Leverage Aversion, Efficient Frontiers, and the Efficient Region*

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* Previously entitled "Leverage Aversion and Portfolio Optimality: An Extension"

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Abstract

We previously proposed that portfolio theory and mean-variance optimization be augmented to incorporate investor aversion to leverage, and illustrated optimal levels of portfolio leverage. We suggest here a new specification for leverage aversion, which may better capture the unique risks of leverage. We also introduce mean-variance-leverage efficient frontiers, comparing them with conventional mean-variance efficient frontiers, and develop the mean-variance-leverage efficient region, which shows that leverage aversion can have a large impact on portfolio choice.

Leverage Aversion, Efficient Frontiers, and the Efficient Region

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In Jacobs and Levy (2012), we proposed that portfolio theory and mean-variance utility (Markowitz 1952) be augmented to incorporate investor aversion to leverage. Mean-variance optimization determines optimal security weights by considering portfolio expected return and variance of portfolio return. To the extent that leverage increases a portfolio's volatility (the square root of variance), mean-variance optimization captures some of the risk associated with leverage. But it fails to capture other components of risk that are unique to using leverage, including the risk of margin calls and forced liquidations (possibly at adverse prices), losses beyond the capital invested, and the risks and costs of bankruptcy.

For an investor with no tolerance for leverage, optimal mean-variance portfolios are unleveraged ("long-only"), and mean-variance optimization is appropriate. But, for an investor able to tolerate leverage, using mean-variance optimization is equivalent to assuming that the investor has an infinite tolerance for leverage or, stated differently, has no aversion to leverage. In practice, however, investors *are* leverage averse. For example, if offered a choice between a portfolio having a particular expected return and variance *without* leverage and another portfolio that offers the same expected return and variance *with* leverage, investors would prefer the portfolio without leverage. The conventional mean-variance utility function cannot distinguish between these two portfolios because it does not represent an important aspect of investor behavior, namely, investor aversion to leverage.

When investors employ leverage, they generally constrain it in an *ad hoc* manner; that is, they choose a level of leverage with which they feel comfortable and impose it on the portfolio. Jacobs and Levy (2012) suggested determining

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the optimal level of leverage by using a utility function that includes an explicit leverage tolerance term in addition to the traditional volatility tolerance term. That article provided one way to specify the leverage tolerance term and illustrated optimal portfolio leverage levels when both volatility and leverage aversion are included in the utility function.

In this article, we provide an alternative specification of the leverage tolerance term, which may better capture the unique risks of leverage. We introduce mean-variance-leverage efficient frontiers and compare them with conventional mean-variance efficient frontiers. We also develop the concept of a mean-variance-leverage efficient region. An analysis of the mean-variance-leverage efficient frontiers and the efficient region shows that leverage aversion can have a large impact on portfolio choice.

Specifying the Leverage Aversion Term

The leverage aversion term that augments a mean-variance utility function can be specified in different ways. Jacobs and Levy (2012) suggested the following:

$$U = \alpha_P - \frac{1}{2\tau_V}\sigma_P^2 - \frac{1}{2\tau_L}c\Lambda^2. \qquad (1)$$

where α_P is the portfolio's expected active return relative to benchmark, σ_P^2 is the variance of the portfolio's active return, Λ is the portfolio's leverage, and c is a constant defined below.[1] With this specification, risk tolerance essentially changes from a one-dimensional attribute (as in mean-variance optimization) to a two-dimensional attribute, with the first dimension being the traditional risk tolerance, renamed as volatility tolerance τ_V, and the second dimension being leverage tolerance, τ_L. We used a squared term for leverage so that both risk components would have similar functional forms. Leverage was defined as:

$$\Lambda = \sum_{i=1}^{N} |h_i| - 1. \tag{2}$$

where h_i is the portfolio holding weight of security i for each of the N securities in the selection universe.[2]

To investigate this utility function, illustrative ranges for the tolerances were determined. As one reference point, a value of $\tau_V = 0$ corresponds to an investor who is completely intolerant of active volatility risk. Such an investor would choose an index fund. As another reference point, a value of $\tau_V \approx 1$ causes quadratic utility of return to be equivalent to log-utility of wealth, a utility function often used in the finance literature (Levy and Markowitz 1979). Thus, we chose $\tau_V \in [0,2]$. For illustrative purposes, we chose τ_L to span the same range as τ_V.

A constant c was selected that would result in the two risk terms (volatility risk, σ_P^2, and leverage risk, $c\Lambda^2$) having similar orders of magnitude. In particular, c was chosen to be the cross-sectional average of the variances of the securities' active returns. That is,

$$c = \frac{1}{N} \sum_{i=1}^{N} \omega_i^2, \tag{3}$$

where ω_i^2 is the variance of the active return of security i. Because portfolios in practice generally have leverage levels ranging from zero to about two (very highly leveraged portfolios are relatively few in number, but can be large in asset size), the product $c\Lambda^2$ should be of a similar order of magnitude to σ_P^2, so that similar values of τ_V and τ_L lead to similar levels of disutility.

