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LIE ALGEBRA MODULES WITH FINITE DIMENSIONAL WEIGHT SPACES, I

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ABSTRACT. Let $\mathfrak{g}$ denote a reductive Lie algebra over an algebraically closed field of characteristic zero, and let $\mathfrak{h}$ denote a Cartan subalgebra of $\mathfrak{g}$. In this paper we study finitely generated $\mathfrak{g}$-modules that decompose into direct sums of finite dimensional $\mathfrak{h}$-weight spaces. We show that the classification of irreducible modules in this category can be reduced to the classification of a certain class of irreducible modules, those we call torsion free modules. We also show that if $\mathfrak{g}$ is a simple Lie algebra that admits a torsion free module, then $\mathfrak{g}$ is of type A or C.

1. INTRODUCTION

Let $\mathfrak{g}$ be a finite-dimensional, reductive Lie algebra over an algebraically closed field, k, of characteristic zero, and let $\mathscr{U}(\mathfrak{g})$ be its enveloping algebra. If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, then let $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$ denote the category of all finitely generated $\mathscr{U}(\mathfrak{g})$-modules that decompose into direct sums of finite dimensional $\mathfrak{h}$-weight spaces. This paper, and a sequel [Fe] in preparation, are devoted to the study of the category $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$. A primary focus of the current paper is the problem of classifying all irreducible modules in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$.

Our approach to classifying modules in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$ that are irreducible involves the study of a rather select class of modules in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$, those we call torsion free modules. A module $M \in \mathscr{M}(\mathfrak{g}, \mathfrak{h})$ is said to be a torsion free module if $\dim k[x] \cdot m = \infty$, for every $x \in \mathfrak{g}\backslash\mathfrak{h}$ and $m \in M\backslash(0)$. In this work, torsion free modules play a role similar to the role of highest weight spaces in the classification of irreducible highest weight modules. Let $\mathfrak{p}_{B,S}$ be a parabolic subalgebra of $\mathfrak{g}$ that contains $\mathfrak{h}$, let $\mathfrak{u}$ be the nilradical of $\mathfrak{p}_{B,S}$, and let $\mathfrak{l}$ be the reductive complement to $\mathfrak{u}$ in $\mathfrak{p}_{B,S}$ (sometimes called a Levi complement to $\mathfrak{u}$ in $\mathfrak{p}_{B,S}$) that is ad($\mathfrak{h}$)-stable. Suppose M is a module in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$, and suppose X denotes the set $M^{\mathfrak{u}} = \{ m \in M : \mathfrak{u} \cdot m = 0 \}$ of $\mathfrak{u}$-invariants in M. Then X has a natural structure of a $\mathscr{U}(\mathfrak{p}_{B,S})$-module. Now assume that M is irreducible, and that X is nontrivial. Then X is irreducible as a $\mathscr{U}(\mathfrak{p}_{B,S})$-module, and also as a $\mathscr{U}(\mathfrak{l})$-module. Furthermore, in this situation, M can be recovered as the unique simple quotient, $M_{B,S}(X)$, of the "generalized Verma

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module", $M_{B,S}(X)$, induced from the $\mathscr{U}(\mathfrak{p}_{B,S})$-module X (see Propositions 3.3 and 3.8). It turns out that every irreducible module $M \in \mathscr{M}(\mathfrak{g}, \mathfrak{h})$ determines a certain canonical choice of a parabolic subalgebra $\mathfrak{p}_{B,S}(\supset \mathfrak{h})$ of $\mathfrak{g}$ (see Proposition 4.17 for the choice of $\mathfrak{p}_{B,S}$) that can be used to reduce the classification problem to the special case of describing torsion free modules that are irreducible. In brief, this reduction comes about as follows. In Theorem 4.18 we show that, with the appropriate choice of $\mathfrak{p}_{B,S}$, the Levi complement $\mathfrak{l}$ has a decomposition, $\mathfrak{l} = \mathfrak{r} \oplus \mathfrak{t}$, where $\mathfrak{r}$ is an ad($\mathfrak{h}$)-stable reductive ideal of $\mathfrak{l}$ that is locally finite on X, and $\mathfrak{t}$ is an ad($\mathfrak{h}$)-stable semisimple ideal of $\mathfrak{l}$ such that X is a torsion free $\mathscr{U}(\mathfrak{t})$-module. This decomposition of $\mathfrak{l}$ yields a corresponding decomposition of the $\mathscr{U}(\mathfrak{l})$-module X into a tensor product $X_{\text{fin}} \otimes X_{\text{fr}}$, where X_{fin} is an irreducible, finite-dimensional $\mathscr{U}(\mathfrak{r})$-module, and X_{fr} is an irreducible, torsion free module in $\mathscr{M}(\mathfrak{t}, \mathfrak{t} \cap \mathfrak{h})$. The irreducible module M is completely determined by the parabolic subalgebra $\mathfrak{p}_{B,S}$ of $\mathfrak{g}$, the decomposition $\mathfrak{l} = \mathfrak{r} \oplus \mathfrak{t}$ of $\mathfrak{l}$, and the modules X_{fin} and X_{fr}. In light of classical results of Cartan and Weyl, this reduces the classification of irreducibles in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$ to the question of classifying irreducible, torsion free modules in $\mathscr{M}(\mathfrak{t}, \mathfrak{t} \cap \mathfrak{h})$, where $\mathfrak{t} \subset \mathfrak{g}$ is a simple Lie algebra. In this connection, we show in Theorem 5.2 that if $\mathfrak{g}$ is a simple Lie algebra that admits a torsion free module, then $\mathfrak{g}$ is either of type A, or of type C. The point of view adopted in this paper was influenced by the main result in [BL1]. In [BL2], using results proved in the present paper, Britten and Lemire classified all irreducible modules in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$ that have a one-dimensional weight space. The author has recently completed a classification of irreducible modules in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$. This work will appear in [Fe].

We close this introduction by briefly describing the contents of this paper, section by section. §2 is devoted to some generalities on Lie algebra representations. §3 is devoted to a collection of results that relate the representation theory of $\mathfrak{g}$ to the representation theory of Levi factors of certain parabolic subalgebras of $\mathfrak{g}$. In §4, the problem of classifying irreducible modules in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$ is reduced to the question of classifying irreducible torsion free modules in an appropriate category $\mathscr{M}(\mathfrak{t}, \mathfrak{t} \cap \mathfrak{h})$, where $\mathfrak{t}$ is a simple Lie subalgebra of $\mathfrak{g}$. It is also shown that every module in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$ has finite length (Theorem 4.21). The main result of §5 is Theorem 5.2, which was described in the preceding paragraph. This paper is partly based on the author's Ph. D. dissertation. Advice and encouragement from Professors G. Benkart, R. Block and A. Joseph are gratefully acknowledged.

We fix the following notation for the rest of this paper. We use $\mathbf{N}^*$ (resp. $\mathbf{N}$, resp. $\mathbf{Z}$) to denote the set of positive integers (resp. the set of nonnegative integers, resp. the set of all integers). We use k to denote a fixed algebraically closed field of characteristic zero, $\mathfrak{g}$ denotes a finite-dimensional reductive Lie algebra over k, $\mathfrak{s}$ denotes a finite-dimensional semisimple Lie algebra over k, $\mathfrak{h}$ denotes a Cartan subalgebra of $\mathfrak{g}$ (or $\mathfrak{s}$), $\mathfrak{p}$ denotes a parabolic subalgebra of $\mathfrak{g}$ (or $\mathfrak{s}$) such that $\mathfrak{h} \subset \mathfrak{p}$, $\mathfrak{u}$ denotes the nilpotent radical of $\mathfrak{p}$, and

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$\mathfrak{l}$ denotes an ad($\mathfrak{h}$)-stable reductive complement to $\mathfrak{u}$ in $\mathfrak{p}$. $\mathscr{M}(\mathfrak{a})$ denotes the category of all (left) $\mathscr{U}(\mathfrak{a})$-modules, where $\mathfrak{a}$ is a k-Lie algebra. $\overline{\mathscr{M}}(\mathfrak{g}, \mathfrak{h})$ denotes the category of all $\mathscr{U}(\mathfrak{g})$-modules that decompose into direct sums of $\mathfrak{h}$-weight spaces, when restricted to $\mathscr{U}(\mathfrak{h})$. $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$ denotes the category of all finitely generated modules in $\overline{\mathscr{M}}(\mathfrak{g}, \mathfrak{h})$ that decompose into direct sums of finite dimensional $\mathfrak{h}$-weight spaces. Throughout, the terms "ring", "algebra", and "module" will mean "ring with identity", "algebra with identity", and "unital module", respectively.

2. Generalities

Several results on Lie algebra modules that are valid in a setting broader than the category $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$, described above, will be recalled in this section.

Suppose $\mathfrak{a}$ is a finite-dimensional Lie algebra over k. Let $\mathscr{U}(\mathfrak{a})$ denote the universal enveloping algebra of $\mathfrak{a}$, and let $\{\mathscr{U}^j(\mathfrak{a})\}_{j\in\mathbf{N}}$ be the standard filtration on $\mathscr{U}(\mathfrak{a})$. Recall that one version of the Poincaré-Birkhoff-Witt Theorem states that the corresponding graded algebra is isomorphic to the symmetric algebra $S(\mathfrak{a})$. We shall view $S(\mathfrak{a})$ as the coordinate ring of the affine space $\mathfrak{a}^*$. If E is a subset of $\mathfrak{a}^*$, then let $\mathscr{I}(E)$ denote the vanishing ideal of E: $\mathscr{I}(E) = \{ p \in S(\mathfrak{a}) : p(x) = 0, \text{ for every } x \in E \}$. Let $\mathbf{I}$ denote the set of all ideals in $S(\mathfrak{a})$ of the form $\mathscr{I}(E)$, for some $E \subset \mathfrak{a}^*$. If $J \in \mathbf{I}$, then let $\mathscr{V}(J) \subset \mathfrak{a}^*$ denote the set of zeros of J: $\mathscr{V}(J) = \{ x \in \mathfrak{a}^* : p(x) = 0, \text{ for every } p \in J \}$. If J is an ideal of $S(\mathfrak{a})$, then let $\sqrt{J}$ denote the radical $\{ p \in S(\mathfrak{a}) : p^n \in J, \text{ for some } n \in \mathbf{N} \}$ of J. J is said to be a radical ideal if $\sqrt{J} = J$. Recall that, by Hilbert's zeros theorem [ZS, Chapter VII, §3], (a) $\mathbf{I}$ is equal to the set of radical ideals in $S(\mathfrak{a})$, and (b) the map $\mathscr{V} \mapsto \mathscr{I}(\mathscr{V})$ sets up a bijective map between the set of all subvarieties of $\mathfrak{a}^*$ and the set $\mathbf{I}$. The inverse map is $J \mapsto \mathscr{V}(J)$. Under this correspondence, irreducible subvarieties of $\mathfrak{a}^*$ are associated with prime ideals in $S(\mathfrak{a})$.

Now suppose $u \in \mathscr{U}^j(\mathfrak{a})\backslash\mathscr{U}^{j-1}(\mathfrak{a})$. Then the associated homogeneous polynomial $\mathrm{gr}(u) \in S_j(\mathfrak{a})$ will be called the symbol of u. When $u \in \mathfrak{a}$, we shall often identify u with its symbol, by using the canonical embedding $\mathfrak{a} \hookrightarrow S(\mathfrak{a})$. Next suppose M is a finitely generated $\mathscr{U}(\mathfrak{a})$-module, and suppose $\mathscr{F} = \{M^j\}_{j\in\mathbf{Z}}$ is a filtration on M. Then say that $\mathscr{F}$ is a good filtration, if the associated graded $\mathrm{gr}\,\mathscr{U}$-module $\mathrm{gr}_{\mathscr{F}} M$ is finitely generated. If M^0 is a finite-dimensional generating subspace of M, $M^i = (0)$, for $i < 0$, and $M^j = \mathscr{U}^j(\mathfrak{a}) \cdot M^0$, for each $j \in \mathbf{N}^*$, then $\mathscr{F} = \{M^j\}_{j\in\mathbf{N}}$ is a good filtration on M [KL, Lemma 6.7]. This filtration is called the standard filtration on M determined by M^0. If M is a finitely generated $\mathscr{U}(\mathfrak{a})$-module, then let $J(M)$ denote the ideal $\sqrt{\mathrm{Ann}\,\mathrm{gr}_{\mathscr{F}} M}$, where $\mathscr{F}$ is any good filtration on M. The notation introduced makes no mention of the filtration $\mathscr{F}$ because $\sqrt{\mathrm{Ann}\,\mathrm{gr}_{\mathscr{F}} M}$ is independent of the choice of the good filtration $\mathscr{F}$ [Ga, p. 448]. We shall simplify notation by writing $\mathscr{V}(M)$ for $\mathscr{V}(J(M))$. $\mathscr{V}(M)$ is called the associated variety of M.

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If $s \in \mathscr{U}(\mathfrak{a})$, then let $\langle s \rangle$ denote the associative subalgebra of $\mathscr{U}(\mathfrak{a})$ generated by s. If $\mathfrak{a}$ is a k-Lie algebra, S is a subset of $\mathscr{U}(\mathfrak{a})$, and M is a $\mathscr{U}(\mathfrak{a})$-module, then define subsets $M^{[S]}$ and $M^{(S)}$ of M by

$$M^{[S]} = \{ m \in M : \langle s \rangle \cdot m \text{ is finite dimensional, for every } s \in S \}$$

and

$$M^{(S)} = \{ m \in M : \text{ if } s \in S, \text{ then } s^r \cdot m = 0, \text{ for some } r = r(s, m) \in \mathbf{N} \}.$$

If $M^{[S]} = M$, then we say S is locally finite on M. If the center, $\mathscr{Z}(\mathfrak{a})$, of $\mathscr{U}(\mathfrak{a})$ is locally finite on M, then following accepted terminology we shall say that M is $\mathscr{Z}(\mathfrak{a})$-finite. If $M^{(S)} = M$, then we say S is locally nilpotent on M. Observe that if $S \subset \mathscr{U}(\mathfrak{a})$ is locally finite on M, then $\operatorname{gr}(S) \subset J(M)$. Note also that $\{0\} \subset \mathscr{V}(M)$, because $J(M)$ is a graded ideal. The following lemma describes the case where $\mathscr{V}(M) = \{0\}$. Although the result is well known, we include a proof, for lack of a suitable reference.

