

Lie Algebra of Unit Tangent Bundle in Minkowski 3-Space

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ABSTRACT

In this paper, a one-to-one correspondence between the set of unit split semi-quaternions and unit tangent bundle of semi-Euclidean plane is given. It is shown that the set of unit split semi-quaternions based on the group operation of multiplication is a Lie group. The Lie algebra of this group, consisting of the vector space matrix of the angular velocity vectors, is also considered. Planar rotations in Euclidean plane are expressed using split semi-quaternions. Some examples are given to illustrate the findings.

Keywords: Lie algebra; pseudo-rotation (boost); split semi-quaternion; unit tangent bundle.

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1. Introduction

Real-quaternions were first described by Irish mathematician Sir William Rowan Hamilton in 1843. The space of real-quaternions coincides with the 4-dimensional vector space over the real-numbers. Quaternions provide a simple and elegant way especially for describing rotations and reflections in space, and therefore have wide applications in Engineering. Moreover, they can be used alongside other methods, such as Euler angles, matrices and mechanics.

There is a one-to-one correspondence between the set of unit semi-quaternions and the unit tangent bundle of Euclidean plane \mathbb{R}^2 , see [8]. The aim of this paper is to generalize the idea of the paper [8] to Minkowski 3-space \mathbb{R}_1^3 by constructing an isomorphism between the unit tangent bundle of semi-Euclidean plane \mathbb{R}_1^2 and the unit split semi 3-sphere $T\mathbb{R}_1^2$. It is shown that $T\mathbb{R}_1^2$ is a Lie group and its Lie algebra is non-compact. As a geometric comment, planar rotations in Euclidean plane \mathbb{R}^2 are expressed in terms of split semi-quaternions and some examples are given to illustrate the main results.

2. Preliminaries

In this section, definitions of the algebras real-quaternions and split semi-quaternions are given. Then, some basic properties of these algebras are briefly introduced. To distinguish real-quaternions from split semi-quaternions, a real-quaternion is expressed by $q = q_0 + q_1i + q_2j + q_3k$ and a split semi-quaternion by $q = q_0 + q_1i + q_2j + q_3k$.

2.1. Real-Quaternions

The real algebra of real-quaternions, denoted by \mathbb{H} , is in one-to-one correspondence with the linear Euclidean space \mathbb{R}^4 having a standard basis $\{1, i, j, k\}$. So, a *real-quaternion* q exists from a scalar part and three imaginary parts:

$$q = q_0 + q_1i + q_2j + q_3k,$$

where $q_i \in \mathbb{R}$ ($i = 0, \dots, 3$) and i, j, k are the *imaginary units*. $S_q = q_0 \in \mathbb{R}$ is called the *scalar part* and $\mathbf{V}_q = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ the *vector part*. The *multiplication rules* between the imaginary units i, j, k are:

$$\begin{aligned} i^2 = j^2 = k^2 = ijk = -1, \\ ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j. \end{aligned}$$

Thus, the multiplication of real-quaternions is not commutative. If $S_q = 0$, then q is said to be a *pure* and is denoted by \mathbf{q} .

The *conjugate* of a real-quaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} = S_q + \mathbf{V}_q$ is

$$\bar{q} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k} = S_q - \mathbf{V}_q,$$

see [3, 4, 5, 2].

In the scalar-vector representation, *multiplication* of the real-quaternions $q = S_q + \mathbf{V}_q$ and $p = S_p + \mathbf{V}_p$ is

$$qp = S_q S_p - \langle \mathbf{V}_q, \mathbf{V}_p \rangle + S_q \mathbf{V}_p + S_p \mathbf{V}_q + \mathbf{V}_q \times \mathbf{V}_p,$$

where $\langle \mathbf{V}_q, \mathbf{V}_p \rangle$ and $\mathbf{V}_q \times \mathbf{V}_p$ are, respectively, the usual *inner* and *vector products* of \mathbf{V}_q and \mathbf{V}_p in \mathbb{R}^3 . In the real quantities $q_i \in \mathbb{R}$ and $p_i \in \mathbb{R}$ ($i = 0, \dots, 3$) representation, the multiplication of $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ and $p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$ can be given also by the matrix product

$$qp := \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}. \quad (2.1)$$

The *norm* of $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ is

$$N_q = q\bar{q} = \bar{q}q = q_0^2 + q_1^2 + q_2^2 + q_3^2$$

and its *modulus* is

$$\|q\| = \sqrt{N_q} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$

Real-quaternions with $N_q = 1$ are called *unit*.

