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LÉVY RANDOM MEASURES

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A Lévy random measure is characterized by a conditional independence structure analogous to the Markov property. Here we introduce Lévy random measures and present their basic properties. Preservation of the Lévy property under transformations of random measures (e.g., change of variable, passage to a limit) and under transformations of the probability laws of random measures is investigated. One random measure is said to be a submeasure of a second random measure if its probability law is absolutely continuous with respect to that of the second. We show that if the second measure is a Lévy random measure then the submeasure is Lévy if and only if the Radon–Nikodym derivative satisfies a natural factorization condition. These results are applied to extend the theories of Gibbs states on bounded sets in $\mathbb{R}^\nu$ and Z^ν.

0. Introduction. In this paper we introduce a class of random measures, Lévy random measures, that possess a conditional independence structure reminiscent of the Markov property. Our results mainly concern preservation of the Lévy property under transformations of random measures and their probability laws. Section 1 of the paper contains the basic definition, examples, and some elementary results. In Section 2 we deal with preservation of the Lévy property under certain transformations of random measures, namely change of variable, construction of product measures, compounding, and passage to a limit. Results in Section 3 concern construction of submeasures, which is effected by an absolutely continuous change of probability law. A natural factorization condition akin to the defining property of a multiplicative functional of a Markov process is shown to be necessary and sufficient in order that a submeasure of a Lévy random measure itself be a Lévy random measure. Finally, in Section 4 we apply the results of the other sections to Gibbs random measures, which are of interest in statistical mechanics as models of distributions of particles in space. Our treatment extends and simplifies previous treatments.

Throughout our notations for measurability are based on [2]. Hence, $Y \in \mathscr{F}$ means that Y is $\mathscr{F}$-measurable; if Y is bounded we write $Y \in b\mathscr{F}$. A nonnegative $Y \in \mathscr{F}$ is distinguished by writing $Y \in p\mathscr{F}$.

1. Lévy random measures. Let $(E, \mathscr{E})$ be a LCCB space and let $(\Omega, \mathscr{M}, P)$ be a probability space. We shall deal with random measures on $(E, \mathscr{E})$, in the following sense.

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(1.1) DEFINITION. A *random measure* on $(E, \mathscr{E})$ over the probability space $(\Omega, \mathscr{M}, P)$ is a mapping $M: \mathscr{E} \times \Omega \to \bar{\mathbb{R}}_+ = [0, \infty]$ such that

(a) For each $A \in \mathscr{E}$ the mapping $\omega \to M(A, \omega)$ is a random variable;
(b) Almost surely, $A \to M(A, \omega)$ is a measure on $\mathscr{E}$.

Of particular importance are random measures corresponding to point processes without multiple points; we call these *simple* random measures. A random measure M on $(E, \mathscr{E})$ is said to be simple provided that

(a) M is σ-finite and purely atomic;
(b) All atoms of M are of mass one.

A simple random measure M can be represented in the form

$$M = \sum_{i \in Z} \varepsilon_{X_i}$$

where for each $i \in Z$, $X_i \in \mathscr{M}/\mathscr{E}$. Here ε_x denotes the measure which has an atom of mass one at x, and no other mass. Among simple random measures are certain Poisson random measures [3] and the Gibbs random measures discussed in Section 4.

Below we sometimes write $M(f)$ for $\int f\, dM$.

We now formulate the Lévy property of conditional independence for random measures. Let M be a random measure on $(E, \mathscr{E})$ and for each $A \in \mathscr{E}$ let

$$\mathscr{F}_A = \sigma(M(B): B \in \mathscr{E}, B \subset A);$$

that is, $\mathscr{F}_A$ is the σ-algebra describing the behavior of M on A and its subsets.

(1.2) DEFINITION. Let $\mathscr{B}$ be a class of proper subsets of E in $\mathscr{E}$; M will be said to be a *Lévy random measure* with respect to $\mathscr{B}$ if for each $A \in \mathscr{B}$ there exists a measurable subset A' of A^c such that

(a) $A' \neq A^c$;
(b) $\mathscr{F}_A$ and $\mathscr{F}_{A^c}$ are conditionally independent given $\mathscr{F}_{A'}$;
(c) If $B \subset A^c$ and $\mathscr{F}_A$ and $\mathscr{F}_{A^c}$ are conditionally independent given $\mathscr{F}_B$, then $A' \subset B$.

The structure of $\mathscr{B}$ is discussed in Proposition (1.5) below.

REMARKS. (1) The meaning of the three conditions in (1.2) is the following. Condition (a) is to eliminate trivialities; without it every random measure satisfies (b) and (c) with $A' = A^c$. Condition (b) is the main restriction: it imposes on the random measure M a conditional independence structure analogous to the Markov property; cf. Example (1.4) below. Condition (c) expresses minimality of the sets A'.

(2) Conditional independence of $\mathscr{F}_A$ and $\mathscr{F}_{A^c}$ given $\mathscr{F}_{A'}$ means, of course, that

$$E[XZ \mid \mathscr{F}_{A'}] = E[X \mid \mathscr{F}_{A'}]E[Z \mid \mathscr{F}_{A'}]$$

for all $X \in p\mathscr{F}_A$, $Z \in p\mathscr{F}_{A^c}$. Equivalent forms of the definition (appearing in [8, page 30]) and various monotone class theorem simplifications thereof, are used below without special comment.

(3) As a convention we take $\mathscr{F}_\phi = \{\phi, \Omega\}$; it is explicitly permitted that A' be empty for some sets A, in which case (1.2b) requires that $\mathscr{F}_A$ and $\mathscr{F}_{A^c}$ be independent. See Example (1.3).

