

LIE GROUP REPRESENTATIONS ON POLYNOMIAL RINGS¹

BY BERTRAM KOSTANT²

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0. Introduction. 1. Let G be a group of linear transformations on a finite dimensional real or complex vector space X . Assume X is completely reducible as a G -module. Let S be the ring of all complex-valued polynomials on X , regarded as a G -module in the obvious way, and let $J \subseteq S$ be the subring of all G -invariant polynomials on X .

Now let J^+ be the set of all $f \in J$ having zero constant term and let $H \subseteq S$ be any graded subspace such that $S = J^+S + H$ is a G -module direct sum. It is then easy to see that

$$(0.1.1) \quad S = JH.$$

(Under mild assumptions H may be taken to be the set of all G -harmonic polynomials on X . That is, the set of all $f \in S$ such that $\partial f = 0$ for every homogeneous differential operator ∂ with constant coefficients, of positive degree, that commutes with G .)

One of our main concerns here is the structure of S as a G -module. Regard S as a J -module with respect to multiplication. Matters would be considerably simplified if S were free as a J -module. One shows easily that S is J -free if and only if $S = J \otimes H$. This, however, is not always the case. For example S is not J -free if G is the two element group $\{I, -I\}$ and $\dim X \geq 2$. On the other hand one has

EXAMPLE 1. It is due to Chevalley (see [2]) that if G is a finite group generated by reflections then indeed $S = J \otimes H$. Furthermore the action of G on H is equivalent to the regular representation of G .

EXAMPLE 2. S is J -free in case G is the full rotation group (with respect to some Euclidean metric on X . For convenience assume in this example that $\dim X \geq 3$). Note that the decomposition of a polynomial according to the relation $S = J \otimes H$ is just the so-called "separation of variables" theorem for polynomials. This is so because J is the ring of radial polynomials and H is the space of all harmonic polynomials (in the usual sense).

Now, for any $x \in X$, let $O_x \subseteq X$ denote the G -orbit of x and let $S(O_x)$ be the ring of all functions on O_x defined by restricting S to O_x . Since J reduces to constants on any orbit it follows that (0.1.1) in-

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duces a G -module epimorphism

$$(0.1.2) \quad H \rightarrow S(O_x).$$

Since our major concern is the case where X is a reductive Lie algebra and G is the adjoint group and since the methods used there belong to algebraic geometry we will assume now that X is complex and that G is algebraic and reductive. All varieties considered are over \mathbf{C} . If Y has an algebraic structure $R(Y)$ will denote the ring of everywhere defined rational functions in Y . Obviously one always has

$$(0.1.3) \quad S(O_x) \subseteq R(O_x).$$

On the other hand if $G^x \subseteq G$ is the isotropy group defined by $x \in X$ then one has a G -module isomorphism

$$(0.1.4) \quad R(G/G^x) \rightarrow R(O_x).$$

The significance of (0.1.4) is that one knows the G -module structure of $R(G/G^x)$ completely by a very simple algebraic Frobenius reciprocity theorem (even though G^x may not be reductive). In fact if V^λ is any irreducible G -module with respect to the representation ν^λ and V_λ is the dual module then one has

$$(0.1.5) \quad \text{mult. of } \nu^\lambda \text{ in } R(G/G^x) = \dim V_\lambda^{G^x}$$

where $V_\lambda^{G^x}$ is the space of vectors in V_λ fixed under G^x .

Now in Examples 1 and 2 (assume complexified) the following three optimum situations occur:

- (a) S is J -free so that $S = J \otimes H$,
- (b) the map $H \rightarrow S(O_x)$ is an isomorphism for certain $x \in X$ and for those x ,
- (c) $R(O_x) = S(O_x)$.

But one observes that if in any general case (b) and (c) hold then, clearly, upon combining (0.1.4) and (0.1.5) one gets the G -module structure of H . If one gets in addition the "graded" G -module structure of H and knows the structure of J then one gets the full graded G -module structure of S in case (a) also holds.