The *multiplicative-inverse* of a non-zero real-quaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ is

$$q^{-1} = \frac{\bar{q}}{N_q}.$$

Hence the real-quaternion algebra \mathbb{H} forms a *division algebra*. For a unit real-quaternion q , we have $q^{-1} = \bar{q}$. For more information about real-quaternions see [11, 12, 14, 17, 15, 6].

The division algebra of real-quaternions \mathbb{H} is isomorphic to the Clifford algebra $Cl_{0,2}$ spanned by the basis $\{1, e_1, e_2, e_1e_2\}$ in 2-dimension by identifying the quaternionic units i, j, k with $e_1, e_2, e_1e_2 = e_3$ in $Cl_{0,2}$, respectively. The standard anti-commuting orthonormal basis elements e_1, e_2 satisfy

$$(e_1)^2 = (e_2)^2 = (e_3)^2 = -1 \quad \text{and} \quad e_1e_2 = -e_2e_1,$$

see [1].

2.2. Split Semi-Quaternions and Unit Tangent Bundle of \mathbb{R}_1^2

Like real-quaternions a *split semi-quaternion* q exists from a scalar part and three imaginary parts:

$$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k},$$

where $q_i \in \mathbb{R}$ ($i = 0, \dots, 3$) and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the *imaginary units*. $S_q = q_0 \in \mathbb{R}$ is called the *scalar part* and $\mathbf{V}_q = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ the *vector part*. The *multiplication rules* between the imaginary units $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are:

$$\begin{aligned} \mathbf{i}^2 = +1, \quad \mathbf{j}^2 = \mathbf{k}^2 = 0, \\ \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = 0, \quad \mathbf{ki} = -\mathbf{ik} = -\mathbf{j}. \end{aligned}$$

Like real-quaternions, the multiplication of split semi-quaternions is not commutative. We denote the algebra of split semi-quaternions by \mathbb{H}_S . If $S_q = 0$, then q is said to be a *pure* and is denoted by \mathbf{q} .

The *conjugate* of a split semi-quaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} = S_q + \mathbf{V}_q$ is

$$\bar{q} = S_q - \mathbf{V}_q.$$

In the scalar-vector representation, *multiplication* of the split semi-quaternions $q = S_q + \mathbf{V}_q$ and $p = S_p + \mathbf{V}_p$ is

$$qp = S_q S_p - \langle \mathbf{V}_q, \mathbf{V}_p \rangle^* + S_q \mathbf{V}_p + S_p \mathbf{V}_q + \mathbf{V}_q \times^* \mathbf{V}_p,$$

where $\langle \mathbf{V}_q, \mathbf{V}_p \rangle^* = -q_1 p_1$ is the *inner product* and $\mathbf{V}_q \times^* \mathbf{V}_p = 0\mathbf{i} - (q_3 p_1 - q_1 p_3)\mathbf{j} + (q_1 p_2 - q_2 p_1)\mathbf{k}$ the *vector product*.

In the real quantities $q_i \in \mathbb{R}$ and $p_i \in \mathbb{R}$ ($i = 0, \dots, 3$) representation, the multiplication of the split semi-quaternions $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ and $p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$ can be given also by the matrix product

$$qP := \begin{pmatrix} q_0 & q_1 & 0 & 0 \\ q_1 & q_0 & 0 & 0 \\ q_2 & -q_3 & q_0 & q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}. \tag{2.2}$$

And thus, the *matrix representation* of a split semi-quaternion $r = r_0 + r_1\mathbf{i} + r_2\mathbf{j} + r_3\mathbf{k}$ can be given as

$$R := \begin{pmatrix} r_0 & r_1 & 0 & 0 \\ r_1 & r_0 & 0 & 0 \\ r_2 & -r_3 & r_0 & r_1 \\ r_3 & -r_2 & r_1 & r_0 \end{pmatrix}. \tag{2.3}$$

The *norm* of $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ is

$$N_q = q\bar{q} = \bar{q}q = q_0^2 - q_1^2$$

and its *modulus* is

$$\|q\| = \sqrt{|N_q|} = \sqrt{|q_0^2 - q_1^2|}.$$

Split semi-quaternions with $N_q = \pm 1$ are called *unit*.