Using the constant c to specify the leverage tolerance term has certain intuitive appeal. In addition to resulting in similar orders of magnitude for the volatility and leverage terms, the use of active returns in computing c is

congruent with the use of active returns in the computation of portfolio expected active return and variance. Moreover, from an implementation perspective, the use of a constant means that the utility function can, if desired, be restated as a quadratic optimization problem, which is advantageous because quadratic solvers are readily available.

However, the unique risks of leverage may relate more to a portfolio's total volatility than to the volatility of its active returns. That is, the risk that portfolio losses will trigger a margin call or exceed the capital invested depends on the portfolio's total volatility. Furthermore, this leverage dimension of risk will not be constant, but will vary across different portfolios having different volatilities.

Specification of the Leverage Aversion Term Using Portfolio Total Volatility

We introduce here another possible specification of an augmented mean-variance utility function that includes a leverage aversion term:

$$U = \alpha_P - \frac{1}{2\tau_V}\sigma_P^2 - \frac{1}{2\tau_L}\sigma_T^2\Lambda^2. \tag{4}$$

where σ_T^2 is the variance of the leveraged portfolio's total return. This leverage aversion term assumes that the risks of leverage rise with the product of the variance of the leveraged portfolio's total return and the square of leverage. This specification may better capture the portfolio's risk of margin calls and forced liquidations.

If α_i is the expected active return of security i, b_i is the weight of security i in the benchmark, x_i is the active weight of security i (and by definition $x_i = h_i - b_i$), σ_{ij} is the covariance between the active returns of securities i and

j, and q_{ij} is the covariance between the total returns of securities i and j, then Equation (4) can be written as:

$$U = \sum_{i=1}^{N} \alpha_i x_i - \frac{1}{2\tau_V} \sum_{i=1}^{N}\sum_{j=1}^{N} x_i \sigma_{ij} x_j - \frac{1}{2\tau_L}\left(\sum_{i=1}^{N}\sum_{j=1}^{N} h_i q_{ij} h_j\right)\Lambda^2. \tag{5}$$

Using Equation (2), and since $h_i = b_i + x_i$, Equation (5) becomes:

$$U = \sum_{i=1}^{N} \alpha_i x_i - \frac{1}{2\tau_V} \sum_{i=1}^{N}\sum_{j=1}^{N} x_i \sigma_{ij} x_j - \frac{1}{2\tau_L}\left(\sum_{i=1}^{N}\sum_{j=1}^{N} (b_i + x_i) q_{ij} (b_j + x_j)\right)\left(\sum_{i=1}^{N} |b_i + x_i| - 1\right)^2. \tag{6}$$

Equation (6) is the utility function to be maximized. In practice, the utility function in Equation (6) is more difficult to optimize than that in Equation (1) because Λ is a function of the x_is, so the leverage risk term requires powers up to and including the fourth order in the x_i terms. We show below a method to solve for optimal portfolios using this utility function.

Optimal Portfolios with Leverage Aversion Based on Portfolio Total Volatility

To examine the effects of leverage aversion using this new specification, we used the enhanced active equity (EAE) portfolio structure, as in Jacobs and Levy (2012). An EAE portfolio has 100% exposure to a underlying market benchmark while permitting short sales equal to some percentage of capital and use of the short-sale proceeds to buy additional long positions. For expository purposes, we assumed the strategy is self-financing and entails no financing costs.[3] An enhanced active 130–30 portfolio, for instance, has leverage of 60% and an enhancement of 30%.

We found EAE portfolios that maximize the utility function represented by Equation (6) for a range of volatility and leverage tolerance pairs (τ_V, τ_L),

subject to standard constraints. The standard constraint set for an EAE portfolio is

$$\sum_{i=1}^{N} x_i = 0 \tag{7}$$

and

$$\sum_{i=1}^{N} x_i \beta_i = 0. \tag{8}$$

Equation (7) says the sum of security active underweights relative to benchmark (including short positions) equals the sum of security active overweights—the full investment constraint. Equation (8) says that the sum of the product of security active weights and security betas equals zero; that is, the net (long-short) portfolio beta equals the benchmark beta. In addition to these standard constraints, we constrained each security's active weight to be between -10% and +10%.