In Example 2 the conditions (b) and (c) hold for any $x \neq 0$ (even if $(x, x) = 0$). In fact, classically, one has exploited (b) and (c) for $(x, x) > 0$ to solve the Dirichlet problem with the sphere as boundary. That is, if f is any continuous function on the sphere one first expands f as a Fourier development of spherical harmonics f_n . The sphere is $O_x \cap \mathbf{R}^m$ and the f_n are in $R(O_x)$. The equality $R(O_x) = S(O_x)$ and the isomorphism $H \rightarrow S(O_x)$ then yields the extension of f_n uniquely as

harmonic polynomials h_n on X . But this yields the desired extension of f .

In Example 1 the conditions (b) and (c) are satisfied for any "regular" element $x \in X$.

Our first concern in this paper is to give criteria for (a), (b) and (c) to hold in general. Since our interest is in the continuous case we will assume G is connected (and hence a variety). Thus Example 2 rather than Example 1 serves as a model.

Now let $P \subseteq X$ be the cone of common zeros defined by the ideal $J+S$ in S . Let X^* be the dual space to X and let $P^* \subseteq X^*$ be defined in a similar way with the roles of X and X^* interchanged. As a criterion to establish (a) and more we prove

PROPOSITION 0.1. *Assume (1) that $J+S$ is a prime ideal in S and (2) there exists an orbit $O_e \subseteq P$ which is dense in P . Then $S = J \otimes H$. Furthermore if G is a subgroup of the complex rotation group then H may be taken as the space of all G -harmonic polynomials. Moreover H then coincides with the space spanned by all powers f^* where $f \in P^*$.*

It may be observed that the criterion is satisfied in Example 2.

An element $x \in X$ is called quasi-regular if $P \subseteq \text{Cl}(\mathbf{C}^* \cdot O_x)$. A criterion to establish (b) is given by

PROPOSITION 0.2. *Assume conditions (1) and (2) of Proposition 0.1 are satisfied. Then the G -module epimorphism $H \rightarrow S(O_x)$ is an isomorphism for any quasi-regular element $x \in X$.*

It may be observed that in Example 2 every nonzero $x \in X$ is quasi-regular.

From known facts in algebraic geometry one has the following criterion to insure (c).

PROPOSITION 0.3. *Let $x \in X$ and assume (1) the closure $\text{Cl}(O_x)$ is a normal variety and (2) $\text{Cl}(O_x) - O_x$ has a codimension of at least 2 in $\text{Cl}(O_x)$. Then $R(O_x) = S(O_x)$.*

It may be observed that the conditions of Proposition 0.3 are satisfied for every $x \in X$ in Example 2.

Now assume that $X = \mathfrak{g}$ is a complex reductive Lie algebra and G is the adjoint group. Here the structure of J is given by a theorem of Chevalley. This asserts that J is a polynomial ring in l (the rank of \mathfrak{g}) homogeneous generators $u_i, i = 1, 2, \dots, l$, with $\deg u_i = m_i + 1$ where the m_i are the exponents of \mathfrak{g} .

Now one knows that here P is the set of all nilpotent elements of \mathfrak{g} [13, Theorem 9.1]. But then by [13, Corollary 5.5], P does contain a dense orbit O_e , namely, the set of all principal nilpotent elements in \mathfrak{g} . Thus to apply Propositions 0.1 and 0.2 one must prove that J^+S is a prime ideal.

If $n = \dim \mathfrak{g}$ (all dimensions are over \mathbf{C}) then one sees easily that $n-l$ is the maximal dimension of any orbit. Let $\mathfrak{r} = \{x \in \mathfrak{g} \mid \dim O_x = n-l\}$. Any regular element $x \in \mathfrak{g}$ belongs to \mathfrak{r} . But also $e \in \mathfrak{r}$ for any principal nilpotent element e . These in fact are extreme cases.

