Life at the Landau Pole

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If a quantum field theory has a Landau pole, the theory is usually called 'sick' and dismissed as a candidate for an interacting UV-complete theory. In a recent study on the interacting 4d O(N) model at large N, it was shown that at the Landau pole, observables remain well-defined and finite. In this work, I study both relevant and irrelevant deformations of the said model at the Landau pole, finding that physical observables remain unaffected. Apparently, the Landau pole in this theory is benign. I speculate about a relation between the 4d O(N) model and a Landau pole in QCD.

I. MOTIVATION

In the early days of quantum field theory, Landau and collaborators studied quantum electrodynamics in perturbation theory [1]. They found that QED has positive β-function in perturbation theory, recognizing that this would lead to an uncontrolled growth of the theory's coupling constant as a function of energy. In modern notation, the QED running coupling to leading order in perturbation theory becomes

$$\alpha(\bar{\mu}) = \frac{1}{\frac{2}{3}\ln\frac{\Lambda_{LP}}{\bar{\mu}}}\,, \tag{1}$$

where it is customary to fix $\Lambda_{LP} = m_e e^{\frac{3\pi}{2\alpha_0}}$ with m_e the electron mass and $\alpha_0 \simeq \frac{1}{137}$.

Landau noted that besides the dependence of the fine-structure constant α on the momentum scale $\bar{\mu}$, the form (1) implied that the running coupling is diverging (has a pole) at a finite momentum scale $\bar{\mu} = \Lambda_{LP}$. Since the coupling diverges at this scale, it seems that one cannot meaningfully probe momentum scales $\bar{\mu} > \Lambda_{LP}$ in QED, so the theory does not have a well-defined continuum limit. It has even been suggested that Landau was so disturbed by this feature that he quit working on quantum field theory.

Modern physics deals with the issue of the Landau pole through a mix of denial and shoulder shrugging. Denial adherents will rightly point out that (1) was derived in perturbation theory, requiring $\alpha \ll 1$, so that as a consequence (1) cannot be expected to correctly capture features such as the Landau pole, where by definition $\alpha \to \infty$. Shrugging adherents will (also rightly) point out that

$$\Lambda_{LP} = m_e e^{\frac{3\pi}{2\alpha_0}} \simeq 10^{280}\,\text{MeV} \tag{2}$$

puts the scale of the Landau pole beyond the Planck scale, so that in practice it is entirely pointless to understand QED in that regime anyway. (However, it should be noted that in the full Standard Model, $\Lambda_{LP} \simeq 10^{34}$ GeV is much lower than (2), but still extremely high [2].) The prevailing dogma in both cases is that theories with a Landau pole should be viewed as 'UV-incomplete', or cut-off theories, that cannot be used to describe continuum physics.

In this work, I will entertain an entirely different perspective, namely that physical observables of a quantum field theory could be well-behaved even when the coupling diverges at the Landau pole, and beyond.

Unfortunately, I am unable to test my perspective in QED (yet), even though other groups have proposed similar ideas [3, 4]. Instead, I will focus on a quantum field theory which can be solved non-perturbatively in the limit of a large number of components, namely the O(N) model [5], using critically important input from $\mathcal{PT}$-symmetric field theory [6]. I will work in 3+1 dimensions where it is known to possess a Landau pole in the large N limit.

In a quantum field theory, the renormalized coupling is not directly observable – as is apparent through the fact that it will depend on the fictitious renormalization scale $\bar{\mu}$. For this reason, finite and well-defined physical observables at infinite renormalized coupling are certainly possible. Indeed, the idea that physical observables turn out to be finite even when the coupling diverges is well supported by several quantum field theory examples, such as $\mathcal{N}=4$ SYM in 3+1 dimensions [7, 8] as well as bosonic and fermionic large N field theories in 2+1 dimensions [9–12].

In this work, I build upon and extend this idea: if physical observables remain finite when the renormalized coupling parameter diverges, maybe observables remain finite and well-defined when the renormalized coupling parameter becomes negative or even complex. After all, the renormalized coupling parameter is not directly observable, so nothing should protect it from becoming complex as long as physical observables remain well-defined. Of course this type of idea cannot be tested in perturbation theory, which is inherently a weak-coupling expansion around non-interacting field theory. For this reason, I heavily employ large N expansion techniques (which do not rely on a small perturbative coupling in order to be applicable) as well as results from $\mathcal{PT}$-symmetric field theory (which allow calculations for negative or complex couplings via analytic continuation).