We maximized the utility function in Equation (6) using a fixed-point iteration. To explain this procedure, we rewrite Equation (6) as the following set of two equations:

$$U = \sum_{i=1}^{N} \alpha_i x_i - \frac{1}{2\tau_V} \sum_{i=1}^{N} \sum_{j=1}^{N} x_i \sigma_{ij} x_j - \frac{1}{2\tau_L} \sigma_T^2 \left(\sum_{i=1}^{N} |b_i + x_i| - 1 \right)^2$$
$$\sigma_T^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} (b_i + x_i) q_{ij} (b_j + x_j). \tag{9}$$

We chose an initial estimate of σ_T^2, and used this as a constant to maximize the utility function in Equation set (9). This maximization provided estimates of the x_is, which were used to compute a new estimate of σ_T^2 using the second equation in Equation set (9). With the new estimate of σ_T^2, we repeated the optimization to find new estimates of the x_is. This iteration was repeated until successive estimates of σ_T^2 differed by a *de minimis* amount.

Using the same data (for stocks in the S&P 100 Index) and estimation procedures used in Jacobs and Levy (2012), and the same range of leverage and volatility tolerances, we derived the enhancement surface for the optimal levels of portfolio leverage using the new specification of leverage aversion. The optimal levels of enhancement were slightly higher than, but substantially similar to, those of the earlier specification. The appendix explains the reasons for the small differences in the optimal levels of enhancement between the two specifications.

Efficient Frontiers with and without Leverage Aversion

Figures 1 and 2 illustrate, in a two-dimensional volatility risk-return framework, how consideration of leverage aversion can affect the investor's choice of optimal portfolio. Figure 1, for example, plots the efficient frontiers (the optimal portfolios) for four cases. The frontiers are computed as discussed in the previous section, with the fourth frontier computed without the 10% constraint on active security weights. The portfolios on these frontiers offer the highest expected active return at each given level of volatility (whether measured as variance or as standard deviation of active return). The frontier in each separate chart is mapped out by varying the level of volatility tolerance from zero to two while holding the level of leverage tolerance constant.

In all the cases illustrated in Figure 1, the efficient frontier begins at the origin, which corresponds to the optimal portfolio when volatility tolerance is zero. In such a situation, the investor cannot tolerate any active volatility, so the optimal portfolio is an index fund, which provides zero standard deviation of active return and thus zero expected active return. As the investor's volatility tolerance rises, the optimal portfolio moves out along the efficient frontier.

The first panel of the figure illustrates the efficient frontier when leverage tolerance is 0, meaning the investor is unwilling, or unable, to use leverage,

hence holds a long-only portfolio. As the investor's tolerance for volatility increases, the optimal portfolio moves out along the frontier, taking on higher levels of standard deviation of active return in order to earn higher levels of expected active return. These portfolios take more concentrated positions in higher-expected-return securities as volatility tolerance increases. The efficient frontier when leverage (including shorting) is not permitted can be derived from either a conventional mean-variance optimization or from a mean-variance-leverage optimization with zero tolerance for leverage. As noted on the figure, every portfolio along the frontier is a "100-0" portfolio, meaning it is invested 100% long, with no short positions.

