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LIBERA AND HILBERT MATRIX OPERATOR ON LOGARITHMICALLY WEIGHTED BERGMAN, BLOCH AND HARDY-BLOCH SPACES

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Abstract. We show that if $\alpha > 1$, then the logarithmically weighted Bergman space $A^2_{\log^\alpha}$ is mapped by the Libera operator $\mathcal{L}$ into the space $A^2_{\log^{\alpha-1}}$, while if $\alpha > 2$ and $0 < \varepsilon \leqslant \alpha-2$, then the Hilbert matrix operator H maps $A^2_{\log^\alpha}$ into $A^2_{\log^{\alpha-2-\varepsilon}}$.

We show that the Libera operator $\mathcal{L}$ maps the logarithmically weighted Bloch space $\mathcal{B}_{\log^\alpha}$, $\alpha \in \mathbb{R}$, into itself, while H maps $\mathcal{B}_{\log^\alpha}$ into $\mathcal{B}_{\log^{\alpha+1}}$.

In Pavlović's paper (2016) it is shown that $\mathcal{L}$ maps the logarithmically weighted Hardy-Bloch space $\mathcal{B}^1_{\log^\alpha}$, $\alpha > 0$, into $\mathcal{B}^1_{\log^{\alpha-1}}$. We show that this result is sharp. We also show that H maps $\mathcal{B}^1_{\log^\alpha}$, $\alpha \geqslant 0$, into $\mathcal{B}^1_{\log^{\alpha-1}}$ and that this result is sharp also.

Keywords: Libera operator; Hilbert matrix operator; Hardy space; Bergman space; Bloch space; Hardy-Bloch space

MSC 2010: 47B38, 47G10, 30H25

1. Introduction

We consider the action of the Libera and Hilbert matrix operators on logarithmically weighted Bergman, Bloch and Hardy-Bloch spaces.

We show that if $\alpha > 1$, then the logarithmically weighted Bergman space $A^2_{\log^\alpha}$ is mapped by the Libera operator $\mathcal{L}$ into the space $A^2_{\log^{\alpha-1}}$. In [4] it is shown that if $f \in A^2_{\log^\alpha}$, where $\alpha > 3$, then $Hf \in A^2$. Here H is the Hilbert matrix operator. Also, in [1] it is shown that $H\colon A^2_{\log^\alpha} \to A^2$ for $\alpha > 2$. We improve this result by showing that if $\alpha > 2$ and $0 < \varepsilon \leqslant \alpha - 2$, then H is well defined on $A^2_{\log^\alpha}$ and maps this space into $A^2_{\log^{\alpha-2-\varepsilon}}$.

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We show that the Libera operator $\mathcal{L}$ maps the logarithmically weighted Bloch space $\mathcal{B}_{\log^\alpha}$, $\alpha \in \mathbb{R}$, into itself, while H maps $\mathcal{B}_{\log^\alpha}$ into $\mathcal{B}_{\log^{\alpha+1}}$.

In [8] it is shown that $\mathcal{L}$ maps the logarithmically weighted Hardy-Bloch space $\mathcal{B}^1_{\log^\alpha}$, $\alpha > 0$, into $\mathcal{B}^1_{\log^{\alpha-1}}$. We note that this result is sharp. Our main results are given in Theorem 5.3. Among other things, we show that H maps $\mathcal{B}^1_{\log^\alpha}$, $\alpha \geqslant 0$, into $\mathcal{B}^1_{\log^{\alpha-1}}$ and that this result is sharp.

The definitions of logarithmically weighted Bergman, Bloch and Hardy-Bloch spaces will be given in Sections 3, 4 and 5, respectively.

For $0 < p \leqslant \infty$, Hardy space H^p is the space of all functions f holomorphic in the unit disk $\mathbb{D} = \{z \in \mathbb{C}\colon\ |z| < 1\}$ for which

$$\|f\|_{H^p} = \|f\|_p = \sup_{0 \leqslant r < 1} M_p(r, f) < \infty,$$

where

$$M_p(r, f) = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f(r\mathrm{e}^{\mathrm{i}t})\right|^p \,\mathrm{d}t\right)^{1/p} \qquad \text{if } \ 0 < p < \infty,$$

and

$$M_\infty(r, f) = \sup_{0 \leqslant t < 2\pi} |f(r\mathrm{e}^{\mathrm{i}t})|.$$

The Lebesgue measure on $\mathbb{D}$ will be denoted by A and will be normalized so as to have $A(\mathbb{D}) = 1$. That is,

$$\mathrm{d}A(z) = \frac{1}{\pi}\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{\pi} r\,\mathrm{d}r\,\mathrm{d}t, \qquad \text{where } \ z = x + \mathrm{i}y = r\mathrm{e}^{\mathrm{i}t}.$$

The Bergman space A^p, $0 < p < \infty$, is the space of holomorphic functions in $L^p(\mathbb{D}, dA)$, that is,

$$A^p = \left\{f \in \mathcal{H}(\mathbb{D})\colon\ \|f\|^p_{A^p} = \int_{\mathbb{D}} |f(z)|^p \,\mathrm{d}A(z) < \infty\right\}.$$

A function f holomorphic in the unit disk $\mathbb{D}$ belongs to the Hardy-Bloch space $\mathcal{B}^{p,q}_0$, $0 < p \leqslant \infty$, $0 < q \leqslant \infty$ (notation from [7]) if

$$\int_0^1 M_p^q(r, f')(1-r)^{q-1}\,\mathrm{d}r < \infty \quad \text{for } 0 < q < \infty,$$

and

$$\sup_{0 \leqslant r < 1} (1-r) M_p(r, f') < \infty \quad \text{for } q = \infty.$$

Let $\mathcal{H}(\mathbb{D})$ denote the space of all functions holomorphic in the unit disk $\mathbb{D}$ of the complex plane endowed with the topology of uniform convergence on compact subsets of $\mathbb{D}$. The dual of $\mathcal{H}(\mathbb{D})$ is equal to $\mathcal{H}(\overline{\mathbb{D}})$, where $g \in \mathcal{H}(\overline{\mathbb{D}})$ means that g is holomorphic in a neighborhood of $\overline{\mathbb{D}}$ (depending on g). The duality pairing is given by

$$\langle f, g\rangle = \sum_{n=0}^{\infty} \widehat{f}(n)\overline{\widehat{g}(n)},$$

where $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n)z^n \in \mathcal{H}(\mathbb{D})$ and $g(z) = \sum_{n=0}^{\infty} \widehat{g}(n)z^n \in \mathcal{H}(\overline{\mathbb{D}})$.

It is easy to see that the Libera operator defined by

$$\overline{\mathcal{L}}g(z) = \sum_{n=0}^{\infty}\Bigl(\sum_{k=n}^{\infty} \frac{\widehat{g}(k)}{k+1}\Bigr)z^n = \int_0^1 g(t + (1-t)z)\,\mathrm{d}t,$$

$$g(z) = \sum_{n=0}^{\infty} \widehat{g}(n)z^n \in \mathcal{H}(\overline{\mathbb{D}})$$

maps $\mathcal{H}(\overline{\mathbb{D}})$ into $\mathcal{H}(\overline{\mathbb{D}})$.

