Lie Algebras of Finite and Affine Type

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Preface

Lie algebras were originally introduced by S. Lie as algebraic structures used for the study of Lie groups. The tangent space of a Lie group at the identity element has the natural structure of a Lie algebra, called by Lie the infinitesimal group. However, Lie algebras also proved to be of interest in their own right. The finite dimensional simple Lie algebras over the complex field were investigated independently by E. Cartan and W. Killing and the classification of such algebras was achieved during the decade 1890–1900. Basic ideas on the structure and representation theory of these Lie algebras were also contributed at a later stage by H. Weyl. Since then the theory of finite dimensional simple Lie algebras has found many and varied applications both in mathematics and in mathematical physics, to the extent that it is now generally regarded as one of the classical branches of mathematics.

In 1967 V. G. Kac and R. V. Moody independently introduced the Lie algebras now known as Kac–Moody algebras. The finite dimensional simple Lie algebras are examples of Kac–Moody algebras; but the theory of Kac–Moody algebras is much broader, including many infinite dimensional examples. The Kac–Moody theory has developed rapidly since its introduction and has also turned out to have applications in many areas of mathematics, including among others group theory, combinatorics, modular forms, differential equations and invariant theory. It has also proved important in mathematical physics, where it has applications to statistical physics, conformal field theory and string theory. The representation theory of affine Kac–Moody algebras has been particularly useful in such applications.

In view of these applications it seems clear that the theory of Lie algebras, of both finite and affine types, will continue to occupy a central position in mathematics into the twenty-first century. This expectation provides the motivation for the present volume, which aims to give a mathematically rigorous development of those parts of the theory of Lie algebras most relevant to the understanding of the finite dimensional simple Lie algebras and the Kac–Moody algebras of affine type. A number of books on Lie algebras are confined to the finite dimensional theory, but this seemed too restrictive for the present volume in view of the many current applications of the Kac–Moody theory. On the other hand the Kac–Moody theory needs a prior knowledge of the finite dimensional theory, both to motivate it and to supply many technical details. For this reason I have included an account both of the Cartan–Killing–Weyl theory of finite dimensional simple Lie algebras and of the Kac–Moody theory, concentrating particularly on the Kac–Moody algebras of affine type. We work with Lie algebras over the complex field, although any algebraically closed field of characteristic zero would do equally well.

I was introduced to the theory of Lie algebras by an inspiring course of lectures given by Philip Hall at Cambridge University in the late 1950s. I have given a number of lecture courses on finite dimensional Lie algebras at Warwick University, and also two lecture courses on Kac–Moody algebras. The present book has developed as a considerably expanded version of the lecture notes of these courses. The main prerequisite for study of the book is a sound knowledge of linear algebra. I have in fact aimed to make this the sole prerequisite, and to explain from first principles any other techniques which are used in the development.

The most influential book on Kac–Moody algebras is the volume *Infinite-Dimensional Lie Algebras*, third edition (1990), by V. Kac. That formidable treatise contains a development of the Kac–Moody theory presupposing a knowledge of the finite dimensional theory, and includes information on several of the applications. The present volume will not rival Kac' account for experts on Kac–Moody algebras. About half of the theory covered in the 3rd edition of Kac' book has been included. However, for those new to the Kac–Moody theory, our account may be useful in providing a gentler introduction, making use of ideas from the finite dimensional theory developed earlier in the book.

The content of the book can be summarised as follows. The basic definitions of Lie algebras, their subalgebras and ideals, representations and modules, are given in Chapter 1. In Chapter 2 the standard results are proved on the representation theory of soluble and nilpotent Lie algebras. The results on representations of nilpotent Lie algebras are used extensively in the subsequent development. The key idea of a Cartan subalgebra is introduced in Chapter 3, where the existence and conjugacy of Cartan subalgebras are proved. We make use of some ideas from algebraic geometry to prove the conjugacy of Cartan subalgebras. In Chapter 4 the Killing form is introduced and used to describe the Cartan decomposition of a semisimple Lie algebra into root spaces with respect to a Cartan subalgebra. The well-known example of the special linear Lie algebra is used to illustrate the general ideas. In Chapter 5 the Weyl group is introduced and shown to be a Coxeter group. This leads on to the definition of the Cartan matrix and the Dynkin diagram. The possible Dynkin diagrams and Cartan matrices are classified in Chapter 6, and in Chapter 7 the existence and uniqueness of a semisimple Lie algebra with a given Cartan matrix are proved. In Chapter 8 the finite dimensional simple Lie algebras are discussed individually and their root systems determined.