Lemma 2.1. *Let $\mathfrak{a}$ be a finite-dimensional Lie algebra over k, and let M be a finitely generated $\mathscr{U}(\mathfrak{a})$-module. Then the following statements are equivalent.*

(a) *M is finite dimensional.*
(b) *$J(M)$ is equal to the augmentation ideal of $S(\mathfrak{a})$.*
(c) *The associated variety $\mathscr{V}(M) = \{0\}$.*

Proof. We begin by noting that M is finite dimensional if and only if $\operatorname{gr}_{\mathscr{F}} M$ is finite dimensional. Let S_M denote the graded algebra $S(\mathfrak{a})/\operatorname{Ann} \operatorname{gr}_{\mathscr{F}} M$. Then $\operatorname{gr}_{\mathscr{F}} M$ is finite dimensional if and only if S_M is finite dimensional, since $\operatorname{gr}_{\mathscr{F}} M$ is a finitely generated, faithful S_M-module. On the other hand, S_M is finite dimensional if and only if the graded ideal $\operatorname{Ann} \operatorname{gr}_{\mathscr{F}} M$ has finite codimension in $S(\mathfrak{a})$. But, $\operatorname{Ann} \operatorname{gr}_{\mathscr{F}} M$ has finite codimension if and only if $J(M) = \sqrt{\operatorname{Ann} \operatorname{gr}_{\mathscr{F}} M}$ is the augmentation ideal of $S(\mathfrak{a})$. Finally observe that the augmentation ideal corresponds to $\mathscr{V}(M) = \{0\}$, under the correspondence between radical ideals and subvarieties of $\mathfrak{a}^*$ described by Hilbert's zeros theorem. Therefore $J(M)$ is equal to the augmentation ideal of $S(\mathfrak{a})$ if and only if $\mathscr{V}(M) = \{0\}$. □

If A is an associative algebra, then the Lie algebra with underlying vector space A and bracket operation $[a, b] = ab - ba$ will be denoted by LA. Let $\mathfrak{a}$ be a k-Lie algebra, and let M be a $\mathscr{U}(\mathfrak{a})$-module. Then let U_M denote the quotient algebra $\mathscr{U}(\mathfrak{a})/\operatorname{Ann} M$, and let $u \mapsto \bar{u}$ be the natural projection from $\mathscr{U}(\mathfrak{a})$ to U_M.

Lemma 2.2. *Let $\mathfrak{a}$ be a k-Lie algebra, let S be a subset of $\mathscr{U}(\mathfrak{a})$, and let M be a finitely generated $\mathscr{U}(\mathfrak{a})$-module. Then,*

(a) *The set $M^{[S]} = \{ m \in M : \langle s \rangle \cdot m$ is finite dimensional, for every $s \in S \}$ is a $\mathscr{U}(\mathfrak{a})$-submodule of M, if the adjoint action of the image, $\overline{S}$, of S in LU_M is locally finite on LU_M.*

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(b) *The set* $M^{(S)} = \{m \in M\colon$ *if* $s \in S$, *then* $s^r \cdot m = 0$, *for some* $r = r(s, m) \in \mathbf{N}\}$ *is a* $\mathscr{U}(\mathfrak{a})$*-submodule of* M, *if* $\overline{S}$ *is locally nilpotent on* LU_M.

Proof. Since

$$M^{[S]} = \bigcap_{s \in S} M^{[s]},$$

clearly it is adequate to prove the lemma in the case where $S = \{s\}$, for some $s \in \mathscr{U}(\mathfrak{a})$. Therefore, assume $S = \{s\}$, for some $s \in S$, and let m be an element of $M^{[s]}$. Then, for every $a \in \mathfrak{a}$ and every positive integer i, there is a finite-dimensional subspace F of M and a finite-dimensional subspace U_0 of U_M such that $\bar{s}^i \cdot m \in F$ and $\operatorname{ad}(\bar{s})^i(\bar{a}) \subset U_0$. Now recall the identity [Ja, p. 38]

$$\bar{s}^n \bar{a} = \sum_{i=0}^{n} \binom{n}{i} (\operatorname{ad} \bar{s})^{n-i}(\bar{a}) \bar{s}^i, \quad \text{where } a \in \mathfrak{a}, \tag{2.3}$$

which holds in U_M. It follows from 2.3 that $\bar{s}^n \bar{a} \cdot m \in U_0 \cdot F$. But $U_0 \cdot F$ is finite dimensional, and so $\bar{a} \cdot m \in M^{[s]}$, whenever $m \in M^{[s]}$, thus proving (a). The proof of (b) is similar. We leave it to the reader to fill in the details. □

Next, we state a theorem proved by Gabber [Ga]. To do so, we must introduce a Poisson bracket structure on $S(\mathfrak{a})$. Suppose $p \in \operatorname{gr}_i \mathscr{U}(\mathfrak{a})$ and $q \in \operatorname{gr}_j \mathscr{U}(\mathfrak{a})$ are homogeneous elements in $\operatorname{gr} \mathscr{U}(\mathfrak{a})$. Choose $x \in \mathscr{U}^i(\mathfrak{a})$ and $y \in \mathscr{U}^j(\mathfrak{a})$ such that $p = \operatorname{gr}_i(x)$ and $q = \operatorname{gr}_j(y)$. Observing that $xy - yx \in \mathscr{U}^{i+j-1}(\mathfrak{a})$, since $\operatorname{gr} \mathscr{U}(\mathfrak{a})$ is commutative, we define the Poisson bracket $\{p, q\}$ of p and q by

$$\{p, q\} = \operatorname{gr}_{i+j-1}(xy - yx).$$

The definition of $\{p, q\}$ is independent of the choices of x and y. Now extend the definition to nonhomogeneous elements of $S(\mathfrak{a})$, by using bilinearity. The Poisson bracket is a Lie algebra structure on $S(\mathfrak{a})$ that is an extension of the bracket operation on $\mathfrak{a}$. Observe that

$$\{pq, r\} = p\{q, r\} + \{p, r\}q, \quad \text{for all } p, q, r \in S(\mathfrak{a}). \tag{2.4}$$

Theorem 2.5 (Gabber [Ga]). *Let* $\mathfrak{a}$ *be a finite-dimensional* k*-Lie algebra, let* M *be a finitely generated* $\mathscr{U}(\mathfrak{a})$ *module, and let* $\mathscr{F}$ *be a good filtration on* M. *Then the graded ideal* $J(M) = \sqrt{\operatorname{Ann} \operatorname{gr}_{\mathscr{F}} M}$ *of* $S(\mathfrak{a})$ *is closed under the Poisson bracket operation defined above.*

Remark 2.6. It follows easily from the commutativity of $S(\mathfrak{a})$ that $J(M)$ is a linear subspace of $S(\mathfrak{a})$. Therefore, $J(M)$ is a Lie subalgebra of $S(\mathfrak{a})$.

A subvariety $\mathscr{V}$ of $\mathfrak{a}^*$ is said to be involutive if the ideal $\mathscr{I}(\mathscr{V})$ is closed under the Poisson bracket. Theorem 2.5 asserts that the associated variety of a finitely generated $\mathscr{U}(\mathfrak{a})$-module is involutive.

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Corollary 2.7. *The set*

$$\mathfrak{a}[M] = \{x \in \mathfrak{a} : M^{[x]} = M\}$$

of all elements in $\mathfrak{a}$ *that are locally finite on* M *is a Lie subalgebra of* $\mathfrak{a}$. *It is in fact the largest subalgebra of* $\mathfrak{a}$ *that is locally finite on* M.

Proof. We shall view the Lie algebra $\mathfrak{a}$ as a subspace of $S(\mathfrak{a})$, by using the canonical embedding $\mathfrak{a} \hookrightarrow S(\mathfrak{a})$ that maps elements of $\mathfrak{a}$ onto their symbols. Let F be a finite-dimensional generating subspace of M. Then, using 2.6, define a decreasing sequence $\{\mathfrak{a}_i\}_{i\in\mathbf{N}}$ of subalgebras of $\mathfrak{a}$, by setting

$$\mathfrak{a}_0 = \mathfrak{a}, \quad M_i = \mathscr{U}(\mathfrak{a}_i)\cdot F, \quad \text{and} \quad \mathfrak{a}_{i+1} = J(M_i) \cap \mathfrak{a}_i, \quad \text{for each } i \in \mathbf{N}.$$

Clearly $\mathfrak{a}[M] \subset \mathfrak{a}_i$, for every $i \in \mathbf{N}$. Since $\mathfrak{a}$ is finite dimensional, and the sequence $\{\mathfrak{a}_i\}_{i\in\mathbf{N}}$ is decreasing, there is a positive integer r such that $\mathfrak{a}_{r+1} = \mathfrak{a}_r$. But, if $\mathfrak{a}_{r+1} = \mathfrak{a}_r$, then $J(M_r)$ is the augmentation ideal of $S(\mathfrak{a}_r)$. By 2.1, it follows therefore that $M_r = \mathscr{U}(\mathfrak{a}_r)\cdot F$ is finite dimensional. Consequently, $F \subset M^{[\mathfrak{a}_r]}$, and so 2.2(a) implies $M^{[\mathfrak{a}_r]} = M$. This means that $\mathfrak{a}[M]$ is equal to the subalgebra $\mathfrak{a}_r$, thus proving the first assertion of the corollary.

It is clear from the definition of $\mathfrak{a}[M]$ that every Lie subalgebra of $\mathfrak{a}$ that is locally finite on M is contained in $\mathfrak{a}[M]$. In other words, $\mathfrak{a}[M]$ is maximal among all Lie subalgebras of $\mathfrak{a}$ that are locally finite on M. This completes the proof of the corollary. □

Remark 2.9. By 2.1, M is finite dimensional if and only if $\mathfrak{a}[M] = \mathfrak{a}$.

Suppose M is a $\mathscr{U}(\mathfrak{a})$ module. Then say that M is a pure module if, for every $x \in \mathfrak{a}$, $M^{[x]}$ is equal to either (0) or M. Irreducible modules clearly are pure modules. If $\mathfrak{s}$ is a Lie subalgebra of $\mathfrak{a}$ and N is a pure $\mathscr{U}(\mathfrak{s})$-module, then the induced $\mathscr{U}(\mathfrak{a})$-module $\operatorname{ind}_{\mathfrak{s}}^{\mathfrak{a}}(N) = \mathscr{U}(\mathfrak{a}) \otimes_{\mathscr{U}(\mathfrak{s})} N$ is a pure module. The following result will be used in the proof of 4.21.

Corollary 2.10. *Suppose* M *is a finitely generated* $\mathscr{U}(\mathfrak{a})$*-module. Then there is a finite chain*

$$(0) = M^0 \subset M^1 \subset \cdots \subset M^{r-1} \subset M^r = M$$

of $\mathscr{U}(\mathfrak{a})$*-submodules of* M *such that* M^i/M^{i-1} *is a pure module, for* $i = 1, \ldots, r$.

Proof. Since M is Noetherian, and the zero module satisfies the proposition, there is a maximal submodule M_0 of M that also satisfies the proposition. If $N = M/M_0$, then, by 2.7, $\mathfrak{a}[N]$ is a linear subspace of $\mathfrak{a}$. Let r denote the codimension of $\mathfrak{a}[N]$ in $\mathfrak{a}$. The proof uses induction on r. If $r = 0$, then $\mathfrak{a}$ is locally finite on N, and so by the maximality of M_0, it follows that $M_0 = M$. This takes care of the case $r = 0$. Next, assuming that the proposition holds whenever the codimension of $\mathfrak{a}[N]$ in $\mathfrak{a}$ is less than some positive integer r, consider the case where the codimension is equal to r. Since N is not pure, there is a nonzero $x \in \mathfrak{a}$ such that $N_1 = N^{[x]}$ lies strictly between (0) and N. Observe that the codimension of $\mathfrak{a}[N_1]$ in $\mathfrak{a}$ is less than

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r, because $kx + \mathfrak{a}[N] \subset \mathfrak{a}[N_1]$. Therefore, by the induction hypothesis, the proposition holds for the inverse image, M_1, of N_1 in M. But this contradicts the maximality of M_0, and hence the proof of the corollary is complete. □

Recall that a subvariety $\mathscr{V}$ of $\mathfrak{a}^*$ is said to be involutive if the vanishing ideal $\mathscr{I}(\mathscr{V})$ is closed under the Poisson bracket. Joseph [Jo1, p. 62] notes the following consequence of 2.5.

