that $\dim R_{\mathcal{D}}/J_{\mathcal{D}_0} R_{\mathcal{D}} \geq \dim \Lambda_L$. The existence of the base change Π_L implies the existence of a homomorphism (after possibly enlarging K) $R_{\mathcal{D}} \rightarrow \mathcal{O}$ whose kernel contains $J_{\mathcal{D}_0} R_{\mathcal{D}}$. Let $r : G_{L^+} \rightarrow \mathcal{G}_n(\mathcal{O})$ denote a representative of the corresponding deformation. (Thus we have $r|_{G_L} \otimes_{\mathcal{O}} \bar{\mathbb{Q}}_l \cong r_l(\Pi_L)$.)

For each $v \in R_1$, let $J_v = \ker(R_v^1 \rightarrow R_v^{\mathfrak{m}})$, and let Q_v denote the unique minimal prime in the kernel of the homomorphism $R_v^1 \rightarrow \mathcal{O}$ induced by $r|_{G_{L_{\bar{v}}}}$, cf. Lemma 5.2. Thus $J_v \subset Q_v$. Let $Q_0 \subset \mathbb{T}_{\mathcal{D}_0}$ denote a minimal prime in the kernel of the map $\mathbb{T}_{\mathcal{D}_0} \rightarrow \mathcal{O}$ induced by $\iota^{-1} \Pi_L^\infty$, and let $Q \subset R_{\mathcal{D}_0}$ be a prime of dimension $\dim \Lambda_L = \dim \mathbb{T}_{\mathcal{D}_0}/Q_0$, minimal over $Q_0 R_{\mathcal{D}_0}$, and contained in the kernel of the map $R_{\mathcal{D}_0} \rightarrow \mathcal{O}$.

We show that Q contains the kernel of the surjective homomorphism $R_{\mathcal{D}_0} \rightarrow R_{\mathcal{D}}$. Since this kernel is generated by the ideals $J_v, v \in R_1$, it suffices to show that for each $v \in R_1$, we have $J_v \subset Q$ or even $Q_v \subset Q$. However, if $Q'_v \subset R_v^1$ is a minimal prime in the kernel of the homomorphism $R_v^1 \rightarrow R_{\mathcal{D}_0}/Q$ then it is contained in the kernel of the homomorphism $R_v^1 \rightarrow R_{\mathcal{D}_0} \rightarrow \mathcal{O}$, hence equal to Q_v . The result follows. \square

Lemma 5.5. *Let $\mathcal{D}_0^\chi = (L_0, \{R_v^{\chi_v}\}_{v \in R})$ be a deformation datum. Suppose that for each $v \in R$, the characters $\chi_{v,1}, \dots, \chi_{v,n}$ are pairwise distinct. Let L be a good extension containing L_0 and such that for each place*

\tilde{w} of L dividing a place \tilde{v} of L_0 , $v \in R$, the restriction $\chi_{v,i}|_{I_{L_{\tilde{w}}}}$ is trivial. Let $\mathcal{D} = (L, \emptyset)$. Then there is a commutative diagram

$$\begin{array}{ccccc} R_{\mathcal{D}} & \longleftarrow & P_{\mathcal{D}} & \longrightarrow & \mathbb{T}_{\mathcal{D}} \\ \downarrow & & \downarrow & & \downarrow \\ R_{\mathcal{D}_0^{\times}}^{\text{nil}} & \longleftarrow & P_{\mathcal{D}_0^{\times}}^{\text{nil}} & \longrightarrow & \mathbb{T}_{\mathcal{D}_0^{\times}}. \end{array}$$

(Here we write A^{nil} for the quotient of A by its nilradical, the superscript A^{red} already being reserved for reducible deformations.)

Proof. It is easy to see that the morphism $R_{\mathcal{D}} \rightarrow R_{\mathcal{D}_0^{\times}}^{\text{nil}}$ exists provided that for each $v \in R$ and each place \tilde{w} of L dividing \tilde{v} , the universal lifting $\rho : G_{L_0, \tilde{v}} \rightarrow \text{GL}_n(R_v^{\chi_v, \text{nil}})$ becomes unramified on restriction to $G_{L_{\tilde{w}}}$. Since $R_v^{\chi_v, \text{nil}}$ is a domain ([Tho12, Proposition 3.16]), this can be checked after extending scalars to $\mathcal{K} = \text{Frac}(R_v^{\chi_v, \text{nil}})$, a field of characteristic zero. Let $\sigma \in G_{L_0, \tilde{v}}$ denote a generator of tame inertia. The element $\rho(\sigma) \in \text{GL}_n(\mathcal{K})$ is regular semi-simple, because its characteristic polynomial $\prod_{i=1}^n (X - \chi_{v,i}(\sigma))$ has distinct roots, by hypothesis. If $N \geq 1$ is an integer such that $\rho(\sigma^N)$ is unipotent, it must therefore be trivial. The existence of the left hand commutative square above is now immediate.

For the right hand square, we must construct the arrow $\mathbb{T}_{\mathcal{D}} \rightarrow \mathbb{T}_{\mathcal{D}_0^{\times}}$. (We remark that these Hecke algebras are reduced.) As in the proof of [Tho15, Proposition 4.18], this can be reduced to the existence of soluble base change and descent for the unitary group G , in the guise of [CT14, Proposition 2.9]. Our appeal to *loc. cit.* is the reason for our assumption at the beginning of this section that n_1 and n_2 are distinct, and that $n_1 n_2$ is even. \square

Lemma 5.6. *Suppose there exists a deformation datum $\mathcal{D} = (L, \emptyset)$ and a prime $\mathfrak{p} \subset R_{\mathcal{D}}$ of dimension 1 and characteristic l satisfying the following conditions:*