We denote by $\mathcal{L}$ the operator

$$\mathcal{L}g(z) = \int_0^1 g(t + (1-t)z)\,\mathrm{d}t,$$
$$g(z) = \sum_{n=0}^{\infty} \widehat{g}(n)z^n \in \mathcal{H}(\mathbb{D}),$$

whenever the integral converges uniformly on compact subsets of $\mathbb{D}$. Uniform convergence means that the limit

$$\lim_{r\to 1^-} \int_0^r g(t + (1-t)z)\,\mathrm{d}t$$

is uniform with respect to z in any compact subset of $\mathbb{D}$. This hypothesis guarantees that $\mathcal{L}g$ is a holomorphic function in $\mathbb{D}$. We call $\mathcal{L}$ also the Libera operator since $\mathcal{L} = \overline{\mathcal{L}}$ on $\mathcal{H}(\overline{\mathbb{D}})$.

The Hilbert matrix is an infinite matrix $H = [h_{n,k}]_{n,k=0}^{\infty}$ whose entries are $h_{n,k} = 1/(n+k+1)$ for all nonnegative integers n and k. It can be viewed as an operator on spaces of holomorphic functions by its action on their Taylor coefficients. If

$$f(z) = \sum_{n=0}^{\infty} \widehat{f}(n)z^n$$

is a holomorphic function in the unit disk $\mathbb{D}$, then we define the transformation H by

$$Hf(z) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} h_{n,k}\widehat{f}(k)z^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\widehat{f}(k)}{n+k+1}z^n.$$

It is possible to write Hf, $f \in H^p$, $1 \leqslant p \leqslant \infty$, in an integral form, which is quite convenient for analyzing the operator. More specifically, by looking at the Taylor series expansion of the function f, we have the following integral representation:

$$Hf(z) = \int_0^1 \frac{f(t)}{1 - tz}\,\mathrm{d}t.$$

It is well known that the Libera operator $\mathcal{L}$ acts as a bounded operator from H^p into H^p if and only if $1 < p \leqslant \infty$ and that $\mathcal{L}$ acts as a bounded operator from A^p into A^p if and only if $2 < p < \infty$ (see [2], [6]). On the other hand, it is well known that the Hilbert matrix operator H acts as a bounded operator from H^p into H^p if and only if $1 < p < \infty$ and that H acts as a bounded operator from A^p into A^p if and only if $2 < p < \infty$ (see [3]).

2. Some preliminary results

In this section we shall collect some results which will be needed in our work. We start with one useful result.

Sublemma 2.1. *Let $\alpha \in \mathbb{R}$ and $a \geqslant 2$. Then*

$$\int_{\log a}^{\infty} t^{\alpha} \mathrm{e}^{-t}\,\mathrm{d}t \leqslant C_\alpha \frac{\log^\alpha a}{a},$$

where C_α is a constant independent of a.

Proof. (1) Case $\alpha \leqslant 0$.

$$\int_{\log a}^{\infty} t^{\alpha} \mathrm{e}^{-t}\,\mathrm{d}t \leqslant \log^\alpha a \int_{\log a}^{\infty} \mathrm{e}^{-t}\,\mathrm{d}t = \frac{\log^\alpha a}{a}.$$

(2) Case $\alpha > 0$. In this case, partial integration gives

$$\begin{aligned}\int_{\log a}^{\infty} t^{\alpha} \mathrm{e}^{-t}\,\mathrm{d}t &= \frac{\log^\alpha a}{a} + \alpha \int_{\log a}^{\infty} t^{\alpha-1} \mathrm{e}^{-t}\,\mathrm{d}t \\ &= \frac{\log^\alpha a}{a} + \alpha \frac{\log^{\alpha-1} a}{a} + \alpha(\alpha-1) \int_{\log a}^{\infty} t^{\alpha-2} \mathrm{e}^{-t}\,\mathrm{d}t \\ &\leqslant C_\alpha \frac{\log^\alpha a}{a} + \alpha(\alpha-1) \int_{\log a}^{\infty} t^{\alpha-2} \mathrm{e}^{-t}\,\mathrm{d}t.\end{aligned}$$

Continuing on this way, we find that

$$\begin{aligned}\int_{\log a}^{\infty} t^{\alpha}\mathrm{e}^{-t}\,\mathrm{d}t &\leqslant C_{\alpha}\frac{\log^{\alpha} a}{a} + \alpha(\alpha-1)\dots(\alpha-\lfloor\alpha\rfloor)\int_{\log a}^{\infty} t^{\alpha-\lfloor\alpha\rfloor-1}\mathrm{e}^{-t}\,\mathrm{d}t\\ &\leqslant C_{\alpha}\frac{\log^{\alpha} a}{a} + \alpha(\alpha-1)\dots(\alpha-\lfloor\alpha\rfloor)\frac{\log^{\alpha-\lfloor\alpha\rfloor-1} a}{a}\\ &\leqslant C_{\alpha}\frac{\log^{\alpha} a}{a},\end{aligned}$$

where $\lfloor\alpha\rfloor$ is the largest integer less then or equal to α. □

Consequently, we get the following result.

Lemma 2.2. *Let $\alpha\in\mathbb{R}$ and let n be a nonnegative integer. Then*

$$\int_0^1 r^n \log^{\alpha}\frac{2}{1-r}\,\mathrm{d}r \asymp \frac{\log^{\alpha}(n+2)}{n+1},$$

where the corresponding constant is independent of n, i.e., there is a constant C_{α} independent of n such that

$$\frac{1}{C_{\alpha}}\frac{\log^{\alpha}(n+2)}{n+1} \leqslant \int_0^1 r^n \log^{\alpha}\frac{2}{1-r}\,\mathrm{d}r \leqslant C_{\alpha}\frac{\log^{\alpha}(n+2)}{n+1}.$$

Proof. (1) Case $\alpha\geqslant 0$. First, we find that

$$\begin{aligned}\int_0^1 r^n \log^{\alpha}\frac{2}{1-r}\,\mathrm{d}r &\geqslant \int_{1-1/(n+1)}^1 r^n \log^{\alpha}\frac{2}{1-r}\,\mathrm{d}r\\ &\geqslant \log^{\alpha}(n+2)\int_{1-1/(n+1)}^1 r^n\,\mathrm{d}r\\ &= \frac{\log^{\alpha}(n+2)}{n+1}\Big(1-\Big(1-\frac{1}{n+1}\Big)^{n+1}\Big)\\ &\geqslant C\frac{\log^{\alpha}(n+2)}{n+1}.\end{aligned}$$