The second panel illustrates the efficient frontier when the investor's leverage tolerance is 1. It is derived from a mean-variance-leverage optimization, where leverage entails a disutility, as specified in Equation (4). Again, the investor with no tolerance for volatility risk would hold the index fund located at the origin. But as investor tolerance for volatility increases, the optimal portfolio moves out along the efficient frontier, achieving higher levels of expected return with higher levels of volatility.
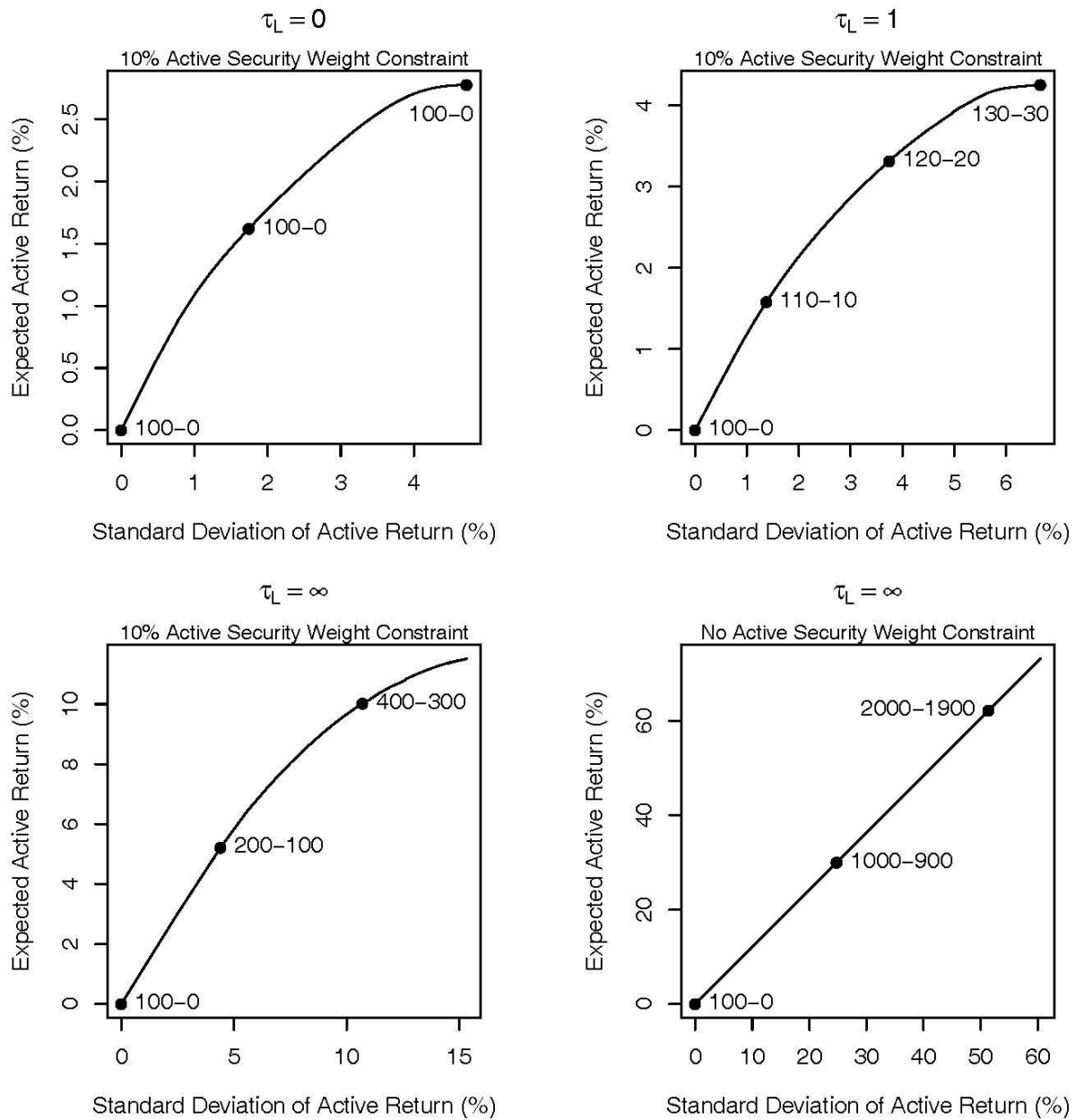
As the plot indicates, increasing volatility is accompanied by increasing leverage. The optimal portfolio ranges from a 100-0 long-only portfolio to a 130-30 enhanced active portfolio. For the investor with a leverage tolerance of 1, any of these portfolios can be optimal, depending on volatility tolerance. Higher risk-return portfolios can be achieved with less concentration of positions when leverage is allowed than when leverage is not allowed.

The third panel illustrates the efficient frontier for an investor with infinite leverage tolerance. As discussed earlier, mean-variance optimization implicitly assumes an infinite tolerance for leverage—that is, no aversion to leverage—so it provides the same result as mean-variance-leverage optimization with infinite leverage tolerance. In this case, as the investor's volatility tolerance increases, the optimal portfolio goes from zero leverage to

enhanced active portfolios of 200-100 to 400-300, etc. For the investor with infinite leverage tolerance, leverage does not give rise to any disutility, so this investor takes on much more leverage than the investors in the prior examples and achieves higher expected returns along with higher standard deviations of return, albeit with increasing leverage risk.

The last panel is identical to the third, except that it removes the 10% constraint on individual security active weights. Because there is no disutility to leverage, and no constraint on individual position sizes, the optimal portfolios all hold the same proportionate active security weights but apply increasing levels of leverage as volatility tolerance increases. Because each portfolio is just a levered version of the same set of active positions, and because we have assumed the EAE structure provides costless self-financing (i.e., short proceeds are used to finance additional long positions), the efficient frontier is simply a straight line. In this case, ever higher levels of leverage are used to achieve ever higher expected returns along with higher standard deviations of return. As with the third panel, the same efficient frontier is derived whether the investor uses conventional mean-variance optimization or mean-variance-leverage optimization, since no disutility is associated with leverage.

Figure 1. Optimal Leverage for Various Leverage Tolerance (τ_L) Cases

$\tau_L = 0$
10% Active Security Weight Constraint

$\tau_L = 1$
10% Active Security Weight Constraint

$\tau_L = \infty$
10% Active Security Weight Constraint

$\tau_L = \infty$
No Active Security Weight Constraint

Efficient Frontiers for Various Leverage Tolerance Cases

Figure 2 displays five different efficient frontiers on one chart. Each frontier corresponds to a different level of leverage tolerance within the leverage tolerance range of 0 to 2. Here zero leverage tolerance again represents an

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investor unwilling or unable to use leverage, and higher efficient frontiers correspond to investors with greater tolerances for leverage.