(4) The pair

$$\mathbf{L}_M = (\mathscr{B}, \{(A, A') : A \in \mathscr{B}\})$$

is called the *Lévy space* of a Lévy random measure M. If $\mathbf{L}_{M_1}$ and $\mathbf{L}_{M_2}$ are Lévy spaces such that

(i) $\mathscr{B}_1 \subset \mathscr{B}_2$;
(ii) $A \in \mathscr{B}_1$ implies $A_1' \supset A_2'$,

where A_i' corresponds to A in $\mathbf{L}_{M_i}$, then we write $\mathbf{L}_{M_1} < \mathbf{L}_{M_2}$. Introduction of this notation simplifies statements of some of the theorems below. Observe that to show that M_2 is a Lévy random measure with $\mathbf{L}_{M_1} < \mathbf{L}_{M_2}$ it suffices to show that $\sigma(M_2(B) : B \subset A_1)$ and $\sigma(M_2(B) : B \subset A_1^c)$ are conditionally independent given $\sigma(M_2(B) : B \subset A_1')$ for every $A_1 \in \mathscr{B}_1$.

We now give two examples.

(1.3) EXAMPLE. *Additive random measures.* Let M be an additive random measure on $(E, \mathscr{E})$; that is, $M(A_1), \cdots, M(A_n)$ are independent whenever $A_1, \cdots, A_n$ are disjoint. Then M is a Lévy random measure with the maximal Lévy space

$$\mathbf{L}_M = (\{A : A \neq \phi, E\}, \{(A, \phi) : A \neq \phi, E\}) .$$

Additive random measures thus have a maximal Lévy property. In particular, every deterministic measure is a Lévy random measure.

(1.4) EXAMPLE. *Markov random fields on Z^ν.* These processes are studied in detail in [4], [13], and [14]; most of the results are summarized in [11]. In [14] a Markov random field on a finite subset E of Z^ν, taking values in $\{0, 1\}$, is a stochastic process $\{X_t : t \in E\}$ such that if Λ is a subset of E not containing a point t but containing the set η_t of its 2ν nearest neighbors, then

$$P\{X_t = 1 \mid X_s, s \in \Lambda\} = P\{X_t = 1 \mid X_s, s \in \eta_t\} .$$

By defining

$$M(A) = \textstyle\sum_{t \in A} X_t$$

we obtain a random measure M on E which is a Lévy random measure with Lévy space

$$\mathbf{L}_M = (\{\{t\} : t \in E\}, \{(\{t\}, \eta_t) : t \in E\}) .$$

In many respects the random measure view of these processes is the most natural and general. We refer the reader to Section 4 for further discussion of these random measures and the corresponding random measures on $\mathbb{R}^\nu$.

The structure of the family $\mathscr{B}_M$ corresponding to a Lévy random measure M is elucidated in the following proposition.

(1.5) PROPOSITION. *If M is a Lévy random measure then*

(a) *$\mathscr{B}_M$ is hereditary in the sense that if $A_1 \in \mathscr{B}_M$ and $A_2 \subset A_1$ then $A_2 \in \mathscr{B}_M$;*
(b) *$\mathscr{B}_M$ is closed under countable intersections;*
(c) *If $A_1, A_2 \in \mathscr{B}_M$ and $(A_1 \cap A_2') \cup (A_1' \cap A_2) = \phi$ then $A_1 \cup A_2 \in \mathscr{B}_M$.*

PROOF.

(a) It is required, of course, that A_2 belong to $\mathscr{E}$. We shall show that $\mathscr{F}_{A_2}$ and $\mathscr{F}_{A_2^c}$ are conditionally independent given $\mathscr{H} = \mathscr{F}_{(A_1-A_2)\cup A_1'}$, from which it follows that $A_2 \in \mathscr{B}_M$ and that $A_2' \subset A_1' \cup (A_1 - A_2)$. Let X, Y, Z and W be nonnegative and measurable with respect to $\mathscr{F}_{A_2}$, $\mathscr{F}_{A_1-A_2}$, $\mathscr{F}_{A_1'}$ and $\mathscr{F}_{A_1^c}$, respectively. Then

$$\begin{aligned} E[XW(YZ)] &= E[E[XY \mid \mathscr{F}_{A_1'}]E[W \mid \mathscr{F}_{A_1'}]Z] \\ &= E[E[E[XY \mid \mathscr{H}] \mid \mathscr{F}_{A_1'}]E[W \mid \mathscr{F}_{A_1'}]Z] \\ &= E[E[YE[X \mid \mathscr{H}] \mid \mathscr{F}_{A_1'}]E[W \mid \mathscr{F}_{A_1'}]Z] \\ &= E[E[X \mid \mathscr{H}]E[W \mid \mathscr{F}_{A_1'}]YZ] \\ &= E[E[X \mid \mathscr{H}]E[W \mid \mathscr{H}]YZ] , \end{aligned}$$

where the first and fifth equalities are by the Lévy property for A_1. By the monotone class theorem we conclude that $E[XW \mid \mathscr{H}] = E[X \mid \mathscr{H}]E[W \mid \mathscr{H}]$ and hence the Lévy property holds for A_2.

(b) is an immediate consequence of (a).

(c) We shall show that $(A_1 \cup A_2)' = A_1' \cup A_2'$ under the stated assumptions. Let X, U, Y, Z_1, Z_2, and W be nonnegative and measurable, respectively, with respect to $\mathscr{F}_{A_1-A_2}$, $\mathscr{F}_{A_1\cap A_2}$, $\mathscr{F}_{A_2-A_1}$, $\mathscr{F}_{A_1'}$, and $\mathscr{F}_B$, where $B = (A_1 \cup A_2)^c - (A_1' \cup A_2')$. Let $\mathscr{H} = \mathscr{F}_{A_1' \cup A_1'}$. By straightforward application of the Lévy properties for A_1 and A_2,

$$E[XUY \mid \mathscr{H}] = E[XU \mid \mathscr{F}_{A_1'}]E[Y \mid \mathscr{F}_{A_1'}] . \tag{1.6}$$

Calculations using (1.6) then show that

$$E[(XUY \cdot W)Z_1Z_2] = E[E[W \mid \mathscr{H}]E[XUY \mid \mathscr{H}]Z_1Z_2] ,$$

which shows that $\mathscr{F}_{A_1\cup A_2}$ and $\mathscr{F}_{(A_1\cup A_2)^c}$ are conditionally independent given $\mathscr{H}$.

