# Lie Groups and Geometric Aspects of Isometric Actions

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*Dedicated to our families.*

# Preface

This book is intended for advanced undergraduates, graduate students, and young researchers in geometry. It was written with two main goals in mind. First, we give a gentle introduction to the classical theory of Lie groups, using a concise geometric approach. Second, we provide an overview of topics related to isometric actions, exploring their relations with the research areas of the authors and giving the main ideas of proofs. We discuss recent applications to active research areas, such as isoparametric submanifolds, polar actions and polar foliations, cohomogeneity one actions, and positive curvature via symmetries. In this way, the text is naturally divided in two interrelated parts.

Let us give a more precise description of such parts. The goal of the first part (Chaps. 1 and 2) is to introduce the concepts of Lie groups, Lie algebras and adjoint representation, relating these objects. Moreover, we give basic results on closed subgroups, bi-invariant metrics, Killing forms, and splitting of Lie algebras in simple ideals. This is done concisely due to the use of Riemannian geometry, whose fundamental techniques are also quickly reviewed.

The second part (Chaps. 3–6) is slightly more advanced. We begin with some results on proper and isometric actions in Chap. 3, presenting a few research comments. In Chap. 4, classical results on adjoint and conjugation actions are presented, especially regarding maximal tori, roots of compact Lie groups, and Dynkin diagrams. In addition, the connection with isoparametric submanifolds and polar actions is explored. In Chap. 5, we survey on the theory of polar foliations, which generalizes some of the objects studied in the previous chapter. Finally, Chap. 6 briefly discusses basic aspects of homogeneous spaces and builds on all the previous material to explore the geometry of low cohomogeneity actions and its interplay with manifolds with positive (and nonnegative) sectional curvature.

Prerequisites expected from the reader are a good knowledge of advanced calculus and linear algebra, together with rudiments of calculus on manifolds. Nevertheless, a brief review of the main definitions and essential results is given in the Appendix A.

This book can be used for a one-semester graduate course (of around 3 h per week) or an individual study, as it was written to be as self-contained as possible.

Part of the material in Chap. 3, as well as Chaps. 5 and 6, may be skipped by students in a first reading. Most sections in the book are illustrated with several examples, designed to convey a geometric intuition on the material. These are complemented by exercises that are usually accompanied by a hint. Some exercises are labeled with a star ($\star$), indicating that they are slightly more involved than the others. We encourage the reader to think about them, in an effort to develop a good working knowledge of the material and practice active reading.

The present book evolved from several lecture notes that we used to teach graduate courses and minicourses. In 2007, 2009, and 2010, graduate courses on Lie groups and proper actions were taught at the University of São Paulo, Brazil, exploring mostly the first four chapters of the text. Graduate students working in various fields followed these courses, with very positive results. During this period, the same material was used in a graduate course at the University of Parma, Italy. Relevant contributions also originated from short courses given by the authors during the XV Brazilian School of Differential Geometry (Fortaleza, Brazil, July 2008), the Second São Paulo Geometry Meeting (São Carlos, Brazil, February 2009), and the Rey Pastor Seminar at the University of Murcia (Murcia, Spain, July 2009). In 2009, a preliminary draft of this text was posted on the arXiv (0901.2374), which prompted instructors in various universities to list it as complementary study material. Since then, we have substantially improved and updated the text, particularly the last chapters, featuring many recent advances in the research areas discussed.

There are several important research areas related to the content of this book that are not treated here. We would like to point out two of these, for which we hope to give the necessary background: first, *representation theory* and *harmonic analysis*, for which we recommend Bröken and Tom Dieck [56], Deitmar [78], Fulton and Harris [90], Gangolli and Varadarajan [94], Helgason [125], Katznelson [136], Knapp [144], and Varadarajan [217], and, second, *symmetries in differential equations* and *integrable systems*, for which we recommend Bryant [57], Fehér and Pusztai [86, 87], Guest [119], Noumi [175], and Olver [176].

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