Lévy-driven CARMA Processes

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3rd Workshop Analysis, Geometry and Probability
Ulm University
September 29th, 2015

Based on joint works with Tina Marquardt, Eckhard Schlemm, Florian Fuchs and Żywilla Fechner

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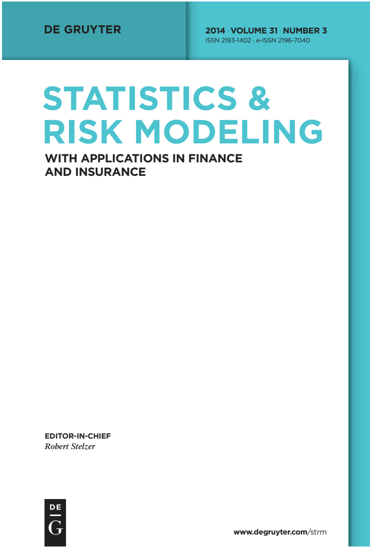

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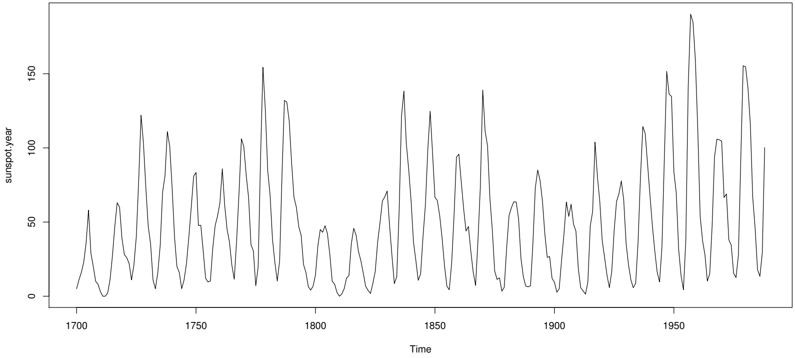

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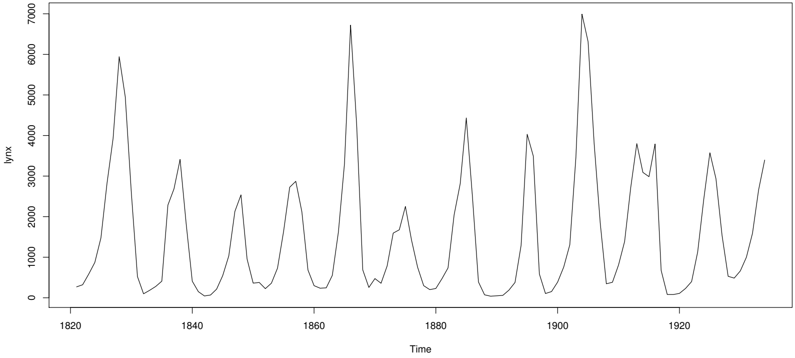
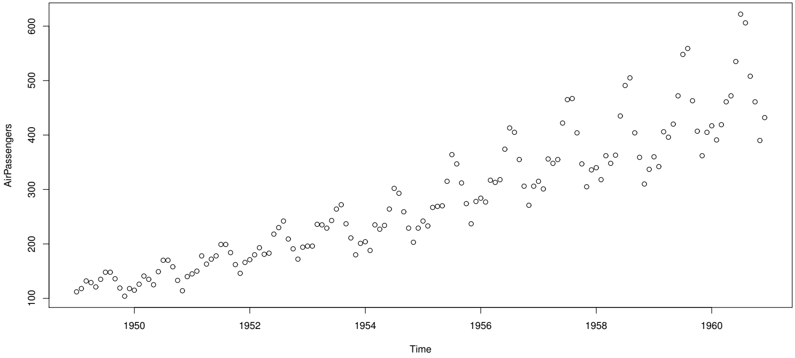
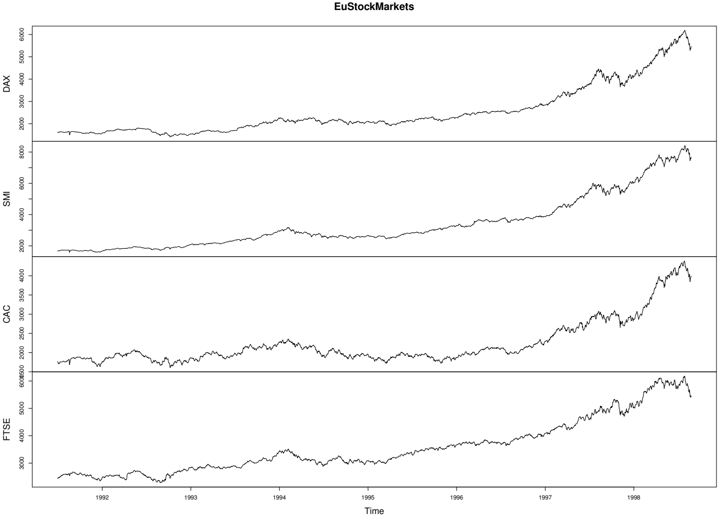


Time Series: Air Passengers



Time Series: European Stock Market Indices



ARMA, VARMA, (M)CARMA

Versatile class of auto-regressive moving-average processes

$$X_n - \varphi_1 X_{n-1} - \ldots - \varphi_p X_{n-p} = \varepsilon_n + \theta_1 \varepsilon_{n-1} + \ldots + \theta_q \varepsilon_{n-q}$$

Extensions to

- multivariate models (Vector ARMA)
- continuous-time models (CARMA), e.g. various papers by Brockwell, Lindner & Co.

Now multivariate CARMA.

Advantages:

- Modelling of dependent time series
- Allow handling of irregularly spaced data and missing observations (thus suitable for high-frequency data).
- Allows consistent estimation and inference at different frequencies

Problem: Definition, Properties and Estimation

Multivariate CARMA processes

$\mathbb{R}^m$-valued Lévy process $\boldsymbol{L}$ satisfying $\mathbb{E}||\boldsymbol{L}(1)||^2 < \infty$.

An $\mathbb{R}^d$-valued second-order MCARMA(p,q) process solves

$$P(D)\boldsymbol{Y}(t) = Q(D)D\boldsymbol{L}(t), \quad D \equiv \frac{d}{dt}.$$

Auto-regressive polynomial

$$P(z) := \mathbf{1}z^p + A_1 z^{p-1} \ldots + A_p \in M_d(\mathbb{R}[z])$$

Moving-average polynomial

$$Q(z) := B_0 z^q + B_1 z^{q-1} \ldots + B_q \in M_{d,m}(\mathbb{R}[z])$$

In the univariate Gaussian case first considered by Doob (1944)

Lévy processes

- Examples: Brownian motion, Poisson process, α-stable motions (Lévy flights)
- The natural tractable class of stochastic processes including the above examples
- Continuous time processes with stationary independent increments, i.e. $L(t_1) - L(t_0), L(t_2) - L(t_1), \ldots, L(t_n) - L(t_{n-1})$ are independent for all $0 \leq t_0 < t_1 < \ldots < t_n$ and identical in law for an equally spaced grid, which are continuous in probability
- Characterised by the Lévy-Khintchine triplet:
  - the drift γ_L
  - the Brownian covariance matrix C_L
  - and the Lévy measure ν_L describing the jumps
- Can be represented as the sum of three independent components:
  - a deterministic linear function
  - a Brownian motion
  - a "pure jump part"

Why go beyond Brownian motion?

- Usage of non-Gaussian Lévy processes allows to reproduce certain features of the data:
  - Heavy tails (extreme events occur more frequently than under a normal distribution)
  - jumps (abrupt changes)
  - positivity
- Models are to simplify reality. General Lévy processes may lead to better linear models of (moderately) non-linear processes.