While this study is exploratory and somewhat speculative, I nevertheless hope that certain aspects merit further consideration when trying to interpret quantum field theory in four dimensions.

II. CALCULATION – THE O(N) MODEL IN 3+1 DIMENSIONS AT LARGE N

Let me first consider the case of the massless theory with quartic interaction, which is essentially a repeat of the calculation in Ref. [5], but included here for completeness. The partition function for this theory is defined through the path integral

$$Z(\lambda,\beta)=\int \mathcal{D}\phi e^{-S_E}\,, \tag{3}$$

with the Euclidean action

$$S_E=\int d^3x\int_0^\beta d\tau\left[\frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial_\mu\vec{\phi}+\frac{\lambda}{N}\left(\vec{\phi}\cdot\vec{\phi}\right)^2\right]\,, \tag{4}$$

where $\vec{\phi}=(\phi_1,\phi_2,\ldots,\phi_N)$ is an N-component scalar field and the theory is defined on the thermal cylinder with $\beta=\frac{1}{T}$ the inverse temperature.

The partition function may be rewritten in a more convenient form by introducing two auxiliary fields σ,ζ with

$$1=\int\mathcal{D}\sigma\delta(\sigma-\vec{\phi}^2)=\int\mathcal{D}\sigma\int\mathcal{D}\zeta e^{i\int\zeta(\sigma-\vec{\phi}^2)}\,. \tag{5}$$

The resulting path integral for σ has quadratic action, such that σ can be integrated out. One finds

$$Z(\lambda,\beta)=\int\mathcal{D}\phi\mathcal{D}\zeta e^{-S_{\rm eff}}\,,\quad S_{\rm eff}=\int d^3x\int_0^\beta d\tau\left[\frac{1}{2}\vec{\phi}\left[-\Box+2i\zeta\right]\vec{\phi}+N\frac{\zeta^2}{4\lambda}\right]\,. \tag{6}$$

Separating the auxiliary field into zero modes and fluctuations $\zeta(x)=\frac{\zeta_0}{2}+\zeta'(x)$, one can verify that the path integral over fluctuations does not contribute to leading order in large N to the partition function. Since ζ_0 is a constant, the path integral over fields $\vec{\phi}$ is quadratic and can be done in closed form. One finds $Z(\lambda,\beta)=\int d\zeta_0 e^{N\beta V p(\sqrt{2i\zeta_0})}$, with the pressure per component in dimensional regularization

$$p(m)=\frac{m^4}{16\lambda}+\frac{m^4}{64\pi^2}\left(\frac{1}{\varepsilon}+\ln\frac{\bar{\mu}^2e^{\frac{3}{2}}}{m^2}\right)+\frac{m^2T^2}{2\pi^2}\sum_{n=1}^{\infty}\frac{K_2(n\beta m)}{n^2}\,, \tag{7}$$

where βV is the space-time volume, $\bar{\mu}$ is the $\overline{\rm MS}$ renormalization scale, $K_i(x)$ denotes modified Bessel functions of the second kind and I have rewritten $i\zeta_0=\frac{m^2}{2}$ to simplify the appearance.

The expression (7) is divergent in the continuum $\lim\varepsilon\to 0$. However, it may be non-perturbatively renormalized as

$$\frac{1}{\lambda}+\frac{1}{4\pi^2\varepsilon}=\frac{1}{\lambda_R(\bar{\mu})}\,, \tag{8}$$

which is standard procedure for large N field theories [13]. The resulting running coupling is given by

$$\lambda_R(\bar{\mu})=\frac{4\pi^2}{\ln\frac{\Lambda_{LP}^2}{\bar{\mu}}}\,, \tag{9}$$