The *multiplicative-inverse* of a split semi-quaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ is defined to be

$$q^{-1} = \frac{\bar{q}}{N_q}$$

provided $N_q \neq 0$, i.e. $q_0 \neq \pm q_1$. Thus, a non-zero split semi-quaternion $p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$ provided $p_0 = \pm p_1$ does not have an inverse. If q has an inverse, then $qq^{-1} = q^{-1}q = 1$. Unlike real-quaternion algebra, algebra of split semi-quaternions does not forms a division algebra. For more information about split semi-quaternions see [13].

A unit split semi-quaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ can also be expressed by one of the following two matrices:

1. If $q_0^2 - q_1^2 = 1$, then

$$Q_1 := \begin{pmatrix} \cosh \alpha & \sinh \alpha & 0 & 0 \\ \sinh \alpha & \cosh \alpha & 0 & 0 \\ q_2 & -q_3 & \cosh \alpha & \sinh \alpha \\ q_3 & -q_2 & \sinh \alpha & \cosh \alpha \end{pmatrix} \tag{2.4}$$

2. If $q_0^2 - q_1^2 = -1$, then

$$Q_2 := \begin{pmatrix} \sinh \alpha & \cosh \alpha & 0 & 0 \\ \cosh \alpha & \sinh \alpha & 0 & 0 \\ q_2 & -q_3 & \sinh \alpha & \cosh \alpha \\ q_3 & -q_2 & \cosh \alpha & \sinh \alpha \end{pmatrix} \tag{2.5}$$

where $\alpha \in \mathbb{R}$. Since $\varepsilon = Q_1^t \varepsilon Q_1$ (resp. $\varepsilon = Q_2^t \varepsilon Q_2$) and $\det Q_1 = \cosh^2 \alpha - \sinh^2 \alpha = 1$ (resp. $\det Q_2 = \sinh^2 \alpha - \cosh^2 \alpha = -1$), the matrix Q_1 (resp. Q_2) is semi-orthogonal for the metric tensor

$$\varepsilon := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here Q_1^t (resp. Q_2^t) is the transpose of Q_1 (resp. Q_2). Therefore, unit split semi-quaternions can be used to represent rotations or reflections, see [7].

Let us denote the set of all unit split semi-quaternions (i.e., *unit split semi 3-sphere*) by $T\mathbb{R}_1^2$. A unit split semi-quaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ can be written in two parts $q^\bullet = (q_0, q_1) \in S_1^1$ and $q^* = (q_2, q_3) \in \mathbb{R}_1^2$ as $q = (q^\bullet, q^*) \in S_1^1 \times \mathbb{R}_1^2 \cong T\mathbb{R}_1^2$, where S_1^1 denotes the *Lorentzian unit circle* in semi-Euclidean plane \mathbb{R}_1^2 (that is, $q_0^2 - q_1^2 = \pm 1$) and $S_1^1 \times \mathbb{R}_1^2$ a *unit tangent bundle* of \mathbb{R}_1^2 , see Fig. 1 for the case $q_0^2 - q_1^2 = 1$.

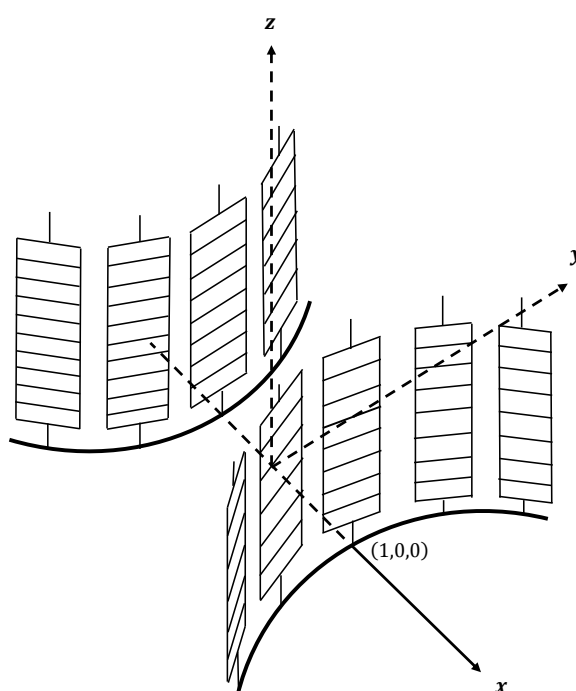


Figure 1. Unit split semi 3-sphere $T\mathbb{R}_1^2 (\cong S_1^1 \times \mathbb{R}_1^2)$.