Making sense of the differential equations: Spectral representation of Lévy processes

Theorem (Marquardt and St. (2007))

Let L be a square integrable m-dimensional Lévy process with mean $E[L(1)] = 0$ and variance $E[L(1)L(1)^*] = \Sigma_L$. Then there exists a unique m-dimensional random orthogonal measure Φ_L with spectral measure F_L such that $E[\Phi_L(\Delta)] = 0$ for any bounded Borel set Δ, $F_L(dt) = \frac{\Sigma_L}{2\pi} dt$ and

$$L(t) = \int_{-\infty}^{\infty} \frac{e^{i\mu t} - 1}{i\mu} \Phi_L(d\mu), \ t \in \mathbb{R}.$$

$$\Longrightarrow \quad DL(t) = \int_{-\infty}^{\infty} e^{i\mu t} \Phi_L(d\mu)$$

Multivariate CARMA processes – Definition

Interpreting differentiation as linear filtering in the spectral domain, we obtain “$Y(t) = P(D)^{-1}Q(D)DL(t)$”:

Definition (Marquardt and St. (2007))

Let L be a square integrable m-dimensional Lévy-process with $E[L(1)] = 0$ and associated random measure Φ_L and $p, q \in \mathbb{N}_0$ with $p > q$. Set

$$P(z) := I_d z^p + A_1 z^{p-1} + ... + A_p,\ Q(z) := B_0 z^q + B_1 z^{q-1} + ... + B_q.$$

Assume $B_q \neq 0$ and $\mathcal{N}(P) := \{z \in \mathbb{C} : det(P(z)) = 0\} \subset \mathbb{R}\backslash\{0\} + i\mathbb{R}$. Then the stationary process Y defined as

$$Y(t) = \int_{-\infty}^{\infty} e^{i\mu t} P(i\mu)^{-1} Q(i\mu)\, \Phi_L(d\mu),\ t \in \mathbb{R}$$

is called d-dimensional Lévy-driven *MCARMA(p, q) process*.

An SDE representation

Theorem (Marquardt and St. (2007))

Let the Lévy process L and P, Q be as before. Define:

- $\beta_{p-j} = -\sum_{i=1}^{p-j-1} A_i \beta_{p-j-i} + B_{q-j}$, $j = 0, 1, \ldots, q$, $\beta_1 = \ldots = \beta_{p-q-1} = 0$
- $\boldsymbol{\beta}^* = (\beta_1^*, \beta_2^*, \ldots, \beta_p^*)$ *and* $\mathbf{A} = \left(\begin{array}{c|ccc} 0 & & I_{d(p-1)} & \\ \hline -A_p & -A_{p-1} & \ldots & -A_1 \end{array}\right)$.

Denote by $G(t) = (G_1(t)^*, \ldots, G_p(t)^*)^*$ *a* $p \times d$*-dimensional process and assume that* $\mathcal{N}(P) := \{z \in \mathbb{C} : det(P(z)) = 0\} \subset (-\infty, 0) + i\mathbb{R}$*. Then* $dG(t) = \mathbf{A}G(t)dt + \boldsymbol{\beta} dL(t)$ *has a unique stationary solution G given by*

$$G(t) = \int_{-\infty}^{t} e^{\mathbf{A}(t-s)} \boldsymbol{\beta}\, L(ds), \quad t \in \mathbb{R}.$$

An SDE representation

Theorem (continued)

It holds that

$$G_1(t) = \int_{-\infty}^{\infty} e^{i\mu t} P(i\mu)^{-1} Q(i\mu) \Phi_L(d\mu) = Y(t), \; t \in \mathbb{R}.$$

So the first d-components are the MCARMA process Y.

- An MCARMA process satisfying $\mathcal{N}(P) := \{z \in \mathbb{C} : \det(P(z)) = 0\} \subset (-\infty, 0) + i\mathbb{R}$ is called **causal**.
- Causal stationary MCARMA processes can be defined via the above state space representation (OU SDE) as soon as $E(\max(\ln(\|L_1\|, 1)) < \infty$.
- For regularly varying Lévy processes with finite mean one can again give a spectral representation in terms of a random content which allows to define also non-causal MCARMA processes with infinite second moment (see Fuchs and St. (2013b)).

Second order structure

Assume $E(L(1)L(1)^*) = \Sigma_L < \infty$.

- MCARMA process Y:
$$\mathbb{C}\mathrm{ov}(Y(t+h), Y(t)) = \int_{-\infty}^{\infty} \frac{e^{i\mu h}}{2\pi} P(i\mu)^{-1} Q(i\mu)\Sigma_L Q(i\mu)^* (P(i\mu)^{-1})^* \, d\mu,$$
$h \in \mathbb{R}$.
- State Space Representation G:
$$\begin{aligned} \mathrm{Var}(G(t)) &= \int_0^{\infty} e^{\mathbf{A}u} \beta \Sigma_L \beta^* e^{\mathbf{A}^* u} du \\ \mathbf{A}\mathrm{Var}(G(t)) + \mathrm{Var}(G(t))\mathbf{A}^* &= -\beta \Sigma_L \beta^* \\ \mathbb{C}\mathrm{ov}(G(t+h), G(t)) &= e^{\mathbf{A}h} \mathrm{Var}(G(t)), \; h \geq 0. \end{aligned}$$

Stationary distribution

If L has characteristic triplet (γ, σ, ν), then the stationary distribution of the state space representation G of a causal MCARMA process is infinitely divisible with characteristic triplet $(\gamma_G^\infty, \sigma_G^\infty, \nu_G^\infty)$, where

- $\gamma_G^\infty = \int_0^\infty e^{\mathbf{A}s}\beta\gamma\, ds + \int_0^\infty \int_{\mathbb{R}^d} e^{\mathbf{A}s}\beta x [I_{\{\|e^{\mathbf{A}s}\beta x\| \le 1\}} - I_{\{\|x\| \le 1\}}]\, \nu(dx)\, ds$
- $\sigma_G^\infty = \int_0^\infty e^{\mathbf{A}s}\beta\, \sigma \beta^* e^{\mathbf{A}^* s}\, ds,$
- $\nu_G^\infty(B) = \int_0^\infty \int_{\mathbb{R}^d} I_B(e^{\mathbf{A}s}\beta x)\, \nu(dx)\, ds.$

Projection on the first d coordinates gives the characteristic triplet of the stationary distribution of Y.
The result extends to higher dimensional marginal distributions and a similar result is possible for non-causal MCARMA processes.
(Essentially an application of results of Rajput and Rosinski (1989))

Existence of Moments

For causal MCARMA processes the existence of moments is determined by the driving Lévy process:

- If $E(\|L(1)\|^r) < \infty$ for some $r \in \mathbb{R}^+$, then$E(\|Y(t)\|^r), E(\|G(t)\|^r) < \infty$.
- If β is injective, then $E(\|G(t)\|^r) < \infty \Longrightarrow E(\|L(1)\|^r) < \infty$.
- A slightly weaker result holds for exponential moments.

Dependence structure

- **Markov properties:**
  As a solution to a stochastic differential equation, the state space representation G of a causal MCARMA process is a strong Markov process.
- **Mixing properties:**
  For a causal MCARMA process the state space representation G is β-mixing and Y is strongly mixing, both with exponentially decaying mixing coefficients.
  In particular, G and Y are ergodic (G is also geometrically ergodic).

  Every stationary MCARMA process is mixing (see Fuchs and St. (2013a)).