The preceding shows that $(A_1 \cup A_2)' \subset A_1' \cup A_2'$; if the inclusion is strict then either A_1' or A_2' fails to satisfy the minimality property (1.2c). □

REMARK. In general $\mathscr{B}_M$ is not closed under even finite unions.

We denote by K the family of all measures m on $\mathscr{E}$ such that $m(A) < \infty$ for every compact set A; elements of K are called *Radon measures* on $\mathscr{E}$. We endow K with the topology of vague convergence and associated Borel σ-algebra $\mathscr{K}$; cf. [7] for details. It will often be convenient to choose K as a canonical sample space for random measures; the proposition below indicates that this does not destroy the Lévy property. The proof is direct and omitted.

(1.7) PROPOSITION. *Let M be a Lévy random measure such that $P\{M \in K\} = 1$. Then the coordinate random measure N on K is a Lévy random measure over $(K, \mathscr{K}, PM^{-1})$ and $\mathbf{L}_N = \mathbf{L}_M$.*

COROLLARY. *If M and M' are identically distributed random measures on $(E, \mathscr{E})$, each of which lies in K almost surely, then M is a Lévy random measure with Lévy space $\mathbf{L}$ if and only if N is.*

The final result of this section shows that the asymmetry between A and A^c in the Definition (1.2) is essentially only apparent.

(1.8) PROPOSITION. *Let M be a Lévy random measure. Then for each $A \in \mathscr{B}_M$, $\mathscr{F}_{A \cup A'}$ and $\mathscr{F}_{A^c}$ are conditionally independent given $\mathscr{F}_{A'}$.*

Straightforward application of the Lévy property and the monotone class theorem yields the proof of (1.8).

2. Preservation of the Lévy property. Most transformations and limiting processes discussed in the context of random measures do not, when applied to Lévy random measures, lead to another random measure that is Lévy at all, let alone with the same Lévy space, except possibly in the case of additive random measures. For example, the superposition of two Lévy random measures with the same Lévy space is not, in general, a Lévy random measure. The reason is the same reason why the sum of Markov processes is not always Markov. Similarly, a mixture of Lévy random measures is not a Lévy random measure; indeed a mixture of additive random measures need not be additive. Simplification (i.e., transformation of a purely atomic random measure to a simple random measure with atoms at the same locations) also does not preserve the Lévy property, although a particular inverse to this operation (namely, compounding) does; see Theorem (2.3) below. The first two results of this section show that change of variable preserves the Lévy property and that the product of independent Lévy random measures is a Lévy random measure with a particular Lévy space. Finally, we consider preservation of the Lévy property under limiting operations.

(2.1) THEOREM. *Let M be a Lévy random measure on $(E, \mathscr{E})$, let E_1 be another* LCCB *space, and let $f\colon E \to E_1$ be a one-to-one function measurable with respect to $\mathscr{E}$ and the Borel σ-algebra $\mathscr{E}_1$ on E_1. Then $N = Mf^{-1}$ is a Lévy random measure on $\mathscr{E}_1$ with*

$$\mathscr{B}_N = \{f(A) : A \in \mathscr{B}_M\}$$

and

$$f(A)' = f(A'), \qquad A \in \mathscr{B}_M .$$

PROOF. By the Kuratowski theorem [10, page 21], $f(A) \in \mathscr{E}_1$ for each $A \in \mathscr{E}$.

Define

$$\mathscr{G}_D = \sigma(N(C) : C \subset D)$$

and observe that $\mathscr{G}_D = \mathscr{F}_{f^{-1}(D)}$, where $\mathscr{F}_A = \sigma(M(B) : B \subset A)$. Choose $f(A) \in \mathscr{B}_N$, $Y \in p\mathscr{G}_{f(A)}$ and $Z \in p\mathscr{G}_{f(A)^c} = p\mathscr{G}_{f(A^c)}$. Then

$$\begin{aligned} E[YZ \mid \mathscr{G}_{f(A)'}] &= E[YZ \mid \mathscr{G}_{f(A')}] \\ &= E[YZ \mid \mathscr{F}_{f^{-1}(f(A'))}] \\ &= E[YZ \mid \mathscr{F}_{A'}] \\ &= E[Y \mid \mathscr{F}_{A'}]E[Z \mid \mathscr{F}_{A'}] \\ &= E[Y \mid \mathscr{G}_{f(A)'}]E[Z \mid \mathscr{G}_{f(A)'}] . \end{aligned}$$

Here the third equality uses the fact that f is one-to-one, which implies that $f^{-1}(f(A')) = A'$, the fourth equality is by the Lévy property of M and the fifth equality traces in reverse order the first three equalities. ☐

In the next result we see that the product of independent Lévy random measures itself possesses the Lévy property with a particular Lévy space. We omit the proof, which follows the patterns used above; the reader may find a picture useful in understanding how one arrives at $(A \times C)'$.

(2.2) THEOREM. *Let M and N be independent Lévy random measures on* LCCB *spaces E and E' respectively, with Lévy spaces $\mathbf{L}_M$ and $\mathbf{L}_N$. Then the product random measure $M \times N$ is a Lévy random measure with $\mathscr{B}_{M\times N} \subset \{A \times C : A \in \mathscr{B}_M, C \in \mathscr{B}_N\}$ and for such A and C*

$$(A \times C)' \subset (A \times C)^c - (A \cup A')^c \times (C \cup C')^c .$$

When M and N are additive, equality holds in the preceding expression for $(A \times C)'$.

The compounding transformation discussed below is that used to construct certain classes of compound Poisson random measures [3] and in particular a compound Poisson process from a simple Poisson process.