On the other hand, by using Sublemma 2.1, we have that

$$\begin{aligned}\int_{1-1/(n+1)}^1 r^n \log^{\alpha}\frac{2}{1-r}\,\mathrm{d}r &\leqslant \int_{1-1/(n+1)}^1 \log^{\alpha}\frac{2}{1-r}\,\mathrm{d}r\\ &= 2\int_{\log(2n+2)}^{\infty} t^{\alpha}\mathrm{e}^{-t}\,\mathrm{d}t\\ &\leqslant 2\int_{\log(n+2)}^{\infty} t^{\alpha}\mathrm{e}^{-t}\,\mathrm{d}t\\ &\leqslant C_{\alpha}\frac{\log^{\alpha}(n+2)}{n+1},\end{aligned}$$

and

$$\int_0^{1-1/(n+1)} r^n \log^\alpha \frac{2}{1-r}\,\mathrm{d}r \leqslant C_\alpha \log^\alpha(n+2) \int_0^{1-1/(n+1)} r^n\,\mathrm{d}r$$
$$= C_\alpha \frac{\log^\alpha(n+2)}{n+1}\Big(1-\frac{1}{n+1}\Big)^{n+1}$$
$$\leqslant C_\alpha \frac{\log^\alpha(n+2)}{n+1}.$$

Therefore,

$$\int_0^1 r^n \log^\alpha \frac{2}{1-r}\,\mathrm{d}r \leqslant C_\alpha \frac{\log^\alpha(n+2)}{n+1}.$$

(2) Case $\alpha < 0$. Let $\varphi(r) = r\log^\alpha(2/r)$, $0 < r \leqslant 1$. Then φ is a nonnegative, increasing function on the interval $(0,1]$ and

$$t^{1-2\alpha}\varphi(r) \leqslant \varphi(tr) \leqslant t\varphi(r)$$

for all $0 < t < 1$. By using Lemma 4.1 in [5], we find that

$$\int_0^1 r^n \frac{\varphi(1-r)}{1-r}\,\mathrm{d}r \asymp \varphi\Big(\frac{1}{n+1}\Big).$$

Hence,

$$\int_0^1 r^n \log^\alpha \frac{2}{1-r}\,\mathrm{d}r \asymp \frac{\log^\alpha(n+2)}{n+1}.$$

□

The following auxiliary result will be useful.

Theorem 2.3. (a) *For every real α, the Taylor coefficients $\widehat{F}(n)$ of the function*

$$F(z) = \frac{1}{1-z}\log^\alpha \frac{2}{1-z}$$

have the property

$$\widehat{F}(n) \asymp \log^\alpha(n+2),$$

where the corresponding constant is independent of n.

(b) *For every real α, the Taylor coefficients $\widehat{G}(n)$ of the function*

$$G(z) = \log^\alpha \frac{2}{1-z}$$

have the property

$$\widehat{G}(n) \asymp \frac{\log^{\alpha-1}(n+2)}{n+1},$$

where the corresponding constant is independent of n.

This theorem is a consequence of Theorem 2.31 on page 192 of the classical monograph [10], hence we omit its proof. Now, we are ready to prove our next result.

Lemma 2.4. *Let $\alpha \in \mathbb{R}$ and let k be a nonnegative integer. Then*

$$\sum_{n=0}^{\infty} \frac{\log^{\alpha}(n+2)}{(n+1)(n+k+1)} \asymp \frac{\log^{\alpha+1}(k+2)}{k+1},$$

where the corresponding constant is independent of k, i.e., there is a constant C_α independent of k such that

$$\frac{1}{C_\alpha} \frac{\log^{\alpha+1}(k+2)}{k+1} \leqslant \sum_{n=0}^{\infty} \frac{\log^{\alpha}(n+2)}{(n+1)(n+k+1)} \leqslant C_\alpha \frac{\log^{\alpha+1}(k+2)}{k+1}.$$

Proof. By using Lemma 2.2 and Theorem 2.3 (b), we find that

$$\begin{aligned} \frac{\log^{\alpha+1}(k+2)}{k+1} &\asymp \int_0^1 r^k \log^{\alpha+1} \frac{2}{1-r} \,\mathrm{d}r \\ &\asymp \int_0^1 r^k \sum_{n=0}^{\infty} \frac{\log^{\alpha}(n+2)}{n+1} r^n \,\mathrm{d}r \\ &= \sum_{n=0}^{\infty} \frac{\log^{\alpha}(n+2)}{n+1} \int_0^1 r^{n+k} \,\mathrm{d}r \\ &= \sum_{n=0}^{\infty} \frac{\log^{\alpha}(n+2)}{(n+1)(n+k+1)}, \end{aligned}$$

where the corresponding constant is independent of k. □

3. Libera and Hilbert matrix operator on logarithmically weighted Bergman spaces

For $\alpha \in \mathbb{R}$ we define the logarithmically weighted Bergman spaces $A^2_{\log^\alpha}$ as follows:

$$A^2_{\log^\alpha} = \Big\{ f \in \mathcal{H}(\mathbb{D}) \colon \|f\|^2_{A^2_{\log^\alpha}} = \int_{\mathbb{D}} |f(z)|^2 \log^{\alpha} \frac{2}{1-|z|^2} \,\mathrm{d}A(z) < \infty \Big\}.$$

Note that $A^2_{\log^\alpha} \subset A^2$ for $\alpha > 0$ and $A^2_{\log^0} = A^2$.

Let $f(z) = \sum\limits_{n=0}^{\infty} \widehat{f}(n) z^n \in A^2_{\log^\alpha}$. By using Parseval's formula and Lemma 2.2, we find that

$$\|f\|^2_{A^2_{\log^\alpha}} = \sum_{n=0}^{\infty} |\widehat{f}(n)|^2 \int_0^1 r^n \log^{\alpha} \frac{2}{1-r} \,\mathrm{d}r \asymp \sum_{n=0}^{\infty} |\widehat{f}(n)|^2 \frac{\log^{\alpha}(n+2)}{n+1},$$

where the corresponding constant is independent of function f, i.e., there is a constant C independent of function f such that

$$\frac{1}{C}\|f\|^2_{A^2_{\log^\alpha}} \leqslant \sum_{n=0}^{\infty}|\widehat{f}(n)|^2\frac{\log^\alpha(n+2)}{n+1} \leqslant C\|f\|^2_{A^2_{\log^\alpha}}.$$

3.1. Libera operator on logarithmically weighted Bergman spaces. Our next result describes the action of the Libera operator $\mathcal{L}$ on the logarithmically weighted Bergman space $A^2_{\log^\alpha}$ for $\alpha > 1$.

