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LÉVY–KHINTCHINE REPRESENTATION OF THE GEOMETRIC MEAN OF MANY POSITIVE NUMBERS AND APPLICATIONS

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Abstract. In the paper, the authors establish, by Cauchy integral formula in the theory of complex functions, Lévy-Khintchine representation for the geometric mean of many positive numbers, find that the geometric mean of many positive numbers is a complete Bernstein function, and supply a new proof of the well known arithmetic-geometric mean inequality.

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