(2.3) THEOREM. *Let $M = \sum \varepsilon_{X_i}$ be a simple Lévy random measure on $\mathscr{E}$. Let (W_i) be a sequence of independent, identically distributed, strictly positive random variables such that (W_i) and M are independent. Then*

(2.4) $$N = \sum W_i \varepsilon_{X_i}$$

is a Lévy random measure and $\mathbf{L}_M < \mathbf{L}_N$.

PROOF. Let λ denote the common distribution of the W_i and let $\{W_x; x \in E\}$ be independent, identically distributed random variables with common distribution λ, such that $\{W_x\}$ and M are independent, and let N' be the random measure defined by

(2.5) $$N'(A) = \sum_{x \in A} W_x M(\{x\}) .$$

Since $M(\{x\}) \neq 0$ for at most countably many x, there is no difficulty with existence and measurability of the right-hand side of (2.5); the sum may, of course, be infinite. We assert that N defined by (2.4) and N' defined by (2.5) are identically distributed. To establish this, it suffices (cf. [7]) to show that

$$E[e^{-N(f)}] = E[e^{-N'(f)}]$$

for each $f \in p\mathscr{E}$, which can be shown by direct computations.

It is evident that $P\{N \in K\} = 1$, so by Proposition (1.7) the theorem follows if N' is a Lévy random measure with $\mathbf{L}_M < \mathbf{L}_{N'}$. For each $A \in \mathscr{E}$ let $\mathscr{F}_A = \sigma(M(B) : B \subset A)$, $\mathscr{G}_A = \sigma(\mathscr{F}_A; W_x : x \in A)$, $\mathscr{H}_A = \sigma(N'(B) : B \subset A)$; then $\mathscr{F}_A \subset \mathscr{H}_A \subset \mathscr{G}_A$. Choose $A \in \mathscr{B}_M$, disjoint sets $A_1, \cdots, A_k \subset A$, and functions $g_1, \cdots, g_k$ from $[0, \infty]$ to $[0, \infty)$. Then

$$\begin{aligned} E[\textstyle\prod_1^k g_i \circ N'(A_i) \mid \mathscr{H}_{A^c}] &= E[\textstyle\prod_1^k g_i(\sum_{x \in A_i} W_x M(x)) \mid \mathscr{H}_{A^c}] \\ &= E[E[\textstyle\prod_1^k g_i(\sum_{x \in A_i} W_x M(x)) \mid \mathscr{G}_{A^c}] \mid \mathscr{H}_{A^c}] \\ &= E[E[\textstyle\prod_1^k g_i(\sum_{x \in A_i} W_x M(x)) \mid \mathscr{F}_{A'}] \mid \mathscr{H}_{A^c}] \\ &= E[\textstyle\prod_1^k g_i(\sum_{x \in A_i} W_x M(x)) \mid \mathscr{F}_{A'}] \\ &= E[\textstyle\prod_1^k g_i(\sum_{x \in A_i} W_x M(x)) \mid \mathscr{H}_{A'}] , \end{aligned}$$

which completes the proof. □

REMARKS. (1) The assumption that the W_i be strictly positive cannot be suppressed in general, as the following example indicates. Let $E = \{1, 2, 3\}$ and let $\{M(1), M(2), M(3)\}$ be a Markov chain with state space $\{0, 1\}$, initial distribution $(\frac{1}{2}, \frac{1}{2})$ and transition kernel

$$K = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

The Lévy property holds for M with $A = \{3\}$ and $A' = \{2\}$. Let W_1, W_2, W_3 be independent and identically distributed with $P\{W_i = 1\} = P\{W_i = 0\} = \frac{1}{2}$. Straightforward computations then verify that

$$P\{N(3) = 1 \mid N(2) = 0\} = \tfrac{1}{4} ,$$

while

$$P\{N(3) = 1 \mid N(1) = 0, N(2) = 0\} = \tfrac{5}{16} .$$

Positivity of the W_i can be suppressed if M is additive.

(2) The inverse operation to that of Theorem (2.3), namely *simplification*, which transforms a purely atomic random measure $M = \sum Y_i \varepsilon_{X_i}$ into the simple random measure $N = \sum \varepsilon_{X_i}$ does not, in general, preserve the Lévy property; examples substantiating this assertion are easily constructed. The transformation $M \to N$ is deterministic but not one-to-one, so should not be expected to preserve the Lévy property.

The final result of this section concerns preservation of the Lévy property under limits. For simplicity we assume that all random measures involved lie in K almost surely.

(2.6) THEOREM. *Let $M^1, M^2, \cdots$ be Lévy random measures on $\mathscr{E}$, all with the same Lévy space* $\mathbf{L}$, *and let M be another random measure on $\mathscr{E}$. Let*

$$\mathscr{F}_A^n = \sigma(M^n(B) : B \subset A)$$
$$\mathscr{F}_A = \sigma(M(B) : B \subset A) .$$

If

(a) *$M^n \to M$ in probability as random elements of K;*
(b) *$\mathscr{F}_A^n \uparrow \mathscr{F}_A$ for each $A \in \mathscr{E}$,*
then M is a Lévy random measure and $\mathbf{L} < \mathbf{L}_M$.

Previously established patterns yield a proof of (2.6) as well; the only subtlety involved is an appeal to a generalized form of the martingale convergence theorem [9, page 143].

REMARKS. (1) If one assumes that $\mathscr{F}_{A'}$ be complete for each A' then (2.6b) need hold only for the sets A^c and A' arising from the Lévy space $\mathbf{L}$. In fact, one then need not assume that $\mathscr{F}_{A'}^n \uparrow \mathscr{F}_{A'}$ but only that $\mathscr{F}_{A'}^n \subset \mathscr{F}_{A'}$, for each A'.

(2) A situation in which the hypotheses hold is the following. Suppose that each A and A' are compact, let A_n be a sequence of compact sets increasing to E and put $M^n(B) = M(B \cap A_n)$. Verification of the Lévy property of M may be difficult to effect directly; by Theorem (2.6) it follows from the Lévy property of each M^n.