Chapters 9 to 13 are concerned with the representation theory of finite dimensional semisimple Lie algebras. We begin in Chapter 9 with the introduction of the universal enveloping algebra, of free Lie algebras and of Lie algebras defined by generators and relations. The finite dimensional irreducible modules for semisimple Lie algebras are obtained in Chapter 10 as quotients of infinite dimensional Verma modules with dominant integral highest weight. In Chapter 11 the enveloping algebra is studied in more detail. Its centre is shown to be isomorphic to the algebra of polynomial functions on a Cartan subalgebra invariant under the Weyl group, and to the algebra of polynomial functions on the Lie algebra invariant under the adjoint group. This algebra is shown to be isomorphic to a polynomial algebra. The properties of the Casimir element of the centre of the enveloping algebra are also discussed. These are important in subsequent applications to representation theory. Characters of modules are introduced in Chapter 12, and Weyl's character formula for the irreducible modules is proved. The fundamental irreducible modules for the finite dimensional simple Lie algebras are discussed individually in Chapter 13. Their discussion involves exterior powers of modules, Clifford algebras and spin modules, and contraction maps.

This concludes the development of the structure and representation theory of the finite dimensional Lie algebras. This development has concentrated particularly on the properties necessary to obtain the classification of the simple Lie algebras and their finite dimensional irreducible modules. Among the significant results omitted from our account are Ado's theorem on the existence of a faithful finite dimensional module, the radical splitting theorem of Levi, the theorem of Malcev and Harish-Chandra on the conjugacy of complements to the radical, and the cohomology theory of Lie algebras.

The theory of Kac–Moody algebras is introduced in Chapter 14, where the Kac–Moody algebra associated to a generalised Cartan matrix is defined. In fact there are two slightly different definitions of a Kac–Moody algebra which have been used. There is a definition in terms of generators and relations which appears the more natural, but there is a different definition, given by Kac in his book, which is more convenient when one wishes to show that a given Lie algebra is a Kac–Moody algebra. I have used the latter definition, but have included a proof that, at least for symmetrisable generalised Cartan matrices, the two definitions are equivalent.

The trichotomy of indecomposable generalised Cartan matrices into those of finite, affine and indefinite types is obtained in Chapter 15. The Kac–Moody algebras of finite type turn out to be precisely the non-trivial finite dimensional simple Lie algebras, and a classification of those of affine type is given. The important special case of symmetrisable Kac–Moody algebras is also introduced. This class includes all those of finite and affine types, and some of those of indefinite type. In Chapter 16 it is shown that symmetrisable algebras have an invariant bilinear form, which plays a key role in the subsequent development. The Weyl group and root system of a Kac–Moody algebra are also discussed. The roots divide into real roots and imaginary roots, and a remarkable theorem of Kac is proved which characterises the set of positive imaginary roots. Kac–Moody algebras of affine type are singled out for more detailed discussion in Chapter 17. In Chapter 18 it is shown how some of them can be realised in terms of a central extension of a loop algebra of a finite dimensional simple Lie algebra, whereas the remainder can be obtained as fixed point subalgebras of these under a twisted graph automorphism.

Chapters 19 and 20 are devoted to the representation theory of Kac–Moody algebras. The representations considered are those from the category $\mathcal{O}$ introduced by Bernstein, Gelfand and Gelfand. In Chapter 19 the irreducible modules in this category are classified, and their characters are obtained in Kac' character formula, a generalisation to the Kac–Moody situation of Weyl's character formula. In Chapter 20 the representations of affine Kac–Moody algebras are discussed. The remarkable identities of I. G. Macdonald are obtained by specialising the denominator of Kac' character formula, interpreted in two different ways; one as an infinite sum and the other as an infinite product. The phenomenon of strings of weights with non-decreasing multiplicities is investigated inside an irreducible module for an affine algebra.

Many of the applications of the representation theory of affine Kac–Moody algebras use the theory of vertex operators. This theory lies beyond the scope of the present volume. However, we have introduced the idea of a vertex operator in Chapter 20 with the aim of encouraging the reader to explore the subject further.

A theory of generalised Kac–Moody algebras was introduced in 1988 by R. Borcherds. These Lie algebras were introduced as part of Borcherds' proof of the Conway–Norton conjectures on the representation theory of the Monster simple group. They are now frequently called Borcherds algebras. In Chapter 21 we have given an account of Borcherds algebras, including the definition and statements of the main results concerning their structure and representation theory, but detailed proofs are not given. Many of the results on Borcherds algebras are quite similar to those for Kac–Moody algebras, but there are examples of Borcherds algebras which are quite different from Kac–Moody algebras. The best known such example is the Monster Lie algebra, which we describe in the final section.

We conclude with an appendix containing one section for each of the algebras of finite and affine types, in which the most important pieces of information about the algebra concerned are collected.

I would like to express my thanks to Roger Astley of Cambridge University Press for his encouragement to complete the half finished manuscript of this book. This was eventually achieved after I had reached the status of Emeritus Professor, and therefore had more time to devote to it. I would also like to thank my colleague Bruce Westbury for the sustained interest he has shown in this work.