Corollary 2.11. *The irreducible components of $\mathscr{V}(M)$ are involutive.*

This result follows from 2.4 and the fact that the vanishing ideal of an irreducible subvariety of $\mathfrak{a}^*$ is a prime ideal in $S(\mathfrak{a})$. Leaving the details of the proof to the reader, we recall the definition of the Gelfand-Kirillov dimension of a finitely generated $\mathscr{U}(\mathfrak{a})$-module, where $\mathfrak{a}$ is a finite-dimensional k-Lie algebra. Let $\{\mathscr{U}^j(\mathfrak{a})\}_{j\in\mathbf{N}}$ be the standard filtration on $\mathscr{U}(\mathfrak{a})$, and let M be a finitely generated $\mathscr{U}(\mathfrak{a})$-module. Then it turns out that the limit

$$\overline{\lim_{m\to+\infty}} \log_m \dim_k \mathscr{U}^m(\mathfrak{a}) \cdot F$$

is a nonnegative integer independent of the choice of F [KL, pp. 90–91]. This invariant of M is called the Gelfand-Kirillov dimension of M and is denoted by $\mathrm{GKdim}(M)$. If $\mathscr{F}$ is the standard filtration on M associated with the generating set F, then $\mathrm{GKdim}(M)$ is equal to the degree of the Hilbert-Samuel polynomial of $\mathrm{gr}_{\mathscr{F}} M$ [KL, p. 91], and so, $\mathrm{GKdim}(M)$ can be identified with the dimension of the associated variety $\mathscr{V}(M)$ [Ha, Chapter I, 7.5]. We now record a result of Gabber and Joseph.

Theorem 2.12. *Let $\mathfrak{a}$ be a finite-dimensional algebraic Lie algebra over k. If M is a finitely generated $\mathscr{U}(\mathfrak{a})$-module and U_M denotes the quotient algebra $\mathscr{U}(\mathfrak{a})/\mathrm{Ann}\,(M)$, viewed as a left $\mathscr{U}(\mathfrak{a})$-module, then*

$$\mathrm{GKdim}(U_M) \leq 2 \cdot \mathrm{GKdim}(M).$$

See [KL, p. 135] for a proof of 2.12.

3. Reductive Lie algebras and parabolic induction

In this section we present the necessary background on representations of reductive Lie algebras. Most of the results we state here are natural extensions of well-known properties of highest weight modules. Our primary tools are two functors, $N \mapsto M_{B,S}(N)$ and $M \mapsto M^{\mathfrak{u}_{B,S}}$, that are frequently used in the theory of highest weight modules. These functors relate the representation theory of $\mathfrak{g}$ and the representation theory of certain reductive subalgebras of $\mathfrak{g}$.

Let $\mathfrak{g}$ be a reductive Lie algebra and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We introduce notation to describe parabolic subalgebras of $\mathfrak{g}$ that contain $\mathfrak{h}$. Denote the set of nonzero roots of the pair ($\mathfrak{g}$, $\mathfrak{h}$) by R, and denote the root lattice of R by $\mathbf{Z}R$. If $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha$ is the root space decomposition of

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$(\mathfrak{g}, \mathfrak{h})$, then choose a Chevalley basis

$$\{ E_\alpha : \alpha \in R \} \cup \{ H_i \in \mathfrak{h} : 1 \leq i \leq n \}$$

for $\mathfrak{g}$ such that $E_\alpha \in \mathfrak{g}_\alpha$, for every $\alpha \in R$. Suppose B is a base of R, and suppose S is a subset of B. Let R_B^+ (respectively R_B^-) denote the set of roots in R that are positive (respectively negative) with respect to the ordering on $\mathfrak{h}^*$ determined by B, and let R_S denote the subsystem $R \cap \sum_{\alpha \in S} \mathbf{Z}\alpha$ of R. Then every parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ that contains $\mathfrak{h}$ is of the form $\mathfrak{p}_{B,S} = \mathfrak{g}_P \oplus \mathfrak{h}$, where $P = R_B^+ \cup R_S$, for some base B of R, and some subset S of B. If $|B \backslash S| = 1$, then say that $\mathfrak{p}_{B,S}$ is a maximal parabolic subalgebra of $\mathfrak{g}$. If P is a subset of R, then let P^s denote the "symmetric part" $P \cap (-P)$ of P, and let P^a denote the "antisymmetric part" $P \setminus (-P)$ of P. Let σ be the Lie algebra automorphism of $\mathfrak{g}$ defined by $\sigma(E_\alpha) = -E_{-\alpha}$, for all $\alpha \in R$, and $\sigma(h) = -h$, for all $h \in \mathfrak{h}$. We now fix the following notation for the rest of this section. Let $\mathfrak{p}$, $\mathfrak{p}^-$ be a pair of opposite parabolic subalgebras of $\mathfrak{g}$ (this means $\sigma(\mathfrak{p}) = \mathfrak{p}^-$) that contain $\mathfrak{h}$. Choose a subset $S \subset B$ of R such that $(\mathfrak{p}, \mathfrak{p}^-) = (\mathfrak{p}_{B,S}, \mathfrak{p}_{-B,-S})$. Let $\mathfrak{u} = \mathfrak{u}_{B,S}$ (respectively $\mathfrak{u}^- = \mathfrak{u}_{B,S}^-$) be the nilradical of $\mathfrak{p}$ (respectively $\mathfrak{p}^-$), and let $\mathfrak{l}$ denote the reductive Lie algebra $\mathfrak{p} \cap \mathfrak{p}^-$. Let $P = R_B^+ \cup R_S$, so that $\mathfrak{u} = \mathfrak{g}_{P^a}$, and $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{g}_{P^s}$.

If N is a $\mathscr{U}(\mathfrak{l})$-module, then define a $\mathfrak{p}$-module structure on N by letting $\mathfrak{u}$ act trivially on it. Now define a $\mathscr{U}(\mathfrak{g})$-module $M_{B,S}(N)$ by setting

$$M_{B,S}(N) = \operatorname{ind}_{\mathfrak{p}}^{\mathfrak{g}}(N) = \mathscr{U}(\mathfrak{g}) \otimes_{\mathscr{U}(\mathfrak{p})} N.$$

On the other hand, if M is a $\mathscr{U}(\mathfrak{g})$-module, then let $M^{\mathfrak{u}_{B,S}}$ denote the set of $\mathfrak{u}_{B,S}$-invariants of M; i.e., let

$$M^{\mathfrak{u}_{B,S}} = \{ m \in M : \mathfrak{u}_{B,S} \cdot m = 0 \}.$$

Since $\mathfrak{u}$ is invariant under the adjoint action of $\mathfrak{l}$, it follows that $M^{\mathfrak{u}}$ is an $\mathfrak{l}$-invariant subspace of M; thus $M^{\mathfrak{u}}$ has the structure of an $\mathscr{U}(\mathfrak{l})$-module. The functors introduced above are adjoints of one another:

Proposition 3.1. *If $N \in \mathscr{M}(\mathfrak{l})$ and $M \in \mathscr{M}(\mathfrak{g})$, then there are natural vector space isomorphisms*

$$\Phi = \Phi_{M,N} : \operatorname{Hom}_{\mathscr{U}(\mathfrak{g})}(M_{B,S}(N), M) \cong \operatorname{Hom}_{\mathscr{U}(\mathfrak{l})}(N, M^{\mathfrak{u}_{B,S}}).$$

Remark 3.2. If $f : M_{B,S}(N) \to M$ is a $\mathfrak{g}$-module map, then $\Phi(f) : N \to M^{\mathfrak{u}}$ is defined by $\Phi(f) = f^{\mathfrak{u}} \circ (1 \otimes \mathrm{id}_N)$, where $1 \otimes \mathrm{id}_N : N \to (M_{B,S}(N))^{\mathfrak{u}}$ is the canonical $\mathfrak{l}$-invariant inclusion map, and $f^{\mathfrak{u}}$ is the restriction of f to $(M_{B,S}(N))^{\mathfrak{u}}$.

The proof of the proposition will be left to the reader.

Proposition 3.3. *If X is an irreducible $\mathscr{U}(\mathfrak{l})$-module, then $M_{B,S}(X)$ has a unique maximal proper submodule and a unique simple quotient $L_{B,S}(X)$.*

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Proof. Suppose M is a proper submodule of $M_{B,S}(X)$. Since $\mathscr{U}(\mathfrak{g}) \cdot v = \mathscr{U}(\mathfrak{g}) \cdot (\mathscr{U}(\mathfrak{l}) \cdot v) = \mathscr{U}(\mathfrak{g}) \cdot (1 \otimes_k X) = M_{B,S}(X)$, if v is a nonzero element of $M \cap (1 \otimes_k X)$, it follows that $M \cap (1 \otimes_k X) = (0)$. Let $M^-(X)$ denote the union of all proper submodules of $M_{B,S}(X)$. Then, clearly $M^-(X) \cap (1 \otimes_k X) = (0)$. But $1 \otimes_k X \neq (0)$, and so $M^-(X)$ is itself a proper submodule of $M_{B,S}(X)$. Therefore, $M^-(X)$ is the desired maximal submodule of $M_{B,S}(X)$, and $M_{B,S}(X)/M^-(X)$ is the unique simple quotient, $L_{B,S}(X)$, of $M_{B,S}(X)$. □

We now recall certain well-known facts about central characters of $\mathscr{U}(\mathfrak{g})$ and (generalized) central characters of $\mathscr{U}(\mathfrak{g})$-modules. Let $\mathfrak{g}$ be a finite-dimensional reductive Lie algebra over k. Let $\mathscr{Z}(\mathfrak{g})$ denote the center of the enveloping algebra $\mathscr{U}(\mathfrak{g})$, and let $\mathscr{Z}(\mathfrak{g})^\wedge$ denote the set of all central characters of $\mathscr{U}(\mathfrak{g})$; i.e., let $\mathscr{Z}(\mathfrak{g})^\wedge$ denote the set of all k-algebra homomorphisms from $\mathscr{Z}(\mathfrak{g})$ to k. If $\theta \in \mathscr{Z}(\mathfrak{g})^\wedge$, then the θ-primary component

$$M^\theta = \{\, m \in M : (z - \theta(z))^n \cdot m = 0, \text{ for some } n \in \mathbf{N}, \text{ and for every } z \in \mathscr{Z}(\mathfrak{g})\}$$

of M is a $\mathscr{U}(\mathfrak{g})$-submodule of M. We shall denote the set $\{\theta \in \mathscr{Z}(\mathfrak{g})^\wedge : M^\theta \neq (0)\}$ of all generalized central characters of a module M by $\operatorname{ch} M$. If M is a $\mathscr{U}(\mathfrak{g})$-module of finite length, then, by Quillen's Lemma [Qu], M is $\mathscr{Z}(\mathfrak{g})$-finite (recall that this means $\mathscr{Z}(\mathfrak{g})$ is locally-finite on M), and $\operatorname{ch} M$ is a finite set. If M is a $\mathscr{Z}(\mathfrak{g})$-finite $\mathscr{U}(\mathfrak{g})$-module, then we have a direct sum decomposition

$$M = \bigoplus_{\theta \in \operatorname{ch} M} M^\theta$$

of M in $\mathscr{M}(\mathfrak{g})$. If M is also assumed to be finitely generated, then the sum above is finite.

Next, we sketch Harish-Chandra's description of the central characters of $\mathscr{U}(\mathfrak{g})$. We begin by introducing the necessary notation. Let $\rho_{B,S}$ denote the sum $\frac{1}{2}\sum_{\alpha \in P^a} \alpha$, and let $A_{B,S}$ denote the algebra automorphism of $\mathscr{Z}(\mathfrak{h})$ $(= S(\mathfrak{h}) = \mathscr{U}(\mathfrak{h}))$ that restricts to $h \mapsto h - \rho_{B,S}(h)$ on $\mathfrak{h}$. Let $\pi : \mathscr{U}(\mathfrak{g}) \to \mathscr{U}(\mathfrak{l})$ and $\pi' : \mathscr{U}(\mathfrak{g}) \to \mathfrak{u}^-\mathscr{U}(\mathfrak{g}) + \mathscr{U}(\mathfrak{g})\mathfrak{u}$ be the projections associated with the decomposition

$$\mathscr{U}(\mathfrak{g}) = \mathscr{U}(\mathfrak{l}) \oplus (\mathfrak{u}^-\mathscr{U}(\mathfrak{g}) + \mathscr{U}(\mathfrak{g})\mathfrak{u})$$

of $\mathscr{U}(\mathfrak{g})$. The map π restricts to a map $\pi_{B,S}: \mathscr{Z}(\mathfrak{g}) \to \mathscr{Z}(\mathfrak{l})$, while the map π' restricts to a map $\pi'_{B,S}: \mathscr{Z}(\mathfrak{g}) \to \mathscr{U}(\mathfrak{g})\mathfrak{u}$ [Vo, p. 118]. If $S = \varnothing$, then denote $\rho_{B,S}$ by ρ_B, denote $A_{B,S}$ by A_B, and denote $\pi_{B,S}$ by π_B. Finally, let $\Gamma_{\mathfrak{g},\mathfrak{h}}$ denote the Harish-Chandra isomorphism $A_B \circ \pi_B : \mathscr{Z}(\mathfrak{g}) \to \mathscr{Z}(\mathfrak{h})$, and let $\chi_{\mathfrak{g},\mathfrak{h}} : \mathscr{Z}(\mathfrak{h})^\wedge \to \mathscr{Z}(\mathfrak{g})^\wedge$ denote the map $\chi_{\mathfrak{g},\mathfrak{h}}(\lambda) = \lambda \circ \Gamma_{\mathfrak{g},\mathfrak{h}}$.

Observe that the Harish-Chandra isomorphisms of the pairs $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{l}, \mathfrak{h})$ satisfy the relation $A_{B,S}^{-1} \circ \Gamma_{\mathfrak{g},\mathfrak{h}} = \Gamma_{\mathfrak{l},\mathfrak{h}} \circ \pi_{B,S}$. Passing to characters, we obtain

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a commutative diagram:

$$\begin{array}{ccc} \mathscr{Z}(\mathfrak{h})^{\wedge} & \xrightarrow{T_{B,S}} & \mathscr{Z}(\mathfrak{h})^{\wedge} \\ \chi_{\mathfrak{l},\mathfrak{h}}\big\downarrow & & \chi_{\mathfrak{g},\mathfrak{h}}\big\downarrow \\ \mathscr{Z}(\mathfrak{l})^{\wedge} & \xrightarrow{\pi^{\wedge}_{B,S}} & \mathscr{Z}(\mathfrak{g})^{\wedge} \end{array}$$

Here $T_{B,S}$ denotes the translation map $(\lambda \mapsto \lambda + \rho_{B,S})$, and $\pi^{\wedge}_{B,S} = (\theta \mapsto \theta \circ \pi_{B,S})$.