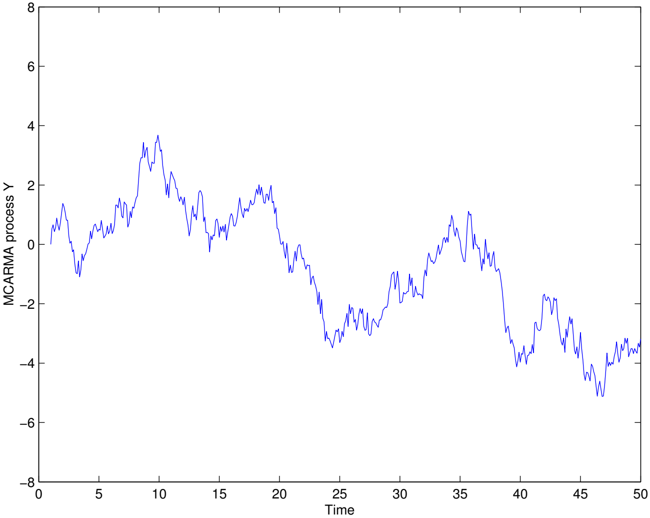
Sample path properties

- The sample paths of a (causal) MCARMA(p, q) process Y with $p > q + 1$ are $(p - q - 1)$-times differentiable with

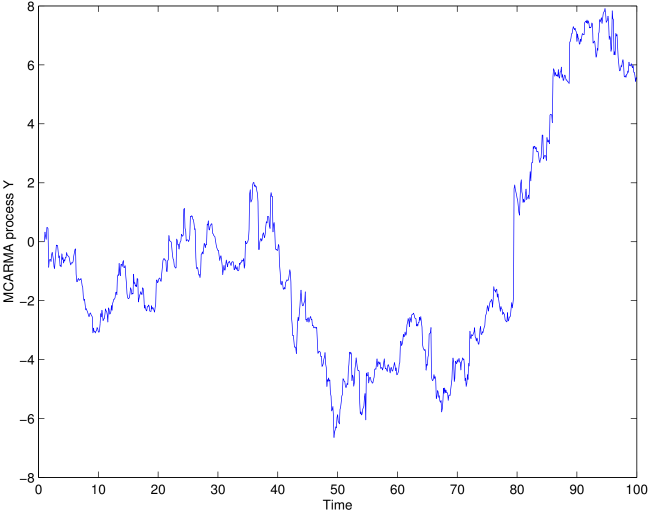
$$\frac{d^i}{dt^i} Y(t) = G_{i+1}(t), \quad i = 1, 2, \ldots, p - q - 1.$$

  (To be precise the $p - q - 1$-th derivative has to be understood in the sense of a unique càdlàg "density" in the case of jumps. Its $p - q - 1$-th derivative exists only a.e.)
- If $p = q + 1$, then $\Delta Y(t) = \beta_1 \Delta L(t) = B_0 \Delta L(t)$.
- If the driving Lévy process L is a Brownian motion, then the sample paths of Y are continuous and $(p - q - 1)$-times continuously differentiable, provided $p > q + 1$.

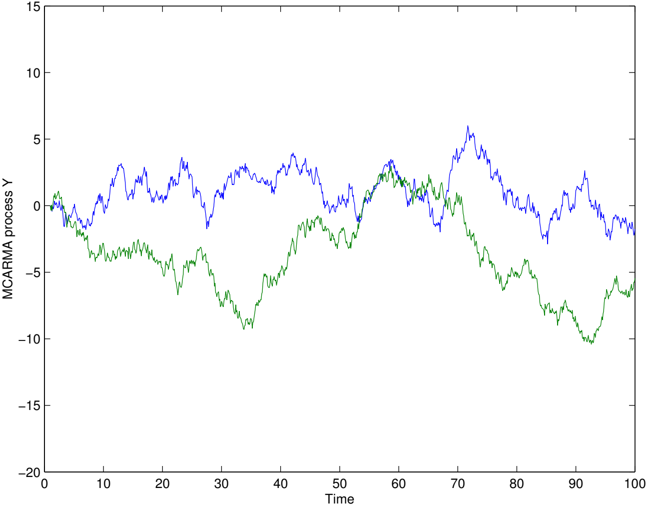
One dimensional Gaussian OU



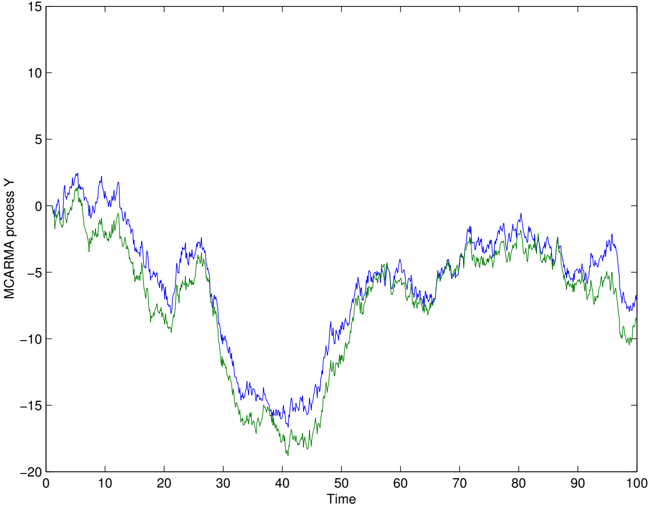
One dimensional NIG OU



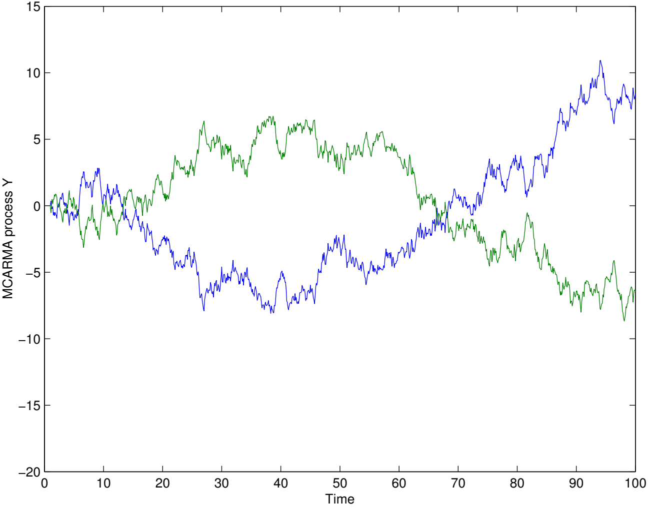
Two dimensional OU process with zero correlation



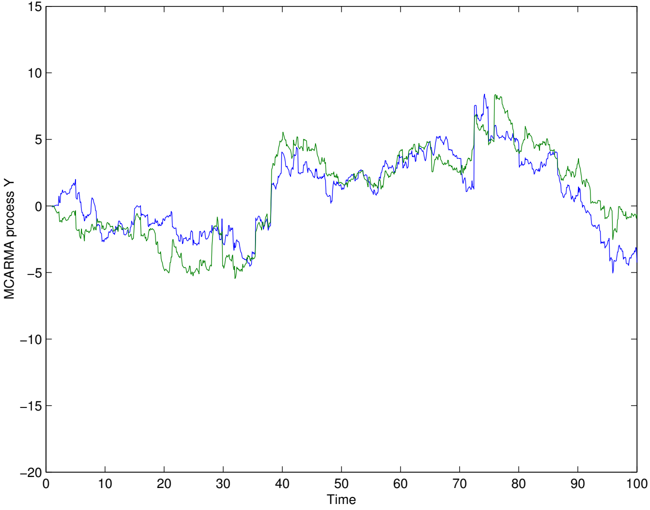
Two dimensional OU process with strong positive correlation