1. *Let A denote the normalization of the quotient ring $R_{\mathcal{D}}/\mathfrak{p}$. Then the induced representation $r_{\mathfrak{p}} : G_L \rightarrow \text{GL}_n(\text{Frac } A)$ is absolutely irreducible.*
2. *For each $v \in S_{l,L}$, the characters $\psi_1^v, \dots, \psi_n^v : I_{L_{\tilde{v}}} \rightarrow A^{\times}$ induced by the homomorphism $\Lambda_L \rightarrow A$ are pairwise distinct.*
3. *There exists a place $v \in S_{l,L}$ and an element $\sigma \in I_{L_{\tilde{v}}}$ such that the eigenvalues of $r_{\mathfrak{p}}(\sigma)$ satisfy no non-trivial \mathbb{Z} -linear relation.*
4. *The prime \mathfrak{p} contains $J_{\mathcal{D}}$.*
5. *There exists a place $u \in T_L \setminus (S_{l,L} \cup S_{a,L})$ such that $r_{\mathfrak{p}}|_{G_{L_{\tilde{u}}}}$ is trivial.*

Then there exists a RACSDC automorphic representation Π_1 of $\text{GL}_n(\mathbb{A}_L)$ satisfying the following conditions:

1. *Π_1 is ι -ordinary.*
2. *$\overline{r_{\iota}(\Pi_1)} \cong \overline{r_{\iota}(\Pi)}|_{G_L}$.*
3. *$\Pi_{1, \tilde{u}}$ is an unramified twist of the Steinberg representation.*

Proof. We can assume that if $\mathfrak{q} = P_{\mathcal{D}} \cap \mathfrak{p}$, then $\text{Frac}(R_{\mathcal{D}}/\mathfrak{p}) = \text{Frac}(P_{\mathcal{D}}/\mathfrak{q})$; otherwise, replace \mathfrak{p} by a suitable twist as in [Tho15, Corollary 4.14].

We introduce the auxiliary deformation data $\mathcal{D}^1 = (L, \{R_u^1\})$ and $\mathcal{D}^{\text{St}} = (L, \{R_u^{\text{St}}\})$. Thus $R_{\mathcal{D}^{\text{St}}}$ is naturally a quotient of $R_{\mathcal{D}^1}$. To prove the lemma, it suffices to show that there is a minimal prime $Q \subset R_{\mathcal{D}^{\text{St}}}$ such that $R_{\mathcal{D}^{\text{St}}}/Q$ is a finite Λ_L -algebra, and $J_{\mathcal{D}^1} R_{\mathcal{D}^{\text{St}}} \subset Q$. Indeed, we also have $\dim R_{\mathcal{D}^{\text{St}}}/Q \geq \dim \Lambda_L$ (see the paragraph before Lemma 5.3), so Q lies above a minimal prime of Λ_L , and then a classicality result in Hida theory (e.g. [Ger, Lemma 2.6.4]) implies the existence of a characteristic zero point of $R_{\mathcal{D}^{\text{St}}}$ which arises from an automorphic representation with the desired properties.

To show the existence of such a minimal prime, we appeal to an $R = \mathbb{T}$ theorem proved in [Tho15]. Let $Q \subset R_{\mathcal{D}^{\text{st}}}$ be a minimal prime inside \mathfrak{p} , and let $Q^1 \subset R_{\mathcal{D}^1}$ be minimal inside Q . Then [Tho15, Corollary 4.20] implies that $J_{\mathcal{D}^1} \subset Q^1$ and $R_{\mathcal{D}^1}/Q^1$ is a finite Λ_L -algebra of dimension $\dim \Lambda_L$. It follows that the natural map $R_{\mathcal{D}^1}/Q^1 \rightarrow R_{\mathcal{D}^{\text{st}}}/Q$ is an isomorphism.

To complete the proof it remains to justify the application of [Tho15, Corollary 4.20], or in other words to check that the hypotheses (1)–(5) of [Tho15, Theorem 4.19] hold. In the notation there, we take $R = \{u\}$ and $S(B) = \emptyset$. By [Tho15, Corollary 5.7], hypothesis (1) of [Tho15, Theorem 4.19] is satisfied when the conditions enumerated at the beginning of [Tho15, §5.2] are satisfied. It is easy to see that these conditions hold in the case considered here. Hypothesis (2) of [Tho15, Theorem 4.19] is satisfied here by assumption. Hypotheses (3) and (4) are satisfied because of the preliminary reductions immediately after the statement of Theorem 5.1. Hypothesis (5) is satisfied by the first paragraph of the current proof. We note that it is assumed in the proof of [Tho15, Theorem 4.19] that the set $S(B)$ is non-empty; however, this assumption plays no role in the proof. \square

We can now complete the proof of Theorem 5.1. By Lemma 5.6, it suffices to find a deformation datum $\mathcal{D} = (L, \emptyset)$ and a prime $\mathfrak{p} \subset R_{\mathcal{D}}$ satisfying the hypotheses of Lemma 5.6. Arguing as in the proof of [Tho15, Theorem 7.1], we can find a deformation datum $\mathcal{D}_0 = (L_0, \{R_v^{\mathbf{m}}\}_{v \in R})$ satisfying the following conditions:

- R consists of the set of places of L_0^+ dividing v_0 and \mathbf{m} is the partition of n corresponding to the Jordan decomposition of N , where $\text{rec}_{E_{\tilde{v}_0}}^T(\Pi_{\tilde{v}_0}) = (r, N)$.
- For each place $v|l$ of L_0^+ , $[L_{0,v}^+ : \mathbb{Q}_l] > n^2$.
- The integer $d = d_{\mathcal{D}_0}$ of Lemma 5.3 satisfies $d > n^2 + 1$.

We introduce also the deformation datum $\mathcal{D}_0^{\chi} = (L_0, \{R_v^{\chi_v}\}_{v \in R})$. Lemma 5.3 then implies that $\dim R_{\mathcal{D}_0^{\chi}}^{\text{red}}/\lambda < n[L_0^+ : \mathbb{Q}] - n^2$.