PROPOSITION 0.4. *Let $x \in \mathfrak{g}$ be arbitrary. Write (uniquely) $x = y + z$ where y is semi-simple, z is nilpotent and $[y, z] = 0$. Let \mathfrak{g}^y be the centralizer of y in \mathfrak{g} so that \mathfrak{g}^y is a reductive Lie algebra and $z \in \mathfrak{g}^y$. Then $x \in \mathfrak{r}$ if and only if z is principal nilpotent in \mathfrak{g}^y .*

Let $x \in \mathfrak{g}$. Consider the values $(du_i)_x$ of the l differential forms $du_i, i = 1, 2, \dots, l$, at x . It is known that these covectors are linearly independent whenever x is regular. (One recalls that the product of the positive roots is an $l \times l$ minor of a suitable $n \times l$ matrix determined by the du_i .) But to prove the primeness of the ideal J^+S one needs to know that these covectors are linearly independent if x is a principal nilpotent element. This fact is contained in

THEOREM 0.1. *Let $x \in \mathfrak{g}$. Then the $(du_i)_x$ are linearly independent if and only if $x \in \mathfrak{r}$.*

Proposition 0.1 may now be applied.

THEOREM 0.2. *One has $S = J \otimes H$ where H is the space of all G -harmonic polynomials on \mathfrak{g} . Furthermore H coincides with the space of all polynomials spanned by all powers of "nilpotent" linear functionals.*

Since Theorem 0.1 shows also that P is a complete intersection the decomposition $S = J \otimes H$ when combined with [15, Proposition 5, §78] gives, in the notation of FAC, all the sheaf cohomology groups $H^i(\mathbf{P}, \mathcal{O}(m))$ where \mathbf{P} is the projective variety defined by P .

Added in proof. Another application of the primeness of J^+S in algebraic geometry is

THEOREM 0.3 (*Added in proof*). *The intersection multiplicity of P , at the origin, with any Cartan subalgebra is w , where w is the order of the Weyl group.*

Next, Proposition 0.2 is put into effect for all orbits of maximal dimension by

THEOREM 0.4. *The set \mathfrak{r} coincides with the set of all quasi-regular ele-*

ments in \mathfrak{g} . (Thus H and $S(O_x)$ are isomorphic as G -modules for any $x \in \mathfrak{r}$.)

As a consequence of Theorems 0.2 and 0.4 one shows that not only is the ideal $J+S$ prime in S but J_1S is prime for any prime ideal $J_1 \subseteq J$. Furthermore one gets the following characterization of all the invariant prime ideals in S which are generated by elements of J .

THEOREM 0.5. *Let $I \subseteq S$ be any G -invariant prime ideal. Let $u \subseteq \mathfrak{g}$ be the affine variety of zeros of I . Then I is of the form $I = J_1S$ for J_1 a prime ideal in J if and only if $u \cap \mathfrak{r}$ is not empty.*

Since $R(O_x) = S(O_x)$ in case O_x is closed and since O_x is closed if x is regular one gets the G -module structure of H by applying Theorem 0.3 and (0.1.5) for x regular. Thus if D denotes the set of dominant integral forms corresponding to a Cartan subgroup A , so that D indexes all the irreducible representations of G as highest weights, then one has

$$(0.1.6) \quad \text{mult. of } \nu^\lambda \text{ in } H = l_\lambda$$

where $l_\lambda = \dim V_\lambda^A$ is the multiplicity of the zero weight of ν_λ .

In order to determine the G -module structure of S^k , the space of homogeneous polynomials on \mathfrak{g} of degree k , one must know more than (0.1.6). In fact using the relation $S = J \otimes H$ what one wants is the multiplicity of ν^λ in $H^j = S^j \cap H$ for any λ and j . As it turns out, for this, one needs $R(O_e) = S(O_e)$ where e is a principal nilpotent element. To show the latter using Proposition 0.3 it is enough to show that P is a normal variety and $P - O_e$ has a codimension of at least 2 in P .

Let $\mathfrak{O}_\mathfrak{r}$ be a set of all orbits of maximal dimension $(n-l)$. The set $\mathfrak{O}_\mathfrak{r}$ may be parameterized by \mathbf{C}^l in the following way. Let

$$u: \mathfrak{g} \rightarrow \mathbf{C}^l$$

be the morphism given by putting $u(x) = (u_1(x), \dots, u_l(x))$ for any $x \in \mathfrak{g}$. Since u reduces to a constant on any orbit it induces a map

$$\eta_\mathfrak{r}: \mathfrak{O}_\mathfrak{r} \rightarrow \mathbf{C}^l.$$

One has

THEOREM 0.6. *$\eta_\mathfrak{r}$ is a bijection.*

Thus to each $\xi \in \mathbf{C}^l$ there exists a unique orbit, $O(\xi)$ of dimension $n-l$ which correspond to ξ under $\eta_\mathfrak{r}$. Now let $P(\xi) = u^{-1}(\xi)$ for any $\xi \in \mathbf{C}^l$ so that

$$\mathfrak{g} = \bigcup_{\xi \in \mathbf{C}^l} P(\xi)$$

is a disjoint union. Note that $P(\xi) = P$ and $O(\xi) = O_e$ if ξ is the origin of \mathbf{C}^l . One proves