Theorem 3.4 (Harish-Chandra [H-C]). *The map* $\chi_{\mathfrak{g},\mathfrak{h}} \colon \mathscr{Z}(\mathfrak{h})^{\wedge} \to \mathscr{Z}(\mathfrak{g})^{\wedge}$ *is surjective and the fibers of* $\chi_{\mathfrak{g},\mathfrak{h}}$ *are finite and invariant under the action of* $\mathscr{W}(\mathfrak{g}, \mathfrak{h})$.

Theorem 3.4 and the commutativity of the diagram above imply that the fibers of $\pi^{\wedge}_{B,S}$ are finite.

Proposition 3.5. (a) *If* $N \in \mathscr{M}(\mathfrak{l})$ *is* $Z(\mathfrak{l})$*-finite, then* $M_{B,S}(N) \in \mathscr{M}(\mathfrak{g})$ *is* $\mathscr{Z}(\mathfrak{g})$*-finite, and* $\operatorname{ch} M_{B,S}(N) = \pi^{\wedge}_{B,S}(\operatorname{ch} N)$. *In particular, if* $\operatorname{ch} N$ *is a finite set, then* $\operatorname{ch} M_{B,S}(N)$ *is a finite set.*

(b) *If* M *is* $\mathscr{Z}(\mathfrak{g})$*-finite, then* $M^{\mathfrak{u}_{B,S}} \in \mathscr{M}(\mathfrak{l})$ *is* $\mathscr{Z}(\mathfrak{l})$*-finite, and*

$$\pi^{\wedge}_{B,S}(\operatorname{ch} M^{\mathfrak{u}_{B,S}}) \subset \operatorname{ch} M.$$

In particular, if $\operatorname{ch} M$ *is a finite set, then* $\operatorname{ch} M^{\mathfrak{u}_{B,S}}$ *is a finite set.*

Part (a) follows from 2.2 and the fact that the range of $\pi'_{B,S}$ is contained in $\mathscr{U}(\mathfrak{g})\mathfrak{u}$. We leave the details to the reader. The first two assertions of part (b) are a special case (corresponding to $i = 0$) of Corollary 3.1.6 in [Vo]. We direct the reader to this reference for a proof. The final assertion of (b) follows now from the observation that the fibers of $\pi^{\wedge}_{B,S}$ are finite.

Before stating the next result, we need to introduce a partial order on $\mathscr{Z}(\mathfrak{l})^{\wedge}$ that is determined by B and S. Let $c(\mathfrak{l})$ denote the center of $\mathfrak{l}$, and let $\lambda \colon \mathscr{Z}(\mathfrak{l})^{\wedge} \to c(\mathfrak{l})^*$ be the restriction map: $\theta \mapsto \theta|_{c(\mathfrak{l})}$ induced by the inclusion $c(\mathfrak{l}) \hookrightarrow \mathscr{Z}(\mathfrak{l})$. Define a partial order $\leq$ on $\mathscr{Z}(\mathfrak{l})^{\wedge}$ by setting $\theta_1 \leq \theta_2$, whenever $\lambda(\theta_1) - \lambda(\theta_2)$ is a sum of roots (or, more accurately, the restriction to $c(\mathfrak{l})$ of a sum of roots) in the set $P^a = \{ \alpha \in R : \mathfrak{g}_\alpha \subset \mathfrak{u} \}$. We shall use $\theta_1 < \theta_2$ to signify that $\theta_1 \neq \theta_2$ and $\theta_1 \leq \theta_2$. If $\Xi \subset \mathscr{Z}(\mathfrak{l})^{\wedge}$, then let $\max \Xi$ denote the set of elements $\theta \in \mathscr{Z}(\mathfrak{l})^{\wedge}$ that are maximal with respect to $\leq$. The following result is a version of Proposition 7.1.8 of [Di], tailored to the present context.

Proposition 3.6. *Suppose* $M \in \mathscr{M}(\mathfrak{g})$ *is generated by a* $\mathscr{U}(\mathfrak{l})$*-submodule,* N, *of* $M^{\mathfrak{u}_{B,S}}$. *Then,*

(a) *There is a unique* $\mathscr{U}(\mathfrak{g})$*-module homomorphism* $\phi \colon M_{B,S}(N) \to M$ *such that, for every* $n \in N$, $\phi(1 \otimes n) = n$. *This map is a surjection.*

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(b) $M = \mathscr{U}(\mathfrak{u}^-) \cdot N$.

(c) *If N is θ-primary, for some $\theta \in \mathscr{Z}(\mathfrak{l})^\wedge$, and M_1 is a $\mathscr{U}(\mathfrak{l})$-submodule of* $\operatorname{res}_{\mathfrak{l}}^{\mathfrak{g}} M$, *then we have a direct sum decomposition*

$$M_1 = \bigoplus_{\theta' \in \mathscr{Z}(\mathfrak{l})^\wedge} M_1^{\theta'}$$

of M_1 in $\mathscr{M}(\mathfrak{l})$. Moreover, $\theta' \leq \theta$, whenever $\theta' \in \operatorname{ch} M_1$ and $M_1^\theta = N \cap M_1$; in particular, $(M^{\mathfrak{u}})^\theta = N$.

Proof. The first assertion of (a) essentially amounts to a special case of 3.1. On the other hand, ϕ is surjective because $\phi(M_{B,S}(N)) = \phi(\mathscr{U}(\mathfrak{g}) \cdot (1 \otimes N)) = \mathscr{U}(\mathfrak{g}) \cdot N = M$. Part (b) follows from the Poincaré-Birkhoff-Witt Theorem:

$$\begin{aligned} M = \mathscr{U}(\mathfrak{g}) \cdot N &= \mathscr{U}(\mathfrak{u}^-) \otimes_k \mathscr{U}(\mathfrak{l}) \otimes_k (\mathscr{U}(\mathfrak{u})\mathfrak{u} + k \cdot 1) \cdot N \\ &= \mathscr{U}(\mathfrak{u}^-) \cdot (\mathscr{U}(\mathfrak{l}) \cdot N), \end{aligned}$$

and so,

$$M = \mathscr{U}(\mathfrak{u}^-) \cdot N. \tag{3.7}$$

Since N is $\mathscr{Z}(\mathfrak{l})$-finite, and the adjoint action of $\mathscr{Z}(\mathfrak{l})$ on $\mathscr{U}(\mathfrak{u}^-)$ is locally finite, 3.7 implies that $\mathscr{Z}(\mathfrak{l})$ is locally finite on M. This yields the first assertion of part (c). Finally, we note that $M = \mathscr{U}(\mathfrak{u}^-)\mathfrak{u}^- \cdot N + N$, by 3.7 and the Poincaré-Birkhoff-Witt Theorem. But observe that $\mathscr{U}(\mathfrak{u}^-)\mathfrak{u}^- \cdot N$ is $\mathscr{U}(\mathfrak{l})$-invariant, and that $\theta' < \theta$, for every $\theta' \in \operatorname{ch}(\mathscr{U}(\mathfrak{u}^-)\mathfrak{u}^- \cdot N)$. Therefore, we have a direct sum decomposition

$$\operatorname{res}_{\mathfrak{l}}^{\mathfrak{g}} M = \mathscr{U}(\mathfrak{u}^-)\mathfrak{u}^- \cdot N \oplus N$$

in $\mathscr{M}(\mathfrak{l})$, where $\mathscr{U}(\mathfrak{u}^-)\mathfrak{u}^- \cdot N = \bigoplus_{\theta' < \theta} (\operatorname{res}_{\mathfrak{l}}^{\mathfrak{g}} M)^{\theta'}$, and $N = (\operatorname{res}_{\mathfrak{l}}^{\mathfrak{g}} M)^\theta$. The remaining assertions of (c) follow from these observations. □

Recall that if X is an irreducible $\mathscr{U}(\mathfrak{l})$-module, then, by 3.3, $M_{B,S}(X)$ has a unique irreducible quotient $L_{B,S}(X)$.

Proposition 3.8. *The maps $L_{B,S}: X \mapsto L_{B,S}(X)$ and $F_{B,S}: V \mapsto V^{\mathfrak{u}_{B,S}}$ determine a bijective correspondence between the set of irreducible modules in $\mathscr{M}(\mathfrak{l})$ and the set of irreducible modules, $V \in \mathscr{M}(\mathfrak{g})$, that have nontrivial $\mathfrak{u}_{B,S}$-invariants.*

Proof. We show

(i) that $V^{\mathfrak{u}}$ is an irreducible $\mathscr{U}(\mathfrak{l})$-module, whenever $V \in \mathscr{M}(\mathfrak{g})$ is irreducible and $V^{\mathfrak{u}}$ is nonzero,

(ii) that $X \cong L_{B,S}(X)^{\mathfrak{u}}$, for every irreducible $\mathscr{U}(\mathfrak{l})$-module X, and

(iii) that $L_{B,S}(V^{\mathfrak{u}}) \cong V$, for every irreducible V in $\mathscr{M}(\mathfrak{g})$ such that $V^{\mathfrak{u}}$ is nonzero.

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Suppose $V \in \mathscr{M}(\mathfrak{g})$ is irreducible, suppose $V^{\mathfrak{u}}$ is nonzero, and suppose V_1 is a nonzero submodule of $V^{\mathfrak{u}}$. We wish to show that $V_1 = V^{\mathfrak{u}}$. Now, $V^{\mathfrak{u}}$ is $\mathscr{Z}(\mathfrak{l})$-finite, by 3.5(b). Let $\theta \in \operatorname{ch} V^{\mathfrak{u}}$, and let V_1 be a θ-primary submodule of $V^{\mathfrak{u}}$. Then, since V is an irreducible $\mathscr{U}(\mathfrak{g})$-module, and $V_1 \subset V^{\mathfrak{u}}$, we see that $V = \mathscr{U}(\mathfrak{g}) \cdot V_1$. Thus, it follows from 3.6(c) that $\max \operatorname{ch} V^{\mathfrak{u}} = \{\theta\}$. But this holds for any $\theta \in \operatorname{ch} V^{\mathfrak{u}}$. Therefore, $\operatorname{ch} V^{\mathfrak{u}}$ has exactly one element θ, and so, by 3.6(c), we have $V^{\mathfrak{u}} = (V^{\mathfrak{u}})^{\theta} = V_1$. Hence, $V^{\mathfrak{u}}$ is irreducible, thus proving (i).

Next, suppose X is an irreducible module in $\mathscr{M}(\mathfrak{l})$. Then observe that the map, $\Phi(p) : X \mapsto L_{B,S}(X)^{\mathfrak{u}}$, determined by the isomorphism, Φ, of 3.1 and the canonical projection, $p : M_{B,S}(X) \to L_{B,S}(X)$, is nonzero. Since X is irreducible by hypothesis and $L_{B,S}(X)$ is irreducible by (i), it follows that $\Phi(p)$ is an isomorphism, thus proving (ii). To prove (iii), suppose V is an irreducible module in $\mathscr{M}(\mathfrak{g})$ such that $V^{\mathfrak{u}}$ is nonzero. In the notation of 3.1, let f be the map $\Phi^{-1}(\mathrm{id})$ from $M_{B,S}(V^{\mathfrak{u}})$ to V that corresponds to the identity map from $V^{\mathfrak{u}}$ to $V^{\mathfrak{u}}$. Then f is nonzero, and hence surjective, by the irreducibility of V. Since $L_{B,S}(V^{\mathfrak{u}})$ is the unique irreducible quotient of $M_{B,S}(V^{\mathfrak{u}})$, it follows that $L_{B,S}(V^{\mathfrak{u}}) \cong V$. □

Proposition 3.9. *If $M \in \mathscr{M}(\mathfrak{g})$ has finite length, then so does $M^{\mathfrak{u}} \in \mathscr{M}(\mathfrak{l})$.*

Proof. Suppose $M \in \mathscr{M}(\mathfrak{g})$ has finite length. Then M is $\mathscr{Z}(\mathfrak{g})$-finite, and $\operatorname{ch} M$ is a finite set. Therefore 3.5(b) implies that $M^{\mathfrak{u}}$ is $\mathscr{Z}(\mathfrak{l})$-finite and that $\operatorname{ch} M^{\mathfrak{u}}$ is a finite set. Clearly it is adequate to show that $(M^{\mathfrak{u}})^{\theta}$ has finite length, for every $\theta \in \operatorname{ch} M^{\mathfrak{u}}$. Suppose $\theta \in \operatorname{ch} M^{\mathfrak{u}}$. Then let Y denote the $\mathscr{U}(\mathfrak{l})$-module $(M^{\mathfrak{u}})^{\theta}$, and let $X = \mathscr{U}(\mathfrak{g}) \cdot Y$. We need to show that Y is Noetherian and Artinian. If

$$\cdots \subset N^{i-1} \subset N^{i} \subset N^{i+1} \subset \cdots$$

is a chain of $\mathscr{U}(\mathfrak{l})$-modules contained in Y, then

$$\cdots \subset \mathscr{U}(\mathfrak{g}) \cdot N^{i-1} \subset \mathscr{U}(\mathfrak{g}) \cdot N^{i} \subset \mathscr{U}(\mathfrak{g}) \cdot N^{i+1} \subset \cdots$$

is a chain of $\mathscr{U}(\mathfrak{g})$-submodules of M. Since M has finite length, the number of distinct modules in the chain $\{\mathscr{U}(\mathfrak{g}) \cdot N^{i}\}_{i \in \mathbf{N}}$ is finite. Now suppose $\mathscr{U}(\mathfrak{g}) \cdot N^{i} = \mathscr{U}(\mathfrak{g}) \cdot N^{i+1}$, for some $i \in \mathbf{Z}$. Then 3.6(c) implies that $N^{i} = N^{i+1}$. This means $\{N^{i}\}_{i \in \mathbf{N}}$ has a finite, maximal refinement, thus proving that Y is Noetherian and Artinian. □