Let u be a place of L_0 above v_0 . By Lemma 5.4, $R_{\mathcal{D}_0}/J_{\mathcal{D}_0^1}R_{\mathcal{D}_0}$ is a finite Λ_{L_0} -algebra of dimension $\dim \Lambda_{L_0} = 1 + n[L_0^+ : \mathbb{Q}]$, hence $\dim R_{\mathcal{D}_0}/(J_{\mathcal{D}_0^1}, \lambda) \geq n[L_0^+ : \mathbb{Q}]$, hence $\dim R_{\mathcal{D}_0}/(J_{\mathcal{D}_0^1}, \lambda, \mathbf{m}_{R_u^1}) \geq n[L_0^+ : \mathbb{Q}] - n^2 > \dim R_{\mathcal{D}_0^{\chi}}^{\text{red}}/\lambda$. We can therefore choose a dimension 1 prime $\mathfrak{p}_0 \subset R_{\mathcal{D}_0}/(J_{\mathcal{D}_0^1}, \lambda, \mathbf{m}_{R_u^1})$ which does not come from $R_{\mathcal{D}_0^{\chi}}^{\text{red}}/\lambda$, and this prime will satisfy conditions 1, 4, and 5 of Lemma 5.6. Conditions 2 and 3 can be ensured by choosing \mathfrak{p}_0 also to avoid countably many quotients of $R_{\mathcal{D}_0}/(J_{\mathcal{D}_0^1}, \lambda, \mathbf{m}_{R_u^1})$ which have dimension bounded above by $\sup_{v|l} (n[L_0^+ : \mathbb{Q}] - [L_{0,v}^+ : \mathbb{Q}_l]) < n[L_0^+ : \mathbb{Q}] - n^2$. The possibility of finding a dimension 1 prime missing these countably many quotients is guaranteed by [Tho15, Lemma 1.9]. These quotients are described explicitly in [Tho15, p. 861, Proof of Theorem 6.1], where [Tho15, Lemma 1.9] is employed for the same purpose. Having chosen a prime \mathfrak{p}_0 satisfying conditions 1–5 of the lemma, we identify it with its pullback to a prime of $R_{\mathcal{D}_0^1}$.

The rings $R_{\mathcal{D}_0^{\chi}}/\lambda$ are canonically identified as χ varies. Similarly, the modules $S_{\mathcal{D}_0^{\chi}}/\lambda$ are canonically identified as χ varies; cf. [Ger, Lemma 2.2.6]. It follows that a prime $\mathfrak{p}^{\chi} \subset R_{\mathcal{D}_0^{\chi}}/\lambda$ contains $J_{\mathcal{D}_0^1}$ if and only if the corresponding prime $\mathfrak{p}^1 \subset R_{\mathcal{D}_0^1}/\lambda$ contains $J_{\mathcal{D}_0^1}$. Fix a choice of χ such that for each $v \in R$, the characters $\chi_{v,1}, \dots, \chi_{v,n}$ are pairwise distinct, and let $\mathfrak{p}^{\chi} \subset R_{\mathcal{D}_0^{\chi}}/\lambda$ be the prime corresponding in this way to \mathfrak{p}_0 .

By Lemma 5.5, we can find a good extension L and a deformation datum $\mathcal{D} = (L, \emptyset)$ such that L contains L_0 , the places in $\tilde{S}_{l,L}$ all split in L_0 , and there exists a commutative diagram

$$\begin{array}{ccccc} R_{\mathcal{D}} & \longleftarrow & P_{\mathcal{D}} & \longrightarrow & \mathbb{T}_{\mathcal{D}} \\ \downarrow & & \downarrow & & \downarrow \\ R_{\mathcal{D}_0^{\chi}}^{\text{nil}} & \longleftarrow & P_{\mathcal{D}_0^{\chi}}^{\text{nil}} & \longrightarrow & \mathbb{T}_{\mathcal{D}_0^{\chi}}. \end{array}$$

We let $\mathfrak{p} = \ker(R_{\mathcal{D}} \rightarrow R_{\mathcal{D}_0^{\chi}}/\mathfrak{p}^{\chi}) \subset R_{\mathcal{D}}$. The existence of the above diagram implies that $J_{\mathcal{D}} \subset \mathfrak{p}$, so that hypothesis 4 of Lemma 5.6 holds. The representation $r_{\mathfrak{p}} = r_{\mathfrak{p}_0}|_{G_L}$ is irreducible because $r_{\mathfrak{p}_0}$ is primitive and $r_{\mathfrak{p}}$ is multiplicity-free (because of the existence of σ), so hypothesis 1 holds. The other hypotheses hold by construction. This completes the proof.

5.2 An automorphy lifting theorem

We now use Theorem 5.1 to improve the main theorem of [Tho15]. Let E be an imaginary CM field with maximal totally real subfield F . Fix a prime l and an isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$.

Theorem 5.7. *Let $\rho : G_E \rightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ be a continuous representation satisfying the following hypotheses.*

1. $\rho^c \cong \rho^\vee \epsilon^{1-n}$.
2. ρ is ramified at only finitely many places.
3. ρ is ordinary of weight λ , for some $\lambda \in (\mathbb{Z}_+^n)^{\mathrm{Hom}(E, \overline{\mathbb{Q}}_l)}$, in the sense of [Tho15, Definition 2.5].
4. $l > 3$ and $E(\zeta_l)$ is not contained in the extension of E cut out by $\mathrm{ad} \bar{\rho}$.
5. $\bar{\rho} \cong \bar{\rho}_1 \oplus \bar{\rho}_2$, where the $\bar{\rho}_i|_{G_{E(\zeta_l)}}$ are adequate, in the sense of [Tho12, §2]. Writing $n_i = \dim \bar{\rho}_i$, we have $n_1 \neq n_2$, $n_1 n_2$ is even, and l does not divide $n_1 n_2 (n_1 + n_2)$. The representation $\bar{\rho}$ is primitive.
6. There exists a place w_0 of E , prime to l , such that $\rho|_{G_{E_{w_0}}}^{\mathrm{ss}} \cong \bigoplus_{i=1}^n \psi \epsilon^{n-i}$ for some unramified character $\psi : G_{E_{w_0}} \rightarrow \overline{\mathbb{Q}}_l^\times$.
7. There exists a RACSDC automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_E)$ such that:
 - (a) Π is ι -ordinary.
 - (b) $r_\iota(\Pi) \cong \bar{\rho}$.
 - (c) Let $\mathrm{rec}_{E_{w_0}}^T(\Pi_{w_0}) = (r, N)$. Then N has a Jordan block of length strictly greater than $\max(n_1, n_2)$.
8. There exists a CM extension E_0/E linearly disjoint from the extension of $E(\zeta_l)$ cut out by $\bar{\rho}$, and RAESDC automorphic representations (π_1, χ_1) and (π_2, χ_2) of $\mathrm{GL}_{n_1}(\mathbb{A}_{E_0})$ and $\mathrm{GL}_{n_2}(\mathbb{A}_{E_0})$, respectively, such that π_1, π_2 are ι -ordinary, and $r_\iota(\pi_i) \cong \bar{\rho}_i|_{G_{E_0}}$ for $i = 1, 2$.