THEOREM 0.7. *For any $\xi \in \mathbf{C}^l$ one has*

$$P(\xi) = \text{Cl}(O(\xi))$$

so that $P(\xi)$ is a variety of dimension $n - l$. Moreover $P(\xi)$ is a complete intersection and $O(\xi)$ coincides with the set of simple points on $P(\xi)$. Finally $P(\xi)$ is a finite union of orbits so that $\text{Cl}(O_x)$ is a finite union of orbits for any $x \in \mathfrak{g}$.

Since $P(\xi)$ is a complete intersection and since its singular locus is the complement (a finite union of orbits) of $O(\xi)$ in $P(\xi)$ one would get the normality of $P(\xi)$ by a theorem of Seidenberg if one knew the dimension of the other orbits in $P(\xi)$ were at most $n - l - 2$.

Now it is well known that $\dim O_x$ is even (and hence $\dim_{\mathbf{R}} O_x$ is a multiple of 4) for any semi-simple element $x \in \mathfrak{g}$. Less known is the following proposition observed independently by the author, Borel, and (most simply proved by) Kirillov.

PROPOSITION 0.5. *The dimension of O_x is even for any $x \in \mathfrak{g}$.*

Combining Theorem 0.6 and Proposition 0.5 one obtains

THEOREM 0.8. *Let $\xi \in \mathbf{C}^l$ be arbitrary. Then $P(\xi)$ is a normal variety and the codimension of $P(\xi) - O(\xi)$ in $P(\xi)$ is at least 2.*

Applying Proposition 0.3 one then has

THEOREM 0.9. *Let $x \in \mathfrak{r}$. Then $R(O_x) = S(O_x)$. (This implies that all $R(O_x)$ for $x \in \mathfrak{r}$ are isomorphic as G -modules; even though they are not in general isomorphic as rings.) Let $\xi = u(x)$. Then $R(O_x)$ ($= R(G/G^x)$) is an affine algebra (even though O_x is not necessarily an affine variety) and $P(\xi)$ is the variety of all maximal ideals of $R(O_x)$. Thus the embedding of G/G^x in \mathfrak{g} as O_x is special in that any morphism of G/G^x (or O_x) into any affine variety extends uniquely to a morphism of $P(\xi) = \text{Cl}(O_x)$ into the variety. (In particular this holds for O_e and $\text{Cl}(O_e) = P$.) Finally (using (0.1.5) and the equality $R(O_x) = S(O_x)$) one has, for any $\lambda \in D$*

$$(0.1.7) \quad \dim V_{\lambda}^{G^x} = l_{\lambda}$$

so that the left side of (0.1.7) is independent of $x \in \mathfrak{r}$.

Now let e_-, x_0, e be a principal S -triple (that is, a “canonical” basis

of a principal three dimensional simple Lie subalgebra). In particular then e is a principal nilpotent element. Used heavily in the theorems above is the result of [13] which asserts that \mathfrak{g}^e is l -dimensional and has a basis $z_i, i = 1, 2, \dots, l$, such that

$$(0.1.8) \quad [x_0, z_i] = m_i z_i$$

where, we recall, the m_i are the exponents of \mathfrak{g} . But now since $\mathfrak{g}^e = \mathfrak{g}^{\mathfrak{g}^e}$ (because \mathfrak{g}^e is commutative) and since (0.1.7) holds for $x = e$ this suggests a generalization of the notion of exponent. Let V be any finite dimensional G -module with respect to a representation ν . If l_ν is the multiplicity of the zero weight of ν then by (0.1.7) one has $\dim V^{\mathfrak{g}^e} = l_\nu$. It follows therefore that there exists a unique nondecreasing sequence of non-negative integers $m_i(\nu), i = 1, 2, \dots, l_\nu$, such that one has