4. Modules with weight space decompositions

Retain the notation introduced in the second paragraph of the preceding section. In particular, let $\mathfrak{g}$ be a finite-dimensional reductive Lie algebra over k, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. If M is a $\mathscr{U}(\mathfrak{h})$-module, and $\lambda \in \mathfrak{h}^*$,

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then set

$$M_\lambda = \{ m \in M : h \cdot m = \lambda(h)m, \text{ for all } h \in \mathfrak{h} \}.$$

More generally, if $\Lambda \subset \mathfrak{h}^*$, then let M_Λ denote $\sum_{\lambda\in\Lambda} M_\lambda$. If $M_\lambda \neq (0)$, then M_λ is said to be a weight space of M. The elements of any such M_λ are called weight vectors, and λ is called the weight of M_λ. Let $\overline{\mathscr{M}}(\mathfrak{g}, \mathfrak{h})$ be the category of $\mathscr{U}(\mathfrak{g})$-modules that decompose into direct sums of weight spaces, when restricted to $\mathfrak{h}$. If M is a module in $\overline{\mathscr{M}}(\mathfrak{g}, \mathfrak{h})$, then let wt M denote the set of weights of M:

$$\operatorname{wt} M = \{ \lambda \in \mathfrak{h}^* : M_\lambda \neq (0) \}.$$

Let $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$ be the full subcategory of $\overline{\mathscr{M}}(\mathfrak{g}, \mathfrak{h})$ determined by the conditions that $M \in \mathscr{M}(\mathfrak{g}, \mathfrak{h})$ if and only if M is finitely generated and M_λ is finite dimensional, for every $\lambda \in \operatorname{wt} M$. If $M \in \mathscr{M}(\mathfrak{g}, \mathfrak{h})$, then M is said to be a $(\mathfrak{g}, \mathfrak{h})$-weight module. The category $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$ is the main focus of this paper. Suppose B is a base of the root system R and let $\leq$ be the partial order on $\mathfrak{h}^*$ determined by B. Then say that a module M in $\overline{\mathscr{M}}(\mathfrak{g}, \mathfrak{h})$ is a B-highest weight module, if there is a $\lambda \in \operatorname{wt} M$ such that $\mu \leq \lambda$, for every $\mu \in \operatorname{wt} M$.

Lemma 4.1. *The categories $\overline{\mathscr{M}}(\mathfrak{g}, \mathfrak{h})$ and $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$ are closed under the operations of taking submodules, taking quotients, and tensoring with finite dimensional $\mathscr{U}(\mathfrak{g})$-modules; in particular, if $M \in \overline{\mathscr{M}}(\mathfrak{g}, \mathfrak{h})$, M' is a submodule of M, and $m \in M'$ decomposes, in M, into a sum $\sum_{\lambda\in\Lambda} m_\lambda$ of weight vectors m_λ, then $m_\lambda \in M'$, for every $\lambda \in \Lambda$. If $M \in \overline{\mathscr{M}}(\mathfrak{g}, \mathfrak{h})$ and $\Lambda \in \mathfrak{h}^*/\mathbf{Z}R$, then $M_\Lambda \in \overline{\mathscr{M}}(\mathfrak{g}, \mathfrak{h})$, and we have a direct sum decomposition*

$$M = \bigoplus_{\Lambda\in\mathfrak{h}^*/\mathbf{Z}R} M_\Lambda,$$

of M, in $\overline{\mathscr{M}}(\mathfrak{g}, \mathfrak{h})$. If M is finitely generated, then the number of nonzero components in the decomposition above is finite.

We leave it to the reader to verify that the proofs of the corresponding assertions for highest weight modules go through with obvious modifications.

Lemma 4.2. *If $M \in \overline{\mathscr{M}}(\mathfrak{g}, \mathfrak{h})$ and the weight spaces of M are all finite dimensional, then M is $\mathscr{Z}(\mathfrak{g})$-finite. If $M \in \mathscr{M}(\mathfrak{g}, \mathfrak{h})$, then* ch M *is a finite set.*

Proof. If $\mathscr{C}(\mathfrak{h})$ denotes the commutant of $\mathfrak{h}$ in $\mathscr{U}(\mathfrak{g})$, then M is $\mathscr{C}(\mathfrak{h})$-finite, because every nonzero element in M belongs to the direct sum of a finite collection of weight spaces, and the weight spaces of M are preserved under the action of $\mathscr{C}(\mathfrak{h})$. But $\mathscr{Z}(\mathfrak{g})$ is a subalgebra of $\mathscr{C}(\mathfrak{h})$, and so, M is $\mathscr{Z}(\mathfrak{g})$-finite. The second assertion of the lemma follows immediately from the fact that all modules in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$ are finitely generated. □

Combining 4.1 and 4.2 we get the following result.

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Lemma 4.3. *If $M \in \overline{\mathscr{M}}(\mathfrak{g}, \mathfrak{h})$ and the weight spaces of M are all finite dimensional, then we have a direct sum decomposition,*

$$M = \bigoplus_{\Lambda, \theta} M_\Lambda^\theta,$$

of M, in $\overline{\mathscr{M}}(\mathfrak{g}, \mathfrak{h})$, where the pairs (Λ, θ) run over a subset of $(\mathfrak{h}^*/\mathbf{Z}R) \times \mathscr{Z}(\mathfrak{g})^\wedge$, and for each pair (Λ, θ), M_Λ^θ is a θ-primary module such that $\operatorname{wt} M_\Lambda^\theta \subset \Lambda$. If $M \in \mathscr{M}(\mathfrak{g}, \mathfrak{h})$, then the direct sum above is a sum in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$, and the number of nonzero components in the sum is finite.

We state one more result, 4.5, on decompositions of modules in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$. The proof of 4.5 will be based on the following lemma.

Lemma 4.4. *Let R be a k-algebra, and let R_1 and R_2 be a pair of commuting subalgebras of R_1 with the same identity element. If R_1 is simple and its center is contained in the center of R_2, then $R_1 R_2 \cong R_1 \otimes_k R_2$.*

This lemma is a special case of Theorem 7.1D in [ANT]. We direct the reader to this reference for a proof.

Lemma 4.5. *Suppose $M \in \mathscr{M}(\mathfrak{g}, \mathfrak{h})$ is irreducible, and suppose $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a decomposition of $\mathfrak{g}$ into a sum of ideals. Set $\mathfrak{h}_i = \mathfrak{g} \cap \mathfrak{h}$, for $i = 1, 2$. Then there are irreducible modules $M_i \in \mathscr{M}(\mathfrak{g}_i, \mathfrak{h}_i)$, such that $M \cong M_1 \otimes_k M_2$, as $\mathscr{U}(\mathfrak{g})$-modules.*

Proof. The proof will be based on the following correspondence of Lemire [Le1, Theorem 1], between irreducible modules in $\overline{\mathscr{M}}(\mathfrak{g}, \mathfrak{h})$ and irreducible modules of the commutant, $\mathscr{C}(\mathfrak{h})$, of $\mathfrak{h}$ in $\mathscr{U}(\mathfrak{g})$. Let V be a module in $\overline{\mathscr{M}}(\mathfrak{g}, \mathfrak{h})$ generated by a weight space V_λ. Then V_λ is a $\mathscr{C}(\mathfrak{h})$-module, and V is irreducible as a $\mathscr{U}(\mathfrak{g})$-module if and only if V_λ is irreducible as a $\mathscr{C}(\mathfrak{h})$-module. The map $V \mapsto V_\lambda$ sets up a bijective correspondence between irreducible modules $V \in \overline{\mathscr{M}}(\mathfrak{g}, \mathfrak{h})$ with weight λ, and irreducible modules of $\mathscr{C}(\mathfrak{h})$ with the same weight λ. This correspondence has been central to Lemire's approach to the study of irreducible modules in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$. It is assumed in his result cited above that $\mathfrak{g}$ is a simple, but the proof extends trivially to the reductive case.

Set $E = M_\lambda$, and let R (respectively R_i, for $i = 1, 2$) denote the image of $\mathscr{C}(\mathfrak{h})$ (respectively $\mathscr{C}(\mathfrak{h}_i)$) under the representation $\sigma : \mathscr{C}(\mathfrak{h}) \to \operatorname{End}_k(E)$ determined by the action of $\mathscr{C}(\mathfrak{h})$ on E. Then, since M is irreducible, the correspondence described above implies that $R = \operatorname{End}_k(E)$. Furthermore, R_1 and R_2 are commuting associative subalgebras of R such that $R = R_1 R_2$. Now observe that the restriction, $\operatorname{res}_{R_1}^R E$, of E to R_1 has a nontrivial submodule, because R and R_1 have the same identity element and E is unital. This fact, and the finite dimensionality of E imply that $\operatorname{res}_{R_1}^R E$ has an irreducible submodule E_1. The irreducibility of E_1 implies that the representation $R_1 \to \operatorname{End}_k(E_1)$ is surjective. On the other hand, since R_1 and R_2 are commuting subalgebras such that $R = R_1 R_2$, and $E = R \cdot E_1$ is a faithful R-module, we

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see that the map $R_1 \to \mathrm{End}_k(E_1)$ is injective. Hence $R_1 \cong \mathrm{End}_k(E_1)$ is simple. Likewise R_2 is simple. It follows that the centers of R_1 and R_2 are the one-dimensional images of $k \cdot 1 \subset \mathscr{C}(\mathfrak{h}_i)$ in R. This means that R_1 and R_2 satisfy the hypotheses of 4.4. Therefore $R \cong R_1 \otimes_k R_2$. But, up to isomorphism, R has a unique irreducible module, and so the R-module E is isomorphic to $E_1 \otimes_k E_2$. Now set $M_i = \mathscr{U}(\mathfrak{g}_i) \cdot E_i$, for $i = 1, 2$. Then, by Lemire's correspondence, M_i is an irreducible $\mathscr{U}(\mathfrak{g}_i)$-module, for $i = 1, 2$, and hence $M_1 \otimes_k M_2$ is an irreducible $\mathscr{U}(\mathfrak{g})$-module. Since $M_\lambda = E \cong E_1 \otimes_k E_2 = (M_1 \otimes_k M_2)_\lambda$, as $\mathscr{C}(\mathfrak{h})$-modules, we see that $M \cong M_1 \otimes_k M_2$ by appealing to Lemire's result once more. □

We need to introduce some notation before stating the next lemma. Suppose M is a module in $\overline{\mathscr{M}}(\mathfrak{g}, \mathfrak{h})$, $\alpha \in R$, and $s = E_\alpha$. Then denote the $\mathscr{U}(\mathfrak{g})$-module $M^{[s]}$ by $M^{[\alpha]}$. If $M^{[\alpha]} = M$, then say M is α-finite, and if $M^{[\alpha]} = (0)$, then say M is α-free. If P is a subset of R, then denote the subspaces $\bigoplus_{\alpha \in P} \mathfrak{g}_\alpha$ and $\sum_{\alpha \in P} [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ of $\mathfrak{g}$, by $\mathfrak{g}_P$ and $\mathfrak{h}_P$, respectively. Now assume $M \in \overline{\mathscr{M}}(\mathfrak{g}, \mathfrak{h})$ is finitely generated. Then, by virtue of 2.7, the set

$$F(M) = \{\alpha \in R : M \text{ is } \alpha\text{-finite}\}$$

is a closed subset of R; that is, $\alpha + \beta \in F(M)$, whenever α, $\beta \in F(M)$ and $\alpha + \beta \in R$. Clearly, $\mathfrak{h} \oplus \mathfrak{g}_{F(M)}$ is contained in the set, $\mathfrak{g}[M]$, of all elements in $\mathfrak{g}$ that are locally finite on M. In fact, equality holds in the preceding statement. To see this, observe that $\mathfrak{g}[M] = \mathfrak{h} \oplus \mathfrak{g}_S$, for some subset S of R, since $\mathfrak{g}[M]$ is a subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. But $\alpha \in F(M)$ if and only if $\mathfrak{g}_\alpha \subset \mathfrak{g}[M]$. This implies $S = F(M)$, and hence $\mathfrak{g}[M] = \mathfrak{h} \oplus \mathfrak{g}_{F(M)}$. Combining this fact with Remark 2.9, we have the following lemma.

Lemma 4.6. *Let M be a finitely generated module in $\overline{\mathscr{M}}(\mathfrak{g}, \mathfrak{h})$. Then $\mathfrak{g}[M] = \mathfrak{h} \oplus \mathfrak{g}_{F(M)}$, where $F(M)$ is the closed subset $\{\alpha \in R : M \text{ is } \alpha\text{-finite}\}$ of R. M is finite dimensional if and only if $F(M) = R$.*

Remark 4.7. If M is a module in $\overline{\mathscr{M}}(\mathfrak{g}, \mathfrak{h})$, then $M^{[\alpha]} = M^{(E_\alpha)}$, for every $\alpha \in R$; in particular, E_α is locally finite on M if and only if E_α is locally nilpotent on M. To see this, observe that if $m \in M_\lambda$ is a weight vector in M, then $E_\alpha \cdot m \in M_{\lambda+\alpha}$. It follows therefore that zero is the only possible eigenvalue of E_α, in its action on M.

Lemma 4.8. *Let M be a module in $\overline{\mathscr{M}}(\mathfrak{g}, \mathfrak{h})$, and let $\alpha \in R$. Then,*

(a) $(\mathrm{wt}\, M) + \mathbf{N}\alpha \subset \mathrm{wt}\, M$, *if M is α-free.*
(b) *If $(\lambda + \mathbf{N}\alpha) \cap \mathrm{wt}\, M$ is a finite set, for every $\lambda \in \mathrm{wt}\, M$, then M is α-finite.*

We leave the proof of the lemma to the reader.