(3) As usual, additivity constitutes a special case. If $M_1, M_2, \cdots$ are additive random measures on $\mathscr{E}$ and $M_n \to M$ in distribution, then M is additive.

3. Construction of submeasures. In this section we discuss in some detail a construction that transforms (the probability law of) one random measure into (that of) another and which, subject to a factorization condition analogous to the defining property of a multiplicative functional of a Markov process, preserves the Lévy property. We also prove a converse: if both random measures are Lévy, the factorization condition is satisfied. As applications, we study in Section 4 the Lévy properties of a class of random measures on $\mathbb{R}^\nu$ and Z^ν arising in statistical mechanics.

For the remainder of this section we denote by N a Lévy random measure on $(E, \mathscr{E})$ with Lévy space $\mathbf{L}_N$. It is assumed throughout that N is a Radon measure almost surely.

(3.1) DEFINITION. Let W be a nonnegative random variable in $\mathscr{F} = \sigma(N)$ such that

$$0 < Z = E[W] < \infty . \tag{3.2}$$

A random measure M on $\mathscr{E}$ such that for each $A \in \mathscr{K}$

$$P\{M \in A\} = \frac{1}{Z} E[W; \{N \in A\}] , \tag{3.3}$$

where Z is given by (3.2), is called the *submeasure* of N generated by W. Here $E[X; \Lambda] = E[X \cdot 1_\Lambda]$. Examples are discussed in detail in Section 4, so we present none at this point.

(3.4) DEFINITION. Let $\mathbf{L} = (\mathscr{B}, \{(A, A') : A \in \mathscr{B}\})$ be a Lévy space such that $\mathbf{L} < \mathbf{L}_N$. A nonnegative random variable $W \in \mathscr{F}$ is an $\mathbf{L}$-*multiplicative functional* of N if for every $A \in \mathscr{B}$, W admits a factorization of the form

$$W = W_1 W_2 W_3 \quad \text{a.s.,} \tag{3.5}$$

where $W_1 \in p\mathscr{F}_{A \cup A'}$, $W_2 \in p\mathscr{F}_{A'}$, $W_3 \in p\mathscr{F}_{A^c}$. Of course, $\mathscr{F}_B = \sigma(N(D) : D \subset B)$.

REMARKS. (1) Observe that we do not require that $W_1 \in \mathscr{F}_A$. Theorems (3.6) and (3.12) below show that in order that the submeasure M of N generated by W be a Lévy random measure with $\mathbf{L} < \mathbf{L}_M$ it is necessary and sufficient that W be an $\mathbf{L}$-multiplicative functional of N. This would not be true if one required that $W_1 \in \mathscr{F}_A$. See also Proposition (1.8).

(2) The null set in (3.5) is permitted to depend on A.

We now present the main results of this section.

(3.6) THEOREM. *Let N be a Lévy random measure and let M be the submeasure of N generated by an $\mathbf{L}$-multiplicative functional W. Then M is a Lévy random measure with $\mathbf{L} < \mathbf{L}_M$.*

PROOF. By (3.3) and Proposition (1.7) we may take N and M to be defined on the same sample space $(K, \mathscr{K})$, but with respect to different probability measures, in the following manner. Let L denote the coordinate random measure on $(K, \mathscr{K})$: that is, $L(A, \omega) = \omega(A)$, and let P be the probability measure on $\mathscr{K}$ with respect to which L is the underlying Lévy random measure N. It follows from (3.3) that with respect to the probability measure Q defined by

$$Q(A) = \frac{1}{Z} \int_A W \, dP \,, \qquad A \in \mathscr{K} \,, \tag{3.7}$$

L is the submeasure M generated by N and W. It then follows from (3.7) that for any $Y \in p\mathscr{K}$ and any sub-σ-algebra $\mathscr{G}$ of $\mathscr{K}$

$$E_Q[Y \mid \mathscr{G}] = \frac{E_P[YW \mid \mathscr{G}]}{E_P[W \mid \mathscr{G}]} \,. \tag{3.8}$$

$E_P[\cdot]$ denotes expectation with respect to P; $E_Q[\cdot]$ is that with respect to Q. Since

$$\begin{aligned} Q\{E_P[W \mid \mathscr{G}] = 0\} &= Z^{-1} E_P[W; \{E_P[W \mid \mathscr{G}] = 0\}] \\ &= Z^{-1} E_P[E_P[W \mid \mathscr{G}]; \{E_P[W \mid \mathscr{G}] = 0\}] \\ &= 0 \,, \end{aligned}$$

we may safely ignore, in (3.8) and similar expressions below, the set on which the denominator of the right-hand side of (3.8) is zero.

Suppose now that $A \in \mathscr{B}$, that $Y \in p\mathscr{F}_A$ and that $Z \in p\mathscr{F}_{A^c}$, where $\mathscr{F}_A = \sigma(L(B) : B \subset A)$. Then in the notation of (3.5),

$$
\begin{aligned}
E_Q[YZ \mid \mathscr{F}_{A'}] &= \frac{E_P[YZW \mid \mathscr{F}_{A'}]}{E_P[W \mid \mathscr{F}_{A'}]} \\
&= \frac{E_P[YZW_1W_2W_3 \mid \mathscr{F}_{A'}]}{E_P[W_1W_2W_3 \mid \mathscr{F}_{A'}]} \\
&= \frac{W_2E_P[YW_1 \mid \mathscr{F}_{A'}]E_P[ZW_3 \mid \mathscr{F}_{A'}]}{W_2E_P[W_1 \mid \mathscr{F}_{A'}]E_P[W_3 \mid \mathscr{F}_{A'}]} \\
&= \frac{E_P[YW_1 \mid \mathscr{F}_{A'}]E_P[ZW_3 \mid \mathscr{F}_{A'}]}{E_P[W_1 \mid \mathscr{F}_{A'}]E_P[W_3 \mid \mathscr{F}_{A'}]},
\end{aligned} \tag{3.9}
$$

where the first equality is by (3.8), the second is by (3.5) and the third is by the Lévy property of L with respect to P. We have also invoked Proposition (1.8).