$$\nu(x_0)z_i = m_i(\nu)z_i$$

for a basis z_i of $V^{\mathfrak{g}^e}$. If ν is the adjoint representation the $m_i(\nu)$ are the usual exponents. If $\nu = \nu^\lambda$ we will write $m_i(\lambda)$ for $m_i(\nu^\lambda)$ and note (because the highest weight has multiplicity one) that

$$m_j(\lambda) = o(\lambda) \quad \text{for } j = l_\lambda$$

where $o(\lambda)$ is the sum of the coefficients of λ relative to the simple roots and that this highest value occurs with multiplicity one among the generalized exponents $m_i(\lambda)$. (This specializes to the familiar relation $m_l = o(\psi)$ when \mathfrak{g} is simple and ψ is the highest root.)

The following theorem now gives the G -module structure of H^i and hence S^k for any j and k .

THEOREM 0.10. *Let $\lambda \in D$ be arbitrary and let $H(\lambda)$ be the set of G -harmonic polynomials which transform under G according to ν^λ . Let (by (0.1.6)) $H(\lambda) = \sum_{j=1}^{l_\lambda} H_j(\lambda)$ be a decomposition into irreducible components so that $H_j(\lambda) \subseteq H^{n_j}$ where $n_j, j = 1, 2, \dots, l_\lambda$, is a nondecreasing sequence of integers. Then $n_j = m_j(\lambda)$ for all j . In particular then $k = o(\lambda)$ is the highest degree k such that ν^λ occurs in H^k . Moreover it occurs with multiplicity one for this value of k .*

Assume for convenience that \mathfrak{g} is simple and let $\psi \in D$ be the highest root. Let $x_i, i = 1, 2, \dots, n$, be a basis of \mathfrak{g} . If the $u_j \in J$ are chosen properly one sees that $\partial u_j / \partial x_i, i = 1, 2, \dots, n$, is a basis of $H_j(\psi)$. One notes then that Theorem 0.10 is a generalization of the result in [13] given by (0.1.8).

H. S. Coxeter observed and A. J. Coleman proved in [4] that if W is the Weyl group and $\sigma \in W$ is the Coxeter-Killing transformation

then the eigenvalues of σ operating on the Cartan subalgebra are $e^{2\pi im_j/s}$, $j=1, 2, \dots, l$, where s is order of σ . Now more generally W operates on the zero weight space of V^λ for any $\lambda \in D$ according (say) to some representation π^λ of W . As a generalization of the Coxeter-Coleman theorem one now has

THEOREM 0.11. *For any $\lambda \in D$ the eigenvalues of $\pi^\lambda(\sigma)$ are $e^{2\pi im_j(\lambda)/s}$, $j=1, 2, \dots, l_\lambda$.*

0.2. By applying the Birkhoff-Witt theorem the results above carry over from S to U , the universal enveloping of \mathfrak{g} (U is obviously a G -module in a natural way).

THEOREM 0.12. *Let U be the universal enveloping algebra over \mathfrak{g} and let $Z \subseteq U$ be the center of U . Then U is free as a Z -module (under multiplication). In fact*

$$(0.2.1) \quad U = Z \otimes E$$

where E is the subspace (and G -submodule) of U spanned by all powers x^k for all nilpotent elements $x \in \mathfrak{g}$. Moreover E is equivalent to H as a G -module so that every irreducible representation of G occurs with finite multiplicity in E (in fact v^λ occurs l_λ times in E for any $\lambda \in D$).

Let V be a finite dimensional irreducible U -module so that one has a G -module algebra epimorphism

$$\rho: U \rightarrow \text{End } V.$$

Since $\rho(Z)$ reduce to the scalars it follows from (0.2.1) that $\rho(E) = \text{End } V$. Now let Y be any subspace of U . If Y is one-dimensional then it is due to Harish-Chandra that there exists an irreducible U -module V such that ρ is faithful on Y . This is not true in general if $\dim Y \geq 2$. However it is true if $Y \subseteq E$.

THEOREM 0.13. *Let $Y \subseteq E$ be any finite dimensional subspace. Then there exists an irreducible U -module V such that ρ is faithful on Y .*

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