We now state an application of Corollary 2.11 to modules in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$. This result was used in our original proof of Proposition 5.1. Although the proof of 5.1 given below does not rely on Lemma 4.9, we have retained the lemma because we believe that it is of independent interest.

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Lemma 4.9. *Let $M \in \mathscr{M}(\mathfrak{g}, \mathfrak{h})$, and let I be the vanishing ideal of an irreducible component of $\mathscr{V}(M)$. Then $I \cap \mathfrak{g}$ is a parabolic subalgebra of $\mathfrak{g}$ that contains $\mathfrak{h}$.*

Proof. By 2.11 $I \cap \mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$. Since $\mathfrak{h}$ is locally finite on M, it follows that $\mathfrak{h}$ is contained in the Lie algebra $I \cap \mathfrak{g}$. Therefore, $I \cap \mathfrak{g}$ takes the form $\mathfrak{h} \oplus \mathfrak{g}_P$ for some closed subset P of R. Notice $E_\alpha E_{-\alpha} \cdot M_\lambda \subset M_\lambda$, for every α in R and every $\lambda \in \mathrm{wt}\, M$. Since the weight spaces of M are finite dimensional, this implies that $E_\alpha E_{-\alpha}$ is locally finite on M, for every α in R. Thus, the symbol of $E_\alpha E_{-\alpha}$ belongs to $J(M) \subset I$. But I is a prime ideal, and so, either $E_\alpha \in I \cap \mathfrak{g}$, or $E_{-\alpha} \in I \cap \mathfrak{g}$, whenever $\alpha \in R$. Consequently, $I \cap \mathfrak{g}$ is of the form $\mathfrak{h} \oplus \mathfrak{g}_P$, where P is a closed subset of R with the property that either $\alpha \in R$, or $-\alpha \in R$. Therefore, $I \cap \mathfrak{g}$ is a parabolic subalgebra of $\mathfrak{g}$ that contains $\mathfrak{h}$. □

For each $\alpha \in R$, denote the dual root $2\alpha/(\alpha, \alpha)$ by $\check{\alpha}$. Suppose B is a base of R, and suppose $\mu \in \mathfrak{h}^*$ is such that $\mu + \rho_B \not\leq s_\alpha(\mu + \rho_B)$, for every $\alpha \in B$. Then we shall say μ is B-dominant. Equivalently, $\mu \in \mathfrak{h}^*$ is B-dominant whenever $-(\mu, \check{\alpha}) \notin \mathbf{N}^*$, for all $\alpha \in B$. We now state an elementary property of highest weight modules over simple Lie algebras of rank one.

Lemma 4.10. *Suppose $\mathfrak{s}_1$ is a simple k-Lie algebra of type A_1, and suppose $\mathfrak{h}_1$ is a Cartan subalgebra of $\mathfrak{s}_1$. Let $\{\pm\alpha\} \subset \mathfrak{h}_1^*$ be the set of nonzero roots of the pair $(\mathfrak{s}_1, \mathfrak{h}_1)$. Let B be the base $\{\alpha\}$ of R, let $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ denote the simple reflection corresponding to α, and let $\leq$ denote the partial order on $\mathfrak{h}_1^*$ determined by B. If N is a cyclic B-highest weight $\mathscr{U}(\mathfrak{s}_1)$-module that is generated by a vector in the highest weight space N_μ, where $\mu \in \mathfrak{h}_1^*$ is B-dominant, then*

$$\{ \lambda \in \mathfrak{h}_1^* : \lambda \leq \mu \text{ and } \lambda + \rho_B \not\leq s_\alpha(\mu + \rho_B) \} \subset \mathrm{wt}\, N.$$

In particular, if $\lambda \in \mathfrak{h}_1^*$ is B-dominant and $\lambda \leq \mu$, then $\lambda \in \mathrm{wt}\, N$.

As mentioned above, this is an elementary property of highest weight modules. The first assertion of the lemma also follows easily from 3.6, and the second assertion follows from the fact that $\lambda + \rho_B \not\leq s_\alpha(\mu + \rho_B)$, if $\lambda, \mu \in \mathfrak{h}_1^*$ are B-dominant and $\lambda \leq \mu$. Leaving the details of the proof to the reader, we state a partial converse of 4.8(b). The essential content of the following lemma is the fact that if $\mathfrak{g}$ is of type A_1, $M \in \overline{\mathscr{M}}(\mathfrak{g}, \mathfrak{h})$ is α-finite, and all the weight spaces of M are finite dimensional, then M is Noetherian. Indeed, 4.11 will be used (via 4.12) in the proof of Theorem 4.21, which asserts that all modules in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$ are Noetherian and Artinian. With this application in mind, in the proof below, we have not assumed finiteness of length, even for modules with a highest weight.

Lemma 4.11. *Suppose the weight spaces of a module $M \in \overline{\mathscr{M}}(\mathfrak{g}, \mathfrak{h})$ are all finite dimensional, and suppose M is an α-finite module, for some $\alpha \in R$. Then $(\lambda + \mathbf{N}\alpha) \cap \mathrm{wt}\, M$ is a finite set, for every $\lambda \in \mathrm{wt}\, M$.*

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Proof. Let $\mathfrak{s}_1$ be a three-dimensional subalgebra $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ of $\mathfrak{g}$, let $\mathfrak{h}_1 = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$, and set $B = \{\alpha\}$. Suppose M is α-finite, and $(\lambda + \mathbf{N}\alpha) \cap \operatorname{wt} M$ is not a finite set. Then we may assume, without loss of generality, that λ is B-dominant. Moreover, there is an increasing sequence $\{j_i\}_{i\in\mathbf{N}}$ of positive integers, and a sequence $\{m_i\}_{i\in\mathbf{N}}$ of nonzero vectors in M, such that for each i, $\mu_i = \lambda + j_i\alpha$ is the weight of m_i, and $E_\alpha \cdot m_i = 0$. Set $M(i) = \mathscr{U}(\mathfrak{s}_1) \cdot m_i$. Then, for each $i \in \mathbf{N}$, the $\mathscr{U}(\mathfrak{s}_1)$-module $N = M(i)$ satisfies the hypotheses of Lemma 4.10, and so $M(i)_\lambda \neq (0)$. Observe that the modules $M(i)$ have distinct central characters, because the set $\{\ \mu_i : i \in \mathbf{N}\ \}$ is collection of distinct, B-dominant weights. It follows that $\mathscr{U}(\mathfrak{s}_1) \cdot M_\lambda$ contains the direct sum $\bigoplus_i M(i)$, and hence, M_λ contains the direct sum $\bigoplus_i M(i)_\lambda$. But this means that M_λ is infinite dimensional, because $M(i)_\lambda \neq (0)$, for every $i \in \mathbf{N}$. This contradicts the assumption that $M \in \mathscr{M}(\mathfrak{g}, \mathfrak{h})$, thus completing the proof of the lemma. □

The following lemma serves as a complement to the assertion made in 4.6 that if $M \in \overline{\mathscr{M}}(\mathfrak{g}, \mathfrak{h})$ is finitely generated, then the subset $\{\ \alpha \in R : M \text{ is } \alpha\text{-finite}\ \}$ is a closed subset of R. However, note that in 4.12, M is assumed to belong to $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$.

Lemma 4.12. *Suppose* $\{\alpha, \beta, \alpha+\beta\} \subset R$, *and suppose* $M \in \mathscr{M}(\mathfrak{g}, \mathfrak{h})$ *is* α*-free and* β*-free. Then* M *is* $(\alpha + \beta)$*-free.*

Proof. Suppose M is not $(\alpha+\beta)$-free. Then, by 2.2, there is no loss of generality in assuming that M is $(\alpha+\beta)$-finite. Since M is α-free and β-free, 4.8(a) implies that if $\lambda \in \operatorname{wt} M$, then $\lambda + \mathbf{N}(\alpha + \beta) \subset \operatorname{wt} M$. But this contradicts 4.11, because we have assumed M is $(\alpha + \beta)$-finite. □

Definition 4.13. Suppose $M \in \overline{\mathscr{M}}(\mathfrak{g}, \mathfrak{h})$, and suppose $\mathfrak{s}$ is a subalgebra of $\mathfrak{g}$. Then say that $\mathfrak{s}$ is torsion free on M, if $M^{[x]} = (0)$, for every $x \in \mathfrak{s}\backslash\mathfrak{h}$. If $M^{[x]} = (0)$, for every $x \in \mathfrak{g}\backslash\mathfrak{h}$, then say that M is a torsion free module.

The following example illustrates the definition above.

Example 4.14. Let $\mathfrak{g}$ be the three-dimensional simple k-Lie algebra with basis vectors E_α, $E_{-\alpha}$, and H_α that satisfy the commutation relations

$$[H_\alpha, E_\alpha] = 2E_\alpha, \quad [H_\alpha, E_{-\alpha}] = -2E_{-\alpha}, \quad \text{and} \quad [E_\alpha, E_{-\alpha}] = H_\alpha,$$

and let $\mathfrak{h}$ be the Cartan subalgebra kH_α of $\mathfrak{g}$. Let V be an infinite-dimensional k-vector space with basis $\{v_i\}_{i\in\mathbf{Z}}$, and let $t \in k$. Define a $\mathscr{U}(\mathfrak{g})$-module structure on V by setting

$$E_\alpha \cdot v_i = (t+i)\,v_{i-1}, \quad E_{-\alpha} \cdot v_i = -(t+i)\,v_{i+1}, \quad \text{and} \quad H_\alpha \cdot v_i = -2\,(t+i)\,v_i.$$

The resulting module is in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$; it is irreducible and torsion free, provided $t \notin \mathbf{Z}$. Notice also that if $N \in \mathscr{M}(\mathfrak{l}, \mathfrak{h})$ is torsion free, then $\mathfrak{p}^-$ is torsion free on $M_{B,S}(N)$ (here, $\mathfrak{l}$, $\mathfrak{p}^-$, B, and S are as described in the second paragraph of §3).

Remark 4.15. Let $\mathfrak{p}_{B,S}$ be a parabolic subalgebra of $\mathfrak{g}$, let $\mathfrak{l} = \bigoplus_{\alpha\in R_S} \mathfrak{g}_\alpha$ be the Levi complement of $\mathfrak{u}$ in $\mathfrak{p}_{B,S}$, and let $M \in \overline{\mathscr{M}}(\mathfrak{g}, \mathfrak{h})$. Suppose all the

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weight spaces of M are finite dimensional, suppose $\mathfrak{l}$ is torsion free on M, and suppose $\Lambda_S \cap \mathrm{wt}\, M$ is nonempty, for some $\Lambda_S \in \mathbf{Z}R_S/\mathfrak{h}^*$. Then, by 4.8(a), $\Lambda_S \subset \mathrm{wt}\, M$, and there is a positive integer $c^M_{\Lambda_S}$ such that

$$\dim_k M_\mu = c^M_{\Lambda_S}, \quad \text{for all } \mu \in \Lambda_S.$$

Consequently, if M' is a subquotient of M and $\mathfrak{g}_\alpha \subset \mathfrak{l}$, where $\alpha \in R$, then every element of $\mathfrak{g}_\alpha \backslash (0)$ acts bijectively on M'; in particular, $\mathfrak{l}$ is torsion free on M'.

Recall that if P is a subset of R, then P^s denotes the set $P \cap (-P)$, and P^a denotes the set $P \backslash (-P)$. P is the disjoint union of its antisymmetric and symmetric parts: $P = P^a \cup P^s$. P is said to be a parabolic subset if $P \cup (-P) = R$, and P is a closed subset of R. A pair (P_1, P_2) of parabolic subsets of R is said to be a pair of opposite parabolic subsets if $P_1^s = P_2^s$, and $P_2^a = -P_1^a$. A subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ that contains $\mathfrak{h}$ is a parabolic subalgebra if and only if $\mathfrak{p}$ is of the form $\mathfrak{g}_P \oplus \mathfrak{h}$, for some parabolic subset P of R. On the other hand, $(\mathfrak{g}_{P'} \oplus \mathfrak{h}, \mathfrak{g}_{P''} \oplus \mathfrak{h})$ is a pair of opposite parabolic subalgebras of $\mathfrak{g}$ if and only if P', P'' are opposite parabolic subsets of R.

Lemma 4.16. *Suppose R is the disjoint union of subsets F and T. Then the following statements are equivalent.*

(a) *F and T are closed subsets of R.*
(b) *$(F \cup T^s, T \cup F^s)$ is a pair of opposite parabolic subsets of R.*
(c) *There is a base B of R, a subset S of B, and a decomposition of S into mutually orthogonal subsets S' and S'', such that*

$$F = (R_B^+ \setminus R_S) \cup R_{S'} \quad \text{and} \quad T = (R_B^- \setminus R_S) \cup R_{S''}$$

are the decompositions of F and T into their antisymmetric and symmetric parts.

The proof of 4.16 is routine, and will be left to the reader.

Recall from §2 that a $\mathscr{U}(\mathfrak{g})$-module M is said to be a pure module if $M^{[x]}$ is equal to (0) or M, for every $x \in \mathfrak{g}$. In the next result, we show that the action of $\mathfrak{g}$ on a pure weight module, M, determines a decomposition of $\mathfrak{g}$ into a sum of subalgebras that are either locally finite or torsion free on M. We make use of this result in our approach to the classification of irreducible modules in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$ and in our proof of the fact that every module in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$ has finite length.

