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Lévy Khinchin Formula on Commutative Hypercomplex System

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Abstract. A commutative hypercomplex system $L_1(Q, m)$ is, roughly speaking, a space which is defined by a structure measure $(c(A, B, r), (A, B \in \beta(Q))$. Such space has been studied by Berezanskii and Krein. Our main purpose is to establish a generalization of convolution semigroups and to discuss the role of the Lévy measure in the Lévy-Khinchin representation in terms of continuous negative definite functions on the dual hypercomplex system.

1. Introduction

The integral representation of negative definite functions is known in the literature as the Lévy-Khinchin formula. This was established for $G = R$ in the late 1930's by Lévy and Khinchin. It had been extended to Lie groups by Hunt [9] and by Parthasarathy et al [13] to locally compact abelian groups with a countable case. In 1969 Harzallah [7] gave a representation formula for an arbitrary locally compact abelian group. Hazod [8] obtained a Lévy-Khinchin formula for an arbitrary locally compact group. The general Lévy-Khinchin formula and the special case, where the involution is identical are due to Berg [4]. Lasser [12] deduced the Lévy-Khinchin formula for commutative hypergroups. Now these contribution may be viewed as a Lévy-Khinchin formula for negative definite functions defined on commutative hypercomplex systems.

Let Q be a complete separable locally compact metric space of points $p, q, r \cdots, \beta(Q)$ be the σ-algebra of Borel subsets, and $\beta_0(Q)$ be the subring of $\beta(Q)$, which consists of sets with compact closure. We will consider the Borel measures; i.e. positive regular measures on $\beta(Q)$, finite on compact sets. The spaces of continuous functions of finite continuous function, of continuous functions vanish-

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ing at infinity, and of continuous functions with compact support are denoted by $C(Q), C_0(Q), C_\infty(Q)$ and $C_c(Q)$, respectively. The space $C_\infty(Q)$ is a Banach space with norm

$$\|.\|_\infty = \sup_{r\in Q} |(.)(r)|.$$

Any continuous linear functional defined on the space $C_0(Q)$ with the inductive topology is called a (complex) Random measure. The space of Radon measures is denoted by $M(Q)$. Let $M_b(Q) = (C_\infty(Q))'$ be the Banach space of bounded Radon measures with norm

$$\|\mu\|_\infty = \sup\{|\mu(f)| \; |f \in C_\infty(Q), |f| \leq 1\},$$

and let $M_c(Q)$ be the space of Radon measures with compact support.
By $M_1(Q), M_b^+(Q)(M_1(Q) \subset M_b^+(Q))$, we denote the set of Radon probability and bounded positive Radon measures on Q, respectively. The topology of simple convergence on functions from $C_0(Q)$ in the space of Radon measures, is called vague topology.

A hypercomplex system with the basis Q is defined by its structure measure $c(A, B, r)(A, B \in \beta(Q); r \in Q)$. A structure measure $c(A, B, r)$ is a Borel measure in A (respectively B) if we fix B, r (respectively A, r) which satisfies the following properties:

(H1) $\forall\, A, B \in \beta_0(Q)$, the function $c(A, B, r) \in C_0(Q)$.

(H2) $\forall\, A, B \in \beta_0(Q)$ and $s, r \in Q$, the following associativity relation holds

$$\int_Q c(A, B, r) d_r c(E_r, C, s) = \int_Q c(B, C, r) d_r c(A, E_r, s), \quad C \in \beta(Q)$$

(H3) The structure measure is said to be commutative if

$$c(A, B, r) = c(B, A, r), \quad (A, B \in \beta_0(Q))$$

A measure m is said to be a multiplicative measure if

$$\int_Q c(A, B, r) dm(r) = m(A)m(B); \quad A, B \in \beta_0(Q)$$

(H4) We will suppose the existence of a multiplicative measure. Under certain relations imposed on the commutative structure measure, multiplicative measure exists. (See [11]).

For any $f, g \in L_1(Q, m)$, the convolution

$$(f * g)(r) = \int_Q \int_Q f(p)g(q) dm_r(p, q) \tag{1.1}$$

is well defined (See [2]).

The space $L_1(Q, m)$ with the convolution (1.1) is a Banach algebra which is commutative if (H3) holds. This Banach algebra is called the hypercomplex system with the basis Q.

A non zero measurable and bounded almost everywhere function $Q \ni r \to \chi(r) \in C$ is said to be a character of the hypercomplex system L_1, if $\forall\, A, B \in \beta_0(Q)$

$$\int_Q c(A, B, r)\chi(r)dm(r) = \chi(A)\chi(B),$$

$$\int_C \chi(r)dm(r) = \chi(C), \quad C \in \beta_0(Q).$$

(H5) A hypercomplex system is said to be normal, if there exists an involution homomorphism $Q \ni r \to r^* \in Q$, such that $m(A) = m(A^*)$, and $c(A, B, C) = c(C, B^*, A), c(A, B, C) = c(A^*, C, B), (A, B \in \beta_0(Q))$ where

$$c(A, B, C) = \int_C c(A, B, r)dm(r)$$

(H6) A normal hypercomplex system possesses a basis unity if there exists a point $e \in Q$ such that $e^* = e$ and

$$c(A, B, e) = m(A^* \cap B), \quad A, B \in \beta(Q),$$

we should remark that, for a normal hypercomplex system, the mapping

$$L_1(Q, m) \ni f(r) \to f^*(r) \in L_1(Q, m)$$

is an involution in the Banach algebra L_1, the multiplicative measure is unique and the characters of such a system are continuous (see [1]). A character χ of a normal hypercomplex system is said to be Hermitian if

$$\chi(r^*) = \overline{\chi(r)} \quad (r \in Q).$$

Let $L_1(Q, m)$ be a hypercomplex system with a basis Q and Φ a space of complex valued functions on Q. Assume that an operator valued function $Q \ni p \to R_p : \Phi \to \Phi$ is given such that the function $g(p) = (R_p f)(q)$ belongs to Φ for any $f \in \Phi$ and any fixed $q \in Q$. The operators $R_p (p \in Q)$ are called generalized translation operators, provided that the following axioms are satisfied:

(T1) Associativity axiom: The equality

$$(R_p^q(R_q f))(r) = (R_q^r(R_p f))(r)$$

holds for any elements $p, q \in Q$.