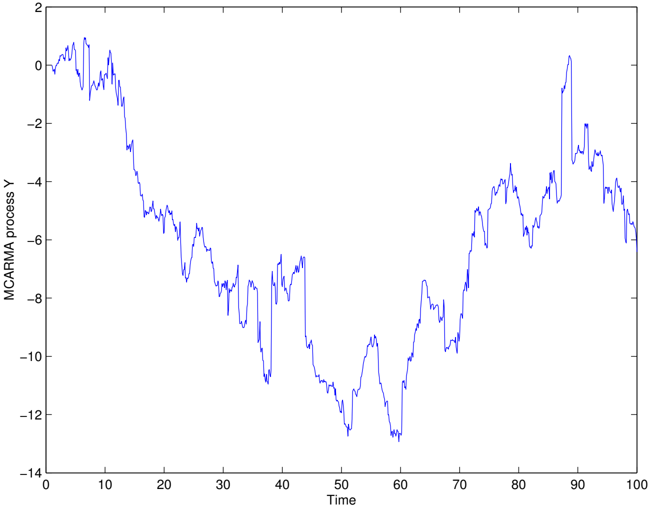
Two dimensional OU process with strong negative correlation



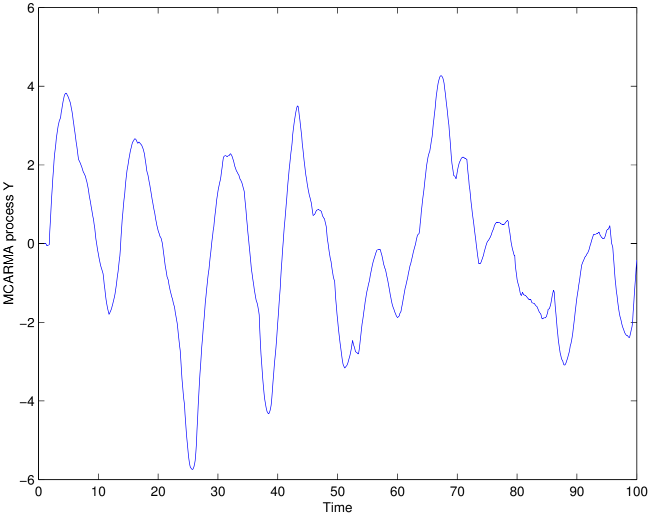
Two dimensional NIG OU process with strong positive correlation



NIG-driven CARMA(1,0)



NIG-driven CARMA(2,0)



Multivariate CARMA processes

Stationary solution to continuous-time state space model

$$\text{state equation} \quad d\boldsymbol{X}(t) = \mathbf{A}\boldsymbol{X}(t)dt + \beta d\boldsymbol{L}(t)$$
$$\text{observation equation} \quad \boldsymbol{Y}(t) = [\mathbf{1}, 0, \ldots, 0]\,\boldsymbol{X}(t),$$

$$\mathbf{A} = \begin{bmatrix} 0 & \mathbf{1} & 0 & \ldots & 0 \\ 0 & 0 & \mathbf{1} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & 0 & \mathbf{1} \\ -A_p & -A_{p-1} & \ldots & \ldots & -A_1 \end{bmatrix},$$

$$\beta = \begin{bmatrix} \beta_1^T & \cdots & \beta_p^T \end{bmatrix}^T, \quad \beta_{p-j} = -I_{[0:q]}(j)\left[\sum_{i=1}^{p-j-1} A_i\beta_{p-j-i} + B_{q-j}\right]$$

We only consider stationary causal MCARMA processes,i.e. $\sigma(\mathbf{A}) \subset (-\infty, 0) + i\mathbb{R}$.

State space models

General N-dimensional continuous-time state space model:

$$\begin{aligned} \text{state equation} \quad d\boldsymbol{X}(t) &= A\boldsymbol{X}(t)dt + Bd\boldsymbol{L}(t) \\ \text{observation equation} \quad \boldsymbol{Y}(t) &= C\boldsymbol{X}(t), \end{aligned}$$

$$A \in M_N^-(\mathbb{R}), \quad B \in M_{N,m}(\mathbb{R}), \quad C \in M_{d,N}(\mathbb{R}).$$

$\boldsymbol{X}$ satisfies

$$\boldsymbol{X}(t) = e^{A(t-s)}\boldsymbol{X}(s) + \int_s^t e^{A(t-u)}Bd\boldsymbol{L}(u).$$

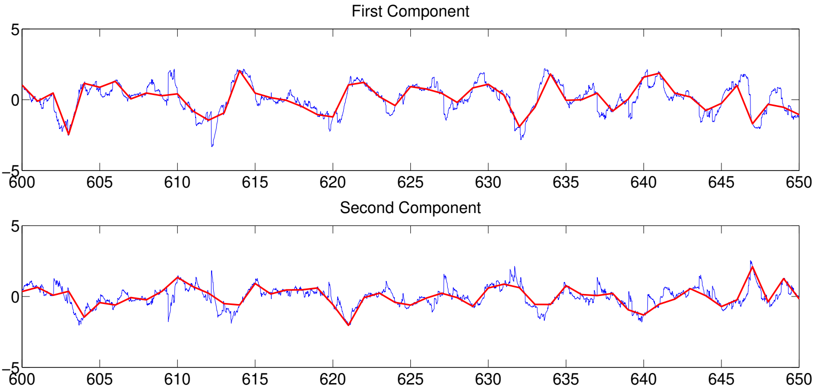
$$\boxed{\text{MCARMA}(P,Q) \Leftrightarrow C(zI_N - A)^{-1}B = P(z)^{-1}Q(z)}$$

Such P, Q exist for any state space model. (see Schlemm and St. (2012a))

Definition of the sampled process

We observe the process $\boldsymbol{Y}$ at discrete, equally spaced times

$$\boldsymbol{Y}_n^{(h)} := \boldsymbol{Y}(nh), \quad n \in \mathbb{Z}, \quad h > 0.$$



State-space representation of $\boldsymbol{Y}^{(h)}$

The sampled process $\boldsymbol{Y}^{(h)}$ satisfies the discrete-time state space model

$$\boldsymbol{X}_n^{(h)} = e^{Ah}\boldsymbol{X}_{n-1}^{(h)} + \boldsymbol{Z}_n,$$
$$\boldsymbol{Y}_n^{(h)} = C\boldsymbol{X}_n^{(h)},$$

where the i.i.d. sequence $(\boldsymbol{Z}_n)_{n\in\mathbb{Z}}$ is given by

$$\boldsymbol{Z}_n = \int_{(n-1)h}^{nh} e^{A(nh-u)} B d\boldsymbol{L}(u).$$

VARMA structure of $\boldsymbol{Y}^{(h)}$

Assume the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A are distinct. We define the polynomial

$$\varphi(z) = \prod_{\nu=1}^{N} \left[1 - e^{-\lambda_\nu h} z\right] \in \mathbb{C}[z].$$

Theorem

There exists a stable monic polynomial $\Theta \in M_d(\mathbb{C}[z])$ of degree at most $N-1$ such that

$$\varphi(B)\boldsymbol{Y}_n^{(h)} = \Theta(B)\varepsilon_n^{(h)}, \qquad B^j \boldsymbol{Y}_n^{(h)} = \boldsymbol{Y}_{n-j}^{(h)},$$

holds with $\varepsilon^{(h)}$ being white noise.