Theorem 3.1. *If $\alpha > 1$, then the operator $\mathcal{L}$ is well defined on $A^2_{\log^\alpha}$ and maps this space into $A^2_{\log^{\alpha-1}}$.*

Proof. Let $f(z) = \sum\limits_{n=0}^{\infty}\widehat{f}(n)z^n \in A^2_{\log^\alpha}$. Then, by using the Cauchy-Schwarz inequality, we find that

$$\sum_{n=0}^{\infty}\frac{|\widehat{f}(n)|}{n+1} \leqslant \left(\sum_{n=0}^{\infty}|\widehat{f}(n)|^2\frac{\log^\alpha(n+2)}{n+1}\right)^{1/2}\left(\sum_{n=0}^{\infty}\frac{1}{(n+1)\log^\alpha(n+2)}\right)^{1/2} < \infty,$$

because $\alpha > 1$. From this, we get that if $f(z) = \sum\limits_{n=0}^{\infty}\widehat{f}(n)z^n \in A^2_{\log^\alpha}$, then $\sum\limits_{n=0}^{\infty}|\widehat{f}(n)| \times (n+1)^{-1} < \infty$. Hence, the operator $\mathcal{L}$ is well defined on $A^2_{\log^\alpha}$. Using inequality (59) from [6], we find that

$$rM_2^2(r,\mathcal{L}f) \leqslant C(1-r)^{-1}\int_r^1 M_2^2(s,f)\,\mathrm{d}s$$

for all $0 \leqslant r < 1$. Therefore,

$$\begin{aligned}
\int_{\mathbb{D}}|\mathcal{L}f(z)|^2\log^{\alpha-1}\frac{2}{1-|z|^2}\,\mathrm{d}A(z) &= 2\int_0^1 rM_2^2(r,\mathcal{L}f)\log^{\alpha-1}\frac{2}{1-r^2}\,\mathrm{d}r\\
&\leqslant C\int_0^1\frac{1}{1-r}\int_r^1 M_2^2(s,f)\,\mathrm{d}s\log^{\alpha-1}\frac{2}{1-r^2}\,\mathrm{d}r\\
&= C\int_0^1 M_2^2(s,f)\int_0^s\frac{1}{1-r}\log^{\alpha-1}\frac{2}{1-r^2}\,\mathrm{d}r\,\mathrm{d}s\\
&\leqslant C\int_0^1 M_2^2(s,f)\int_0^s\frac{1}{1-r}\log^{\alpha-1}\frac{2}{1-r}\,\mathrm{d}r\,\mathrm{d}s\\
&= C\int_0^1 M_2^2(s,f)\Big(\log^\alpha\frac{2}{1-s}-\log^\alpha 2\Big)\,\mathrm{d}s\\
&\leqslant C\int_0^1 M_2^2(s,f)\log^\alpha\frac{2}{1-s}\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&= C\int_0^1 uM_2^2(u^2, f)\log^\alpha \frac{2}{1-u^2}\,\mathrm{d}u \\
&\leqslant C\int_0^1 uM_2^2(u, f)\log^\alpha \frac{2}{1-u^2}\,\mathrm{d}u \\
&= C\int_{\mathbb{D}} |f(z)|^2\log^\alpha \frac{2}{1-|z|^2}\,\mathrm{d}A(z) < \infty.
\end{aligned}
$$

Hence, $\mathcal{L}f \in A^2_{\log^{\alpha-1}}$. □

3.2. Hilbert matrix operator on logarithmically weighted Bergman spaces. In [4] it is shown that if $f \in A^2_{\log^\alpha}$, where $\alpha > 3$, then $Hf \in A^2$. Also, in [1] it is shown that $H\colon A^2_{\log^\alpha} \to A^2$ for $\alpha > 2$. Our next theorem improves this result.

Theorem 3.2. *If $\alpha > 2$ and $0 < \varepsilon \leqslant \alpha - 2$, then H is well defined on $A^2_{\log^\alpha}$ and maps this space into $A^2_{\log^{\alpha-2-\varepsilon}}$.*

Proof. For $\alpha > 2$, we have that if $f(z) = \sum\limits_{n=0}^{\infty} \widehat{f}(n)z^n \in A^2_{\log^\alpha}$, then we get $\sum\limits_{n=0}^{\infty} |\widehat{f}(n)|/(n+1) < \infty$. Therefore, the operator H is well defined on $A^2_{\log^\alpha}$. On the other hand, if $f(z) = \sum\limits_{n=0}^{\infty} \widehat{f}(n)z^n \in A^2_{\log^\alpha}$, then by using the Cauchy-Schwarz inequality and Lemma 2.4, we find that

$$
\begin{aligned}
\|Hf\|^2_{A^2_{\log^{\alpha-2-\varepsilon}}} &\asymp \sum_{n=0}^{\infty} |\widehat{Hf}(n)|^2 \frac{\log^{\alpha-2-\varepsilon}(n+2)}{n+1} \\
&= \sum_{n=0}^{\infty} \left|\sum_{k=0}^{\infty} \frac{\widehat{f}(k)}{n+k+1}\right|^2 \frac{\log^{\alpha-2-\varepsilon}(n+2)}{n+1} \\
&\leqslant \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{|\widehat{f}(k)|^2\log^\alpha(k+2)}{n+k+1} \\
&\quad \times \sum_{k=0}^{\infty} \frac{1}{(n+k+1)\log^\alpha(k+2)} \frac{\log^{\alpha-2-\varepsilon}(n+2)}{n+1} \\
&\leqslant \sum_{k=0}^{\infty} |\widehat{f}(k)|^2 \frac{\log^\alpha(k+2)}{k+1} \sum_{k=0}^{\infty} \frac{1}{\log^\alpha(k+2)} \sum_{n=0}^{\infty} \frac{\log^{\alpha-2-\varepsilon}(n+2)}{(n+1)(n+k+1)} \\
&\leqslant C\|f\|^2_{A^2_{\log^\alpha}} \sum_{k=0}^{\infty} \frac{1}{\log^\alpha(k+2)} \frac{\log^{\alpha-1-\varepsilon}(k+2)}{k+1} \\
&= C\|f\|^2_{A^2_{\log^\alpha}} \sum_{k=0}^{\infty} \frac{1}{(k+1)\log^{1+\varepsilon}(k+2)} < \infty.
\end{aligned}
$$

Therefore, $Hf \in A^2_{\log^{\alpha-2-\varepsilon}}$. □

We note that for $\alpha \in (1,2]$ the operator H is well defined on $A^2_{\log^\alpha}$. We do not know whether Theorem 3.2 holds in this case. A natural question is: Does Theorem 3.2 hold for $\varepsilon = 0$?

4. Libera and Hilbert matrix operator on logarithmically weighted Bloch spaces

For $\alpha \in \mathbb{R}$ we define the logarithmically weighted Bloch spaces $\mathcal{B}_{\log^\alpha}$ as follows,

$$\mathcal{B}_{\log^\alpha} = \Big\{ f \in \mathcal{H}(\mathbb{D})\colon\ |f'(z)|(1-|z|) = \mathcal{O}\Big(\log^\alpha \frac{2}{1-|z|} \Big) \Big\}.$$

The norm in the space $\mathcal{B}_{\log^\alpha}$ is defined by

$$\|f\|_{\mathcal{B}_{\log^\alpha}} = |f(0)| + \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|) \log^{-\alpha} \frac{2}{1-|z|}.$$

Note that $\mathcal{B}_{\log^0}$ is the Bloch space $\mathcal{B}_0^{\infty,\infty} = \mathcal{B}$.