(T2) There exists an element $e \in Q$ such that R_e is the identity in Φ. See [3].

Clearly, the convolution (1.1) in the hypercomplex system $L_1(Q,m)$ and the corresponding family of generalized translation operators R_p satisfy the relation

$$(f * g)(p) = \int_Q (R_s f)(p) g(s^*) ds, \quad f, g \in L_1 \tag{1.2}$$

Denote by $\hat{Q}$ the support of the plancherel measure $\hat{m}$ [6]. For any $M, N \in \beta(\hat{Q})$ and $\chi \in \hat{Q}$, we set

$$\hat{c}(M,N,\chi) = \int_Q \overline{\chi(r)} \int_M \varphi(r) d\hat{m}(\varphi) \int_N \psi(r) d\hat{m}(\psi) dr, \quad \varphi, \psi \in \hat{Q}. \tag{1.3}$$

Then, $\hat{c}(M,N,\chi)$ defines a structure measure on the dual hypercomplex system if and only if it belongs to $C_0(\hat{Q})$ for each fixed M, N and the product of $\varphi, \psi \in \hat{Q}$ is a positive definite function.

The dual hypercomplex system $L_1(\hat{Q},\hat{m})$ associated to $\hat{c}(M,N,\chi)$ will be constructed. The Plancherel measure $\hat{m}$ is a multiplicative measure for $\hat{c}(M,N,\chi)$. $L_1(\hat{Q},\hat{m})$ is normal with the involution $\chi^*(r) = \overline{\chi(r)}$ and has a basis unity $\hat{e} \equiv 1(r)$. We denote by $\hat{\hat{m}}$ the Plancherel measure corresponding to the dual hypercomplex system and by $\hat{\hat{Q}} = supp\, \hat{\hat{m}}$. We say that there is a duality if $Q = \hat{\hat{Q}}$. See [1].

2. Negative definite functions

Let $L_1(Q,m)$ be a commutative normal hypercomplex system with basis unity e.

Definition 2.1. A continuous bounded function $\psi : Q \to C$ is called *negative definite* if for any $r_1, \cdots, r_n \in Q$ and $c_1, \cdots, c_n \in C, n \in N$

$$\sum_{i,j=1}^{n} [\psi(r_i) + \overline{\psi(r_j)} - (R_{r_j^*}\psi)(r_i)] c_i \overline{c_j} \geq 0 \tag{2.1}$$

By $P(Q)$ and $N(Q)$, we shall denote the set of all continuous positive definite functions and negative definite functions on Q respectively. For example each constant $c \geq 0$ is a negative definite function. Obviously the following holds for a negative definite function ψ

$$\psi(e) \geq 0, \overline{\psi(r)} = \psi(r^*), (R_{r^*}\psi)(r) \in R$$

and

$$\psi(r) + \psi(r^*) \geq R_{r^*}\psi(r)$$

The following basic properties of negative definite functions on Q are stated without proofs, for details and proofs, you can refer to [15].

Theorem 2.1. *A function $\psi : Q \to C$ is negative definite if and only if the following conditions are satisfied:*

(i) *$\psi(e) \geq 0, \psi$ is a continuous bounded function,*

(ii) *$\overline{\psi(r)} = \psi(r^*)$ for each $r \in Q$, and*

(iii) *for $r_1, \cdots, r_n \in Q$ and $c_1, \cdots, c_n \in C$ with $\sum_{i=1}^{n} c_i = 0$, the summation*

$$\sum_{i,j=1}^{n} (R_{r_j^*}\psi)(r_i)c_i\overline{c_j} \leq 0$$

Corollary 2.1. *Let ψ be a function on Q.*

(i) *If $\psi \in N(Q)$, then $r \mapsto \psi(r) - \psi(e)$ is negative definite.*

(ii) *If $\varphi \in P(Q)$, then $r \mapsto \varphi(e) - \varphi(r)$ is negative definite.*

If the generalized translation operators R_t extended to L_∞ mapping $C_0(Q)$ into $C_0(Q \times Q)$, then inequality (2.1) is equivalent to the inequality

$$\int_Q \int_Q (\psi(r) + \overline{\psi(s)} - (R_{s^*}\psi)(r))x(r)\overline{x(s)}drds \geq 0, \; x \in L_1 \tag{2.2}$$

Theorem 2.2. *Let $\psi : Q \to C$ be a continuous bounded function, $\psi(e) \geq 0$ and $\varphi_t : r \mapsto \exp(-t\psi(r))$ be positive definite for each $t \geq 0$. Then ψ is negative definite.*

Definition 2.2. A continuous function $h : Q \to R$ is called *homomorphism* if $h(r^*) = -h(r)$ and $(R_r h)(s) = h(r) + h(s), r, s \in Q$.