$\Rightarrow$ $\boldsymbol{Y}^{(h)}$ *is a weak* VARMA$(N, N-1)$ *process.*

Idea for the Estimation

- Use a Quasi-Maximum-Likelihood-Approach
- But: There is neither a precise asymptotic theory for the QML estimators for the relevant state space nor the relevant VARMA processes
- Thus: Develop a general theory for QML estimation of strongly mixing state space models
- QML estimation theory for multivariate CARMA processes is obtained as a special case
- Actually: MCARMA processes are parametrised via a state space form, because this leads to feasible identifiability conditions

Parametrisation

There is some parameter set $\Theta \subset \mathbb{R}^r$, and for each $\boldsymbol{\vartheta} \in \Theta$ one is given matrices $A_{\boldsymbol{\vartheta}}$, $B_{\boldsymbol{\vartheta}}$ and $C_{\boldsymbol{\vartheta}}$ of matching dimensions, as well as a Lévy process $\boldsymbol{L}_{\boldsymbol{\vartheta}}$.

$$\begin{aligned} \boldsymbol{X}_n^{(h)} &= e^{A_{\boldsymbol{\vartheta}} h} \boldsymbol{X}_{n-1}^{(h)} + \boldsymbol{Z}_n, \\ \boldsymbol{Y}_n^{(h)} &= C_{\boldsymbol{\vartheta}} \boldsymbol{X}_n^{(h)}, \end{aligned}$$

with

$$\boldsymbol{Z}_n = \int_{(n-1)h}^{nh} e^{A_{\boldsymbol{\vartheta}}(nh-u)} B_{\boldsymbol{\vartheta}} \, d\boldsymbol{L}_{\boldsymbol{\vartheta}}(u).$$

Assumptions I

Assumption (C1)

For each $\vartheta \in \Theta$, it holds that $E\boldsymbol{L}_\vartheta = 0_m$, that $E\left\|\boldsymbol{L}_\vartheta(1)\right\|^2$ is finite, and that the covariance matrix $\Sigma_\vartheta^{\boldsymbol{L}} = E\boldsymbol{L}_\vartheta(1)\boldsymbol{L}_\vartheta(1)^T$ is non-singular.

Assumption (C2)

For each $\vartheta \in \Theta$, the eigenvalues of A_ϑ have strictly negative real parts.

Assumption (C3)

For all $\vartheta \in \Theta$, the triplet $(A_\vartheta, B_\vartheta, C_\vartheta)$ is minimal with McMillan degree N.

Assumptions II

Assumption (C4)

The collection of output processes $K(\Theta) := (\boldsymbol{Y}_\vartheta, \vartheta \in \Theta)$ *corresponding to the state space models* $(A_\vartheta, B_\vartheta, C_\vartheta, \boldsymbol{L}_\vartheta)$ *is identifiable from the spectral density.*

Assumption (C5)

For all $\vartheta \in \Theta$*, the spectrum of* A_ϑ *is a subset of* $\{z \in \mathbb{C} : -\pi/h < \operatorname{Im} z < \pi/h\}$.

The QML estimator

$$\hat{\boldsymbol{\vartheta}}^{L,(h)} = \mathrm{argmin}_{\boldsymbol{\vartheta}\in\Theta}\widehat{\mathscr{L}}^{(h)}(\boldsymbol{\vartheta}, \boldsymbol{y}^{L,(h)}),$$

$$\widehat{\mathscr{L}}^{(h)}(\boldsymbol{\vartheta}, \boldsymbol{y}^{L,(h)}) = \sum_{n=1}^{L}\left[d\log 2\pi + \log\det V_{\boldsymbol{\vartheta}}^{(h)} + \hat{\varepsilon}_{\boldsymbol{\vartheta},n}^{(h)} V_{\boldsymbol{\vartheta}}^{(h),-1}\hat{\varepsilon}_{\boldsymbol{\vartheta},n}^{(h)}\right],$$

where $\hat{\varepsilon}_{\boldsymbol{\vartheta}}^{(h)}$ are the pseudo-innovations of the observed process $\boldsymbol{Y}^{(h)} = \boldsymbol{Y}_{\boldsymbol{\vartheta}_0}^{(h)}$, which are computed from the sample $\boldsymbol{y}^{L,(h)} = (\boldsymbol{Y}_1^{(h)}, \ldots, \boldsymbol{Y}_L^{(h)})$ using the Kalman filter and $V_{\boldsymbol{\vartheta}}^{(h)}$ are their (pseudo-)covariances.

Further Assumptions I

Assumption (C6)

The parameter space Θ *is a compact subset of* $\mathbb{R}^r$.

Assumption (C7)

The functions $\boldsymbol{\vartheta} \mapsto A_{\boldsymbol{\vartheta}}$, $\boldsymbol{\vartheta} \mapsto B_{\boldsymbol{\vartheta}}$, $\boldsymbol{\vartheta} \mapsto C_{\boldsymbol{\vartheta}}$, *and* $\boldsymbol{\vartheta} \mapsto \Sigma^{\boldsymbol{L}}_{\boldsymbol{\vartheta}}$ *are continuous. Moreover, for each* $\boldsymbol{\vartheta} \in \Theta$, *the matrix* $C_{\boldsymbol{\vartheta}}$ *has full rank.*

Assumption (C8)

The true parameter value $\boldsymbol{\vartheta}_0$ *is an element of the interior of* Θ.

Further Assumptions II

Assumption (C9)

The functions $\vartheta \mapsto A_{\vartheta}$, $\vartheta \mapsto B_{\vartheta}$, $\vartheta \mapsto C_{\vartheta}$, and $\vartheta \mapsto \Sigma_{\vartheta}^{\boldsymbol{L}}$ are three times continuously differentiable.

Assumption (C10)

There exists a positive number δ such that $E\left\|\boldsymbol{L}_{\vartheta_0}(1)\right\|^{4+\delta} < \infty$.

Assumption (C11)

There exists a positive index j_0 such that the $\left[(j_0+2)d^2\right] \times r$ matrix

$$\nabla_{\vartheta}\left(\left[\mathbf{1}_{j_0+1} \otimes K_{\vartheta}^{(h),T} \otimes C_{\vartheta}\right] \left[\left(\operatorname{vec} e^{\mathbf{1}_N h}\right)^T \quad \left(\operatorname{vec} e^{A_{\vartheta} h}\right)^T \quad \cdots \quad \left(\operatorname{vec} e^{A_{\vartheta}^{j_0} h}\right)^T \right]^T \underset{\operatorname{vec}V_{\vartheta}}{} \right)_{\vartheta=\vartheta_0}$$

has rank r.

QML - Asymptotic results

Theorem (Schlemm and St. (2012b))

$\mathbf{y}^{L,(h)} = (\mathbf{Y}^{(h)}_{\vartheta_0,1}, \ldots, \mathbf{Y}^{(h)}_{\vartheta_0,L})$ be a sample of length L from the discretely observed output process corresponding to the parameter value $\vartheta_0 \in \Theta$. Under (C1)-(C7) the QML estimator $\hat{\vartheta}^{L,(h)} = \mathrm{argmin}_{\vartheta\in\Theta}\widehat{\mathscr{L}}(\vartheta, \mathbf{y}^{L,(h)})$ is strongly consistent, i. e.

$$\hat{\vartheta}^{L,(h)} \xrightarrow[L\to\infty]{a.s.} \vartheta_0. \tag{1}$$

If, moreover, (C8)-(C11) hold, then $\hat{\vartheta}^{L,(h)}$ is asymptotically normally distributed, i. e. $\sqrt{L}\left(\hat{\vartheta}^{L,(h)} - \vartheta_0\right) \xrightarrow[L\to\infty]{d} \mathscr{N}(\mathbf{0}, \Xi)$, where the asymptotic covariance matrix $\Xi = J^{-1}IJ^{-1}$ is given by

$$I = \lim_{L\to\infty} L^{-1}\mathbb{V}\mathrm{ar}\left(\nabla_{\vartheta}\mathscr{L}\left(\vartheta_0, \mathbf{y}^L\right)\right), \quad J = \lim_{L\to\infty} L^{-1}\nabla^2_{\vartheta}\mathscr{L}\left(\vartheta_0, \mathbf{y}^L\right).$$

QML - Order Selection

Fasen and Kimmig (2015) have extended the results to order selection. They have among other results shown that using the QML estimators with the BIC gives consistent estimators.