Similarly,

$$
\begin{aligned}
E_Q[Y \mid \mathscr{F}_{A'}] &= E_P[YW \mid \mathscr{F}_{A'}]/E_P[W \mid \mathscr{F}_{A'}] \\
&= \frac{W_2E_P[YW_1 \mid \mathscr{F}_{A'}]E_P[W_3 \mid \mathscr{F}_{A'}]}{W_2E_P[W_1 \mid \mathscr{F}_{A'}]E_P[W_3 \mid \mathscr{F}_{A'}]} \\
&= E_P[YW_1 \mid \mathscr{F}_{A'}]/E_P[W_1 \mid \mathscr{F}_{A'}]
\end{aligned} \tag{3.10}
$$

and, in the same way,

$$
E_Q[Z \mid \mathscr{F}_{A'}] = E_P[ZW_3 \mid \mathscr{F}_{A'}]/E_P[W_3 \mid \mathscr{F}_{A'}] .
$$

Comparing (3.9) with (3.10) and (3.11), we see that

$$
E_Q[YZ \mid \mathscr{F}_{A'}] = E_Q[Y \mid \mathscr{F}_{A'}]E_Q[Z \mid \mathscr{F}_{A'}] ,
$$

which establishes the Lévy property of the submeasure M and the inclusion $\mathbf{L} < \mathbf{L}_M$. □

REMARKS. (1) There need be no particular relation between $\mathbf{L}_N$ and $\mathbf{L}_M$ in general. If N is additive and $\mathbf{L} \neq \mathbf{L}_N$ and $P\{W = 0\} = 0$, then M and N are each submeasures of the other (N is the submeasure of M generated by the $\mathbf{L}$-multiplicative functional W^{-1}) and in general $\mathbf{L}_M \neq \mathbf{L}_N$. If $\mathbf{L} = \mathbf{L}_N$ then the Theorem shows that $\mathbf{L}_N < \mathbf{L}_M$. If $\mathbf{L} = \mathbf{L}_N$ and $P\{W = 0\} = 0$ then $\mathbf{L}_N = \mathbf{L}_M$ by symmetry.

(2) Since (3.5) is analogous to the defining property of a multiplicative functional of a Markov process [2, Definition (III. 1.1)], Theorem (3.6) is the analog, in the context of Lévy random measures, of the possession of the Markov property by a subprocess of a Markov process [2, Theorem (III. 3.3)]. This connection furthers the analogy between the Lévy property and the Markov property. See also Theorem 10.4 of [6].

The following result is a converse to Theorem (3.6): if a submeasure of a Lévy random measure is itself a Lévy random measure, then the generating functional is multiplicative. This result is analogous to Theorem (III. 2.3) of [2].

(3.12) THEOREM. *Let N be a Lévy random measure with Lévy space $\mathbf{L}_N$ and let M be a submeasure of N such that M is a Lévy random measure with $\mathbf{L}_M < \mathbf{L}_N$. Then there exists an $\mathbf{L}_M$-multiplicative functional W of N that generates M.*

PROOF. We use the notations of (3.6) and assume for simplicity that $E_P[W]=1$. Suppose that $A \in \mathscr{B}_M$, let $\tilde{A} = A \cup A'$, and choose $X \in p\mathscr{F}_{\tilde{A}}$, $Y \in p\mathscr{F}_A$, and $Z \in p\mathscr{F}_{A^c}$. Then

$$\begin{aligned} \int XYZ \cdot W\,dP &= \int XYZ\,dQ \\ &= \int YE_Q[XZ \mid \mathscr{F}_{A'}]\,dQ \\ &= \int YE_Q[X \mid \mathscr{F}_{A'}]E_Q[Z \mid \mathscr{F}_{A'}]\,dQ \end{aligned}$$

by the Lévy property of the submeasure M. By (3.8) the last expression is equal to

$$\begin{aligned} &\int YE_P[XW \mid \mathscr{F}_{A'}]E_P[ZW \mid \mathscr{F}_{A'}]E_P[W \mid \mathscr{F}_{A'}]^{-2}W\,dP \\ &\qquad = \int YE_P[XW \mid \mathscr{F}_{A'}]E_P[ZW \mid \mathscr{F}_{A'}]E_P[W \mid \mathscr{F}_{A'}]^{-1}\,dP \\ &\qquad = \int YE_P[XE_P[W \mid \mathscr{F}_{\tilde{A}}]ZE_P[W \mid \mathscr{F}_{A^c}] \mid \mathscr{F}_{A'}]E[W \mid \mathscr{F}_{A'}]^{-1}\,dP \\ &\qquad = \int XYZE_P[W \mid \mathscr{F}_{\tilde{A}}]E_P[W \mid \mathscr{F}_{A'}]^{-1}E_P[W \mid \mathscr{F}_{A^c}]\,dP\,. \end{aligned}$$

The second equality above is justified by the Lévy property of the generating random measure N as follows:

$$\begin{aligned} &E_P[XE_P[W \mid \mathscr{F}_{\tilde{A}}]ZE_P[W \mid \mathscr{F}_{A^c}] \mid \mathscr{F}_{A'}] \\ &\qquad = E_P[E_P[XW \mid \mathscr{F}_{\tilde{A}}]E_P[ZW \mid \mathscr{F}_{A^c}] \mid \mathscr{F}_{A'}] \\ &\qquad = E_P[E_P[XW \mid \mathscr{F}_{\tilde{A}}] \mid \mathscr{F}_{A'}]E_P[E_P[ZW \mid \mathscr{F}_{A^c}] \mid \mathscr{F}_{A'}] \\ &\qquad = E_P[XW \mid \mathscr{F}_{A'}]E_P[ZW \mid \mathscr{F}_{A'}]\,, \end{aligned}$$

where the third equality is by the Lévy property and by Proposition (1.8). Note that this computation could not be carried out with A in place of $\tilde{A}$; cf. the remark following (3.5). By the monotone class theorem it follows that (3.5) holds with $W_1 = E_P[W \mid \mathscr{F}_{\tilde{A}}]$, $W_2 = E_P[W \mid \mathscr{F}_{A'}]$ and $W_3 = E_P[W \mid \mathscr{F}_{A^c}]$. □

In general the limit, even in the almost sure sense, of Lévy random measures need not itself be a Lévy random measure. Using submeasures we can establish a set of conditions under which a limit in distribution is Lévy; compare Theorem (2.6).