It might at first appear from the figure that the highest level of leverage tolerance results in the dominant efficient frontier; that is, the higher leverage that results allows the investor to achieve higher returns at any given level of volatility.[4] But one must consider the leverage tolerance levels associated with each efficient frontier. When leverage aversion is considered, it becomes apparent that each frontier consists of the set of optimal portfolios for an investor with the given level of leverage tolerance.

For example, consider the three portfolios represented by the points labeled A, B and C in Figure 2 (whose characteristics are provided in Table 1). Portfolio A is the optimal portfolio for an investor with a leverage tolerance of 1 and a volatility tolerance of 0.24. This is a 125-25 portfolio with a standard deviation of active return of 5% and an expected active return of about 3.93%.

Portfolio B, by contrast, is the optimal portfolio with the same standard deviation as portfolio A, but corresponding to an investor leverage tolerance of 2, rather than 1. Portfolio B dominates portfolio A in an expected active return-standard deviation framework, because it offers a higher expected return at the same level of standard deviation. But it is only optimal for an investor with a leverage tolerance of 2 and volatility tolerance of 0.14; it is suboptimal for an investor with a leverage tolerance of 1. Portfolio B represents a 140-40 enhanced active portfolio; it entails significantly more leverage than the 125-25 portfolio at point A. The disutility of incurring that additional leverage more than offsets the benefit of the incremental expected return for the investor with less tolerance for leverage.

Finally, consider portfolio C, which is the optimal portfolio with an expected active return of about 3.93% (the same as portfolio A) for an investor with a leverage tolerance of 2. This portfolio also dominates portfolio A in an active return-standard deviation framework, because it offers the same

expected return at a lower standard deviation. While portfolio C is optimal for an investor with a leverage tolerance of 2 and a volatility tolerance of 0.09, it is suboptimal for an investor with a leverage tolerance of 1 for the same reason that portfolio B is suboptimal: it entails more leverage than portfolio A, 135-35 versus 125-25. Again, the disutility of the additional leverage more than offsets the benefit of the lower volatility for the investor with less tolerance for leverage.

Figure 2 demonstrates that conventional mean-variance optimization and efficient frontier analysis are inadequate to identify truly optimal portfolios when investors use leverage and are averse to leverage risk. Rather, the efficient frontier differs for investors with different tolerances for leverage, and mean-variance-leverage optimization must be used to solve for optimal portfolios.

For each of the five efficient frontiers in Figure 2, volatility tolerance ranges from 0 (the origin) to 2 (the rightmost point on each frontier). A curve connecting these endpoints would identify portfolios optimal for investors with a volatility tolerance of two and leverage tolerances ranging from zero to two. For such investors, the optimal portfolio along this curve will depend on the leverage tolerance of each particular investor. (Note that, because different security active weight constraints become binding as one moves along each of the constant leverage tolerance frontiers, a curve connecting the endpoints would not be smooth.)

With volatility tolerance of 0 and any level of leverage tolerance, an investor would choose the portfolio located at the origin—an index fund. With leverage tolerance of 0, an investor would choose from the lowest frontier shown the portfolio congruent with the investor's volatility tolerance. With a leverage tolerance of 2, an investor would choose from the highest frontier shown the portfolio congruent with the investor's volatility tolerance. The optimal portfolio for an investor with any pair of leverage tolerance and

volatility tolerance values between 0 and 2 will lie somewhere within the perimeter defined by the leverage and volatility tolerance frontiers of 0 and 2. Both volatility tolerance *and* leverage tolerance must be specified to determine the optimal portfolio for a given investor.