Lemma 2.1. *If h is a homomorphism, then $\psi = ih$ is negative definite.*

Proof. Suppose h is a homomorphism. Then for any $r_1, \cdots, r_n \in Q$ the matrix

$$(\psi(r_i) + \overline{\psi(r_j)} - (R_{r_s^*}\psi)(r_i))$$

is the zero matrix, and it follows that $\psi \in N(Q)$. □

Definition 2.3. A continuous function $q : Q \to R$ is called a *quadratic form*, if

$$(R_s q)(r) + (R_{s^*} q)(r) = 2(q(s) + q(r)), \quad r, s \in Q, \tag{2.3}$$

By using (1.2) and (2.3), clearly a quadratic form q satisfies:

$$q(e) = 0, \quad q(r^*) = q(r),$$

and

$$(\mu * \nu)(q) + (\mu * \overline{\nu})(q) = 2(\mu(Q)\nu(q) + \nu(Q)\mu(q)), \; \mu, \nu \in M_b(Q) \tag{2.4}$$

Lemma 2.2. *Let q be a quadratic form and $\mu \in M_b(Q)$. Then*

$$\mu^{2n}(q) = 4n^2\mu(q)^{2n-1}\mu(q) - n(2n-1)\mu(Q)^{2n-2}\mu * \overline{\mu}(q) \tag{2.5}$$

for each $n \in N$, where μ^n, the n-fold convolution of μ.

Proof. (2.5) can be proved by induction. By (2.4), we obtain

$$\mu^{2n+2}(q) = 2\mu(Q)^{2n+1}\mu(q) + 2\mu(Q)\mu^{2n+1}(q) - \mu^{2n+1} * \overline{\mu}(q) \tag{2.6}$$

and

$$\begin{aligned} \mu^{2n+1} * \overline{\mu}(q) &= \mu^{2n} * \mu * \overline{\mu}(q) \\ &= \frac{1}{2}[\mu^{2n} * \mu * \overline{\mu}(q) + \mu^{2n} * \overline{\mu * \overline{\mu}(q)}] \\ &= \mu(Q)^{2n}\mu * \overline{\mu}(q) + \mu(Q)^2\mu^{2n}(q) \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \mu^{2n+1}(q) &= \mu^{2n} * \mu(q) \\ &= 2[\mu(Q)^{2n}\mu(q) + \mu(Q)\mu^{2n}(q)] - \mu^{2n} * \overline{\mu}(q) \\ &= 2\mu(Q)^{2n}\mu(q) + 2\mu(Q)\mu^{2n}(q) \\ &\quad -\mu(Q)^{2n-1}\mu * \overline{\mu}(q) - \mu(Q)^2\mu^{2n-1}(q) \end{aligned} \tag{2.8}$$

Similar to (2.8) $\mu^{2n}(q) = 2\mu(Q)^{2n-1}\mu(q)$,

$$\mu^{2n}(q) = 2\mu(Q)^{2n-1}\mu(q) + 2\mu(Q)\mu^{2n-1}(q) - \mu(Q)^{2n-2}\mu * \overline{\mu}(q) - \mu(Q)^2\mu^{2n-2}(q)$$

Then

$$\begin{aligned} \mu(Q)\mu^{2n-1}(q) &= \frac{1}{2}\mu^{2n}(q) - \mu(Q)^{2n-1}\mu(q) + \frac{1}{2}\mu(Q)^{2n-2}\mu * \overline{\mu}(q) \\ &\quad + \frac{1}{2}\mu(Q)^2\mu^{2n-2}(q) \end{aligned} \tag{2.9}$$

Substituting by (2.7) and (2.9) in (2.6), we get

$$\mu^{2(n+1)}(q) = 4(n+1)^2\mu(Q)^{2n+1}\mu(q) - (n+1)(2n+1)\mu(Q)^{2n}\mu * \overline{\mu}(q).$$

□

Corollary 2.2. *Let q be a quadratic form. Then*

$$\lim_{n\to\infty} \frac{(R_s^n q)(s)}{4n^2} = q(s) - \frac{1}{2}(R_{s^*}q)(s). \tag{2.10}$$

By using (2.4)and (1.2), the limit (2.10) can be proved easily.

Let $\mathcal{S}$ be a smallest abelian semigroup containing Q with unity e and natural involution $s \mapsto s^*, s \in \mathcal{S}$.

A function $F : \mathcal{S} \to C$ will be called adapted if its restriction $f := F|Q$ is locally bounded, measurable, and

$$F(s * r) = \int \int (R_s f)(r) d\mu(s) d\mu(r), \quad s, r \in Q$$

Lemma 2.3. *Let $\Psi : \mathcal{S} \to C$ be negative definite on $\mathcal{S}$ and*

$$\Psi(s * r) = \int \int (R_s \psi)(r) d\mu(s) d\mu(r), \quad s, r \in Q$$

Then ψ is negative definite on Q.

Proof. Consider $s_1, \cdots, s_n \in \mathcal{S}$, then for $c_1, \cdots, c_n \in C$, we have

$$\begin{aligned} 0 &\leq \sum_{i,j=1}^{n} c_i \overline{c_j} (\Psi(s_i) + \overline{\Psi(s_j)} - \Psi(s_i * s_j^*)) \\ &= \sum_{i,j=1}^{n} c_i \overline{c_j} \left(\int \int \psi(s_i) d\mu(s_j) d\mu(s_i) + \int \int \overline{\psi(s_j)} d\mu(s_i) d\mu(s_j) \right. \\ &\quad \left. - \int \int ((R_{s_i} \psi)(s_j^*) d\mu(s_j) d\mu(s_i) \right) \\ &= \int \int \sum c_i c_j (\Psi(s_j) + \overline{\Psi(s_j)} - (R_{s_i} \psi)(s_j^*)) d\mu(s_i) d\mu(s_i). \end{aligned}$$

Then by (2.2) ψ is negative definite. □

Theorem 2.3. *Nonnegative quadratic forms are negative definite.*

Proof. Let $q : Q \to R^+$ be a quadratic form, and for each $s, r \in \mathcal{S}$, put

$$\mathbf{q}(s * r) = \int \int (R_s q)(r) d\mu(s) d\mu(r).$$

Then

$$\begin{aligned} \mathbf{q}(s * r) + \mathbf{q}(s * r^*) &= \int \int ((R_s q)(r) + (R_s q)(r^*)) d\mu(s) d\mu(r) \\ &= 2 \int \int (q(s) + q(r^*)) d\mu(s) d\mu(r) \\ &= 2 \left[\int \int q(s) d\mu(s) d\mu(r) + \int \int q(r) d\mu(s) d\mu(r) \right] \\ &= 2[\mathbf{q}(s * e) + \mathbf{q}(r * e)] = 2[\mathbf{q}(s) + \mathbf{q}(r)], \end{aligned}$$

which shows that $\mathbf{q}$ is a nonnegative quadratic form on $\mathcal{S}$. By [4] q is negative definite and hence so is q by Lemma 2.3. □

3. Covolution semigroups

Let $L_1(Q,m)$ be a commutative normal hypercomplex system with basis Q and basis unity e.