Multiplication on \mathbb{H}_S is a binary operation on $T\mathbb{R}_1^2$ and is associative, has the identity element $e = 1 + 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$, and has the multiplicative-inverse $q^{-1} = \pm \bar{q}$ for each unit split semi-quaternion q . So, $T\mathbb{R}_1^2$ is a *group* with the multiplication operation on \mathbb{H}_S .

The algebra of split semi-quaternions \mathbb{H}_S is isomorphic to the Clifford algebra $\mathcal{C}\ell_{1,0,1}$ spanned by the basis $\{1, e_1, e_2, e_1e_2 = e_3\}$ of a degenerate quadratic form with degeneracy in 1-dimension by identifying the quaternionic units $\mathbf{i}, \mathbf{j}, \mathbf{k}$ with e_1, e_2, e_3 in $\mathcal{C}\ell_{1,0,1}$, respectively. The standard anti-commuting generators e_1, e_2 satisfy

$$(e_1)^2 = +1, (e_2)^2 = (e_3)^2 = 0 \quad \text{and} \quad e_1e_2 = -e_2e_1.$$

3. Lie Algebra of Unit Split Semi 3-Sphere $T\mathbb{R}_1^2$

In this section, the Lie algebra of unit semi 3-sphere $T\mathbb{R}_1^2$ is considered. Initially, let us define the differentiable mapping

$$\psi : \mathbb{H}_S \rightarrow \mathbb{R}; \quad \psi(q) = q_0^2 - q_1^2$$

for $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \in \mathbb{H}_S$. By restricting the domain set to $T\mathbb{R}_1^2$, we get the *regular value* of ψ as 1. The Jacobian matrix of ψ is

$$J_q(\psi) = \begin{pmatrix} 2q_0 & -2q_1 & 0 & 0 \end{pmatrix},$$

and thus $\text{rank}(\psi) = 1$ (i.e., ψ is surjective) for all $q \in \psi^{-1}(1) = T\mathbb{R}_1^2$. So, $T\mathbb{R}_1^2$ is a submanifold of \mathbb{H}_S . Moreover, since the mappings

$$\alpha : T\mathbb{R}_1^2 \times T\mathbb{R}_1^2 \rightarrow T\mathbb{R}_1^2; \quad \alpha(q, p) = qp$$

and

$$\beta : T\mathbb{R}_1^2 \rightarrow T\mathbb{R}_1^2; \quad \beta(q) = q^{-1} = \pm\bar{q}$$

are differentiable, $T\mathbb{R}_1^2$ is a *Lie group* of dimension three.

The *angular velocity matrix* of a unit split semi-quaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ can be given by one of the following two matrices:

1. If $q_0^2 - q_1^2 = 1$, using Eq.(2.4), we get

$$\mathcal{W}_1 = \dot{Q}_1 Q_1^{-1} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ k & l & 0 & 1 \\ -l & -k & 1 & 0 \end{pmatrix} \tag{3.1}$$

2. If $q_0^2 - q_1^2 = -1$, using Eq.(2.5), we get

$$\mathcal{W}_2 = \dot{Q}_2 Q_2^{-1} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ l & k & 0 & 1 \\ -k & -l & 1 & 0 \end{pmatrix} \tag{3.2}$$

where in both

$$k = (\dot{q}_2 - q_3) \cosh \alpha + (\dot{q}_3 - q_2) \sinh \alpha,$$

$$l = (q_2 - \dot{q}_3) \cosh \alpha + (q_3 - \dot{q}_2) \sinh \alpha.$$

Here, Q_1^{-1} and Q_2^{-1} are, respectively, the inverse matrices of Q_1 and Q_2 , and $\dot{Q}_1, \dot{Q}_2, \dot{q}_2, \dot{q}_3$ are, respectively, the derivatives of Q_1, Q_2, q_2, q_3 .