Echelon state space parametrisations

- Integer $N > 0$ (McMillan degree)
- Non-negative structure indices $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_d)$ satisfying $\sum \nu_i = N$

Canonical parametrisation

$$\psi_{\boldsymbol{\nu}} : \mathbb{R}^{q(\boldsymbol{\nu})} \supset \Theta \ni \boldsymbol{\vartheta} \mapsto (A_{\boldsymbol{\vartheta}}, B_{\boldsymbol{\vartheta}}, C_{\boldsymbol{\vartheta}}, \boldsymbol{L}_{\boldsymbol{\vartheta}}), \quad A_{\boldsymbol{\vartheta}} \in M_N(\mathbb{R})$$

- Every MCARMA process is obtained for some $\boldsymbol{\nu}$.

Examples of canonical parametrisations

$\boldsymbol{\nu} = (1,1)$ (Ornstein-Uhlenbeck type process), 7 parameters:

$$A_{\boldsymbol{\vartheta}} = \begin{bmatrix} \vartheta_1 & \vartheta_2 \\ \vartheta_3 & \vartheta_4 \end{bmatrix}, \quad B_{\boldsymbol{\vartheta}} = \begin{bmatrix} \vartheta_1 & \vartheta_2 \\ \vartheta_3 & \vartheta_4 \end{bmatrix}, \quad C_{\boldsymbol{\vartheta}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\boldsymbol{\nu} = (1,2)$, 10 parameters (CARMA (2,1)):

$$A_{\boldsymbol{\vartheta}} = \begin{bmatrix} \vartheta_1 & \vartheta_2 & 0 \\ 0 & 0 & 1 \\ \vartheta_3 & \vartheta_4 & \vartheta_5 \end{bmatrix}, \quad B_{\boldsymbol{\vartheta}} = \begin{bmatrix} \vartheta_1 & \vartheta_2 \\ \vartheta_6 & \vartheta_7 \\ \vartheta_3 + \vartheta_5\vartheta_6 & \vartheta_4 + \vartheta_5\vartheta_7 \end{bmatrix}, \quad C_{\boldsymbol{\vartheta}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$\boldsymbol{\nu} = (2,1)$, 11 parameters (CARMA (2,1)):

$$A_{\boldsymbol{\vartheta}} = \begin{bmatrix} 0 & 1 & 1 \\ \vartheta_1 & \vartheta_2 & \vartheta_3 \\ \vartheta_4 & \vartheta_5 & \vartheta_6 \end{bmatrix}, \quad B_{\boldsymbol{\vartheta}} = \begin{bmatrix} \vartheta_7 & \vartheta_8 \\ \vartheta_1 + \vartheta_2\vartheta_7 & \vartheta_3 + \vartheta_2\vartheta_8 \\ \vartheta_4 + \vartheta_5\vartheta_7 & \vartheta_6 + \vartheta_5\vartheta_8 \end{bmatrix}, \quad C_{\boldsymbol{\vartheta}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boldsymbol{\nu} = (2,2)$, 15 parameters (CARMA (2,1)):

. . . . . .

Simulation Study I

Bivariate NIG-driven MCARMA process

$$\boldsymbol{X}(t) = \begin{bmatrix} \vartheta_1 & \vartheta_2 & 0 \\ 0 & 0 & 1 \\ \vartheta_3 & \vartheta_4 & \vartheta_5 \end{bmatrix} \boldsymbol{X}(t)dt + \begin{bmatrix} \vartheta_1 & \vartheta_2 \\ \vartheta_6 & \vartheta_7 \\ \vartheta_3 + \vartheta_5\vartheta_6 & \vartheta_4 + \vartheta_5\vartheta_7 \end{bmatrix} d\boldsymbol{L}(t),$$

$$\boldsymbol{Y}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \boldsymbol{X}(t), \quad \text{vech}\, \Sigma^{\boldsymbol{L}} = (\vartheta_8, \vartheta_9, \vartheta_{10}).$$

$$\boldsymbol{L}(1) \stackrel{d}{=} \boldsymbol{\mu} + V\Delta\beta + V^{1/2}N$$

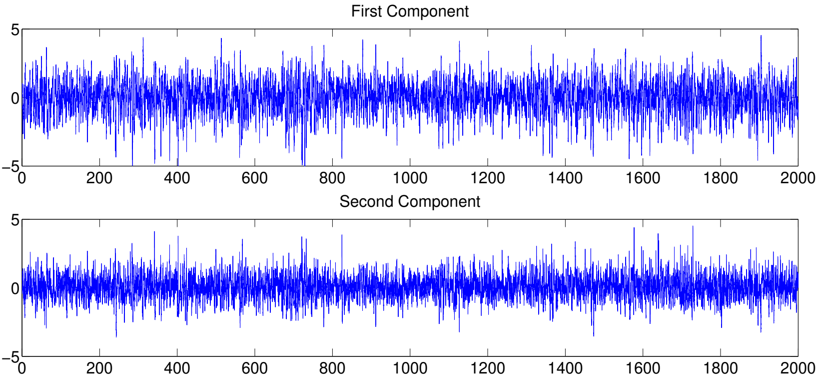
where

- $V \sim \mathrm{IG}(\delta/\kappa, \delta^2)$,
- $N \sim \mathcal{N}(0, \Delta)$.