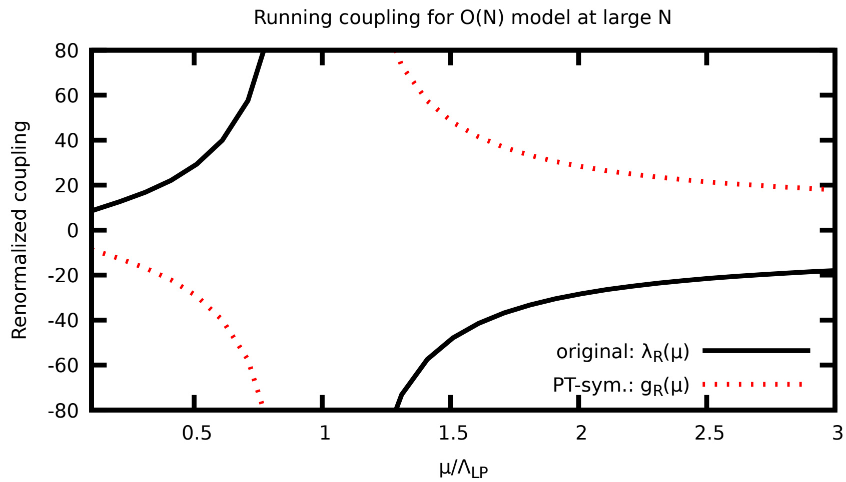

FIG. 1. Running coupling in the O(N) model in 3+1 dimensions. Shown are results for the coupling λ_R in the original theory (9) as well as for the coupling g_R in the analytically continued theory ($\mathcal{PT}$-symmetric theory). Adapted from Ref. [5].

which has a Landau pole at $\bar{\mu} = \Lambda_{LP}$, cf. Fig. 1.

In order to make sense of the theory at the Landau pole, a procedure for analytically continuing the theory *beyond* the Landau pole is necessary. This procedure has been provided in the form of a conjecture in Ref. [6] for so-called $\mathcal{PT}$-symmetric field theory[1]. Naively continuing (9) for $\bar{\mu} > \Lambda_{LP}$, the sign of λ_R becomes negative. So in order to make sense of the O(N) model beyond the Landau pole, one is led to consider a theory where the sign of the coupling is flipped:

$$\lambda \to -g + i0^+ \,, \tag{10}$$

where the small imaginary part has been included in order to be able to 'go around' the Landau pole. Following standard nomenclature [14], the theory with flipped-sign coupling is referred to as $\mathcal{PT}$-symmetric field theory, and its partition function is denoted by $Z_{\mathcal{PT}}(g,\beta)$. Following Ref. [6], the analytic continuation of $Z(\lambda,\beta)$ is given by

$$\ln Z_{\mathcal{PT}}(g,\beta) = \mathrm{Re} \ln Z(\lambda = -g + i0^+, \beta) \,. \tag{11}$$

To evaluate $Z_{\mathcal{PT}}(g,\beta)$, one may directly employ the pressure function (7), where now the sign of the coupling has been flipped. Regardless of the sign of the coupling, the expression for the

[1] Note that the conjecture in Ref. [6] has been formulated at zero temperature and checks exist only for $d = 1$ (quantum mechanics). I thank W. Ai for pointing this out to me.

pressure (7) is divergent in the 4d continuum $\lim \varepsilon \to 0$. The $\mathcal{PT}$-coupling renormalization, given by

$$\frac{1}{g} - \frac{1}{4\pi^2\varepsilon} = \frac{1}{g_R(\bar{\mu})}\,, \tag{12}$$

differs from (8) by a sign, making the $\mathcal{PT}$-symmetric theory asymptotically free. Using the renormalized coupling $g_R(\bar{\mu})$, one can express the renormalized pressure function as

$$p(m) = -\frac{m^4}{16 g_R(\bar{\mu})} + \frac{m^4}{64\pi^2} \ln \frac{\bar{\mu}^2 e^{\frac{3}{2}}}{m^2} + \frac{m^2 T^2}{2\pi^2} \sum_{n=1}^{\infty} \frac{K_2(n\beta m)}{n^2}\,. \tag{13}$$