The *Lie algebra* of the group $T\mathbb{R}_1^2$ can also be given by one of the following two vector spaces:

1. Using Eq.(3.1), we have

$$T_1\mathbb{R}_1^2 = \text{span}\{\lambda\mathcal{W}_1 : \lambda \in \mathbb{R}\} \tag{3.3}$$

2. Using Eq.(3.2), we haven

$$T_2\mathbb{R}_1^2 = \text{span}\{\lambda\mathcal{W}_2 : \lambda \in \mathbb{R}\} \tag{3.4}$$

both equipped with the Lie product operation

$$[Q, P] = QP - PQ$$

where Q and P denotes, respectively, the matrix representations of any two unit split semi-quaternions q and p with $N_q = N_p = \pm 1$, and QP denotes the usual matrix product of Q and P .

We can write

$$\lambda\mathcal{W}_1 := \lambda \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + \lambda l \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} + \lambda k \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\lambda \mathcal{W}_2 := \lambda \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + \lambda k \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} + \lambda l \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

This means that the basis elements both of the vector spaces $T_1\mathbb{R}_1^2$ and $T_2\mathbb{R}_1^2$ are

$$\mathbf{i} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{j} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \mathbf{k} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

satisfying

$$[\mathbf{i}, \mathbf{j}] = -2\mathbf{k}, \quad [\mathbf{j}, \mathbf{k}] = 0, \quad [\mathbf{k}, \mathbf{i}] = 2\mathbf{j}.$$

Left invariant vector fields on $T\mathbb{R}_1^2$ at the point

$$\mathbf{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$

are $X = q\mathbf{i}$, $Y = q\mathbf{j}$ and $Z = q\mathbf{k}$. These vector fields can be expressed as

$$X = q_1 + q_0\mathbf{i} - q_3\mathbf{j} - q_2\mathbf{k},$$

$$Y = q_0\mathbf{j} + q_1\mathbf{k},$$

$$Z = q_1\mathbf{j} + q_0\mathbf{k}$$

when we restrict ourselves to the unit element $e = 1 + 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$ of the Lie group $T\mathbb{R}_1^2$.

Let us denote the Lie algebra of the group $T\mathbb{R}_1^2$ by $T_*\mathbb{R}_1^2$, which is given by $T_1\mathbb{R}_1^2$ for the case $q_0^2 - q_1^2 = 1$, see Eq.(3.3), and by $T_2\mathbb{R}_1^2$ for the case $q_0^2 - q_1^2 = -1$, see Eq.(3.4). We denote the matrix representation of the Lie algebra $T_*\mathbb{R}_1^2$ by

$$\text{Ad } X : T_*\mathbb{R}_1^2 \rightarrow T_*\mathbb{R}_1^2; \quad \text{Ad } X(Y) = [Y, X]$$

where $X, Y \in T_*\mathbb{R}_1^2$. The form $K(X, Y) = \text{Tr}(\text{Ad } X \circ \text{Ad } Y)$ for all $Y, Y \in T_*\mathbb{R}_1^2$ is the *Killing bilinear form* on $X \in T_*\mathbb{R}_1^2$. Here, $\text{Tr}(\text{Ad } X \circ \text{Ad } Y)$ stands for the trace of the mapping

$$\text{Ad } X \circ \text{Ad } Y : T_*\mathbb{R}_1^2 \rightarrow T_*\mathbb{R}_1^2; \quad \text{Ad } X \circ \text{Ad } Y(Z) = [X, [Y, Z]]$$

where "o" deontes the usual matrix product.