4.1. Libera operator on logarithmically weighted Bloch spaces. Now, we have the following theorem.

Theorem 4.1. *Let $\alpha \in \mathbb{R}$. Then $\mathcal{L}$ is well defined on $\mathcal{B}_{\log^\alpha}$ and maps this space into $\mathcal{B}_{\log^\alpha}$.*

Proof. By Theorem 2.1 (a) in [8], we have that if $f(z) = \sum\limits_{n=0}^{\infty} \widehat{f}(n) z^n \in \mathcal{B}_{\log^\alpha}$, then $\sum\limits_{n=0}^{\infty} |\widehat{f}(n)|/(n+1) < \infty$. Therefore, the Libera operator $\mathcal{L}$ is well defined on $\mathcal{B}_{\log^\alpha}$.

By using Lemma 22 from [6], for $\nu = 1$ and $p = \infty$, we find that

$$M_\infty(r, (\mathcal{L}f)') \leqslant (1-r)^{-2} \int_r^1 (1-s) M_\infty(s, f')\,\mathrm{d}s$$

for all $0 \leqslant r < 1$. Then, by using Sublemma 2.1, we have that

$$\begin{aligned}(1-r)M_\infty(r,(\mathcal{L}f)') &\leqslant \frac{1}{1-r}\int_r^1 (1-s)M_\infty(s,f')\,\mathrm{d}s \\ &\leqslant C\frac{1}{1-r}\int_r^1 \log^\alpha \frac{2}{1-s}\,\mathrm{d}s \\ &= C\frac{1}{1-r}\int_{\log(2/(1-r))}^{\infty} t^\alpha \mathrm{e}^{-t}\,\mathrm{d}t \\ &\leqslant C\log^\alpha \frac{2}{1-r}.\end{aligned}$$

Hence, we obtain $\mathcal{L}f \in \mathcal{B}_{\log^\alpha}$. □

4.2. Hilbert matrix operator on logarithmically weighted Bloch spaces. Our next result describes the action of the Hilbert matrix operator H on the logaritmically weighted Bloch space $\mathcal{B}_{\log^\alpha}$ for $\alpha \in \mathbb{R}$. We improve results given in Proposition 5.1 and Proposition 5.2 in [4].

Theorem 4.2. *Let $\alpha \in \mathbb{R}$. Then H is well defined on $\mathcal{B}_{\log^\alpha}$ and maps this space into $\mathcal{B}_{\log^{\alpha+1}}$.*

Proof. By Theorem 2.1 (a) in [8], we have that if $f(z) = \sum\limits_{n=0}^{\infty} \widehat{f}(n) z^n \in \mathcal{B}_{\log^\alpha}$, then $\sum\limits_{n=0}^{\infty} |\widehat{f}(n)|/(n+1) < \infty$. Hence, the Hilbert matrix operator H is well defined on $\mathcal{B}_{\log^\alpha}$.

Now, let $f \in \mathcal{B}_{\log^\alpha}$, where without loss of generality, we can additionally assume that $f(0) = 0$. Then, by Lemma 4.2.8 in [9], we can write

$$f(z) = \int_{\mathbb{D}} \frac{f'(w)(1-|w|^2)}{\overline{w}(1-\overline{w}z)^2} \,\mathrm{d}A(w),$$

for all $z \in \mathbb{D}$. Also, we have that

$$\int_{\mathbb{D}} \frac{z^k |z|^{2n}}{1-\overline{z}} \,\mathrm{d}A(z) = \frac{1}{n+k+1},$$

for all nonnegative integers n and k. Therefore,

$$\int_{\mathbb{D}} f(z) \frac{|z|^{2n}}{1-\overline{z}} \,\mathrm{d}A(z) = \sum_{k=0}^{\infty} \frac{\widehat{f}(k)}{n+k+1} = \widehat{Hf}(n).$$

Consequently,

$$\begin{aligned}
|\widehat{Hf}(n)| &= \left| \int_{\mathbb{D}} f(z) \frac{|z|^{2n}}{1-\overline{z}} \,\mathrm{d}A(z) \right| \\
&= \left| \int_{\mathbb{D}} \frac{f'(w)(1-|w|^2)}{\overline{w}} \int_{\mathbb{D}} \frac{|z|^{2n}}{(1-\overline{w}z)^2(1-\overline{z})} \,\mathrm{d}A(z) \,\mathrm{d}A(w) \right| \\
&= \frac{1}{\pi} \left| \int_{\mathbb{D}} \frac{f'(w)(1-|w|^2)}{\overline{w}} \int_0^1 r^{2n+1} \int_0^{2\pi} \frac{1}{(1-\overline{w}r\mathrm{e}^{\mathrm{i}\theta})^2(1-r\mathrm{e}^{-\mathrm{i}\theta})} \,\mathrm{d}\theta \,\mathrm{d}r \,\mathrm{d}A(w) \right| \\
&= \frac{1}{\pi} \left| \int_{\mathbb{D}} \frac{f'(w)(1-|w|^2)}{\overline{w}} \int_0^1 r^{2n+1} \frac{2\pi}{(1-r^2\overline{w})^2} \,\mathrm{d}r \,\mathrm{d}A(w) \right| \\
&\leqslant C \int_{\mathbb{D}} \frac{|f'(w)|(1-|w|)}{|w|} \int_0^1 \frac{r^{2n+1}}{|1-r^2 w|^2} \,\mathrm{d}r \,\mathrm{d}A(w) \\
&\leqslant C \int_{\mathbb{D}} \frac{\log^\alpha \frac{2}{1-|w|}}{|w|} \int_0^1 \frac{r^{2n+1}}{|1-r^2 w|^2} \,\mathrm{d}r \,\mathrm{d}A(w)
\end{aligned}$$

$$
\begin{aligned}
&= C\int_0^1 r^{2n+1}\int_{\mathbb{D}} \frac{\log^\alpha \frac{2}{1-|w|}}{|w||1-r^2w|^2}\,\mathrm{d}A(w)\,\mathrm{d}r\\
&= C\int_0^1 r^{2n+1}\int_0^1 \log^\alpha\frac{2}{1-\varrho}\int_0^{2\pi}\frac{1}{|1-r^2\varrho\mathrm{e}^{\mathrm{i}\theta}|^2}\,\mathrm{d}\theta\,\mathrm{d}\varrho\,\mathrm{d}r\\
&\leqslant C\int_0^1 r^{2n+1}\int_0^1 \frac{1}{1-r^2\varrho}\log^\alpha\frac{2}{1-\varrho}\,\mathrm{d}\varrho\,\mathrm{d}r\\
&= C\int_0^1 \log^\alpha\frac{2}{1-\varrho}\int_0^1\frac{r^n}{1-r\varrho}\,\mathrm{d}r\,\mathrm{d}\varrho.
\end{aligned}
$$

On the other hand, we find that

$$
\int_0^1 \frac{r^n}{1-r\varrho}\,\mathrm{d}r = \sum_{k=0}^\infty \frac{\varrho^k}{n+k+1}.
$$