The pressure, being a physical observable, cannot depend on the choice of renormalization scale $\bar{\mu}$, so $\frac{dp}{d\ln\bar{\mu}} = 0$. This fixes the running for the renormalized coupling $g_R(\bar{\mu})$ and as a consequence the form of the running coupling itself as

$$g_R(\bar{\mu}) = \frac{4\pi^2}{\ln \frac{\bar{\mu}^2}{\Lambda_{LP}^2}}\,, \tag{14}$$

which is the same as (9) up to a sign. Both running couplings are shown in Fig. 1, where it can be seen that they diverge at $\bar{\mu} = \Lambda_{LP}$. Whereas $\bar{\mu} = \Lambda_{LP}$ is the Landau pole in the O(N) model, $\bar{\mu} = \Lambda_{LP}$ looks a lot like the scale parameter for QCD (cf. section III).

Inserting the form (14) of the running coupling into (13), one obtains

$$p(m) = \frac{m^4}{64\pi^2} \ln \frac{\Lambda_{LP}^2 e^{\frac{3}{2}}}{m^2} + \frac{m^2 T^2}{2\pi^2} \sum_{n=1}^{\infty} \frac{K_2(n\beta m)}{n^2}\,. \tag{15}$$

At this point it is worth to pause and consider the following observations:

- Any dependence of $p(m)$ on the unphysical renormalization scale $\bar{\mu}$ has dropped out

- The expression for the pressure is *identical* to (14) evaluated at a the Landau pole, $\bar{\mu} = \Lambda_{LP}$

- The expression for the pressure in the $\mathcal{PT}$-symmetric theory (with flipped sign coupling and asymptotic freedom) is *identical* to the the pressure in the original O(N) model with Landau pole, as can be seen by inserting (9) into (14)

- For generic values of m, the pressure at the Landau pole will be *finite* rather than infinite.

As a consequence of these observations, the O(N) model and the $\mathcal{PT}$-symmetric O(N) model are identical at large N, even though from Fig. 1 one of these has a Landau pole and the other one is asymptotically free. For this reason I will refer to the scale $\bar{\mu} = \Lambda_{LP}$ as 'the Landau pole' also in the $\mathcal{PT}$-symmetric theory.

One can calculate the pressure for any temperature T by noting that the remaining single integral over ζ_0 in the partition function is again dominated by the saddle points at large N, so that the physical pressure of the theory per component is given by (15) with $m = \bar{m}$ the solution to

$$0 = \frac{dp(m)}{dm^2} = \frac{m^2}{32\pi^2} \ln \frac{\Lambda_{LP}^2 e^1}{m^2} - \frac{mT}{4\pi^2} \sum_{n=1}^{\infty} \frac{K_1(n\beta m)}{n} \,. \tag{16}$$

If (16) has more than one solution (as it generically does), then in the large N limit, the solution with the biggest $\mathrm{Re}\, p(\bar{m})$ will dominate over all others.

At zero temperature (a.k.a. the vacuum), a simple solution to (16) is $\bar{m} = 0$, which corresponds to the usual starting point for perturbative calculations. However, there is a second solution to (16) located at $\bar{m} = \Lambda_{LP}\sqrt{e}$, which is usually dismissed as 'being too close to the Landau pole' [13]. However, the physical pressure per component for this second solution is given by

$$p(m = \Lambda_{LP}\sqrt{e}, T = 0) = \frac{\Lambda_{LP}^4 e^2}{128\pi^2} \,, \tag{17}$$

which is perfectly finite. In addition, since $p(m = \Lambda_{LP}\sqrt{e}) > p(m = 0)$, the solution (17) is thermodynamically preferred over the perturbative vacuum.

Despite the presence of the Landau pole, observables in the bosonic theory seem to make physical sense. For instance, one can calculate thermodynamic properties at finite temperature by tracking solutions to (16) numerically, and evaluating $p(\bar{m})$. As discussed in Ref. [5], at small temperature the numerical solution $\bar{m}$ of the thermodynamically preferred phase is continuously connected to $\bar{m}(T = 0) = \sqrt{e}\Lambda_{LP}$. The numerical solution $\bar{m}$ becomes complex above a critical temperature $T = T_c \simeq \Lambda_{LP}/\sqrt{e}$, but using results from $\mathcal{PT}$-symmetric field theory [6], the analytically continued pressure is continuous across $T = T_c$. A plot of the pressure as a function of temperature from Ref. [5] is reproduced in Fig. 2, and results for the entropy and specific heat can be found in Ref. [5].