Then ρ is automorphic: there exists a RACSDC automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_E)$ such that $\rho \cong r_\iota(\pi)$.

Proof. There exists a soluble extension L/E such that the hypotheses of Theorem 5.1 apply to Π_L ; thus we can find a further soluble extension E_1/E , disjoint from the extension of $E(\zeta_l)$ cut out by $\bar{\rho}$, and a RACSDC automorphic representation Π_1 of $\mathrm{GL}_n(\mathbb{A}_{E_1})$ satisfying the hypotheses of [Tho15, Theorem 7.1] with respect to $\rho|_{E_1}$. This implies that $\rho|_{E_1}$ is automorphic, and irreducible by property (3) in Theorem 5.1. It follows by soluble descent that ρ is itself automorphic. \square

6 Symmetric power functoriality

In this section we use what we have done so far to draw consequences for the existence of symmetric power functoriality for $\mathrm{GL}_2(\mathbb{A}_F)$.

Theorem 6.1. *Let F be a totally real field, and let (π, χ) be a RAESDC automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ which is not automorphically induced from a quadratic CM extension. Then:*

1. Suppose that $F \cap \mathbb{Q}(\zeta_5) = \mathbb{Q}$. Then the 6th symmetric power lifting of π exists, as a cuspidal automorphic representation of $\mathrm{GL}_7(\mathbb{A}_F)$.
2. Suppose that $F \cap \mathbb{Q}(\zeta_7) = \mathbb{Q}$. Then the 8th symmetric power lifting of π exists, as a cuspidal automorphic representation of $\mathrm{GL}_9(\mathbb{A}_F)$.

By a series of reductions which are essentially the same as those of [CT15, §6], it is enough to prove the following result.

Theorem 6.2. *Let $l \in \{5, 7\}$, and let F be a totally real number field. Fix an isomorphism $\iota : \overline{\mathbb{Q}_l} \cong \mathbb{C}$. Let (π, χ) be a RAESDC automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$, and suppose that the following hypotheses hold:*

1. *π is ι -ordinary of weight zero, i.e. at every infinite place $\tau : F \hookrightarrow \mathbb{R}$, the infinity type of π is $a_\tau = (1/2, -1/2)$.*
2. *Let $\rho = r_\iota(\pi)$. Then the residual representation $\bar{\rho} : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}_l})$ is irreducible, and its image contains a conjugate of $\mathrm{SL}_2(\mathbb{F}_{l^a})$ for some $a > 1$. Moreover, writing $\varphi \in \mathrm{Gal}(\overline{\mathbb{Q}_l}/\mathbb{Q}_l)$ for a lift of the arithmetic Frobenius, the representations $({}^\varphi\bar{\rho} \otimes \bar{\rho})|_{G_{F(\zeta_l)}}$ and $(\mathrm{Sym}^{l-3}\bar{\rho})|_{G_{F(\zeta_l)}}$ are adequate.*
3. *There exists a place u_0 of F , coprime to 2, such that π_{u_0} is an unramified twist of the Steinberg representation and q_{u_0} is a primitive root mod l . Moreover, $\bar{\rho}$ is ramified at u_0 .*

Then $\mathrm{Sym}^{l+1}\rho$ is automorphic.

Proof. Let L/F be the extension of $F(\zeta_l)$ cut out by $\bar{\rho}$, and let S be a finite set of finite places of F with the property that for any intermediate extension $L/L_0/F$ which is Galois over F with simple Galois group, there exists a place of S which is not split in L_0 . We assume moreover that S is disjoint from the set of places of F which divide l or at which ρ is ramified.

Let F_0/F be an S -split totally real quadratic extension in which u_0 is inert, and let $\omega_{F_0/F} : G_F \rightarrow \overline{\mathbb{Q}_l}^\times$ denote the corresponding quadratic character. Let E/F be a CM imaginary quadratic extension which is S -split, ramified at u_0 , and in which every place of F dividing l splits. By [CHT08, Lemma 4.1.4], we can find a finite order character $\psi : G_E \rightarrow \overline{\mathbb{Q}_l}^\times$, unramified above u_0 , such that $\psi\psi^c = r_\iota(\chi)|_{G_E}$.

Let F_1/F be a soluble totally real extension satisfying the following conditions:

1. F_1 is $S \cup \{u_0\}$ -split.
2. For each place $v|l$ of F_1 , the local degree $[F_{1,v} : \mathbb{Q}_l]$ is even.
3. The extension $F_1 \cdot F_0/F_1$ is everywhere unramified.
4. Let $E_1 = E \cdot F_1$. Then if $w \nmid u_0$ is a place of E_1 at which π_{E_1} or $\psi|_{G_{E_1}}$ is ramified, then w is split over F_1 . The extension E_1/F_1 is unramified away from the places dividing u_0 .

Let $\pi_1 = \pi_{E_1} \otimes \iota\psi^{-1}|_{E_1}$, a RACSDC automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{E_1})$ which is ι -ordinary of weight zero. (It is cuspidal because of the assumption that π_{u_0} is an unramified twist of the Steinberg representation; this property is preserved under base change.) After possibly replacing π_1 by $\pi_1 \otimes \iota\omega_{F_0/F}|_{G_{E_1}}$, we can assume that $\pi_{1,w} \cong \mathrm{St}_{2,E_{1,w}}$ for each place $w|u_0$ of E_1 .