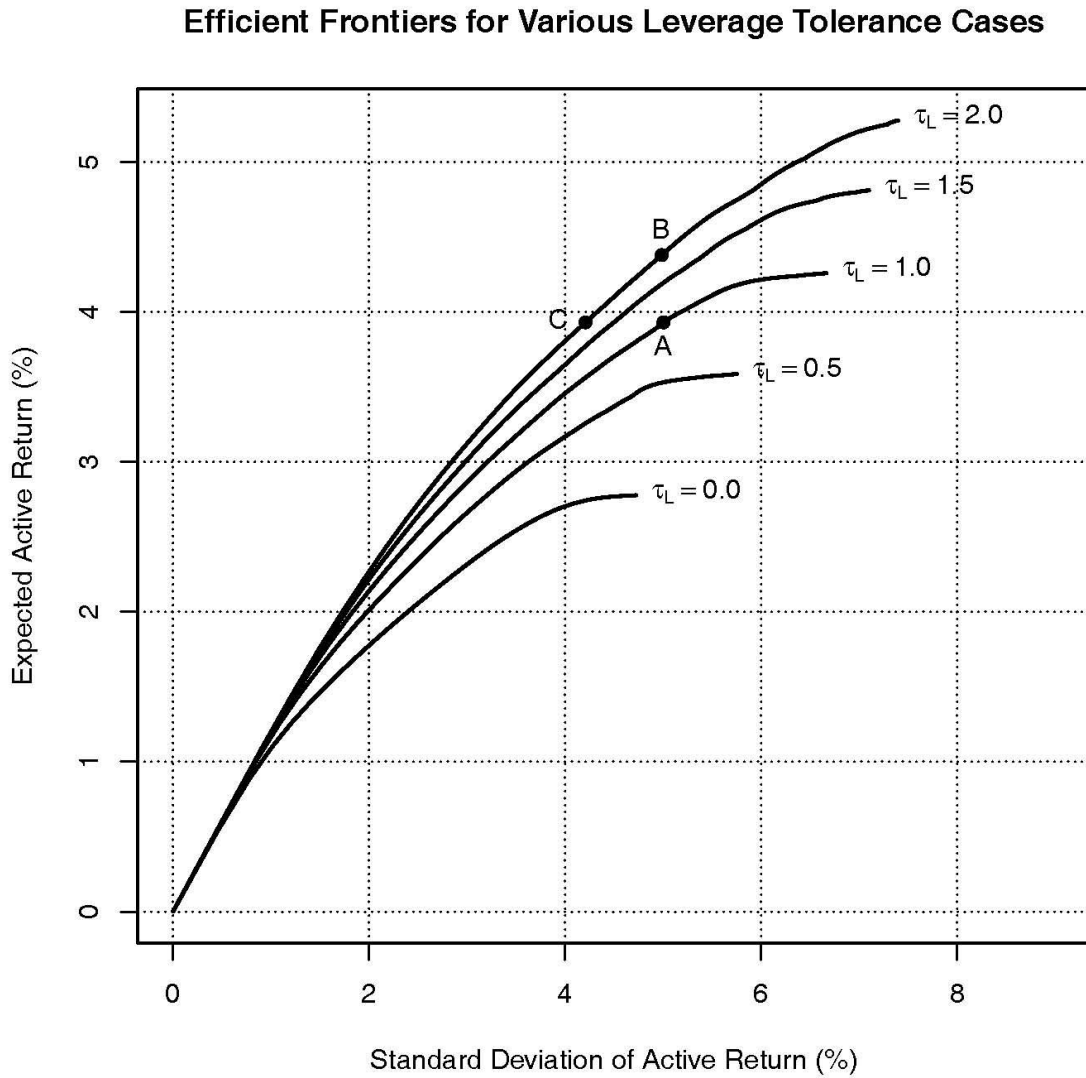**Figure 2. Efficient Frontiers for Various Leverage Tolerance (τ_L) Cases**

Efficient Frontiers for Various Leverage Tolerance Cases

Table 1. Portfolio Characteristics

	τ_L	τ_V	EAE%*	σ_P	α_P
A	1.00	0.24	125-25	5.00	3.93
B	2.00	0.14	140-40	5.00	4.39
C	2.00	0.09	135-35	4.21	3.93

* Rounded to the nearest percent.

The Efficient Region

Figure 3 illustrates the efficient frontiers for various combinations of leverage and volatility tolerance when there is no constraint on the security active weights. Here the curve linking the optimal portfolios for an investor with a volatility tolerance of 2 is smooth (unlike in Figure 2). Furthermore, both the standard deviations of active return and the expected active returns range higher than in Figure 2. Figure 3 also shows efficient frontiers for investors with volatility tolerances of 1.5, 1.0, 0.5, 0.2, 0.1, and 0.05. As volatility tolerance declines from 2, the frontiers shift to the left and downward. In the limit, when volatility tolerance is 0, the optimal portfolio lies at the origin (an index fund). Depending on the investor's leverage and volatility tolerances, the optimal portfolio will lie somewhere in the two-dimensional efficient region shown. Once again, the critical roles of both leverage and volatility tolerance are apparent.

Figure 3. Efficient Frontiers for Various Leverage (τ_L) and Volatility (τ_V) Tolerance Cases with No Security Active Weight Constraint