Hence, by using Lemma 2.2 and Lemma 2.4, we obtain

$$
\begin{aligned}
|\widehat{Hf}(n)| &\leqslant C\sum_{k=0}^\infty \frac{1}{n+k+1}\int_0^1 \varrho^k\log^\alpha\frac{2}{1-\varrho}\,\mathrm{d}\varrho\\
&\leqslant C\sum_{k=0}^\infty \frac{\log^\alpha(k+2)}{(k+1)(n+k+1)}\\
&\leqslant C\frac{\log^{\alpha+1}(n+2)}{n+1}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|(Hf)'(z)| &= \left|\sum_{n=1}^\infty n\widehat{Hf}(n)z^{n-1}\right|\\
&\leqslant \sum_{n=1}^\infty n|\widehat{Hf}(n)||z|^{n-1}\\
&\leqslant C\sum_{n=1}^\infty \frac{n}{n+1}\log^{\alpha+1}(n+2)|z|^{n-1}\\
&\leqslant C\sum_{n=0}^\infty \log^{\alpha+1}(n+3)|z|^n\\
&\leqslant C\sum_{n=0}^\infty \log^{\alpha+1}(n+2)|z|^n.
\end{aligned}
$$

By using Theorem 2.3 (a), we find that

$$
\frac{1}{1-|z|}\log^{\alpha+1}\frac{2}{1-|z|} \asymp \sum_{n=0}^\infty \log^{\alpha+1}(n+2)|z|^n.
$$

Finally,

$$|(Hf)'(z)| \leqslant C \sum_{n=0}^{\infty} \log^{\alpha+1}(n+2)|z|^n$$
$$\leqslant C \frac{1}{1-|z|} \log^{\alpha+1} \frac{2}{1-|z|}.$$

Hence, $Hf \in \mathcal{B}_{\log^{\alpha+1}}$. □

5. Libera and Hilbert matrix operator on logarithmically weighted Hardy-Bloch spaces

For $\alpha \in \mathbb{R}$ we define the logarithmically weighted Hardy-Bloch spaces $\mathcal{B}^1_{\log^\alpha}$ in the following way:

$$\mathcal{B}^1_{\log^\alpha} = \left\{ f \in \mathcal{H}(\mathbb{D})\colon \ \|f\|_{\mathcal{B}^1_{\log^\alpha}} = |f(0)| + \int_{\mathbb{D}} |f'(z)| \log^\alpha \frac{2}{1-|z|} \, \mathrm{d}A(z) < \infty \right\}.$$

For $\alpha = 0$, $\mathcal{B}^1_{\log^0}$ is the Hardy-Bloch space $\mathcal{B}^{1,1}_0$ (notation from [7]). We note that if $\alpha \geqslant 0$, then $\mathcal{B}^1_{\log^\alpha} \subseteq \mathcal{B}^{1,1}_0 \subseteq H^1$ and if $f(z) = \sum\limits_{n=0}^{\infty} \widehat{f}(n) z^n \in H^1$, then $\sum\limits_{n=0}^{\infty} |\widehat{f}(n)|/(n+1) < \infty$.

5.1. Libera operator on logarithmically weighted Hardy-Bloch spaces. Action of the Libera operator on the logarithmically weighted Hardy-Bloch spaces has been considered in [8] and the following two theorems are proved.

Theorem 5.1 ([8])**.** *Let $\alpha \geqslant -1$ and let $f(z) = \sum\limits_{n=0}^{\infty} \widehat{f}(n) z^n \in \mathcal{H}(\mathbb{D})$, where $\widehat{f}(n) \downarrow 0$ as $n \to \infty$. Then*

$$f \in \mathcal{B}^1_{\log^\alpha} \qquad \text{if and only if} \qquad \sum_{n=0}^{\infty} \widehat{f}(n) \frac{\log^\alpha(n+2)}{n+1} < \infty.$$

Moreover, there is a constant C independent of the function f, such that

$$\frac{1}{C} \|f\|_{\mathcal{B}^1_{\log^\alpha}} \leqslant \sum_{n=0}^{\infty} \widehat{f}(n) \frac{\log^\alpha(n+2)}{n+1} \leqslant C \|f\|_{\mathcal{B}^1_{\log^\alpha}}.$$

Theorem 5.2 ([8]). *Let $\alpha > 0$.*

(a) *Then $\mathcal{L}$ is well defined on $\mathcal{B}^1_{\log^\alpha}$ and maps this space into $\mathcal{B}^1_{\log^{\alpha-1}}$.*

(b) *If $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$, where $\widehat{f}(n) \downarrow 0$ as $n \to \infty$ and $\sum_{n=0}^{\infty} \widehat{f}(n)/(n+1) < \infty$, then $\mathcal{L}f \in \mathcal{B}^1_{\log^{\alpha-1}}$ implies $f \in \mathcal{B}^1_{\log^\alpha}$.*

(c) *If $\alpha < 0$, then $\overline{\mathcal{L}}$ cannot be extended to a continuous operator from $\mathcal{B}^1_{\log^\alpha}$ to $\mathcal{H}(\mathbb{D})$.*

For (a) see Theorem 2.3 in [8]. Item (b) follows from Theorem 1.1 and Theorem 1.2 in [8]. For (c) see Theorem 2.1 (c) in [8].

5.2. Hilbert matrix operator on logarithmically weighted Hardy-Bloch spaces. Now we are ready to state the main theorem of this section.

Theorem 5.3. *Let $\alpha \geqslant 0$.*

(a) *Then H is well defined on $\mathcal{B}^1_{\log^\alpha}$ and maps this space into $\mathcal{B}^1_{\log^{\alpha-1}}$.*

(b) *If $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$, where $\widehat{f}(n) \downarrow 0$ as $n \to \infty$ and $\sum_{n=0}^{\infty} \widehat{f}(n)/(n+1) < \infty$, then $Hf \in \mathcal{B}^1_{\log^{\alpha-1}}$ implies $f \in \mathcal{B}^1_{\log^\alpha}$.*

(c) *The result in* (a) *is sharp in the sense that for any $\varepsilon > 0$ there exists $f \in \mathcal{B}^1_{\log^\alpha}$ such that $Hf \notin \mathcal{B}^1_{\log^{\alpha-1+\varepsilon}}$.*

(d) *If $\alpha < 0$, then H cannot be extended to a continuous operator from $\mathcal{B}^1_{\log^\alpha}$ to $\mathcal{H}(\mathbb{D})$.*

Proof. (a) By Theorem 2.1 (b) in [8], we have that if $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n \in \mathcal{B}^1_{\log^\alpha}$, then $\sum_{n=0}^{\infty} |\widehat{f}(n)|/(n+1) < \infty$. Therefore, the operator H is well defined on $\mathcal{B}^1_{\log^\alpha}$.