Proposition 4.17. *Suppose M is a pure module in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$. Then $\mathfrak{g}$ has a unique pair, $(\mathfrak{p}_M, \mathfrak{p}_M^-)$, of opposite parabolic subalgebras such that*

(a) *the nilradical, $\mathfrak{u}_M$, of $\mathfrak{p}_M$ is locally nilpotent on M,*
(b) *the nilradical, $\mathfrak{u}_M^-$, of $\mathfrak{p}_M^-$ is torsion free on M, and*
(c) *the common Levi factor $\mathfrak{l} = \mathfrak{p}_M \cap \mathfrak{p}_M^-$ of $\mathfrak{p}_M$ and $\mathfrak{p}_M^-$ decomposes into a direct sum $\mathfrak{l}_M = c(\mathfrak{l}_M) \oplus \mathfrak{s}_M \oplus \mathfrak{t}_M$ of ideals, where $c(\mathfrak{l}_M)$ is the center*

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of $\mathfrak{l}_M$, $\mathfrak{s}_M$ *is the largest* $\mathrm{ad}(\mathfrak{h})$*-stable semisimple subalgebra of* $\mathfrak{g}$ *that is locally finite on* M, *and* $\mathfrak{t}_M$ *is the largest* $\mathrm{ad}(\mathfrak{h})$*-stable semisimple subalgebra of* $\mathfrak{g}$ *that is torsion free on* M.

In addition, there is a parabolic subalgebra, $\mathfrak{q}_M$, *of* $\mathfrak{g}$ *such that* $\mathfrak{q}_M$ *contains* $\mathfrak{h}$, *the nilradical,* $\mathfrak{v}_M$, *of* $\mathfrak{q}_M$ *is locally nilpotent on* M, *and the* $\mathrm{ad}(\mathfrak{h})$*-stable Levi complement,* $\mathfrak{r}_M$, *of* $\mathfrak{v}_M$ *in* $\mathfrak{q}_M$ *is torsion free on* M.

Proof. Let

$$F = \{\ \alpha \in R : M \text{ is } \alpha\text{-finite}\ \}, \qquad T = \{\ \alpha \in R : M \text{ is } \alpha\text{-free}\ \},$$
$$P = F \cup T^s, \quad \text{and} \quad P^- = T \cup F^s.$$

Then R is a disjoint union of F and T, since M is pure. Furthermore, F and T are closed subsets of F, by 4.6 and 4.12. Thus, by 4.16, $\mathfrak{p}_M = \mathfrak{g}_P \oplus \mathfrak{h}$ and $\mathfrak{p}_M^- = \mathfrak{g}_{P^-} \oplus \mathfrak{h}$ constitute a pair of opposite parabolic subalgebras of $\mathfrak{g}$. Since $\mathfrak{u}_M = \mathfrak{g}_{F^a}$ and $\mathfrak{u}_M^- = \mathfrak{g}_{T^a}$, (b) is a consequence of the definition of T, while (a) follows from 4.7 and the definition of F. To prove (c), define subalgebras $\mathfrak{s}_M$ and $\mathfrak{t}_M$ of $\mathfrak{g}$ by setting

$$\mathfrak{s}_M = \mathfrak{g}_{F^s} \oplus \mathfrak{h}_{F^s} \quad \text{and} \quad \mathfrak{t}_M = \mathfrak{g}_{T^s} \oplus \mathfrak{h}_{T^s},$$

respectively. If $\mathfrak{l}_M = \mathfrak{s}_M + \mathfrak{t}_M + \mathfrak{h}$, then clearly $\mathfrak{l}_M$ is an $\mathrm{ad}(\mathfrak{h})$-stable reductive subalgebra of $\mathfrak{g}$. Moreover, if $c(\mathfrak{l}_M)$ denotes the center of $\mathfrak{l}_M$, then $c(\mathfrak{l}_M)$ is the centralizer of $\mathfrak{s}_M + \mathfrak{t}_M$ in $\mathfrak{l}_M$, the sum $\mathfrak{s}_M + \mathfrak{t}_M$ is direct, and $\mathfrak{s}_M \oplus \mathfrak{t}_M$ is the commutator of $\mathfrak{l}_M$. Furthermore, $\mathfrak{s}_M \oplus \mathfrak{t}_M \oplus c(\mathfrak{l}_M)$ is an $\mathrm{ad}(\mathfrak{h})$-stable decomposition of $\mathfrak{l}_M$. Observe that, if $\mathfrak{m}$ is an $\mathrm{ad}(\mathfrak{h})$-stable semisimple subalgebra of $\mathfrak{g}$, then there is a subset A of R such that $A = A^s$, and $\mathfrak{m} = \mathfrak{h}_A \oplus \mathfrak{g}_A$. Suppose $\mathfrak{s}' = \mathfrak{h}_{A'} \oplus \mathfrak{g}_{A'}$ is an $\mathrm{ad}(\mathfrak{h})$-stable semisimple subalgebra of $\mathfrak{g}$ that is locally finite on M. Then, $A' \subset F^s$, and therefore, $\mathfrak{s}' \subset \mathfrak{s}_M$. In other words, $\mathfrak{s}_M$ is the largest $\mathrm{ad}(\mathfrak{h})$-stable semisimple subalgebra that is locally finite on M. Similarly, one can show that $\mathfrak{t}_M$ is the largest $\mathrm{ad}(\mathfrak{h})$-stable semisimple subalgebra of $\mathfrak{g}$ that is torsion free on M. Next, suppose $(\mathfrak{p}_1, \mathfrak{p}_1^-)$, is another pair of parabolic subalgebras of $\mathfrak{g}$ that satisfy (a), (b), and (c). Then, the characterizations of $\mathfrak{s}_M$, $\mathfrak{t}_M$, and $c(\mathfrak{l}_M)$ in (c) imply that $\mathfrak{p}_1 \cap \mathfrak{p}_1^- = \mathfrak{l}_M$. Let $\mathfrak{u}_1$ and $\mathfrak{u}_1^-$ be the nilradicals of $\mathfrak{p}_1$ and $\mathfrak{p}_1^-$, respectively. Then (a), (b), and (c) imply that $\mathfrak{u}_1 \subset \mathfrak{g}_{F^a} = \mathfrak{u}_M$ and $\mathfrak{u}_1^- \subset \mathfrak{g}_{T^a} = \mathfrak{u}_M^-$. But, $\mathfrak{g} = \mathfrak{u}_M \oplus \mathfrak{l} \oplus \mathfrak{u}_M^- = \mathfrak{u}_1 \oplus \mathfrak{l} \oplus \mathfrak{u}_1^-$, and so, comparing dimensions, we see that $\mathfrak{u}_1 = \mathfrak{u}_M$ and $\mathfrak{u}_1^- = \mathfrak{u}_M^-$.

We use 4.16 to prove the last assertion of the proposition too. Let $P = R_B^+ \cup R_{S''}$, in the notation of 4.16(c), and let $\mathfrak{q}_M$ be the parabolic subalgebra $\mathfrak{h} \oplus \mathfrak{g}_P (= \mathfrak{p}_{B,S''})$. If $\mathfrak{v}_M$ denotes the nilradical of $\mathfrak{q}_M$, then $\mathfrak{v}_M \subset \mathfrak{g}_{P^a} \subset \mathfrak{g}_F$. Therefore, $\mathfrak{v}_M$ is locally finite on M, and so, by 4.7, $\mathfrak{v}_M$ is locally nilpotent on M. On the other hand, if $\mathfrak{r}_M$ denotes the $\mathrm{ad}(\mathfrak{h})$-stable Levi factor of $\mathfrak{q}_M$, then $\mathfrak{r}_M = \mathfrak{h} \oplus \mathfrak{g}_{T^s}$. Therefore, $\mathfrak{r}_M$ is torsion free on M. This completes the proof of the proposition. □

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We now apply Proposition 4.17 to the problem of classifying irreducible modules in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$. Let M denote an irreducible module in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$. Then, clearly, M is a pure module. Therefore, there is a unique pair, $(\mathfrak{p}, \mathfrak{p}^-)$, of opposite parabolic subalgebras of $\mathfrak{g}$ such that $(\mathfrak{p}, \mathfrak{p}^-)$ and M together satisfy conditions (a), (b), and (c) of 4.17. Since M is irreducible, by 3.8, $M^{\mathfrak{u}}$ is an irreducible module in $\mathscr{M}(\mathfrak{l}, \mathfrak{h})$. (We note that if M is torsion free, then $\mathfrak{u} = (0)$, and so in this case, $\mathfrak{l} = \mathfrak{g}$ and $M^{\mathfrak{u}} = M$.) Furthermore, M can be recovered from $M^{\mathfrak{u}}$, since $M \cong L_{B,S}(M^{\mathfrak{u}})$. Therefore, an irreducible module M in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$ is completely determined by the module $M^{\mathfrak{u}}$ in $\mathscr{M}(\mathfrak{l}, \mathfrak{h})$. In other words, the classification of irreducible modules in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$ can be reduced to the problem of classifying irreducible modules in $\mathscr{M}(\mathfrak{l}, \mathfrak{h})$. Now choose a base B of R and a subset S of B such that

$$(\mathfrak{p}_{B,S}, \mathfrak{p}_{-B,-S}) = (\mathfrak{p}, \mathfrak{p}^-),$$

where $(\mathfrak{p}, \mathfrak{p}^-)$ is the pair of opposite parabolic subalgebras described in 4.17; this choice imposes strong constraints on the isomorphism class of $M^{\mathfrak{u}}$. Let $\mathfrak{s}$, $\mathfrak{t}$, and $c(\mathfrak{l})$ be as in 4.17(c), and let $\mathfrak{r} = \mathfrak{s} \oplus c(\mathfrak{l})$. Then, by 4.5, the decomposition $\mathfrak{l} = \mathfrak{r} \oplus \mathfrak{t}$ gives rise to a decomposition of $M^{\mathfrak{u}}$ into a tensor product $X_1 \otimes X_2$, where X_1 is an irreducible $\mathscr{U}(\mathfrak{r})$-module, and X_2 is an irreducible $\mathscr{U}(\mathfrak{t})$-module. If M is in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$, and $F(M)$ is the set $\{ \alpha \in R : M \text{ is } \alpha\text{-finite} \}$, then by 4.6, M is finite dimensional if and only if $F(M) = R$. Consequently, X_1 is finite dimensional. On the other hand, $\mathfrak{t}$ is torsion free on M, and so, in particular, $\mathfrak{t}$ is torsion free on X_2. We summarize this discussion in the following theorem.

Theorem 4.18. *Suppose M is an irreducible module in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$. Let $\mathfrak{p}$, $\mathfrak{p}^-$, $\mathfrak{u}$, $\mathfrak{l}$, $\mathfrak{s}$, $\mathfrak{t}$, and $c(\mathfrak{l})$ be as in 4.17, and let $\mathfrak{r}$ be the $\mathrm{ad}(\mathfrak{h})$-stable reductive subalgebra $\mathfrak{s} \oplus c(\mathfrak{l})$ of $\mathfrak{g}$. Let B be a base of R, and let S be a subset of B such that $(\mathfrak{p}_{B,S}, \mathfrak{p}_{-B,-S}) = (\mathfrak{p}, \mathfrak{p}^-)$. Then $M^{\mathfrak{u}}$ is an irreducible $\mathscr{U}(\mathfrak{l})$-module that decomposes into a tensor product $X_{\mathrm{fin}} \otimes X_{\mathrm{fr}}$ of an irreducible, finite-dimensional $\mathscr{U}(\mathfrak{r})$-module, X_{fin}, and an irreducible, torsion free module X_{fr} in $\mathscr{M}(\mathfrak{t}, \mathfrak{t} \cap \mathfrak{h})$. Furthermore, given the pair $(X_{\mathrm{fin}}, X_{\mathrm{fr}})$, the module M can be recovered as $L_{B,S}(X_{\mathrm{fin}} \otimes X_{\mathrm{fr}})$.*

Since the irreducible, finite-dimensional modules of a reductive Lie algebra can be described using well-known results of Cartan and Weyl, it follows from 4.18 that the classification of irreducible modules in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$ can be reduced to the classification of irreducible torsion free modules M in $\mathscr{M}(\mathfrak{t}, \mathfrak{t} \cap \mathfrak{h})$, where $\mathfrak{t}$ is a uniquely determined $\mathrm{ad}(\mathfrak{h})$-stable semisimple subalgebra of $\mathfrak{g}$. Furthermore, if the subalgebra $\mathfrak{t}$ decomposes into a sum $\bigoplus_{i=0}^{r} \mathfrak{t}_i$ of ideals, then, by 4.5, there is a corresponding decomposition of X_{fr} into a tensor product $\bigotimes_{i=1}^{r} X_{\mathrm{fr},i}$ of irreducible, torsion free $\mathscr{U}(\mathfrak{t}_i)$-modules. Consequently, the problem can be further reduced to the classification of irreducible modules $M \in \mathscr{M}(\mathfrak{t}, \mathfrak{t} \cap \mathfrak{h})$, where $\mathfrak{t}$ is simple, and M is torsion free.

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We devote the rest of this section to showing that every module M in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$ has finite length. Recall that $\overline{\mathscr{M}}(\mathfrak{g}, \mathfrak{h})$ denotes the category of (not necessarily finitely generated) $\mathscr{U}(\mathfrak{g})$-modules that decompose into direct sums of $\mathfrak{h}$-weight spaces, when restricted to $\mathfrak{h}$.