Let $X := \mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ and $Y := \mathbf{y} = y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k}$ be two vectors from $T_*\mathbb{R}_1^2$. The corresponding matrices of the mappings $\text{Ad } X$, $\text{Ad } Y$ and $\text{Ad } X \circ \text{Ad } Y$ are

$$\text{Ad } X := \begin{pmatrix} 0 & 0 & 0 \\ 2x_3 & 0 & -2x_1 \\ 2x_2 & -2x_1 & 0 \end{pmatrix}; \quad \text{Ad } Y := \begin{pmatrix} 0 & 0 & 0 \\ 2y_3 & 0 & -2y_1 \\ 2y_2 & -2y_1 & 0 \end{pmatrix}$$

and

$$\text{Ad } X \circ \text{Ad } Y := \begin{pmatrix} 0 & 0 & 0 \\ -4x_1y_2 & 4x_1y_1 & 0 \\ -4x_1y_3 & 0 & 4x_1y_1 \end{pmatrix}$$

We get

$$K(X, Y) = \text{Tr}(\text{Ad } X \circ \text{Ad } Y) = -8\langle X, Y \rangle^* = 8x_1y_1.$$

If the first component $x_1 \in \mathbb{R}$ of X is non-zero, then

$$K(X, X) = -8\langle X, X \rangle^* = 8x_1^2 > 0$$

and this means that the set

$$\tilde{T}\mathbb{R}_1^2 = \{X = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} : x_1, x_2, x_3 \in \mathbb{R} \text{ and } x_1 \neq 0\}$$

is a *non-compact* Lie group.

4. Planar Rotation and Split Semi-Quaternions

A motion f_α of the semi-Euclidean plane \mathbb{R}_1^2 can be given with respect to an orthonormal coordinate system as

$$f_\alpha : \mathbb{R}_1^2 \rightarrow \mathbb{R}_1^2; \quad \mathbf{x} \mapsto A_\alpha \cdot \mathbf{x} + \mathbf{c}$$

where

$$A_\alpha = \begin{pmatrix} \cosh \alpha & \sinh \alpha \\ \sinh \alpha & \cosh \alpha \end{pmatrix}.$$

Here, A_α is a semi-orthogonal matrix and represents a *pseudo-rotation* (also known as *boost*) of \mathbb{R}_1^2 through a positive oriented hyperbolic angle $\alpha \in \mathbb{R}$, and the vector $\mathbf{c} = (c_1, c_2) \in \mathbb{R}_1^2$ is a 2×1 column matrix representing the *translation part* of the motion, see [16, 10].

Lemma 4.1 ([7], Proposition 3.1). *The non-pure unit split semi-quaternion*

$$q = -\cosh \frac{\alpha}{2} + \sinh \frac{\alpha}{2} \mathbf{i} + q_2 \mathbf{j} + q_3 \mathbf{k}; \quad \sinh \frac{\alpha}{2} \neq 0$$

represents a negative oriented pseudo-rotation in \mathbb{R}_1^2 with an hyperbolic angle $\alpha \in \mathbb{R}$ and center

$$m = \left(\frac{q_2}{\sinh \frac{\alpha}{2}}, \frac{q_3}{\sinh \frac{\alpha}{2}} \right).$$

Example 4.1. Let $q = q_0 + q_1 \mathbf{i} + q_2 \mathbf{j} + q_3 \mathbf{k}$ be a non-pure unit split semi-quaternion defined by

$$q_0 = -\cosh \frac{\pi}{8}; \quad q_1 = \sinh \frac{\pi}{8}; \quad q_2 = 1; \quad q_3 = -\frac{3}{5}.$$

Then, the map

$$f_q : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} q_0^2 + q_1^2 & 2q_0q_1 \\ 2q_0q_1 & q_0^2 + q_1^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2(q_1q_2 + q_0q_3) \\ -2(q_1q_3 + q_0q_2) \end{pmatrix}$$

represents a semi-Euclidean planar motion because

$$\det \begin{pmatrix} q_0^2 + q_1^2 & 2q_0q_1 \\ 2q_0q_1 & q_0^2 + q_1^2 \end{pmatrix} = 1.$$

Moreover, we obtain

$$f_q \left(\frac{1}{\sinh \frac{\pi}{8}}, \frac{-3}{5 \sinh \frac{\pi}{8}} \right) = \left(\frac{1}{\sinh \frac{\pi}{8}}, \frac{-3}{5 \sinh \frac{\pi}{8}} \right).$$

Thus, the map f_q represents a negative oriented *pseudo-rotation* in \mathbb{R}_1^2 with an hyperbolic angle $\pi/4 \in \mathbb{R}$ and center

$$m = \left(\frac{1}{\sinh \frac{\pi}{8}}, \frac{-3}{5 \sinh \frac{\pi}{8}} \right).$$

The pseudo-rotation in semi-Euclidean plane \mathbb{R}_1^2 , which is given by Lem. 4. 1., is obtained using non-pure unit split semi-quaternions. However, by the following Prop. 4. 2, we show that if we use pure unit split semi-quaternions, we obtain the rotations in Euclidean plane \mathbb{R}^2 .