Definition 3.1. A family $(\mu_t)_{t>0}, \mu_t \in M_b^+(Q)$ is called a *convolution semigroup* on Q, if

(i) $\mu_t(Q) \leq 1$, for each $t > 0$,

(ii) $\mu_{t_1} * \mu_{t_2} = \mu_{t_1+t_2}$ for $t_1, t_2 > 0$,

(iii) $\lim\limits_{t\to 0} \mu_t = \varepsilon_e$

with respect to the vague topology on $M_b(Q)$.

Lemma 3.1. *Let $(\mu_t)_{t>0}$ be a convolution semigroup on Q. Then, for $\chi \in \hat{Q}$, the function $t \mapsto \hat{\mu}_t(\chi), R^+ \to C$ is continuous.*

Proof. Using Urysohn's lemma, see [14], there exists $f \in C_c(Q)$ satisfing $0 \leq f \leq 1$ and $f(0) = 1$. By (iii) and (i) above

$$1 = f(0) = \lim_{t\to 0} < \mu_t, f > \leq \lim_{t\to 0} \inf \mu_t(Q) \leq \lim_{t\to 0} \sup \mu_t(Q) \leq 1$$

and this shows that

$$\lim_{t\to 0} \mu_t = \varepsilon_e \quad \text{in the Bernolli topology} \tag{3.1}$$

For $t_1, t_0 > 0$ and $\chi \in \hat{Q}$, we find, as in [5],

$$|\hat{\mu}_t(\chi) - \hat{\mu}_{t_0}(\chi)| \leq |\hat{\mu}_{|t-t_0|}(\chi) - 1|,$$

and since the right-hand side, by (3.1), tends to zero uniformly on compact subsets of $\hat{Q}$, we get

$$\lim_{t\to t_0} \mu_t = \mu_{t_0} \quad \text{in the Bernolli topology.}$$

□

Theorem 3.1. *Assume that $\hat{Q}$ is the dual of Q. If $(\mu_t)_{t>0}$ is a convolution semigroup on Q, then there exists exactly one negative definite function $\psi : \hat{Q} \to C$ with $Re\psi \geq 0$ such that*

$$\hat{\mu}_t(\chi) = \exp(-t\psi(\chi)) \text{ for each } \chi \in \hat{Q}, t > 0.$$

Proof. With some modifications to the proof of Theorem 8.3 in [5], we can prove this theorem easily. □

Theorem 3.2. *Assume that Q has a duality and $\psi : \hat{Q} \to C$ is a negative definite* *function with* $Re\psi \geq 0$*, such that* $\hat{\mu}_t(\chi) = \exp(-t\psi(\chi))$ *is positive definite for* $t < 0$*. Then there exists a unique convolution semigroup* $(\mu_t)_{t>0}$ *on* Q *such that* ψ *is associated to* $(\mu_t)_{t>0}$.

Proof. Since $Re\psi \geq 0$, then $|\exp|(-t\psi(\chi)| \leq 1$. Thus by the duality of Q, there are unique determined measures $\mu_t \in M_b^+(Q), t > 0$, such that $\hat{\mu}_t(\chi) = \exp(-t\psi(\chi))$. Obviously $(\mu_t)_{t>0}$ satisfies properties (i) and (ii). Further using the boundedness of ψ on compact subsets of $\hat{Q}$,

$$\lim_{t\to 0} \hat{\mu}_t(\chi) = \lim_{t\to 0} \exp(-t\psi(\chi)) = 1.$$

The duality of Q defines a structure measure $\hat{c}$ as in (1.3) on the dual hypercomplex system $L_1(\hat{Q}, \hat{m})$. The Plancherel measure $\hat{m}$ is the multiplicative measure for $\hat{c}$.

Let $f \in C_0(Q), \varepsilon > 0$, by [10], there exists $g \in C_0(\hat{Q})$, such that $\|f - \tilde{g}\|_\infty < \varepsilon$. Now we obtain

$$|\mu_t(f) - \varepsilon_e(f)| \leq 2\varepsilon + \int_{\hat{Q}} |g(\chi)||\hat{\mu}_t(\overline{\chi}) - 1| d\hat{m}(\chi)$$

which gives $\lim_{t\to 0} \mu_t = \varepsilon_e$ in the vague topology on $M_b(Q)$. □

4. The Lévy-Khinchin representation of the negative definite function

Throughout this section, we consider $L_1(\hat{Q}, \hat{m})$ to be dual of $L_1(Q, m)$. Let S denote the set of probability and symmetric measures on $\hat{Q}$ with compact support, i.e.

$$S = \{\sigma | \sigma \in M_1^+(\hat{Q}) \cap M_c(\hat{Q}),\ \sigma(\chi) = \sigma(\overline{\chi}) = \check{\sigma}(\chi)\}$$