- Pure jump
- Skewed
- Semi-heavy tailed

Simulation Study II

One realization of a bivariate NIG-driven MCARMA process



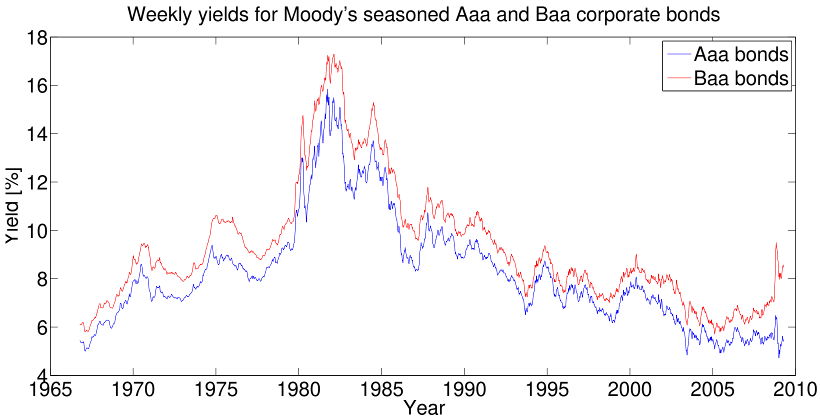
Simulation Study III

QML estimates for a bivariate NIG-driven MCARMA

- Time horizon $[0, 2000]$
- Observed at integer times
- 350 replicates

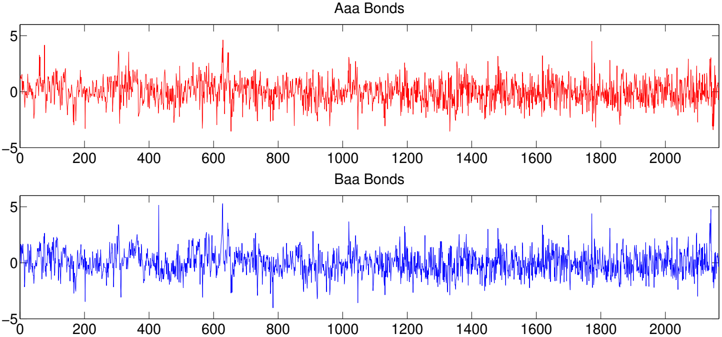
para.	sample mean	bias	sample std. dev.	est. std. dev.
ϑ_1	-1.0001	0.0001	0.0354	0.0381
ϑ_2	-2.0078	0.0078	0.0479	0.0539
ϑ_3	1.0051	-0.0051	0.1276	0.1321
ϑ_4	-2.0068	0.0068	0.1009	0.1202
ϑ_5	-2.9988	-0.0012	0.1587	0.1820
ϑ_6	1.0255	-0.0255	0.1285	0.1382
ϑ_7	2.0023	-0.0023	0.0987	0.1061
ϑ_8	0.4723	-0.0028	0.0457	0.0517
ϑ_9	-0.1654	0.0032	0.0306	0.0346
ϑ_{10}	0.3732	0.0024	0.0286	0.0378

Application to corporate bond yields I



Application to corporate bond yields II

Data show unit roots but no cointegration
Weekly log-yields after differencing and devolatilization using a moving window of width 52 (corresponding to one year)



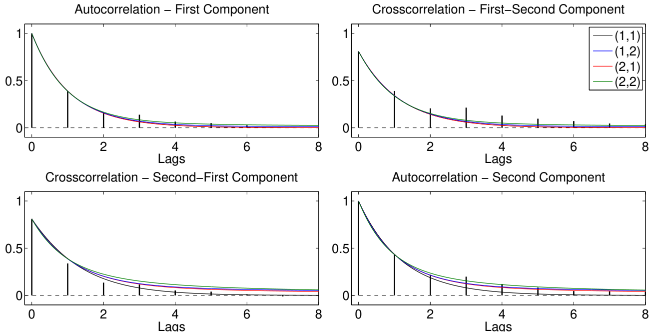
Application to corporate bond yields III

QMLE estimates of the parameters of an $MCARMA_{\alpha,\beta}$ model for weekly yields of Moody's seasoned corporate bonds

(α, β)	(1, 1)		(1, 2)		(2, 1)		(2, 2)	
	$\hat{\vartheta}_i$	$\sigma(\vartheta_i)$	$\hat{\vartheta}_i$	$\sigma(\vartheta_i)$	$\hat{\vartheta}_i$	$\sigma(\vartheta_i)$	$\hat{\vartheta}_i$	$\sigma(\vartheta_i)$
$\hat{\vartheta}_1$	-1.1326	0.1349	-1.1538	0.1401	-1.3776	0.0320	**-0.0010**	**0.0336**
$\hat{\vartheta}_2$	0.2054	0.1171	0.2307	0.1008	-2.4033	0.0197	-1.1601	0.5964
$\hat{\vartheta}_3$	0.3316	0.1206	-0.2528	0.1716	0.0228	0.0050	**-0.0098**	**0.0268**
$\hat{\vartheta}_4$	-1.0935	0.1065	**-0.0362**	**0.0472**	-4.9948	0.1096	**0.1829**	**0.7429**
$\hat{\vartheta}_5$	2.4105	0.2324	-1.2516	0.1286	-4.6276	0.1538	1.4646	0.3931
$\hat{\vartheta}_6$	2.2483	0.2061	-2.5747	0.4595	-0.0153	0.0108	1.3662	0.4039
$\hat{\vartheta}_7$	2.7055	0.2116	1.6345	0.2940	-1.2442	0.0391	-0.7438	0.2387
$\hat{\vartheta}_8$			2.8552	0.1966	0.2573	0.0492	-1.7563	0.7209
$\hat{\vartheta}_9$			3.5702	0.2151	2.4302	0.1370	-2.6936	0.6694
$\hat{\vartheta}_{10}$			4.9076	0.3888	2.9784	0.2766	1.7369	0.5381
$\hat{\vartheta}_{11}$					4.1571	0.5043	-3.6136	3.0265
$\hat{\vartheta}_{12}$							2.8483	2.5122
$\hat{\vartheta}_{13}$							4.4848	0.3327
$\hat{\vartheta}_{14}$							5.5079	0.1803
$\hat{\vartheta}_{15}$							7.0218	1.4357
$-2\log L_{\boldsymbol{\vartheta}}(\boldsymbol{y})$	**9,893.8**		**9,850.4**		**9,853.0**		**9,840.7**	

Application to corporate bond yields IV

Empirical autocorrelations compared to those of the estimated models



Recovery of the driving Lévy process

Minimum-phase assumption:

- $\dim \boldsymbol{L}_t \leq \dim \boldsymbol{Y}_t$,
- B_q, $B_q^T B_0$ full rank,
- $\det B_q^{\sim 1} Q(z) \neq 0, \quad \forall z \in \mathbb{R}_+ + \mathrm{i}\mathbb{R}$.

Proposition (Brockwell and Schlemm (2013))

Let $\boldsymbol{Y}$ be an $\boldsymbol{L}$-driven MCARMA(p,q) process. If $\dim \boldsymbol{L}_t \leq \dim \boldsymbol{Y}_t$, and a minimum-phase assumption is satisfied, then there exist matrices $M_{1,\nu}$, M_2, M_3, and $\boldsymbol{B}$, such that

$$\boldsymbol{L}_t = \sum_{\nu=0}^{p-q-1} M_{1,\nu} \mathrm{D}^\nu \boldsymbol{Y}_t + M_2 \int_{-\infty}^t \mathrm{e}^{\boldsymbol{B}(t-s)} E_q \boldsymbol{Y}_s \mathrm{d}s + M_3 \int_0^t \boldsymbol{Y}_s \mathrm{d}s.$$

Recovery of the driving Lévy process

- Only discrete-time observations available: $\boldsymbol{Y}_0, \boldsymbol{Y}_h, \boldsymbol{Y}_{2h}, \ldots, \boldsymbol{Y}_N$, for some $h > 0$
- Approximate derivatives by forward differences
- Cut off and approximate integrals by trapezoidal numerical integration scheme
- Obtain estimates $\left(\widehat{\Delta \boldsymbol{L}}_1^{(h)}, \ldots, \widehat{\Delta \boldsymbol{L}}_N^{(h)}\right)$ of Lévy increments $\Delta \boldsymbol{L}_n = \boldsymbol{L}_n - \boldsymbol{L}_{n-1}$

Proposition (Brockwell and Schlemm (2013))

If $E\left(\|\boldsymbol{L}_1\|^k\right) < \infty$, *then* $E\left(\left\|\widehat{\Delta \boldsymbol{L}}_n^{(h)} - \Delta \boldsymbol{L}_n\right\|^\kappa\right) \leq Ch^{1/2}$, *for* $\kappa = 1, \ldots, k$.