Efficient Frontiers for Various Leverage and Volatility Tolerance Cases
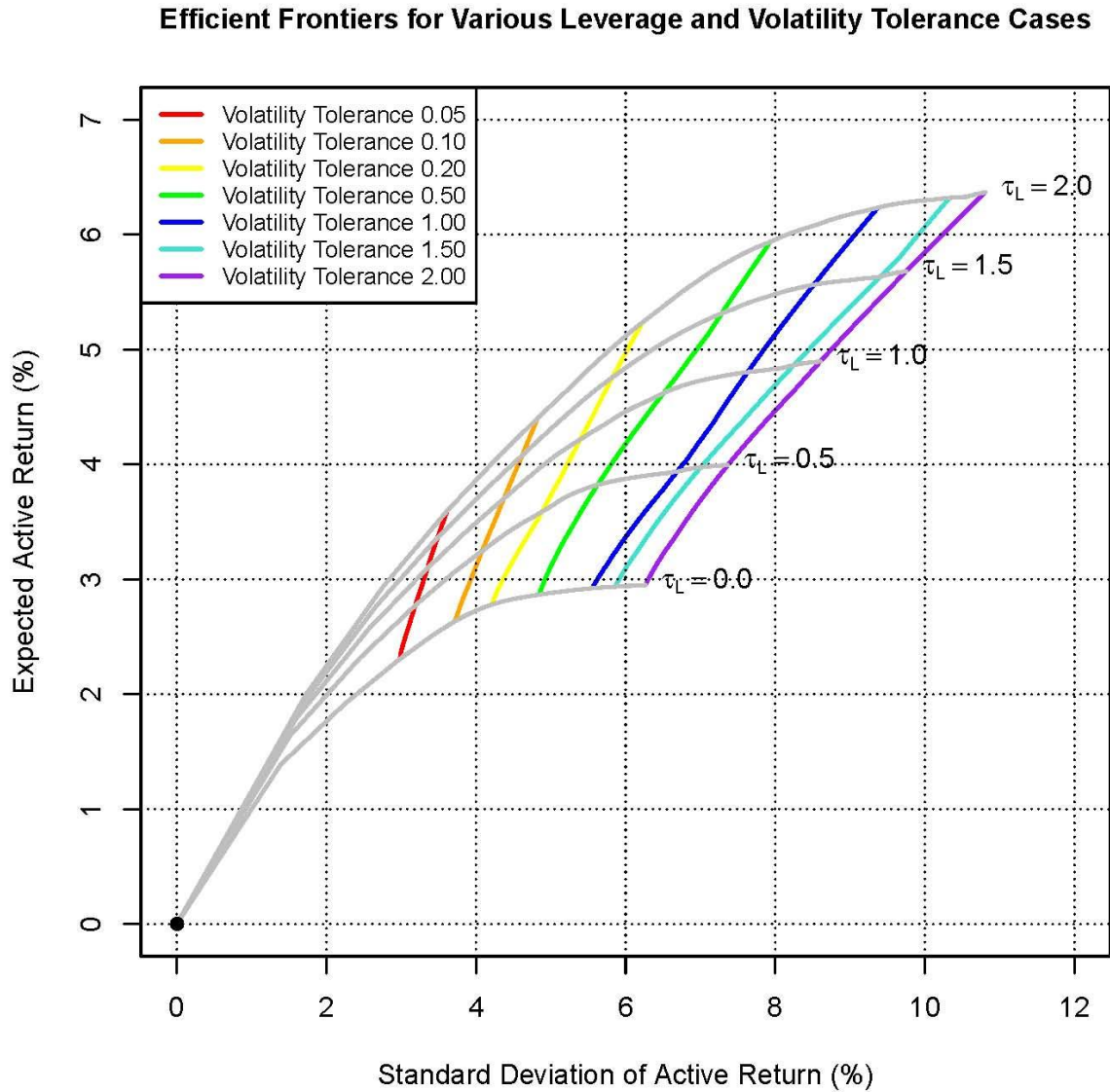
Conclusion

Conventional portfolio theory and mean-variance optimization need to be augmented to incorporate leverage aversion. We propose that a leverage aversion term using the variance of the portfolio's total return be incorporated in the investor's utility function. We use this specification to show the effects of leverage aversion on the efficient frontier.

Conventional mean-variance optimization considers only portfolio expected return and risk as measured by portfolio volatility. It assumes that investors have infinite tolerance for leverage. In a mean-variance framework, highly leveraged portfolios are preferred because they offer the highest expected active return at each level of active risk.

Leverage, however, entails its own unique set of risks distinct from the risks posed by volatility; these include the risk of margin calls and forced liquidations (possibly at adverse prices), losses beyond the capital invested, and the risks and costs of bankruptcy. Investors, in addition to being volatility-averse, are leverage-averse. They do not have an infinite tolerance for leverage. The highly leveraged portfolios that result from conventional mean-variance optimization entail too much leverage risk for leverage-averse investors.

We show that, when leverage aversion is included in portfolio optimization, lower mean-variance-leverage efficient frontiers with less leverage are optimal. The frontier that is optimal for a particular investor depends upon that investor's leverage tolerance. The optimal portfolio on that frontier for that investor depends upon that investor's volatility tolerance.

We develop a two-dimensional mean-variance-leverage efficient region. The location of a given investor's optimal portfolio within that region depends on the investor's leverage and volatility tolerances. The critical roles of both leverage tolerance and volatility tolerance are apparent.

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Appendix: Comparison of the Enhancement Surfaces Using Two Different Specifications

As in Jacobs and Levy (2012), we chose 100×100 pairs of values for the tolerances (τ_V, τ_L) to cover the illustrative range [0.001, 2] for a total of 10,000 optimizations. Tolerances for volatility and leverage can be greater than 2, and as leverage tolerance approaches infinity, the optimal portfolio approaches that determined by a conventional mean-variance utility function.