Lemma 4.1. *Let* V *be a compact neighbourhood of* $e \in Q$*. Then there exists a* $\sigma \in S$ *such that* $\tilde{\sigma}(r) \leq \frac{1}{2}$ *for each* $r \in Q \setminus V$*, where* $\tilde{\sigma}$ *is the Fourier transform of* σ *on* $\hat{Q}$*.*

Proof. By using Urysohn's lemma, there exists a function $\varphi \in C_c(Q)$ such that $0 \leq \varphi \leq 1, \varphi(r) = 1$, for each $r \in V$ and $supp\,\varphi \subset V$. Since φ is a nonnegative constant function, then $\varphi \in P(Q)$. By Theorem 3.1 in [1] which is the analogue of Bochner's theorem for hypercomplex system, there is a positive bounded measure μ on $\hat{Q}$, such that $\tilde{\mu} = \varphi$. One can easily obtain that $\mu \in M_1(\hat{Q})$ and $\mu = \check{\mu}$. Choosing a compact symmetric set $J \subseteq \hat{Q}$ such that $\mu(J) \geq \frac{3}{4}$ and putting $\sigma = \mu(J)^{-1}(\mu|J)$, we find,

$$\|\varphi - \tilde{\sigma}\|_V = \|\tilde{\mu} - \tilde{\sigma}\|_V \leq \|\mu - \sigma\|_\infty \leq \frac{1}{2}$$

and thus $\tilde{\sigma}(r) \leq \frac{1}{2}$ for $r \in Q \setminus V$. □

Theorem 4.1. *Let* (μ_t) *be a convolution semigroup on* Q *and* $\psi : \hat{Q} \to C$ *the* *negative definite function associated to* $(\mu_t)_{t>0}$. *Then the net* $\left(\frac{1}{t}\mu_t|Q\setminus\{e\}\right)_{t>0}$ *of positive measures on* $Q\setminus\{e\}$ *converges vaguely as* $t\to 0$ *to a measure* μ *on* $Q\setminus\{e\}$. *For every* $\sigma\in S$, *the function* $\psi*\sigma-\psi$ *is continuous positive definite on* $\hat{Q}$ *and the positive bounded measure* μ_σ *on* Q *whose Fourier transform is* $\psi*\sigma-\psi$ *satisfies*

$$(1-\tilde{\sigma})\mu=\mu_\sigma|Q\setminus\{e\} \tag{4.1}$$

Proof. Let $\sigma\in S$. The measure $(1-\tilde{\sigma})\frac{1}{t}\mu_t$ is positive bounded on Q, for $t>0$ and

$$\begin{aligned}\left[(1-\tilde{\sigma})\frac{1}{t}\mu_t\right]^{\wedge}(\chi) &= \int(1-\tilde{\sigma}(r))\frac{1}{t}\overline{\chi(r)}d\mu_t(r)\\ &= \frac{1}{t}\left[\int\overline{\chi(r)}d\mu_t(r)-\int\tilde{\sigma}(r)\overline{\chi(r)}d\mu_t(r)\right]\\ &= \frac{1}{t}\left[\hat{\mu}_t(\chi)-\int\int\chi(s)\overline{\chi(r)}d\sigma(s)d\mu_t(r)\right]\\ &= \frac{1}{t}\left[\hat{\mu}_t(\chi)-(\hat{\mu}_t*\sigma)(\chi)\right]\\ &= \frac{1}{t}[1-\exp(-t\psi)*(\sigma-\varepsilon_e)](\chi)\text{ for }\chi\in\hat{Q}\end{aligned}$$

Since $\lim_{t\to 0}\frac{1}{t}(1-e^{-t\psi})=\psi$ uniformaly on compact subsets of $\hat{Q}$, we find that

$$\lim_{t\to 0}\left[(1-\tilde{\sigma})\frac{1}{t}\mu_t\right]^{\wedge}(\chi)=(\psi*(\sigma-\varepsilon_e))(\chi)=\psi*\sigma(\chi)-\psi(\chi),$$

pointwise (or uniformly over compact sets) on $\hat{Q}$. This shows that the function $\chi\mapsto\psi*\sigma(\chi)-\psi(\chi)$ is continuous positive definite, and furthermore, see ([5], 3.13) that

$$\lim_{t\to 0}(1-\tilde{\sigma})\frac{1}{t}\mu_t=\mu_\sigma,$$

in the Bernolli topology on Q, where μ_σ is positive bounded on Q such that

$$\hat{\mu}_\sigma=\psi*\sigma-\psi$$

For, $\varphi\in C_c^+(Q)$ with $supp\,\varphi\subset Q\setminus\{e\}$, we may choose by Lemma 4.1, $\sigma\in S$ such that $\tilde{\sigma}\le\frac{1}{2}$ in neighbourhood of $supp\,\varphi$, let φ' be a function defined by

$$r\overset{\varphi'}{\mapsto}\begin{cases}\dfrac{\varphi(r)}{1-\tilde{\sigma}(r)} & \text{for } r\in supp\,\varphi\\ 0 & \text{for } r\notin supp\,\varphi\end{cases}$$

this function belongs to $C_c^+(Q)$. Since

$$\begin{aligned} < \frac{1}{t}\mu_t, \varphi > &= \int \frac{1}{t}\mu_t(r)\varphi(r)dr \\ &= \int (1-\tilde{\sigma}(r))\frac{1}{t}\mu_t(r)\frac{\varphi(r)}{1-\sigma\check{(}r)}dr \\ &= \left\langle (1-\tilde{\sigma})\frac{1}{t}\mu_t, \varphi' \right\rangle \end{aligned}$$

then

$$\lim_{t\to 0}\left\langle \frac{1}{t}\mu_t, \varphi \right\rangle = \lim_{t\to 0}\left\langle (1-\tilde{\sigma})\frac{1}{t}\mu_t, \varphi' \right\rangle =< \mu_\sigma, \varphi' >$$

This shows, that there exists a positive measure μ on $Q \setminus \{e\}$ satisfies

$$\mu = \lim_{t\to 0}\frac{1}{t}\mu_t | Q\setminus\{e\} \text{ vaguely on } Q\setminus\{e\},$$

and

$$(1-\tilde{\sigma})\mu = \mu_\sigma | Q\setminus\{e\} \qquad \text{for } \sigma \in S.$$

□

Definition 4.1. The positive measure μ on $Q\setminus\{e\}$ defined by Theorem 4.1 in (4.1) is called the *Lévy measure* for the convolution semigrpup $(\mu_t)_{t>0}$ on Q (and also the Lévy measure for the negative definite function ψ on $\hat{Q}$).