Generalized method of moments

- $(\mathbb{P}_\vartheta)_{\vartheta\in\Theta}$ parametric family of probability distributions on $\mathbb{R}^m$
- $X^N = (X_1, X_2, \ldots, X_N)$ i. i. d. sample from $\mathbb{P}_{\vartheta_0}$ of length N
- Moment function $g : \mathbb{R}^m \times \Theta \to \mathbb{R}^q$ satisfying $E(g(X_1, \vartheta)) = 0 \Leftrightarrow \vartheta = \vartheta_0$
- Symmetric positive definite $q \times q$ matrix W_N

The sample analogue of the identifiability condition defines the GMM estimator

$$\hat{\vartheta}^N = \operatorname{argmin}_{\vartheta\in\Theta} \left\| \frac{1}{N} \sum_{n=1}^{N} g(X_n, \vartheta) \right\|_{W_N} .$$

Special cases:

- Maximum likelihood: $\mathbb{P}_\vartheta \sim p_\vartheta(\cdot)$, $g(X, \vartheta) = \nabla_\vartheta \log p_\vartheta(X)$
- Least squares for characteristic functions: $\mathbb{P}_\vartheta$ i. d. $\sim \psi_\vartheta$, $g(X, \vartheta) = \begin{bmatrix} \Re\left(\mathrm{e}^{\mathrm{i}\langle \boldsymbol{u}_k, X\rangle} - \mathrm{e}^{\psi_\vartheta(\boldsymbol{u}_k)}\right) \\ \Im\left(\mathrm{e}^{\mathrm{i}\langle \boldsymbol{u}_k, X\rangle} - \mathrm{e}^{\psi_\vartheta(\boldsymbol{u}_k)}\right) \end{bmatrix}_{k=1,\ldots,q/2}$

Generalized method of moments with noisy data

- Perturbed sample $X^{N,(h)} = (X_1 + \varepsilon_1^{(h)}, \dots, X_N + \varepsilon_N^{(h)})$:

$$\hat{\vartheta}^{N,(h)} = \operatorname{argmin}_{\vartheta \in \Theta} \left\| \frac{1}{N} \sum_{n=1}^{N} g\left(X_n + \varepsilon_n^{(h)}, \vartheta\right) \right\|_{W_{N,h}}$$

Theorem (Brockwell and Schlemm (2013))

In addition to standard smoothness and moment conditions assume that there exists a rate function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\beta(h) \to 0$ as $h \to 0$, such that

$$\sup_n E\left(\left\| g\left(X_n + \varepsilon_n^{(h)}, \vartheta_0\right) - g(X_n, \vartheta_0) \right\|\right) = O\left(\beta(h)\right), \quad \text{as } h \to 0.$$

If $h = h_N$ is chosen dependent on N such that $N^{1/2}\beta(h_N) \to 0$ as $N \to \infty$, then

$$N^{1/2}\left(\hat{\vartheta}^{N,(h_N)} - \vartheta_0\right) \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Sigma).$$

GMM estimation for reconstructed Lévy increments

Setup:

- Parametric family of Lévy processes $(\boldsymbol{L}_\vartheta)_{\vartheta \in \Theta}$
- Moment function $g : \mathbb{R}^m \times \Theta \to \mathbb{R}^q$ satisfying $Eg(\Delta \boldsymbol{L}_{\vartheta_0,1}, \vartheta) = 0 \Leftrightarrow \vartheta = \vartheta_0$
- $\boldsymbol{L}_{\vartheta_0}$-driven MCARMA process $\boldsymbol{Y}$

Procedure:

- Discrete observations $\boldsymbol{Y}_0, \boldsymbol{Y}_h, \ldots, \boldsymbol{Y}_N$
- Sample of approximate increments $\left(\widehat{\Delta \boldsymbol{L}}_1^{(h)}, \ldots, \widehat{\Delta \boldsymbol{L}}_N^{(h)}\right)$
- GMM estimation as

$$\hat{\hat{\vartheta}}^{N,(h)} = \operatorname{argmin}_{\vartheta \in \Theta} \left\| \frac{1}{N} \sum_{n=1}^{N} g\left(\widehat{\Delta \boldsymbol{L}}_n^{(h)}, \vartheta\right) \right\|_{W_{N,h}}$$

GMM estimation for reconstructed Lévy increments

Theorem (Brockwell and Schlemm (2013))

Under standard smoothness assumptions on g, and moment conditions on $\boldsymbol{L}_{\vartheta_0,1}$, the rate function can be taken as $\beta : h \mapsto h^{1/2}$. That is, if $h = h_N$ such that $Nh_N \to 0$ as $N \to \infty$, then the estimator $\hat{\hat{\vartheta}}^{N,(h_N)}$ is consistent and asymptotically normally distributed with .

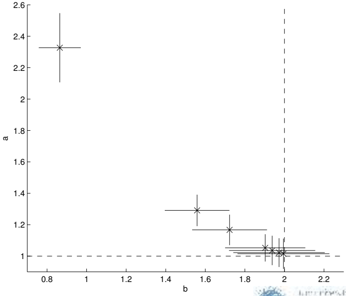
Example: Influence of the sampling frequency

CARMA(3,1):

$$\mathrm{d}\boldsymbol{X}_t = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{2} & -\frac{3}{2} & -2 \end{bmatrix} \boldsymbol{X}_t \mathrm{d}t + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mathrm{d}\Gamma_t^{2,1}$$

$$\boldsymbol{Y}_t = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \boldsymbol{X}_t$$

- Parametric family $(\Gamma^{b,a})_{(b,a)\in\mathbb{R}_+^2}$
- $g(x,b,a) = \nabla_{(b,a)} \log f^{b,a}(x)$ [ML]
- time grid $(0, h, 2h, \ldots, 200)$
- $h \in \{0.5, 0.1, 0.05, \ldots, 0.001, 0.0005\}$
- 500 independent realizations



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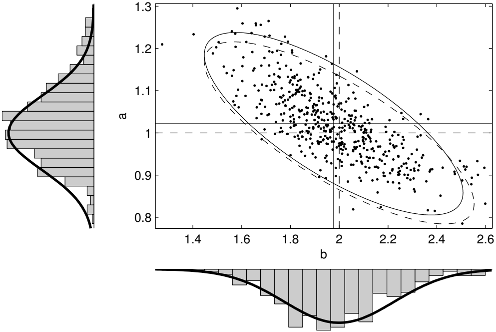
Example continued – Asymptotic normality

- Empirical distribution of $\left(\hat{b}^{200,(0.001)}, \hat{a}^{200,(0.001)}\right)$
- 500 independent realizations



Other questions

- Prediction (see Brockwell and Lindner (2015))
- Extremal behaviour under regularly varying Lévy processes (Moser and St. (2011, 2013)
- Estimation for very heavily tailed/stable MCARMA processes (see Fasen and Fuchs (2013a,b))
- Other methods to reconstruct the driving Lévy process (e.g. Ferrazzano and Fuchs (2013))
- Cointegration (works by Vicky Fasen and coauthors)
- Other estimation techniques (efficient estimators?)
- Different sampling schemes (for (Poisson) random sampling results of e.g. Lii and Masry (1992, 1994) can be applied; high frequency limit for deterministic irregular grids started to be considered in Fechner and St. (2015))
- Use in various applications
- ...

Thank you very much for your attention!

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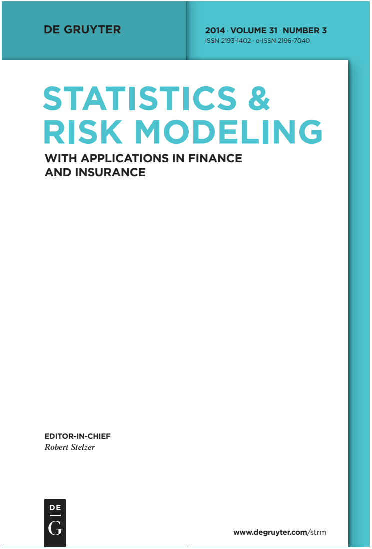

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