(3.13) THEOREM. *Let P be the probability law of a Lévy random measure N with Lévy space $\mathbf{L}_N$. Let $\mathbf{L}$ be a Lévy space with $\mathbf{L} < \mathbf{L}_N$ and let $W, W_1, W_2, \cdots$ be $\mathscr{K}$-measurable and nonnegative. Suppose that*

(a) $0 < Z = E_P[W] < \infty$;
(b) *$W_1, W_2, \cdots$ are $\mathbf{L}$-multiplicative functionals*;
(c) *$W_n \to W$ in $L^1(P)$.*

Let $Q, Q_1, Q_2, \cdots$ be the probability laws of the submeasures $M, M_1, M_2, \cdots$ of

N generated by $W, W_1, W_2, \cdots$, respectively. Then $M_n \to M$ in distribution and M is a Lévy random measure with $\mathbf{L} < \mathbf{L}_M$.

PROOF. By (3.3)

$$Q_n(A) = Z_n^{-1} \int_A W_n \, dP ,$$

where $Z_n = \int W_n \, dP$; the hypotheses (a) and (c) imply that (at least for n sufficiently large) $0 < Z_n < \infty$, so we can assume that Q_n is well defined for each n.

To show that $M_n \to M$ in distribution we may show that $\int f dQ_n \to \int f dQ$ for every bounded, continuous function f on K, which follows from (c) by an obvious triangle inequality estimate.

To show that M is a Lévy random measure with $\mathbf{L} < \mathbf{L}_M$ we need only, by virtue of Theorem (3.6), show that W is $\mathbf{L}$-multiplicative. The computations in Theorem (3.12) show that for each n and each $A \in \mathscr{B}$ we may write

$$W_n = E_P[W_n \mid \mathscr{F}_{\bar{A}}] E_P[W_n \mid \mathscr{F}_{A'}]^{-1} E_P[W_n \mid \mathscr{F}_{A^c}] . \tag{3.14}$$

Let (n') be a subsequence such that $W_{n'} \to W$ almost surely with respect to P. Taking limits along (n') in (3.14) yields, by L^1 continuity of conditional expectations,

$$W = E_P[W \mid \mathscr{F}_{\bar{A}}] E_P[W \mid \mathscr{F}_{A'}]^{-1} E_P[W \mid \mathscr{F}_{A^c}] ,$$

which shows that W is $\mathbf{L}$-multiplicative. □

4. Gibbs random measures. In this section we apply the results of the preceding sections to a class of random measures important in statistical mechanics, namely Gibbs random measures. We consider such random measures, which are models of distributions of particles in space subject to a potential energy of interaction, on both $\mathbb{R}^\nu$ and Z^ν. Throughout, all Poisson random measures on $\mathbb{R}^\nu$ are assumed to have nonatomic mean measures, and are hence simple.

To begin, let E be a bounded convex set in $\mathbb{R}^\nu$ or Z^ν with positive uniform measure $|E|$. Let K denote the set of Radon measures on E. By a *potential* on E we mean a measurable mapping $U: K \to \mathbb{R} \cup \{+\infty\}$ such that $U(0) = 0$, where $0 \in K$ is the measure with no mass.

(4.1) DEFINITION. Let N be a random measure on E. If

$$0 < Z(U, N) = E[e^{-U(N)}] < \infty , \tag{4.2}$$

then the submeasure M of N generated by $W = e^{-U(N)}$ is the *Gibbs submeasure* of N with potential U.

REMARK. The constant $Z(U, N)$ defined by (4.2) is of interest in itself, especially in statistical mechanical applications such as Example (4.6) below. In this context it is known as the *partition function*. If N is the restriction to E of a random measure on $\mathbb{R}^\nu$ then the behavior of $Z(U, N)$ as $E \uparrow \mathbb{R}^\nu$ is also a problem of interest.

We say that a potential U has range $r > 0$ if whenever $m_1, m_2 \in K$ and the supports of m_1 and m_2 are separated by more than r with respect to the Euclidean metric on $\mathbb{R}^\nu$, then

$$U(m_1 + m_2) = U(m_1) + U(m_2)\,. \tag{4.3}$$

A random measure M is said to be a *Lévy random measure of order r* if M is a Lévy random measure with

$$\mathscr{B}_M = \{A \subset E : A \text{ is closed}, \ A \neq \phi, E\}$$

and

$$A' \subset \{y \in A^c : d(y, A) \leqq r\}$$

for each $A \in \mathscr{B}_M$, where d is the Euclidean distance on $\mathbb{R}^\nu$ and $d(y, A) = \inf\{d(y, x) : x \in A\}$. One then has the following result.

(4.4) THEOREM. (a) *Let M be the Gibbs submeasure of a Lévy random measure of order r generated by a potential U. If U has range r then M is a Lévy random measure of order r.*

(b) *If M is the submeasure of a Lévy random measure N of order r; if* $P\{N(A^c) = 0\} > 0$ *for each* $A \in \mathscr{B}_N$; *if the functional of N generating M is positive* a.s. *with respect to the probability law P of N; and if M is a Lévy random measure of order r, then there exist a potential U and a function* $U^1 : K \to \mathbb{R}$ *(which is not necessarily measurable) such that*

(i) *M is the Gibbs submeasure of N with potential U;*
(ii) U^1 *has range r in the sense of* (4.3);
(iii) $\{U \neq U^1\}$ *is a P-null set.*

PROOF. (a) If U has range r and A is closed then by the Möebius inversion formula we may write for $m \in K$

$$U(m) = U(m_{\tilde{A}}) + U(m_{A^c}) - U(m_{A'})\,, \tag{4.5}$$

where we have put $\tilde{A} = A \cup A'$. We then conclude at once from (4.5) and Theorem (3.6) that M is a Lévy random measure of order r. Here $m_A(B) = m(B \cap A)$.