Lemma 4.19. *Suppose $M \in \overline{\mathscr{M}}(\mathfrak{g}, \mathfrak{h})$ is torsion free, suppose all the weight spaces of M are finite dimensional, and suppose*

$$M = \bigoplus_{\Lambda \in \mathfrak{h}^*/\mathbf{Z}R} M_\Lambda$$

is the decomposition of M specified by 4.1. *Then, the following assertions are equivalent.*

(a) *M has finite length.*
(b) *M is finitely generated.*
(c) *The number of nonzero components in the decomposition above is finite.*

Proof. Assertion (b) is a trivial consequence of (a). That (c) is a consequence of (b) is a special case of 4.1. Finally, suppose (c) holds. Then assume, without loss of generality, that $\operatorname{wt} M = \Lambda$, for some $\Lambda \in \mathfrak{h}^*/\mathbf{Z}R$. Notice, if M' is a submodule of M, then by 4.15, $\operatorname{wt} M' = \Lambda$, and $\mathfrak{g}_\alpha \backslash (0)$ maps M'_μ bijectively onto $M'_{\mu+\alpha}$, for every $\alpha \in R$ and every $\mu \in \Lambda$. But this means $M' = \mathscr{U}(\mathfrak{g}) \cdot M'_\lambda$, for any $\lambda \in \Lambda$. Therefore, if

$$\cdots \subset M^{i-1} \subset M^i \subset M^{i+1} \subset \cdots$$

is chain of submodules of M, then

$$\cdots \subset M_\lambda^{i-1} \subset M_\lambda^i \subset M_\lambda^{i+1} \subset \cdots$$

is an order-isomorphic chain of k-subspaces of M_λ. Since the dimension of M_λ is finite, the second chain has finite length and hence, so does the first. This completes the proof. □

In 4.21, we prove that every module in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$ has finite length. As an immediate consequence of this, we show in 4.22 that if $M \in \mathscr{M}(\mathfrak{g}, \mathfrak{h})$, then $M^{\mathfrak{u}} \in \mathscr{M}(\mathfrak{l}, \mathfrak{h})$, a fact we have not established thus far. As preparation for 4.21, we consider a special case of 4.22:

Lemma 4.20. *Suppose $M \in \mathscr{M}(\mathfrak{g}, \mathfrak{h})$, and suppose $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$ such that the nilradical, $\mathfrak{v}$, of $\mathfrak{q}$ is locally nilpotent on M and the* $\operatorname{ad}(\mathfrak{h})$-*stable Levi factor, $\mathfrak{r}$, of $\mathfrak{q}$ is torsion free on M. Then, for every subquotient M_1 of M, $M_1^{\mathfrak{v}} \in \overline{\mathscr{M}}(\mathfrak{r}, \mathfrak{h})$ is a torsion free module of finite length; in particular $M_1^{\mathfrak{v}}$ belongs to $\mathscr{M}(\mathfrak{r}, \mathfrak{h})$.*

Proof. Since $M_1 \in \mathscr{M}(\mathfrak{g}, \mathfrak{h})$, there is no loss of generality in assuming that $\operatorname{wt} M_1 \subset \Lambda_0$, for some $\Lambda_0 \in \mathfrak{h}^*/\mathbf{Z}R$. Let $S \subset B$ be subsets of R such that $\mathfrak{q} = \mathfrak{p}_{B,S}$, and let N denote $M_1^{\mathfrak{v}}$. Observe that M_1 is $\mathscr{Z}(\mathfrak{g})$-finite, and that $\operatorname{ch} M_1$ is a finite set, because M_1 belongs to $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$. Therefore, by 3.5(b), N is a $\mathscr{Z}(\mathfrak{r})$-finite module in $\overline{\mathscr{M}}(\mathfrak{r}, \mathfrak{h})$, and $\operatorname{ch} N$ is a finite set. This means the decomposition $N = \bigoplus_{\theta \in \operatorname{ch} N} N^\theta$ is a finite direct sum in $\overline{\mathscr{M}}(\mathfrak{r}, \mathfrak{h})$. Hence

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it is adequate to show that $N^\theta \in \overline{\mathscr{M}}(\mathfrak{r}, \mathfrak{h})$ has finite length, for each $\theta \in \operatorname{ch} N$. Let $\theta \in \operatorname{ch} N$. Suppose λ belongs to $\operatorname{wt} N^\theta$, and suppose z belongs to the center, $c(\mathfrak{r})$, of $\mathfrak{r}$. Then, since $\mathscr{Z}(\mathfrak{r}) \cap \mathfrak{h} = c(\mathfrak{r})$, one checks easily that $\theta(z)n = z \cdot n = \lambda(z)n$, for every $n \in N_\lambda^\theta$. Therefore, if ν, $\mu \in \operatorname{wt} N^\theta$, then $\nu|_{c(\mathfrak{r})} = \theta|_{c(\mathfrak{r})} = \mu|_{c(\mathfrak{r})}$. It follows that $(\nu - \mu) \perp c(\mathfrak{r})$. On the other hand, ν and μ belong to $\operatorname{wt} M_1 = \Lambda_0$, and so $\nu - \mu \in \mathbf{Z}R$. Hence, $\nu - \mu$ belongs to $\mathbf{Z}R_S = c(\mathfrak{r})^\perp \cap \mathbf{Z}R$. In other words, for a given $\theta \in \operatorname{ch} N$, there is a unique $\Lambda \in \mathfrak{h}^*/\mathbf{Z}R_S$ such that $N_\Lambda^\theta \neq (0)$. Now, by 4.15, $\mathfrak{r}$ is torsion free on M_1, and hence $N^\theta \in \overline{\mathscr{M}}(\mathfrak{r}, \mathfrak{h})$ is torsion free. In light of 4.19, we see therefore that N^θ has finite length, thus completing the proof of the lemma. □

Theorem 4.21. *Every module M in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$ has finite length.*

Proof. In view of 4.3 and 2.10, we assume, without loss of generality, that M is a θ-primary pure module such that $\operatorname{wt} M \subset \Lambda$, for some $\Lambda \in \mathfrak{h}^*/\mathbf{Z}R$. Then, by the last assertion of 4.17, there is a parabolic subalgebra, $\mathfrak{q}$ ($\supset \mathfrak{h}$), of $\mathfrak{g}$ such that the nilradical, $\mathfrak{v}$, of $\mathfrak{q}$ is locally nilpotent on M, and the $\operatorname{ad}(\mathfrak{h})$-stable Levi factor, $\mathfrak{r}$, of $\mathfrak{q}$ is torsion free on M. Since $\mathscr{U}(\mathfrak{g})$ is Noetherian and M is finitely generated, it is adequate to show that M is an Artinian $\mathscr{U}(\mathfrak{g})$-module. Suppose

$$M = M_0 \supset M_1 \supset M_2 \supset \cdots$$

is a descending chain of $\mathscr{U}(\mathfrak{g})$-submodules of M. Now recall that, by 4.15, if $\mathfrak{r}$ is torsion free on a certain module, then it is also torsion free on all subquotients of the module. Therefore, replacing M_i by $M_i/\bigcap_j M_j$ if necessary, we assume without loss of generality that the condition $\bigcap_i M_i = (0)$ is satisfied, in addition to the other conditions on M, described above. But 4.20 implies that the descending chain

$$M^\mathfrak{v} \supset M_1^\mathfrak{v} \supset M_2^\mathfrak{v} \supset \cdots$$

in $\mathscr{M}(\mathfrak{r}, \mathfrak{h})$ has finite length. Therefore, there is an $r \in \mathbf{N}$ such that $M_{r+j}^\mathfrak{v} = M_r^\mathfrak{v}$, for all $j \in \mathbf{N}$. This means $M_r^\mathfrak{v} \subset \bigcap_i M_i = (0)$. Since $\mathfrak{v}$ is locally nilpotent on M_r, it follows from Engel's Theorem that $M_r = (0)$. Consequently M is an Artinian $\mathscr{U}(\mathfrak{g})$-module. □

Corollary 4.22. *If $M \in \mathscr{M}(\mathfrak{g}, \mathfrak{h})$, then $M^\mathfrak{u} \in \mathscr{M}(\mathfrak{l}, \mathfrak{h})$.*

Proof. By 4.21, M has finite length, and so, 3.9 implies that $M^\mathfrak{u} \in \mathscr{M}(\mathfrak{l})$ has finite length. This means $M^\mathfrak{u}$ is finitely generated. Since the weight spaces of $M^\mathfrak{u}$ are finite dimensional, it follows that $M^\mathfrak{u} \in \mathscr{M}(\mathfrak{l}, \mathfrak{h})$. □

5. Torsion free modules

In this section we prove that if $\mathfrak{s}$ is a simple Lie algebra that admits a torsion free module, then $\mathfrak{s}$ is either of type A, or of type C. We begin with a computation of Gelfand-Kirillov dimensions.

Proposition 5.1. *Suppose $M \in \mathscr{M}(\mathfrak{g}, \mathfrak{h})$ is torsion free. Then the Gelfand-Kirillov dimension of M is equal to the rank of the derived Lie algebra $[\mathfrak{g}, \mathfrak{g}]$.*

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We thank Professor A. Joseph for suggesting the proof given below. Our original proof was much longer and obscured the elementary nature of the result.

Proof. If

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is an exact sequence of finitely generated $\mathscr{U}(\mathfrak{s})$-modules, then it follows that

$$\text{GKdim}\, M_2 = \max\{\text{GKdim}\, M_1, \text{GKdim}\, M_3\}$$

[KL, Proposition 5.1(b)]. In light of this fact and 4.3, we assume that M is generated by a weight space M_λ. By 4.15 it follows that $\text{wt}\, M \in \mathfrak{h}^*/\mathbf{Z}R$, and that there is a positive integer, c, such that $\dim_k M_\mu = c$, for all $\mu \in \text{wt}\, M$. Let R be the root system of the pair $([\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{h})$, and let n be the rank of $[\mathfrak{g}, \mathfrak{g}]$. Now choose a base B of R, and set $r = \max_{\alpha \in R}(\rho_B, \check{\alpha})$. Then,

$$\sum_{\substack{|(\mu, \rho_B)| \le m \\ \mu \in \mathbf{Z}R}} \mathscr{U}(\mathfrak{g})_\mu \cdot M_\lambda \subset \mathscr{U}^m(\mathfrak{g}) \cdot M_\lambda \subset \sum_{\substack{|(\mu, \rho_B)| \le mr \\ \mu \in \mathbf{Z}R}} \mathscr{U}(\mathfrak{g})_\mu \cdot M_\lambda,$$

for all $m \in \mathbf{N}$. Hence,

$$c \cdot (2m+1)^n \le \dim_k \mathscr{U}^m(\mathfrak{g}) \cdot M_\lambda \le c \cdot (2mr+1)^n .$$

But this implies $\lim_{m \to +\infty} \log_m \dim_k \mathscr{U}^m(\mathfrak{g}) \cdot M_\lambda = n$, thus proving that $\text{GKdim}(M) = n$. □

Theorem 5.2. *Suppose $\mathfrak{s}$ is a simple Lie algebra, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{s}$, and $M \in \mathscr{M}(\mathfrak{s}, \mathfrak{h})$ is torsion free. Then the algebra $\mathfrak{s}$ is either of type A or of type C.*

Proof. If $\mathfrak{s}$ admits a torsion free module, then by 4.15 and 4.21, $\mathfrak{s}$ admits an irreducible torsion free module. Let M be one such module, let n be the rank of $\mathfrak{s}$, and let U_M denote the algebra $\mathscr{U}(\mathfrak{s})/\text{Ann}(M)$. Now observe that $\mathfrak{s}$ is algebraic, since it is simple. Therefore, 2.12 and 5.1 imply that $\text{GKdim}\,(U_M) \le 2n$. On the other hand, it is easy to see that $\text{GKdim}\,(U_M) > 0$, because M is an infinite-dimensional, finitely generated, faithful U_M-module. Since $\text{Ann}\, M$ is a primitive ideal, $\text{GKdim}(U_M)$ is equal to the dimension of a nilpotent coadjoint orbit, by 7.1 of [BK]. Thus, $\mathfrak{s}^*$ has a nilpotent orbit, $\mathscr{O}$, such that

$$0 < \dim \mathscr{O} \le 2n, \tag{5.3}$$

where n denotes the rank of $\mathfrak{s}$. But, by Table 1 of [Jo2] and Lemma 3.3 of [Jo3], Lie algebras of type A and C are the only simple Lie algebras with nilpotent orbits that satisfy 5.3. This completes the proof of the theorem. □

Remark 5.4. It follows from Table 1 of [Jo2] and Lemma 3.3 of [Jo3], that the nilpotent orbit of 5.3 is in fact the minimal nonzero nilpotent orbit, $\mathscr{O}_{\text{min}}$, in $\mathfrak{s}^*$. Moreover, by [BK, 7.1], the associated variety of the left $\mathscr{U}(\mathfrak{s})$-module U_M is a union of nilpotent coadjoint orbits. Hence, by the minimality of $\mathscr{O}_{\text{min}}$, it follows that the associated variety of U_M is the closure, $\mathscr{O}_{\text{min}} \cup \{0\}$, of $\mathscr{O}_{\text{min}}$

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(the associated variety of any primitive quotient of $\mathscr{U}(\mathfrak{s})$ is the closure of a nilpotent orbit, but we do not have to appeal to that deep result, in the simple situation under consideration).

In his 1975 paper, [Le2], Lemire gave a construction of a class of irreducible torsion free $\mathscr{U}(\mathfrak{g})$-modules with one-dimensional weight spaces, in the case where $\mathfrak{g}$ is a simple Lie algebra of type A. On the other hand, in the author's Ph. D. thesis, a class of irreducible torsion free modules with one-dimensional weight spaces was constructed for Lie algebras of type C (these modules are also described in [Fe]), and Lemire's construction in [Le2] was recovered by restricting to an appropriate subalgebra of type A. Hence, as noted in the author's thesis, 5.2 can be strengthened to the assertion that a simple Lie algebra admits a torsion free module if and only if the algebra is of type A or C. In [BL2], Lemire and Britten prove that the two classes of modules just described are the only irreducible torsion free modules with one-dimensional weight spaces, thereby completing the classification of irreducible modules in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$ that have a one-dimensional weight space.

The work begun here is continued in a second paper, [Fe]. There we complete the classification of irreducible modules in $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$, and give geometric constructions of all irreducible torsion free modules.

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