Theorem 4.2. *Let μ denote the Lévy measure of a given convolution semigroup $(\mu_t)_{t>0}$. Then*

(i) $\int_{Q\setminus\{e\}}(1-\mathrm{Re}\chi(r))d\mu(r) < \infty$ *for each* $\chi \in \hat{Q}$.

(ii) *If V is a compact neighbourhood of e in Q, then $\mu|Q\setminus V \in M^+(Q)$*

Proof. (i) For $\chi \in \hat{Q}$, let $\sigma = \frac{1}{2}(\varepsilon_\chi + \varepsilon_{\overline{\chi}}) \in S$; then $\tilde{\sigma} = \mathrm{Re}\chi(x)$ and by (4.1)

$$\int_{Q\setminus\{e\}}(1-\mathrm{Re}\chi(r))d\mu(r) = \int_{Q\setminus\{e\}}(1-\tilde{\sigma}(r))d\mu(r) = \mu_\sigma|_{Q\setminus\{e\}} < \infty$$

The statement of (ii) follows as in ([5], 18.4). □

The following two lemmas can be proved exactly as in ([5], 18.13 and 18.16).

Lemma 4.2. *Let $h : \hat{Q} \to R$ be continuous and $h(1) = 0$. h be a homomorphism if and only if $h * \sigma - h = 0$ for each $\sigma \in S$.*

Lemma 4.3. *Let $q : \hat{Q} \to R$ be continuous with $q(\chi) = q(\overline{\chi})$, $q(1) = 0$. q is a quadratic form if and only if $q * \sigma - q$ is a constant function for each $\sigma \in S$.*

Moreover q is nonnegative if and only if $q * \sigma - q \geq 0$ for all $\sigma \in S$.

Corollary 4.1. *Let (μ_t) be a convolution semigroup on Q. Assume that ψ is the associated negative definite function. If the Lévy measure μ of $(\mu_t)_{t>0}$ is symmetric, then $Im\psi$ is a homomorphism. In particular $iIm\psi$ is negative definite. Further μ is also the Lévy measure of (v_t), where $v_t = \mu_{t/2} * \overline{\mu}_{t/2}$.*

Proof. $\check{\mu} = \mu$ is equivalent to $\check{\mu}_\sigma = \mu_\sigma$ for each $\sigma \in S$. This is equivalent to $\psi * \sigma - \psi$ being real valued for each $\sigma \in S$. Thus $\mathrm{Im}\psi * \sigma - \mathrm{Im}\psi = 0$ for each $\sigma \in S$, and by Lemma 4.2 $\mathrm{Im}\psi$ is a homomorphism. Thus $i\mathrm{Im}\psi$ is negative definite. Theorem 4.1 yields that $\mu_t)_{t>0}$ and $(v_t)_{t>0}$ define the same class of measures $\mu_\sigma, \sigma \in S$. Therefore the uniqueness of the measure satisfing (4.1) implies the second assertian.

Since Q is locally compact, then for every compact subset k of $\hat{Q}$, there exists a constant $M_k \geq 0$, a neighbourhood U_k of e in Q and a finite subset N_k of k such that for each $r \in U_k$

$$\sup_r\{1 - \mathrm{Re}\chi(r) : \chi \in k\} \leq M_k \sup_r\{1 - \mathrm{Re}\chi(r) : \chi \in N_k\} \tag{4.2}$$

see [13]. □

Lemma 4.4. *Let μ be a positive symmetric measure on $Q \setminus \{e\}$ such that*

$$\int_{Q\setminus\{e\}} (1 - \mathrm{Re}\chi(x))d\mu(r) < \infty, \quad \textit{for } \chi \in \hat{Q}$$

The function $\psi_\mu : \hat{Q} \to R$ defined by

$$\psi_\mu(\chi) = \int_{Q\setminus\{e\}} (1 - \mathrm{Re}\chi(r))d\mu(r) \quad \textit{for } \chi \in \hat{Q}$$

is continuous and negative definite.

Proof. Let $\chi_0 \in \hat{Q}, \varepsilon > 0$ and K be a compact neighbourhood of χ_0, then there exists a constant $M_K \geq 0$, a finite set $N_K = \{\chi_1, \cdots, \chi_n\} \subseteq \hat{Q}$ and a neighbourhood U_K of e in Q such that

$$\begin{aligned}
& \int_{U_K\setminus\{e\}} \sup_{\chi\in K}(1 - \mathrm{Re}\chi(r))d\mu(r) \\
\leq \quad & M_k \int_{U_k\setminus\{e\}} \sup_{\chi\in N_K}(1 - \mathrm{Re}\chi(r))d\mu(r) \\
\leq \quad & M_k \sum_{i=1}^{n} \int_{U_k\setminus\{e\}} (1 - \mathrm{Re}\chi_i(r))d\mu(r) \\
\leq \quad & M_k \sum_{i=1}^{n} \int_{Q\setminus\{e\}} (1 - \mathrm{Re}\chi_i(r))d\mu(r) = M_k \sum_{i=1}^{n} \psi_m(\chi_i)
\end{aligned}$$