Now, let $f \in \mathcal{B}^1_{\log^\alpha}$, where without loss of generality, we can additionally assume that $f(0) = 0$. Then, by Lemma 4.2.8 in [9], we have

$$f(z) = \int_{\mathbb{D}} \frac{f'(w)(1-|w|^2)}{\overline{w}(1-\overline{w}z)^2} \,\mathrm{d}A(w),$$

for all $z \in \mathbb{D}$. Let $S = \int_{\mathbb{D}} |(Hf)'(z)| \log^{\alpha-1}(2/(1-|z|)) \,\mathrm{d}A(z)$. Then, by using the integral representation of the Hilbert matrix operator $Hf(z) = \int_0^1 (f(t)/(1-tz)) \,\mathrm{d}t$, we find that

$$
\begin{aligned}
S &\leqslant \int_{\mathbb{D}}\int_0^1 \frac{|f(t)|}{|1-tz|^2}\log^{\alpha-1}\frac{2}{1-|z|}\,\mathrm{d}t\,\mathrm{d}A(z)\\
&= \int_{\mathbb{D}}\int_0^1 \frac{1}{|1-tz|^2}\left|\int_{\mathbb{D}}\frac{f'(w)(1-|w|^2)}{\overline{w}(1-\overline{w}t)^2}\,\mathrm{d}A(w)\right|\log^{\alpha-1}\frac{2}{1-|z|}\,\mathrm{d}t\,\mathrm{d}A(z)\\
&\leqslant C\int_{\mathbb{D}}\frac{|f'(w)|(1-|w|)}{|w|}\int_{\mathbb{D}}\log^{\alpha-1}\frac{2}{1-|z|}\int_0^1\frac{\mathrm{d}t}{|1-tz|^2|1-tw|^2}\,\mathrm{d}A(z)\,\mathrm{d}A(w)\\
&\leqslant C\int_{\mathbb{D}}\frac{|f'(w)|(1-|w|)}{|w|}\int_{\mathbb{D}}\frac{1}{|1-zw|^3}\log^{\alpha-1}\frac{2}{1-|z|}\,\mathrm{d}A(z)\,\mathrm{d}A(w)\\
&\leqslant C\int_{\mathbb{D}}\frac{|f'(w)|}{|w|}\int_{\mathbb{D}}\frac{1}{|1-zw|^2}\log^{\alpha-1}\frac{2}{1-|z|}\,\mathrm{d}A(z)\,\mathrm{d}A(w)\\
&\leqslant C\int_{\mathbb{D}}\frac{|f'(w)|}{|w|}\int_0^1\int_0^{2\pi}\frac{\mathrm{d}\theta}{|1-r\mathrm{e}^{\mathrm{i}\theta}w|^2}\log^{\alpha-1}\frac{2}{1-r}\,\mathrm{d}r\,\mathrm{d}A(w)\\
&\leqslant C\int_{\mathbb{D}}\frac{|f'(w)|}{|w|}\int_0^1\frac{1}{1-r|w|}\log^{\alpha-1}\frac{2}{1-r}\,\mathrm{d}r\,\mathrm{d}A(w).
\end{aligned}
$$

On the other hand, by using Lemma 2.2 and Theorem 2.3 (b), we have that

$$
\begin{aligned}
\int_0^1\frac{1}{1-r|w|}\log^{\alpha-1}\frac{2}{1-r}\,\mathrm{d}r &= \sum_{n=0}^{\infty}|w|^n\int_0^1 r^n\log^{\alpha-1}\frac{2}{1-r}\,\mathrm{d}r\\
&\asymp \sum_{n=0}^{\infty}\frac{\log^{\alpha-1}(n+2)}{n+1}|w|^n\\
&\asymp \log^{\alpha}\frac{2}{1-|w|}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
S &\leqslant C\int_{\mathbb{D}}\frac{|f'(w)|}{|w|}\log^{\alpha}\frac{2}{1-|w|}\,\mathrm{d}A(w)\\
&\leqslant C\int_{\mathbb{D}}|f'(w)|\log^{\alpha}\frac{2}{1-|w|}\,\mathrm{d}A(w)\\
&< \infty
\end{aligned}
$$

and we get that $Hf\in\mathcal{B}^1_{\log^{\alpha-1}}$.

(b) We have $Hf(z)=\sum\limits_{n=0}^{\infty}\widehat{Hf}(n)z^n$, where $\widehat{Hf}(n)=\sum\limits_{k=0}^{\infty}\widehat{f}(k)/(n+k+1)\downarrow 0$ as $n\to\infty$. Then, by using Theorem 5.1, we find that

$$
\|Hf\|_{\mathcal{B}^1_{\log^{\alpha-1}}}\asymp\sum_{n=0}^{\infty}\widehat{Hf}(n)\frac{\log^{\alpha-1}(n+2)}{n+1},
$$

where the corresponding constant is independent of f. On the other hand, by using Lemma 2.4, we have that

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{Hf}(n)\frac{\log^{\alpha-1}(n+2)}{n+1} &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\widehat{f}(k)}{n+k+1}\frac{\log^{\alpha-1}(n+2)}{n+1}\\ &= \sum_{k=0}^{\infty}\widehat{f}(k)\sum_{n=0}^{\infty}\frac{\log^{\alpha-1}(n+2)}{(n+1)(n+k+1)}\\ &\asymp \sum_{k=0}^{\infty}\widehat{f}(k)\frac{\log^{\alpha}(k+2)}{k+1}.\end{aligned}$$

Therefore,

$$\|Hf\|_{\mathcal{B}^1_{\log^{\alpha-1}}} \asymp \sum_{n=0}^{\infty}\widehat{f}(n)\frac{\log^{\alpha}(n+2)}{n+1},$$

where the corresponding constant is independent of f. Then $\sum_{n=0}^{\infty}\widehat{f}(n)\log^{\alpha}(n+2)\times(n+1)^{-1}<\infty$ and by using Theorem 5.1 we find that $f\in\mathcal{B}^1_{\log^{\alpha}}$.

(c) Let $\varepsilon>0$ and let $\widehat{f}(n)=(\log^{\alpha+1+\varepsilon/2}(n+2))^{-1}$ for all $n\geqslant 0$. Then $\sum_{n=0}^{\infty}\widehat{f}(n)\times(n+1)^{-1}<\infty$ and $\widehat{f}(n)\downarrow 0$ as $n\to\infty$. Also, we find that

$$\sum_{n=0}^{\infty}\widehat{f}(n)\frac{\log^{\alpha}(n+2)}{n+1}<\infty \qquad\text{and}\qquad \sum_{n=0}^{\infty}\widehat{f}(n)\frac{\log^{\alpha+\varepsilon}(n+2)}{n+1}=\infty.$$

Let $f(z)=\sum_{n=0}^{\infty}\widehat{f}(n)z^n$. Then $f\in\mathcal{B}^1_{\log^{\alpha}}$ by Theorem 5.1 and $Hf\notin\mathcal{B}^1_{\log^{\alpha-1+\varepsilon}}$, because otherwise we would have $\sum_{n=0}^{\infty}\widehat{f}(n)\log^{\alpha+\varepsilon}(n+2)(n+1)^{-1}<\infty$ by part (b) of this theorem. A contradiction.