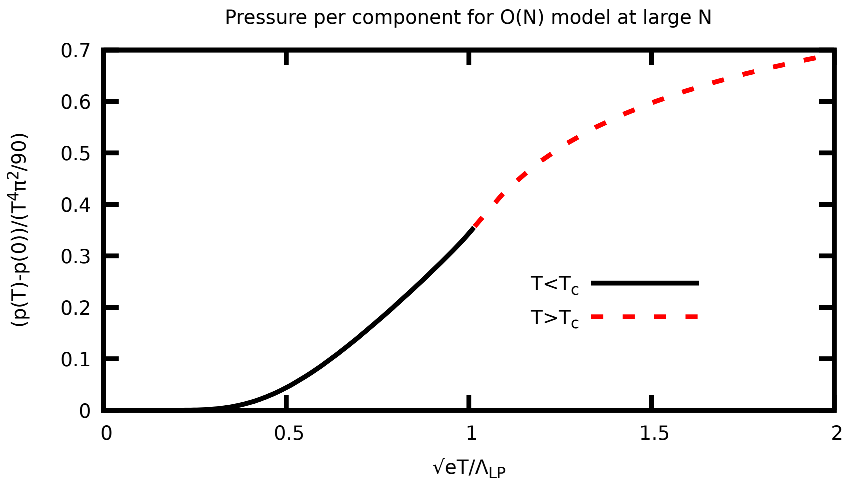

FIG. 2. Pressure per component as a function of temperature for the O(N) model at large N, adapted from Ref. [5]. $T_c \simeq 0.616\Lambda_{LP}$ denotes the location in temperature where the solution to (16) for m becomes complex.

A. Adding Relevant Deformations

One might worry that the results from the previous sections are an artifact of tuning away all relevant and irrelevant operators. For this reason, it is useful to consider repeating the analysis for the Euclidean action

$$S_E = \int d^3x d\tau \left[\frac{1}{2}\partial_\mu \vec{\phi}\cdot\partial_\mu\vec{\phi} + \frac{1}{2}m_{\rm bare}^2\vec{\phi}^2 - \frac{g}{N}\left(\vec{\phi}\cdot\vec{\phi}\right)^2\right]. \tag{18}$$

Introducing the auxiliary fields as before in (5), one may again integrate out σ, and one finds in complete analogy with the previous section the $\mathcal{PT}$-symmetric pressure function

$$p(m) = -\frac{(m^2 - m_{\rm bare}^2)^2}{16g} + \frac{m^4}{64\pi^2}\left(\frac{1}{\varepsilon} + \ln\frac{\bar{\mu}^2 e^{\frac{3}{2}}}{m^2}\right) + \frac{m^2T^2}{2\pi^2}\sum_{n=1}^{\infty}\frac{K_2(n\beta m)}{n^2}. \tag{19}$$

The explicit $\varepsilon \to 0$ divergence can again be taken care of by using the same non-perturbative renormalization as before (12). This leads to

$$p(m) = \frac{2m^2 m_{\rm bare}^2 - m_{\rm bare}^4}{16g} + \frac{m^4}{64\pi^2}\ln\frac{\Lambda_{LP}^2 e^{\frac{3}{2}}}{m^2} + \frac{m^2T^2}{2\pi^2}\sum_{n=1}^{\infty}\frac{K_2(n\beta m)}{n^2}. \tag{20}$$

Since the bare coupling $\frac{1}{g}$ diverges as $\varepsilon \to 0$, there are residual divergences remaining in (20). The first one of these can be taken care of by renormalizing the bare mass parameter as

$$\frac{m_{\rm bare}^2}{g} = \frac{m_R^2(\bar{\mu})}{g_R(\bar{\mu})}\,, \tag{21}$$