Let y_0, \dots, y_s denote the places of F_1 dividing u_0 , and let z_0, \dots, z_s denote the places of E_1 dividing y_0, \dots, y_s , respectively. We observe that, by construction, q_{y_i} is a primitive root modulo l for each $i = 0, \dots, s$. Let b denote the least integer such that $q_{y_0}^b \equiv -1 \pmod{l}$, and choose a cyclic totally real extension F_2/F_1 of degree b which is S -split and in which y_0 is inert and each place y_1, \dots, y_s splits. Let $E_2 = F_2 \cdot F_1$, and let $\pi_2 = \pi_{1,E_2}$.

Let $\rho_2 = r_\iota(\pi_2)$, and let $\chi_2 = \det \rho_2$. We have an isomorphism of residual representations

$$\left(\mathrm{Sym}^{l+1}\bar{\rho}_2\right)^{\mathrm{ss}} \cong ({}^\varphi\bar{\rho}_2 \otimes \bar{\rho}_2) \oplus \bar{\chi}_2^2 \mathrm{Sym}^{l-3}\bar{\rho}_2.$$

By [CT15, Proposition 3.10] and base change for U_2 , we can find a RACSDC automorphic representation π'_2 of $\mathrm{GL}_2(\mathbb{A}_{E_2})$ with the following properties:

1. π'_2 is ι -ordinary and for every embedding $\tau : E_2 \hookrightarrow \overline{\mathbb{Q}_l}$, we have $\mathrm{HT}_\tau(r_\iota(\pi'_2)) = \{(1-l)/2, (1+l)/2\}$.
2. For each $i = 0, \dots, s$, $\pi'_{2,z_i} \cong \mathrm{St}_{2,E_{2,z_i}} \otimes \iota\omega$, where ω denotes the Teichmüller lift of $\epsilon^{(l-1)/2}$.
3. $\overline{r_\iota(\pi'_2)} \cong \epsilon^{(l-1)/2} \otimes \bar{\rho}_2$.

4. If π'_2 is ramified at a place $w \nmid u_0$ of E_2 , then w is split over F_2 .

By [Ram00, Theorem M], there exists a RAESDC automorphic representation $(\Pi'_1, |\cdot|^{2-l})$ of $\mathrm{GL}_4(\mathbb{A}_{E_2})$ such that $r_l(\Pi'_1) \cong \varphi r_l(\pi'_2) \otimes \epsilon^{(1-l)/2} \otimes r_l(\pi_2)$. There also exists a RACSDC automorphic representation Π'_2 of $\mathrm{GL}_{l-2}(\mathbb{A}_{E_2})$ such that $r_l(\Pi'_2) \cong \mathrm{Sym}^{l-3} \rho_2$, by [GJ78] (if $l = 5$) and by [Kim03] (if $l = 7$).

We define $\Pi_1 = \Pi'_1 \otimes |\cdot|^{(l-2)/2}$ and $\Pi_2 = \Pi'_2 \otimes \iota(\epsilon \chi_2)^2$. Then $\Pi_1 \boxplus \Pi_2$ is a conjugate self-dual, regular algebraic automorphic representation of $\mathrm{GL}_{l+2}(\mathbb{A}_{E_2})$ of weight $\lambda = 0$, and satisfies the following conditions:

1. $\overline{r_l(\Pi_1 \boxplus \Pi_2)} \cong (\varphi \bar{\rho}_2 \otimes \bar{\rho}_2) \oplus \bar{\chi}_2^2 \mathrm{Sym}^{l-3} \bar{\rho}_2$.
2. $\Pi_1 \boxplus \Pi_2$ is ι -ordinary (by [CT14, Lemma 2.6]).
3. For each $i = 0, \dots, s$, $\Pi_{1, w_i} \cong \mathrm{St}_{3, E_2, w_i}(\omega) \boxplus \mathrm{St}_{1, E_2, w_i}(\omega)$.
4. For each $i = 0, \dots, s$, $\Pi_{2, w_i} \cong \mathrm{St}_{l-2, E_2, w_i}$.

In particular, the hypotheses of §4 apply to $(\Pi_1 \otimes \omega_{F_0/F}|_{G_{E_2}}) \boxplus \Pi_2$. Let Π denote the RACSDC automorphic representation of $\mathrm{GL}_{l+2}(\mathbb{A}_{E_2})$ whose existence is asserted by Theorem 4.2. Let $E_0 = E_2 \cdot F_0$, and let $\Pi_0 = \Pi_{E_0}$. We have $r_l(\Pi_0) \cong (\mathrm{Sym}^{l+1} \bar{\rho}_2)^{\mathrm{ss}}$, and Theorem 5.7 now implies that $\mathrm{Sym}^{l+1} \rho_2|_{G_{E_0}}$ is automorphic. The desired result now follows by soluble descent, which applies since the representation $\mathrm{Sym}^{l+1} \rho_2|_{G_{E_0}}$ is irreducible. \square

7 Complements: the mixed-parity case

In this section, we show how to extend our results to those cuspidal automorphic representations associated to Hilbert modular forms whose weights are not of constant parity. Recall that if F is a totally real field and π is a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ such that π_∞ is essentially square-integrable, then π is associated to Hilbert modular forms of weights $k_v \geq 2$ at every place $v|\infty$ of F . Conversely, every such π arises in this fashion. The representation π is algebraic (up to twist) if and only if it is associated to a Hilbert modular form of weights k_v of parity independent of $v|\infty$. This condition is vacuous if $F = \mathbb{Q}$, but non-trivial over every other totally real field F , as follows from well-known density results (see for example [Clo86]).

Our proofs in this section rely on the observation (which goes back at least to the work of Blasius–Rogawski [BR93]) that a representation π which does not satisfy this parity condition does in fact become algebraic up to twist after base change to an imaginary CM extension.