Proposition 4.1. *Let $p = \varepsilon_1 \mathbf{i} + p_2 \mathbf{j} + p_3 \mathbf{k}$ and $q = \varepsilon_2 \mathbf{i} + q_2 \mathbf{j} + q_3 \mathbf{k}$ be unit pure split semi-quaternions for $\varepsilon_1 = \pm 1$ and $\varepsilon_2 = \pm 1$. Then,*

$$pqp^{-1} = pqp = \mathbf{q}' = \varepsilon_2 \mathbf{i} + q'_2 \mathbf{j} + q'_3 \mathbf{k} := (q'_2, q'_3) \in \mathbb{R}^2$$

corresponds to the reflection of the point $q = (q_2, q_3) \in \mathbb{R}^2$ in the point $m = (\varepsilon_1 \varepsilon_2 p_2, \varepsilon_1 \varepsilon_2 p_3) \in \mathbb{R}^2$ in Euclidean plane \mathbb{R}^2 .

Proof. Let $p = \varepsilon_1 \mathbf{i} + p_2 \mathbf{j} + p_3 \mathbf{k}$ and $q = \varepsilon_2 \mathbf{i} + q_2 \mathbf{j} + q_3 \mathbf{k}$ be unit pure split semi-quaternions for $\varepsilon_1 = \pm 1$ and $\varepsilon_2 = \pm 1$. Then,

$$\mathbf{q}' = pqp^{-1} = pqp = \varepsilon_2 \mathbf{i} + q'_2 \mathbf{j} + q'_3 \mathbf{k}.$$

We can define a linear map

$$\varphi_p : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

as

$$\begin{pmatrix} q_2 \\ q_3 \end{pmatrix} \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} q_2 \\ q_3 \end{pmatrix} + \begin{pmatrix} 2\varepsilon_1\varepsilon_2p_2 \\ 2\varepsilon_1\varepsilon_2p_3 \end{pmatrix} = \begin{pmatrix} q_2' \\ q_3' \end{pmatrix}.$$

It is straightforward that

$$\varphi_p(\varepsilon_1\varepsilon_2p_2, \varepsilon_1\varepsilon_2p_3) = (\varepsilon_1\varepsilon_2p_2, \varepsilon_1\varepsilon_2p_3)$$

and this means that the linear map φ_p can be used to represent a rotation with an angle 180° and center $m = (\varepsilon_1\varepsilon_2p_2, \varepsilon_1\varepsilon_2p_3)$ (i.e., equivalent to a reflection in the point m) in Euclidean plane \mathbb{R}^2 . \square

Example 4.2. Let us consider the unit pure split semi-quaternions $\mathbf{p} = -\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{q} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$. Then, it can be easily observed that the product

$$\mathbf{p}\mathbf{q}\mathbf{p}^{-1} = \mathbf{p}\mathbf{q}\mathbf{p} = \mathbf{i} + 0\mathbf{j} - \mathbf{k} := (0, -1) \in \mathbb{R}^2$$

represents a reflection of the point $\mathbf{q} = (q_2, q_3) = (-2, 3) \in \mathbb{R}^2$ in the point $m = (\varepsilon_1\varepsilon_2p_2, \varepsilon_1\varepsilon_2p_3) = (-1, 1) \in \mathbb{R}^2$ in Euclidean plane \mathbb{R}^2 .

5. Conclusions

The algebra of real-quaternions can be considered by representing rotations and reflections in Euclidean plane \mathbb{R}^2 , see [9]. The algebra of split semi-quaternions can be considered by representing pseudo-rotations in \mathbb{R}_1^2 and planar rotations in \mathbb{R}^2 . Moreover, we can consider the set of all unit split semi-quaternions (i.e., unit split semi 3-sphere) as a unit tangent bundle of \mathbb{R}_1^2 .

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