The running of the renormalized mass $m_R(\bar{\mu})$ is again fixed by requiring that $\frac{dp(m)}{d\ln\bar{\mu}} = 0$, which leads to

$$m_R^2(\bar{\mu}) = \frac{\rm const}{\ln\frac{\bar{\mu}^2}{\Lambda_{LP}^2}}\,, \quad \text{or} \quad \frac{m_R^2(\bar{\mu})}{g_R(\bar{\mu})} = m_0^2\,, \tag{22}$$

with constant and finite mass scale m_0. This leads to

$$p(m) = \frac{m^2 m_0^2}{8} - \frac{m_{\rm bare}^4}{16g} + \frac{m^4}{64\pi^2}\ln\frac{\Lambda_{LP}^2 e^{\frac{3}{2}}}{m^2} + \frac{m^2T^2}{2\pi^2}\sum_{n=1}^{\infty}\frac{K_2(n\beta m)}{n^2}\,. \tag{23}$$

For the remaining term, note that in the limit $\varepsilon \to 0$

$$\frac{m_{\rm bare}^4}{g} = m_0^2 m_{\rm bare}^2 = m_0^4 g = \frac{m_0^4}{\frac{1}{g_R(\bar{\mu})} + \frac{1}{4\pi^2\varepsilon}} \to 0\,, \tag{24}$$

so that the pressure function becomes

$$p(m) = \frac{m^2 m_0^2}{8} + \frac{m^4}{64\pi^2}\ln\frac{\Lambda_{LP}^2 e^{\frac{3}{2}}}{m^2} + \frac{m^2T^2}{2\pi^2}\sum_{n=1}^{\infty}\frac{K_2(n\beta m)}{n^2}\,. \tag{25}$$

For small values of m_0, the properties of this theory are close to the unmodified version considered in the previous section. The second-order phase transition at finite temperature persists, but is pushed to higher values of $\frac{T_c}{\Lambda_{LP}}$.

B. Adding Irrelevant Deformations

Now let us consider what happens when adding irrelevant operators to the theory. In this case, I study

$$S_E = \int d^3x d\tau \left[\frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial_\mu\vec{\phi} - \frac{g}{N}\left(\vec{\phi}\cdot\vec{\phi}\right)^2 + \frac{\alpha}{N^2}\left(\vec{\phi}\cdot\vec{\phi}\right)^3\right]\,, \tag{26}$$

where α is the bare sextic coupling parameter. Introducing the auxiliary fields as before in (5), it is possible, but not very enlightening, to integrate out σ exactly. Instead, in the large N limit it is again permissible to replace $\sigma(x)$ by just its global zero mode σ_0, so that

$$Z_{\mathcal{PT}}(g,\beta) = \int d\sigma_0 d\zeta_0 e^{N\beta V p(m=\sqrt{i\zeta_0},\sigma_0)}\,, \tag{27}$$

where

$$p(m,\sigma_0) = \frac{g\sigma_0^2}{N^2} - \frac{\alpha\sigma_0^3}{N^3} + \frac{\sigma_0 m^2}{2N} + \frac{m^4}{64\pi^2}\left(\frac{1}{\varepsilon} + \ln\frac{\bar{\mu}^2 e^{\frac{3}{2}}}{m^2}\right) + \frac{m^2T^2}{2\pi^2}\sum_{n=1}^{\infty}\frac{K_2(n\beta m)}{n^2}\,. \tag{28}$$

At large N, the integral over σ_0 is done with the saddle point method, with two saddles located at

$$\sigma_0^{(1,2)} = \frac{gN}{3\alpha}\left(1 \pm \sqrt{1+\frac{3\alpha m^2}{2g^2}}\right) . \tag{29}$$

For small $\frac{3\alpha m^2}{2g^2}$ (justified below), one can expand the square root in this expression to obtain

$$\begin{aligned} \frac{\sigma_0^{(1)}}{N} &= -\frac{m^2}{4g} + \frac{3\alpha m^4}{32g^3} - \frac{9\alpha^2 m^6}{128 g^5} + \sum_{n=3}^{\infty}\mathcal{O}\left(\frac{\alpha^{2n}}{g^{2n+1}}\right) , \\ \frac{\sigma_0^{(2)}}{N} &= \frac{2g}{3\alpha} + \frac{m^2}{4g} - \frac{3\alpha m^4}{32g^3} + \frac{9\alpha^2 m^6}{128 g^5} + \sum_{n=3}^{\infty}\mathcal{O}\left(\frac{\alpha^{2n}}{g^{2n+1}}\right) , \end{aligned} \tag{30}$$