(d) Since $\mathcal{B}^1_{\log^{\alpha}}\subset\mathcal{B}^1_{\log^{\beta}}$ for $\beta<\alpha$, we may assume that $-1<\alpha<0$. Let

$$f(z)=\sum_{n=0}^{\infty}\frac{z^n}{\log(n+2)}.$$

For every $r\in(0,1)$ the function $f_r(z)=f(rz)$ belongs to $\mathcal{H}(\overline{\mathbb{D}})$ and by Theorem 5.1, the set $\{f_r\colon\ r\in(0,1)\}$ is bounded in $\mathcal{B}^1_{\log^{\alpha}}$. On the other hand,

$$Hf_r(0)=\sum_{k=0}^{\infty}\frac{r^k}{(k+1)\log(k+2)}\to\infty\quad\text{as } r\uparrow 1.$$

This contradicts the fact that if a set $X\subset\mathcal{B}^1_{\log^{\alpha}}$ is bounded and H is bounded on $\mathcal{B}^1_{\log^{\alpha}}$, then the set $\{Hf(0)\colon\ f\in X\}$ is bounded, because the functional $h\to h(0)$ is continuous on $\mathcal{H}(\mathbb{D})$. This completes the proof. □

Remark 5.4. We note that the result stated in Theorem 5.2 (a) is sharp in the sense that for any $\varepsilon > 0$ and $\alpha > 0$ there exists $f \in \mathcal{B}^1_{\log^\alpha}$ such that $\mathcal{L}f \notin \mathcal{B}^1_{\log^{\alpha-1+\varepsilon}}$. As above, we have that

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{\log^{\alpha+1+\varepsilon/2}(n+2)} \in \mathcal{B}^1_{\log^\alpha},$$

while $\mathcal{L}f \notin \mathcal{B}^1_{\log^{\alpha-1+\varepsilon}}$.

Corollary 5.5. *Let $\alpha \geqslant 0$ and let $f(z) = \sum\limits_{n=0}^{\infty} \widehat{f}(n) z^n$, where $\widehat{f}(n) \geqslant 0$ for all nonnegative integers n and $\sum\limits_{n=0}^{\infty} \widehat{f}(n)/(n+1) < \infty$. Then*

$$Hf \in \mathcal{B}^1_{\log^{\alpha-1}} \qquad \text{if and only if} \qquad \sum_{n=0}^{\infty} \widehat{f}(n) \frac{\log^\alpha(n+2)}{n+1} < \infty.$$

Moreover, there is a constant C independent of the function f, such that

$$\frac{1}{C} \|Hf\|_{\mathcal{B}^1_{\log^{\alpha-1}}} \leqslant \sum_{n=0}^{\infty} \widehat{f}(n) \frac{\log^\alpha(n+2)}{n+1} \leqslant C \|Hf\|_{\mathcal{B}^1_{\log^{\alpha-1}}}.$$

Proof. We have that $\widehat{Hf}(n) = \sum\limits_{k=0}^{\infty} \widehat{f}(k)/(n+k+1) \downarrow 0$ as $n \to \infty$, because $\widehat{f}(k) \geqslant 0$ for all $k \geqslant 0$. Now the proof follows from the proof of part (b) of Theorem 5.3. $\square$

Corollary 5.6. *Let $\alpha \geqslant 0$ and let $f(z) = \sum\limits_{n=0}^{\infty} \widehat{f}(n) z^n \in \mathcal{H}(\mathbb{D})$, such that $\sum\limits_{n=0}^{\infty} |\widehat{f}(n)| \log^\alpha(n+2)(n+1)^{-1} < \infty$. Then $Hf \in \mathcal{B}^1_{\log^{\alpha-1}}$.*

Proof. Let $x_n = \operatorname{Re} \widehat{f}(n)$ and $y_n = \operatorname{Im} \widehat{f}(n)$ for all nonnegative integers n. Then, functions $g(z) = \sum\limits_{n=0}^{\infty} x_n z^n$ and $h(z) = \sum\limits_{n=0}^{\infty} y_n z^n$ are holomorphic in the unit disk $\mathbb{D}$. Now, let $x_n^+ = (|x_n| + x_n)/2$ and $x_n^- = (|x_n| - x_n)/2$ for all $n = 0, 1, \ldots$. Then, $x_n^\pm \geqslant 0$, $x_n^\pm \leqslant |x_n| \leqslant |\widehat{f}(n)|$ and $x_n^+ - x_n^- = x_n$. Therefore, functions $g^+(z) = \sum\limits_{n=0}^{\infty} x_n^+ z^n$ and $g^-(z) = \sum\limits_{n=0}^{\infty} x_n^- z^n$ are holomorphic in the unit disk $\mathbb{D}$, and

$$\sum_{n=0}^{\infty} x_n^\pm \frac{\log^\alpha(n+2)}{n+1} \leqslant \sum_{n=0}^{\infty} |\widehat{f}(n)| \frac{\log^\alpha(n+2)}{n+1} < \infty.$$

Hence, by using Corollary 5.5, we find that

$$Hg^+ \in \mathcal{B}^1_{\log^{\alpha-1}} \quad \text{and} \quad Hg^- \in \mathcal{B}^1_{\log^{\alpha-1}}.$$

Then, we have

$$\begin{aligned}\int_{\mathbb{D}} |(Hg)'(z)| \log^{\alpha-1} \frac{2}{1-|z|}\, \mathrm{d}A(z) &= \int_{\mathbb{D}} |(Hg^+)'(z) - (Hg^-)(z)| \log^{\alpha-1} \frac{2}{1-|z|}\, \mathrm{d}A(z) \\ &\leqslant \int_{\mathbb{D}} \big(|(Hg^+)'(z)| + |(Hg^-)(z)|\big) \log^{\alpha-1} \frac{2}{1-|z|}\, \mathrm{d}A(z) < \infty,\end{aligned}$$

and we get $Hg \in \mathcal{B}^1_{\log^{\alpha-1}}$. In the same way, we prove that $Hh \in \mathcal{B}^1_{\log^{\alpha-1}}$. Then, we have that $Hf = Hg + \mathrm{i}Hh$, because of $f = g + \mathrm{i}h$. Consequently,

$$\begin{aligned}\int_{\mathbb{D}} |(Hf)'(z)| \log^{\alpha-1} \frac{2}{1-|z|}\, \mathrm{d}A(z) &= \int_{\mathbb{D}} |(Hg)'(z) + \mathrm{i}(Hh)'(z)| \log^{\alpha-1} \frac{2}{1-|z|}\, \mathrm{d}A(z) \\ &\leqslant \int_{\mathbb{D}} (|(Hg)'(z)| + |(Hh)'(z)|) \log^{\alpha-1} \frac{2}{1-|z|}\, \mathrm{d}A(z) \\ &< \infty.\end{aligned}$$

Therefore, $Hf \in \mathcal{B}^1_{\log^{\alpha-1}}$. □

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