Theorem 7.1. *Let F be a totally real field, and let $r \geq 1$ be an integer. Suppose that for every RAESDC automorphic representation (π, χ) of $\mathrm{GL}_2(\mathbb{A}_F)$ which is not automorphically induced from a quadratic CM extension, there exists a RAESDC automorphic representation (Π, Ξ) of $\mathrm{GL}_{r+1}(\mathbb{A}_F)$ such that*

$$\mathrm{Sym}^r \mathrm{rec}_{F_v} \pi_v \cong \mathrm{rec}_{F_v} \Pi_v$$

for all places $v \nmid \infty$ of F . (Equivalently, for all primes l and for all isomorphisms $\iota : \bar{\mathbb{Q}}_l \cong \mathbb{C}$, we have $\mathrm{Sym}^r r_l(\pi) \cong r_l(\Pi)$.)

Then for every cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ such that π_∞ is essentially square integrable and π is not automorphically induced from a quadratic CM extension, there exists a cuspidal automorphic representation Π of $\mathrm{GL}_{r+1}(\mathbb{A}_F)$ such that

$$\mathrm{Sym}^r \mathrm{rec}_{F_v} \pi_v \cong \mathrm{rec}_{F_v} \Pi_v$$

for all places $v \nmid \infty$ of F . In particular, $L(\mathrm{Sym}^r \pi, s) = L(\Pi, s)$ has an analytic continuation to the entire complex plane, and satisfies the expected functional equation.

Corollary 7.2. *Let F be a totally real field, and let $r \geq 1$ be an integer. Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ such that π_∞ is essentially square-integrable. Suppose furthermore that π is*

not automorphically induced from a quadratic CM extension. Then there exists a cuspidal automorphic representation Π of $\mathrm{GL}_{r+1}(\mathbb{A}_F)$ satisfying

$$\mathrm{Sym}^r \mathrm{rec}_{F_v} \pi_v \cong \mathrm{rec}_{F_v} \Pi_v$$

for all places $v \nmid \infty$ in each of the following cases:

1. F arbitrary and $1 \leq r \leq 4$.
2. $F \cap \mathbb{Q}(\zeta_5) = \mathbb{Q}$ and $r \in \{5, 6\}$.
3. $F \cap \mathbb{Q}(\zeta_{35}) = \mathbb{Q}$ and $r = 7$.
4. $F \cap \mathbb{Q}(\zeta_7) = \mathbb{Q}$ and $r = 8$.

Proof. The first point is implied by known cases of symmetric power functoriality [GJ78], [KS02], [Kim03]. The remaining points follow on combining Theorem 7.1 with Theorem 6.1 (case $r = 6, 8$) and [CT15, Theorem 1.2] (case $r = 5, 7$). \square

The remainder of this section is devoted to the proof of Theorem 7.1. We begin with some preliminary results.

Theorem 7.3. 1. Let F be a number field, and π a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Then $\mathrm{Sym}^2 \pi$ exists, and the following are equivalent:

- (a) $\mathrm{Sym}^2 \pi$ is cuspidal.
 - (b) π is not automorphically induced from a quadratic extension of F .
 - (c) For all non-trivial Hecke characters $\eta : F^\times \backslash \mathbb{A}_F^\times \rightarrow \mathbb{C}^\times$, we have $\pi \not\cong \pi \otimes \eta$.
2. Let F be a number field, and π a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Let E/F be a quadratic extension. Suppose that $\mathrm{Sym}^2 \pi$ is cuspidal. Then the representations π_E and $\mathrm{Sym}^2 \pi_E$ are also cuspidal.

Proof. 1. Immediate from [GJ78].

2. Immediate from [AC89, Theorem 3.4.2]. (Note that as $\mathrm{Sym}^2 \pi$ is a cuspidal automorphic representation of $\mathrm{GL}_3(\mathbb{A}_F)$, it cannot be automorphically induced from a quadratic extension.) \square

Lemma 7.4. Let F be a totally real field, and let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ such that π_∞ is essentially square-integrable. Let E/F be an imaginary CM extension such that π_E is cuspidal. (Such a field always exists; for example, one can choose E to be ramified at a place where π is unramified.) Then there exists a continuous character $\chi : E^\times \backslash \mathbb{A}_E^\times \rightarrow \mathbb{C}^\times$ such that $\pi_E \otimes \chi$ is RACSDC.

Proof. Let $\psi : F^\times \backslash \mathbb{A}_F^\times \rightarrow \mathbb{C}^\times$ denote the central character of π . Using the fact, due to Chevalley, that the arithmetic subgroups of the unit group of F are congruence subgroups, we can assume (after passing to a twist) that ψ is of finite order. In particular, π is unitary. If $v \mid \infty$ is a place of F , then the Langlands parameter $\mathrm{rec}_{F_v} \pi_v$ is equal to the induction $\mathrm{Ind}_{\mathbb{C}^\times}^{W_{F_v}} \varphi$, where $\varphi(z) = (z/|z|)^{p_v}$ for some $p_v \in \mathbb{Z}_{>0}$. We have $\psi_v(-1) = (-1)^{p_v+1}$. (We write $|z| = (z\bar{z})^{1/2} = |z|_{\mathbb{C}}^{1/2}$ for the usual absolute value on \mathbb{C} .) If $\chi : E^\times \backslash \mathbb{A}_E^\times \rightarrow \mathbb{C}^\times$ is a continuous unitary character of infinity type

$$\chi_\tau(z) = (z/|z|)^{m_\tau},$$

and $\tau : E \hookrightarrow \mathbb{C}$ is an embedding inducing the place v of F , then the Langlands parameter $\tau^{-1} \circ \mathrm{rec}_{E_v}(\pi_E \otimes \chi)$ is given by the formula:

$$z \mapsto \begin{pmatrix} (z/|z|)^{p_v+m_\tau} & 0 \\ 0 & (z/|z|)^{-p_v+m_\tau} \end{pmatrix}.$$

Thus $\pi_E \otimes \chi$ is regular algebraic if for each $\tau : E \hookrightarrow \mathbb{C}$ as above, $p_v + m_\tau$ is an odd integer; while $\pi_E \otimes \chi$ is conjugate self-dual if and only if $\chi\chi^c = \psi^{-1} \circ \mathbb{N}_{E/F}$.