Thus there exists a neighbourhood V of e such that

$$\int_{V\setminus\{e\}}(1-\text{Re}\chi(r))d\mu(r)<\frac{\varepsilon}{4} \tag{4.3}$$

for each $\chi\in K$. Since $\mu|_{k\setminus v}$ is bounded, then there exists a neighbourhood $W\subseteq K$ of χ_0 in $\hat{Q}$ such that

$$\left|\int_{Q\setminus V}(\chi(r)-\chi_0(r))d\mu(r)\right|<\frac{\varepsilon}{2} \tag{4.4}$$

for each $\chi\in W$. For the continuity of ψ_μ we have

$$\begin{aligned}
& |\psi_\mu(\chi)-\psi_\mu(\chi_0)| \\
= & \left|\int_{Q\setminus\{e\}}(1-\text{Re}\chi(r)d\mu(r)-\int_{Q\setminus\{e\}}(1-\text{Re}\chi_0(r)d\mu(r)\right| \\
= & \left|\int_{V\setminus\{e\}}(1-\text{Re}\chi(r)d\mu(r)-\int_{V\setminus\{e\}}(1-\text{Re}\chi_0(r)d\mu(r)+\int_{Q\setminus V}(\text{Re}\chi_0)(r)-\text{Re}\chi(r)d\mu(r)\right| \\
\leq & \int_{V\setminus\{e\}}(1-\text{Re}\chi(r)d\mu(r)+\int_{V\setminus\{e\}}(1-\text{Re}\chi_0(r)d\mu(r)+\left|\int_{Q\setminus V}(\text{Re}\chi_0)(r)-\text{Re}\chi(r)d\mu(r)\right| \\
< & \ \varepsilon
\end{aligned}$$

for each $\chi\in W$. From (4.3) and (4.4), the continuity of ψ_μ is verified. In order to show that ψ_μ is negative definite, it is sufficient to prove that μ is a Lévy measure for it. For $f\in C_c^+(\hat{Q})$ such that $f(\overline{\chi})=f(\chi)$ and $\int f(\chi)dx=1$, we may apply Fubini's theorem to find

$$\begin{aligned}
(\psi_\mu * f)(\chi) &= \int_{\hat{Q}}(R_\eta f)(\chi)\psi_\mu(\eta)d\eta \\
&= \int_{\hat{Q}} f(\eta)\int_{Q\setminus\{e\}}[1-\text{Re}\chi(r)\eta(r)]d\mu(r) \\
&= \int_{Q\setminus\{e\}}[1-\text{Re}\chi(r)\tilde{f}(r)]d\mu(r)
\end{aligned} \tag{4.5}$$

In particular we have for $\chi=1$

$$\int_{Q\setminus\{e\}}(1-\tilde{f}(r))d\mu(r)=\int f(\eta)\psi_\mu(\eta)d\eta$$

The measure $d\tau(r)=(1-\tilde{f}(r))d\mu(r)$ is thus positive bounded on $Q\setminus\{e\}$. Then can be consider as a positive bounded measure on Q and we have that

$$\hat{\tau}(\chi)=\text{Re}\hat{\tau}(\chi)=\int_{Q\setminus\{e\}}\text{Re}\chi(r)(1-\tilde{f}(r))d\mu(r)\quad\text{for }\ \chi\in\hat{Q}.$$

Put $f = \sigma$ in (4.5). Then

$$\begin{aligned}\psi_\mu * \sigma(\chi) - \psi_\mu(\chi) &= \int_{Q\setminus\{e\}} \mathrm{Re}\chi(r)(1-\tilde{\sigma}(r))d\mu(r)\\ &= \int_{Q\setminus\{e\}} \mathrm{Re}\chi(r)(1-\tilde{\sigma}(r))d\mu(r)\end{aligned}$$

i.e. $\psi_\mu * \sigma - \psi_\mu$ is the Fourier transform of the measure $(1-\tilde{\sigma})(r)|\mu$ and by the result (4.1) of Theorem 4.1, the measure μ is Lévy measure of ψ_μ. □

Theorem 4.3. *Let (μ_t) be a convolution semigroup on Q with an associated negative definite function $\psi : \hat{Q} \to C$, and Lévy measure μ. Assume that μ is positive symmetric on $Q \setminus \{e\}$ such that*

$$\int_{Q\setminus\{e\}} (1-\mathrm{Re}\chi(r)d\mu(r) < \infty \qquad \textit{for } \chi \in \hat{Q}.$$

Then

$$\psi(\chi) = C + ih(\chi) + q(\chi) + \int_{Q\setminus\{e\}} (1-\mathrm{Re}\chi(r))d\mu(r) \tag{4.6}$$

for $\chi \in \hat{Q}$, where C is a nonnegative constant, $h : \hat{Q} \to R$, is a continuous homomorphism, $q : \hat{Q} \to R$, is a nonnegative quadratic form. Moreover c, h, q in (4.6) *are determined uniquely by $(\mu_t)_{t>0}$ such that $c = \psi(1), h = \mathrm{Im}\psi$, and*

$$q(\chi) = \lim_{n\to\infty}\left[\frac{(R^n_\chi\psi)(\chi)}{4n^2} + \frac{(R^n_{\overline{\chi}}\psi)(\chi)}{2n}\right] \tag{4.7}$$