(b) By Theorem (3.12) there exists a multiplicative functional W of N which generates M. Since W is positive a.s. we can write $W = e^{-U(N)}$ for some measurable function $U: K \to \mathbb{R}$; replacing, if necessary, W by $W/W(0)$ we may take $U(0) = 0$. Consider a fixed set $A \in \mathscr{B}_N$ ($= \mathscr{B}_M$ by (3.6)) and factor W according to (3.5):

$$e^{-U(N)} = e^{-V_1(N)} e^{-V_2(N)} e^{-V_3(N)}$$

where V_1, V_2, V_3 are measurable with respect to $\mathscr{F}_{\tilde{A}}$, $\mathscr{F}_{A'}$ and $\mathscr{F}_{A^c}$, respectively; these three functions may all be assumed to be potentials. An easy calculation shows that

$$U(m) = V_1(m_{\tilde{A}}) + V_3(m_{A^c})$$

for P-almost all m such that $m(A') = 0$; as before m_B is the restriction of $m \in K$ to $B \in \mathscr{E}$. Since $P\{N(A') = 0\} \geqq P\{N(A^c) = 0\} > 0$, the set of m in question is not void. Identification of $V_1(m_{\tilde{A}})$ is easily accomplished on the same basis: if $m = m_A$ then $V_1(m) = U(m)$. Consequently

$$\begin{aligned} U(m) &= U(m_{\tilde{A}}) + U(m_{A^c}) \\ &= U(m_A) + U(m_{A^c - A'}) \end{aligned}$$

for P- almost every m such that $m(A') = 0$.

There exists a countable family $\mathscr{A}$ of closed subsets of E such that if m_1 and m_2 have supports separated by r then there exists $A \in \mathscr{A}$ such that m_1 is supported in A and m_2 in $A^c - A'$. Consequently, U has the property that $U(m) = U(m_1) + U(m_2)$ for P- almost every m representable in the form $m = m_1 + m_2$, where the supports of m_1 and m_2 are separated by r. U^1 is then obtained by setting $U = 0$ on a P-null set. □

(4.6) EXAMPLE. Let $N = \sum \varepsilon_{X_i}$ be the Poisson random measure on E with mean the Lebesgue measure and let N' be obtained from N by randomizing the masses of the atoms of N according to the procedure of Theorem (2.3). That is, $N' = \sum W_i \varepsilon_{X_i}$ where the W_i are independent, identically distributed, nonnegative random variables independent of N. By Theorem (2.3) N' is additive. Hence, if U is a potential of range r, then the Gibbs submeasure M' of N' generated by U is a Lévy random measure of order r (provided it be well defined by (4.1)). M' may be regarded as one form of Gibbs state expressing the distribution of particles with random masses subject to a potential energy of interaction. This model and the associated Lévy characterization generalize models and results concerning distributions of identical particles in $\mathbb{R}^\nu$ [5, 12] and Z^ν [11, 13, 14]. Of particular interest are pair potentials, namely potentials U of the form

$$U\left(\textstyle\sum_1^k a_i \varepsilon_{y_i}\right) = \mu \textstyle\sum_1^k a_i + \sum_{i \neq j} a_i a_j V(y_i, y_j) , \tag{4.7}$$

where $V\colon \mathbb{R}^\nu \times \mathbb{R}^\nu \to \mathbb{R} \cup \{+\infty\}$ and $\mu > 0$ is the chemical potential. Conditions on V implying that U satisfy (4.2) may be found in [5, 12]. In general the simple random measure M describing particle positions will not be a Lévy random measure; cf. Remark (2) following Theorem (2.3).

For a specific example consider the hard core potential V_δ with range δ, given by

$$\begin{aligned} V_\delta(x, y) &= \infty , \quad |x - y| < \delta \\ &= 0 , \quad |x - y| \geqq \delta . \end{aligned}$$

The Gibbs submeasure M_δ of the Poisson random measure N generated by the pair potential

$$U_\delta\left(\textstyle\sum \varepsilon_{y_i}\right) = \textstyle\sum_{i \neq j} V_\delta(y_i, y_j) \tag{4.8}$$

is by Theorem (4.4) a Lévy random measure of order δ, since the potential U_δ given by (4.8) is evidently of range δ. It may serve, for example, as a model of the distribution of the centers of balls of diameter δ that cannot overlap. By Theorem (3.13), M_δ converges in distribution to the Poisson random measure N as $\delta \to 0$, confirming that this is the appropriate generalization of the Poisson random measure to bodies of positive diameter. Random masses may be incorporated using (4.7).

To represent balls of random radii, one can transform the Poisson random measure $N = \sum \varepsilon_{X_i}$ to the random measure $N' = \sum \varepsilon_{(X_i, D_i)}$, where the D_i are i.i.d. random variables taking values in $[0, \delta]$ for some $\delta > 0$. The ball at X_i is to have diameter D_i; δ is the maximum diameter. The random measure N' is Poisson with mean measure the product of Lebesgue measure on E and the common distribution of the D_i [3]. The Gibbs submeasure M_δ of N' generated by the potential U which takes the value ∞ at $\sum \varepsilon_{(y_i, d_i)}$ if $|y_i - y_j| < \frac{1}{2}(d_i + d_j)$ for some $i \neq j$ and is zero elsewhere is, again by Theorem (4.4), a Lévy random measure of order δ on $E \times [0, \delta]$. Hence the projection of M_δ onto E, which represents the positions of particle centers, is also Lévy; it is, however, M itself which is of main interest. As before, $M_\delta \to N'$ in distribution as $\delta \to 0$.

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