To prove the lemma, we must therefore show the existence of a character satisfying these two conditions. Choose for each place $v|\infty$ of F an embedding $\tau_v : E \hookrightarrow \mathbb{C}$ inducing the place v of F , and define $m_{\tau_v} = p_v + 1$, $m_{\tau_v^c} = -(p_v + 1)$. Choose a character $\chi_0 : E^\times \backslash \mathbb{A}_E^\times \rightarrow \mathbb{C}^\times$ with infinity type

$$\chi_{0,\tau}(z) = (z/|z|)^{m_\tau}$$

for each embedding $\tau : E \hookrightarrow \mathbb{C}$. Then the character $\chi_0\chi_0^c$ is of finite order; let $\lambda = \chi_0|_{F^\times \backslash \mathbb{A}_F^\times}$, so that $\chi_0\chi_0^c = \lambda \circ \mathbb{N}_{E/F}$. The character $\lambda : F^\times \backslash \mathbb{A}_F^\times \rightarrow \mathbb{C}^\times$ is continuous and of finite order, and we have for each place $v|\infty$

$$\lambda_v(-1) = \chi_{0,\tau_v}(-1) = (-1)^{p_v+1}.$$

In particular, the quantity $(\psi_v\lambda_v)(-1) = 1$ is independent of the choice of $v|\infty$. By [CHT08, Lemma 4.1.4], then, there exists a character $\chi_1 : E^\times \backslash \mathbb{A}_E^\times \rightarrow \mathbb{C}^\times$ of finite order such that $\chi_1\chi_1^c = (\psi^{-1}\lambda^{-1}) \circ \mathbb{N}_{E/F} = (\psi^{-1} \circ \mathbb{N}_{E/F})(\chi_0\chi_0^c)^{-1}$. The character $\chi = \chi_0\chi_1$ now has the desired properties. \square

Lemma 7.5. *Let E be an imaginary CM field, and π a RACSDC automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$. Suppose that $\mathrm{Sym}^2 \pi$ is cuspidal. Then there exists a set \mathcal{L} of rational primes of Dirichlet density 1 such that for all $l \in \mathcal{L}$ and for all isomorphisms $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$, the representation $\bar{r}_l(\pi) : G_E \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ is irreducible, and its image contains a conjugate of $\mathrm{SL}_2(\mathbb{F}_l)$.*

Proof. By [BLGGT14, Theorem 5.5.2], there exists a set \mathcal{L} of rational primes of Dirichlet density 1 such that for all $l \in \mathcal{L}$ and for all isomorphisms $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$, the representations $r_l(\pi) : G_E \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ and $\mathrm{Sym}^2 r_l(\pi) : G_E \rightarrow \mathrm{GL}_3(\overline{\mathbb{Q}}_l)$ are irreducible. Applying [BLGGT14, Proposition 5.3.2] and possibly shrinking \mathcal{L} , we can further assume that the representations $\bar{r}_l(\pi)$ and $\mathrm{Sym}^2 \bar{r}_l(\pi)$ are irreducible. Casting out finitely many elements of \mathcal{L} , we can also assume that for each $l \in \mathcal{L}$, π and E are unramified above l .

To prove the lemma, it therefore suffices (by the classification of finite subgroups of $\mathrm{PGL}_2(\overline{\mathbb{F}}_l)$) to show that for all but finitely many choices of $l \in \mathcal{L}$, the projective image of $\bar{r}_l(\pi)$ contains an element of order strictly greater than 5. (Note that the image cannot be contained in the normalizer of a quadratic torus, since $\mathrm{Sym}^2 \bar{r}_l(\pi)$ is, by hypothesis, irreducible.) To show this, let $\lambda = (\lambda_{\tau,1}, \lambda_{\tau,2})_{\tau:E \hookrightarrow \mathbb{C}}$ denote the weight of π . Let $k = 1 + |\sup_\tau \lambda_{\tau,1} - \inf_\tau \lambda_{\tau,2}|$. If $v|l$ is a place of E , let S_v denote the set of embeddings $E_v \hookrightarrow \overline{\mathbb{Q}}_l$. If $\tau \in S_v$, let $\omega_\tau : I_v \rightarrow \overline{\mathbb{F}}_l^\times$ denote the corresponding fundamental character. If $l > k + 2$ is a prime and $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$ is an isomorphism, then for each place $v|l$ of E there exists a partition $S_v = J_1 \coprod J_2$ such that $(\bar{r}_l(\pi)|_{I_v})^{\mathrm{ss}}$ is a sum of two characters α_1, α_2 , given by the formulae

$$\begin{aligned} \alpha_1(x)^{-1} &= \prod_{\tau \in J_1} \omega_\tau(x)^{\lambda_{\iota^{-1}\tau,1}+1} \prod_{\tau \in J_2} \omega_\tau(x)^{\lambda_{\iota^{-1}\tau,2}}, \\ \alpha_2(x)^{-1} &= \prod_{\tau \in J_2} \omega_\tau(x)^{\lambda_{\iota^{-1}\tau,1}+1} \prod_{\tau \in J_1} \omega_\tau(x)^{\lambda_{\iota^{-1}\tau,2}}. \end{aligned}$$

Indeed, this follows from the fact that $r_l(\pi)$ is crystalline with Hodge-Tate weights $\mathrm{HT}_\tau(r_l(\pi)) = \{\lambda_{\iota^{-1}\tau,1} + 1, \lambda_{\iota^{-1}\tau,2}\}$, together with the results of Fontaine-Laffaille [FL82, §§4–5]. To prove the lemma, it is therefore enough to show that for l sufficiently large, and for all isomorphisms $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$, there exists $v|l$ such that the character α_2/α_1 has order strictly greater than 5.