Proof. Since μ is symmetric, by corollary 4.1, $h = \mathrm{Im}\psi$ is a homomorphism, and $ih \in N(\hat{Q})$. Let $C = \psi(1)$ then by Corollary 2.1, the function $\psi - CI \in N(\hat{Q})$ with the Lévy measure μ. Further the function $\psi' = \psi - CI - ih \in N(\hat{Q})$ associated to the same Lévy measure μ.
By Theorem 4.2, the function

$$\psi_\mu(\chi) = \int_{Q\setminus\{e\}} (1-\mathrm{Re}\chi(r))d\mu(r)$$

is finite at all $\chi \in \hat{Q}$, and by Lemma 4.4, it follows that the function $q = \psi' - \psi_\mu$ is continuous, real valued, symmetric and $q(1) = 0$. Using Lemma 4.2, for $\sigma \in S$ we get

$$\psi' * \sigma - \psi' = \psi * \sigma - \psi$$

and one can easily obtain

$$\psi_\mu * \sigma - \psi_\mu = \int_{Q\{e\}} \mathrm{Re}\chi(r)(1-\tilde{\sigma}(r))d\mu(r), \quad \sigma \in S' \tag{4.8}$$

Then by (4.1) and (4.8), we see that

$$q * \sigma - q = (\psi' - \psi_\mu) * \sigma - (\psi' - \psi_\mu) = \hat{\mu}_\sigma - (\psi_\mu * \sigma - \psi_\mu) = \mu_\sigma(\{e\}) \geq 0$$

By Lemma 4.3 this implies that q is a nonnegative quadratic form on $\hat{Q}$ and the first requirement is proved.
Secondly, given (4.7), of course $C = \psi(1)$, and $h = \mathrm{Im}\psi$. Denote again $\psi_\mu(\chi) = \int_{Q\setminus\{e\}}(1 - \mathrm{Re}\chi(r))d\mu(r)$. By Lemma 4.4 ψ_μ is negative definite. By Corollary 2.2

$$\begin{aligned}(4.9)\qquad & q(\chi) - \frac{1}{2}(R_{\overline{\chi}}q)(\chi) \\ = \; & \lim_{n\to\infty} \frac{(R_\chi^n q)(\chi)}{4n^2} \\ = \; & \lim_{n\to\infty} \frac{(R_\chi^n \psi)(\chi)}{4n^2} - \lim_{n\to\infty} \frac{(R_\chi \psi_\mu)(\chi)}{4n^2} \\ = \; & \lim_{n\to\infty} \frac{(R_\chi^n \psi)(\chi)}{4n^2} - \lim_{n\to\infty} \frac{1}{4n^2}\int (1 - \mathrm{Re}(\chi(r))^{2n})d\mu(r)\end{aligned}$$

Since Q is locally compact, the Fubini's theorem is available by the inequality (4.2).
Obviously

$$\lim_{n\to\infty} \frac{1}{4n^2}(1 - \mathrm{Re}(\chi(r))^{2n}) = 0 \quad \text{for each } r \in Q.$$

If $\chi(r) \neq 0$, let $0 < \rho \leq 1$ and $-\pi \leq \theta \leq \pi$ such that $\chi(r) = \rho \exp i\theta$. Then for $n \in N$, $\dfrac{\sin n\theta}{n\theta}$ is bounded away from Q on $\left[\dfrac{\pi}{2}, \pi\right]$, and

$$\frac{1}{4n^2}(1 - \cos 2n\theta) = \frac{1}{2}\left(\frac{\sin n\theta}{n\theta}\right)^2 \left(\frac{\theta}{\sin\theta}\right)^2 \left(\frac{1 - \cos 2\theta}{2}\right) \leq C(1 - \cos 2\theta)$$

where $C \geq 0$ is a constant.
Also

$$\frac{1 - \rho^{2n}}{4n^2} \leq \frac{1-\rho}{2n} \leq \frac{1-\rho^2}{2}.$$

Then

$$\begin{aligned}\frac{1}{4n^2}(1 - \mathrm{Re}(\chi(r))^{2n}) &= \frac{1}{4n^2}(1 - \rho^{2n}) + \frac{\rho^{2n}}{4n^2}(1 - \cos 2n\theta) \\ &\leq \frac{1}{2}(1 - \rho^2) + \rho^{2n} C(1 - \cos 2\theta) \\ &\leq \frac{1}{2}(1 - \rho^2) + C(\rho^2 - \rho^2 \cos 2\theta) \\ &\leq \frac{1}{2}(1 - \rho^2) + C(1 - \mathrm{Re}(\chi(r))^2)\end{aligned}$$

Then by the theorem of domainated convergence

$$\frac{1}{4n^2}\int (1 - \mathrm{Re}(\chi(r))^{2n})d\mu(r) = 0,$$

and (4.9) yields

$$q(\chi) = \lim_{n\to\infty} \frac{(R^n_\chi \psi)(\chi)}{4n^2} + \frac{1}{2}(R_{\overline{\chi}} q)(\chi). \tag{4.10}$$

By means of (2.4), $(R_{\overline{\chi}} q)(\chi) = \lim_{n\to\infty} \dfrac{(R^n_{\overline{\chi}} q)(\chi)}{2n}$

$$\begin{aligned}(R_{\overline{\chi}} q)(\chi) &= \lim_{n\to\infty} \frac{(R^n_{\overline{\chi}} q)(\chi)}{2n} \\ &= \lim_{n\to\infty} \frac{(R^n_{\overline{\chi}} \psi)(\chi)}{2n} - \lim_{n\to\infty} \frac{1}{2n} \int_{Q\setminus\{e\}} (1 - |\chi(r)|^{2n}) d\mu(r)\end{aligned}$$

Since

$$\frac{1}{2n}(1 - |\chi(r)|^{2n}) \leq (1 - |\chi(r)|^2),$$

Applying the dominated convergence theorem again, we have

$$\lim_{n\to\infty} \frac{1}{2n} \int_{Q\setminus\{e\}} (1 - |\chi(r)|^{2n}) d\mu(r) = 0,$$

and

$$(R_{\overline{\chi}} q)(\chi) = \lim_{n\to\infty} \frac{(R^n_{\overline{\chi}} \psi)(\chi)}{2n}$$

subistituting in (4.10), the equality (4.7) is stablished. □

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