To estimate the required inputs for Equation (6)—security expected active returns, covariances of security active returns, and covariances of security total returns—we used daily return data for the constituent stocks in the S&P 100 Index over the two years (505 trading days) ending on 30 September 2011. For estimating security expected active returns, we used a random transformation of actual active returns while maintaining a skill, or information coefficient (correlation between predicted and actual active returns), of 0.1, representing a manager with strong insight. For a description of the estimation procedure used, see Jacobs and Levy (2012). We assumed the future covariances were known, so we calculated them based on the actual daily active returns and the actual daily total returns respectively.

The results from this specification were broadly similar to the results from using the specification in Jacobs and Levy (2012), which used a constant based on an average of individual securities' active return variances rather than the total variance of individual portfolios. At zero leverage tolerance, the optimal portfolios lie along the volatility tolerance axis and have no leverage and hence no enhancement ("long-only"). At zero volatility tolerance, the portfolios lie along the leverage tolerance axis and have no active return volatility and hence hold benchmark weights in each security ("index fund"). For portfolios above the axes, optimal enhancement is approximately independent of volatility tolerance if the latter is large enough. However, the optimal enhancement is highly dependent on the level of leverage tolerance

chosen. This supports our assertion that leverage tolerance should be considered when selecting an optimal portfolio.

The optimal enhancements using the new specification are slightly higher (by less than 5 percentage points) than those derived under the old specification. This is not surprising giving the relationship between the two specifications. Note that the utility function represented by Equation (4) is equivalent to that of Equation (1) if one multiplies the leverage risk term of Equation (1) by the ratio:

$$R = \frac{\sigma_T^2}{c}. \tag{A1}$$

Using the expression for the variance of the portfolio's total return from Equation (5) and also Equation (3), Equation (A1) can be rewritten as:

$$R = \frac{\sum_{i=1}^{N}\sum_{j=1}^{N} h_i q_{ij} h_j}{\frac{1}{N}\sum_{i=1}^{N} \omega_i^2}. \tag{A2}$$

This expression is the ratio of the portfolio total return variance to the average (across all securities in the selection universe) of the variance of each stock's active returns. Calculating Equation (A2) across the 10,000 optimal portfolios found by using the same constraint set for an enhanced active equity (EAE) portfolio and the same sample of S&P 100 stocks as in Jacobs and Levy (2012), we found $R \approx 0.85$.

As might be expected with a ratio close to 1, the results from optimization using Equation (4) were similar to those from using Equation (1). The major difference is that the new specification indicates that slightly more leverage is optimal than in the earlier specification, within the risk tolerance ranges examined. This is because the ratio R is less than 1, implying a lower penalty for leverage risk in Equation (4) than in Equation (1).

It is difficult to draw general conclusions from this comparison, however, because *R* will vary depending upon the portfolio structure, the level of the enhancement, the sample data, etc. In particular, the optimal portfolios in Jacobs and Levy (2012) derive from a constant estimated from the average of individual securities' active return variances. The results reflected in this article rely on the total return variance of a diversified portfolio. Since total return variance is larger than active return variance, this will raise R, while portfolio diversification effects will lower R. The net effect depends on the particular situation, so R may be greater than or less than one.

Endnotes

[1] The use of σ_P^2 as the measure of volatility risk assumes that active returns are normally distributed and that the investor is averse to the variance of active returns. If the return distribution is not normal, displaying skewness or kurtosis ("fat tails") for instance, or the investor is averse to downside risk (semi-variance) or value at risk (VaR), the conclusions of this article still hold. That is, the investor should include a leverage aversion term in the utility function, along with the appropriate measure of volatility risk, with neither risk term necessarily assuming normality.

      Leverage may give rise to fatter tails in active returns. For example, a drop in a stock's price may trigger margin calls, which may result in additional selling, while an increase in a stock's price may lead investors to cover short positions, which can make the stock's price rise even more.

      Also, note that if volatility risk is measured as the variance of total returns (such as for an absolute return strategy) rather than the variance of active returns, the conclusions of this article still hold.

[2] When the investor's leverage tolerance is zero, portfolio leverage, Λ, will be zero. Note that since short positions entail unlimited liability, they, like leveraged long positions, expose the portfolio to losses beyond the invested capital. Hence, investors with zero leverage tolerance would impose a non-negativity constraint on the h_is--that is, a no shorting constraint.

[3] In practice there would be financing costs (such as stock loan fees); furthermore, hard-to-borrow stocks may entail higher fees. For more on EAE portfolios, see Jacobs and Levy (2007).

[4] Note that the expected active returns shown do not reflect any costs associated with leverage related events, such as forced liquidation at adverse prices or bankruptcy. These costs, however, are reflected in the disutility implied by the leverage aversion term.