This character is given by the formula

$$\alpha_2/\alpha_1 = \prod_{\tau \in J_1} \omega_\tau^{\lambda_{\iota^{-1}\tau,1}+1-\lambda_{\iota^{-1}\tau,2}} \prod_{\tau \in J_2} \omega_\tau^{-(\lambda_{\iota^{-1}\tau,1}+1-\lambda_{\iota^{-1}\tau,2})}.$$

Let f_v denote the absolute residue degree of E_v , and fix an embedding $\tau_0 : E_v \hookrightarrow \overline{\mathbb{Q}}_l$. Writing $\varphi \in \mathrm{Gal}(E_v/\mathbb{Q}_l)$ for the arithmetic Frobenius, this allows us to identify $S_v = \{\tau_0, \varphi \circ \tau_0, \dots, \varphi^{f_v-1} \circ \tau_0\} = \{0, 1, \dots, f_v - 1\}$, so that the order of the character α_2/α_1 is equal to the order of the image of the integer

$$\sum_{i \in J_1} l^i (\lambda_{\iota^{-1}\tau,1} + 1 - \lambda_{\iota^{-1}\tau,2}) - \sum_{i \in J_2} l^i (\lambda_{\iota^{-1}\tau,1} + 1 - \lambda_{\iota^{-1}\tau,2})$$

in the group $\mathbb{Z}/(l^{f_v} - 1)\mathbb{Z}$. This integer is not zero (since it is not even zero modulo l), so to prove the lemma it is enough to show that for l sufficiently large, we have

$$5 \left| \sum_{i \in J_1} l^i (\lambda_{l^{-1}\tau,1} + 1 - \lambda_{l^{-1}\tau,2}) - \sum_{i \in J_2} l^i (\lambda_{l^{-1}\tau,1} + 1 - \lambda_{l^{-1}\tau,2}) \right| < l^{f_v} - 1.$$

However, the left hand side here is less than or equal to

$$5 \sum_{i=0}^{f_v-1} l^i (\lambda_{l^{-1}\tau,1} + 1 - \lambda_{l^{-1}\tau,2}) \leq 5k(l^{f_v} - 1)/(l - 1),$$

so it is enough to take $l > 5k + 1$. \square

Proposition 7.6. *Let $r \geq 1$ be an integer. Let E/F be an imaginary CM quadratic extension of a totally real field, and suppose that for every RAESDC automorphic representation (π, χ) of $\mathrm{GL}_2(\mathbb{A}_F)$ such that π is not automorphically induced from a CM extension, the r^{th} symmetric power $\mathrm{Sym}^r r_l(\pi)$ is automorphic.*

Then for every RACSDC automorphic representation Π of $\mathrm{GL}_2(\mathbb{A}_E)$ which is not automorphically induced from a quadratic extension, the r^{th} symmetric power $\mathrm{Sym}^r r_l(\Pi)$ is automorphic.

Proof. By Lemma 7.5 we can find a prime $l > 2(r+2)$ and an isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$ such that the residual representation $r_l(\Pi)$ is irreducible and its image contains a conjugate of $\mathrm{SL}_2(\mathbb{F}_l)$. We can further assume that E and π are unramified above l , and that for each embedding $\tau : E \hookrightarrow \overline{\mathbb{Q}}_l$, the elements of the set $\mathrm{HT}_\tau(\mathrm{Sym}^r r_l(\Pi))$ lie in an interval of length at most $l-2$, and that l is split in E .

There exists a continuous character $\varphi : G_E \rightarrow \overline{\mathbb{F}}_l^\times$ such that $\overline{(r_l(\Pi) \otimes \varphi)^c} \cong \overline{r_l(\Pi)} \otimes \varphi$. In particular, the representation $\overline{(r_l(\Pi) \otimes \varphi)}$ extends to a homomorphism $\overline{R} : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_l)$, which is odd (cf. the proof of [Tho15, Theorem 8.1]). Applying [BLGGT14, Theorem 4.4.1] and a standard extension of this result to a totally real base field, we can construct a lift $R : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{Z}}_l)$ which is potentially diagonalizable in the sense of [BLGGT14, §1.4], and unramified almost everywhere. The representations $R|_{G_E}$ and $r_l(\Pi)$ are potentially diagonalizable by [BLGGT14, Lemma 1.4.3], and it now follows from [BLGGT14, Theorem 4.2.1] and soluble descent that R itself is automorphic.

By assumption, the representation $\mathrm{Sym}^r R$ is automorphic, hence $\mathrm{Sym}^r \overline{r_l(\Pi)}$ is (in the terminology of [BLGGT14]) residually potentially diagonalizably automorphic. Applying [BLGGT14, Theorem 4.2.1] once more, we deduce that $\mathrm{Sym}^r r_l(\Pi)$ is automorphic. \square

Proof of Theorem 7.1. Let F be a totally real field. Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ such that π_∞ is essentially square integrable and $\mathrm{Sym}^2 \pi$ is cuspidal. We must show that the r^{th} symmetric power lifting of π exists.

Choose an imaginary CM quadratic extension E/F . By Lemma 7.4, there exists a Hecke character $\chi : E^\times \backslash \mathbb{A}_E^\times \rightarrow \mathbb{C}^\times$ such that $\pi_E \otimes \chi$ is RACSDC. Combining our hypothesis with Proposition 7.6, we deduce that the r^{th} symmetric power lifting of $\pi_E \otimes \chi$ exists, as a cuspidal automorphic representation of $\mathrm{GL}_{r+1}(\mathbb{A}_E)$. The representation $\mathrm{Sym}^r \pi_E$ thus exists and descends to a cuspidal automorphic representation Π of $\mathrm{GL}_{r+1}(\mathbb{A}_F)$.

It remains to show that for each place $v \nmid \infty$ of F , Π_v has the correct form. Let v be a finite place of F , and let E_0 be a quadratic imaginary CM extension of F in which v splits. Applying Lemma 7.4 once more, we can find a Hecke character $\chi_0 : E_0^\times \backslash \mathbb{A}_{E_0}^\times \rightarrow \mathbb{C}^\times$ such that $\pi_{E_0} \otimes \chi_0$ is RACSDC. Applying local-global compatibility for $\mathrm{Sym}^r(\pi_{E_0} \otimes \chi_0)$, a twist of Π_{E_0} , we deduce that

$$\mathrm{Sym}^r \mathrm{rec}_{F_v} \pi_v = \mathrm{rec}_{F_v} \Pi_v.$$

Since v was arbitrary, this completes the proof. \square

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