for the two solutions. Inserting $\sigma_0^{(1)}$ into (28), one finds

$$p(m) = -\frac{m^4}{16g} + \frac{\alpha m^6}{64 g^3} + \frac{m^4}{64\pi^2}\left(\frac{1}{\varepsilon} + \ln\frac{\bar{\mu}^2 e^{\frac{3}{2}}}{m^2}\right) + \frac{m^2T^2}{2\pi^2}\sum_{n=1}^{\infty}\frac{K_2(n\beta m)}{n^2} + \sum_{n=1}^{\infty}\mathcal{O}\left(\frac{m^{2n+6}\alpha^{2n}}{g^{2n+3}}\right) . \tag{31}$$

Renormalizing the coupling g as in (12), leads to

$$p(m) = -\frac{m^4}{16 g_R(\bar{\mu})} + \frac{\alpha m^6}{64 g^3} + \frac{m^4}{64\pi^2}\ln\frac{\bar{\mu}^2 e^{\frac{3}{2}}}{m^2} + \frac{m^2T^2}{2\pi^2}\sum_{n=1}^{\infty}\frac{K_2(n\beta m)}{n^2} + \sum_{n=1}^{\infty}\mathcal{O}\left(\frac{m^{2n+6}\alpha^{2n}}{g^{2n+3}}\right) , \tag{32}$$

but this implies that $\frac{\alpha}{g^3}$ is divergent. Thus the bare sextic coupling parameter also needs to be renormalized as

$$\frac{\alpha}{g^3} = \frac{\alpha_R(\bar{\mu})}{g_R^3(\bar{\mu})} . \tag{33}$$

This leaves the whole tower of additional terms $\frac{\alpha^{2n}}{g^{2n+3}}$, $n \geq 1$ that are potentially divergent. However, one finds that in the $\varepsilon \to 0$ limit

$$\frac{\alpha^{2n}}{g^{2n+3}} = \frac{\alpha_R^{2n}(\bar{\mu})}{g_R^{6n}(\bar{\mu})}\frac{1}{g^{3-4n}} = \frac{\alpha_R^{2n}(\bar{\mu})}{g_R^{6n}(\bar{\mu})}\frac{1}{\left(\frac{1}{g_R(\bar{\mu})} + \frac{1}{4\pi^2\varepsilon}\right)^{4n-3}} \to 0 , \tag{34}$$

because $n \geq 1$. Therefore, none of these terms contribute, and one is left with

$$p(m) = \frac{m^6}{M^2} + \frac{m^4}{64\pi^2}\ln\frac{\Lambda_{LP}^2 e^{\frac{3}{2}}}{m^2} + \frac{m^2T^2}{2\pi^2}\sum_{n=1}^{\infty}\frac{K_2(n\beta m)}{n^2} , \tag{35}$$

where I have used the renormalization group invariance of the pressure to express

$$\frac{\alpha_R(\bar{\mu})}{64 g_R^3(\bar{\mu})} = \frac{1}{M^2} , \tag{36}$$

with constant mass scale M. One observes that for the same reason as (34), expanding the square root in (29) is justified for $\sigma_0^{(1)}$.

For the second solution $\sigma_0 = \sigma_0^{(2)}$, (28) becomes

$$p(m) = \frac{4g^3}{27\alpha^2} + \frac{gm^2}{3\alpha} + \frac{m^4}{16g} - \frac{\alpha m^6}{64g^3} + \frac{m^4}{64\pi^2}\left(\frac{1}{\varepsilon} + \ln\frac{\bar{\mu}^2 e^{\frac{3}{2}}}{m^2}\right) + \frac{m^2T^2}{2\pi^2}\sum_{n=1}^{\infty}\frac{K_2(n\beta m)}{n^2} + \dots \quad (37)$$

The explicit $\frac{1}{\varepsilon}$ divergence can by renormalizing the coupling g, but the sign of the counterterm must be flipped (and as a consequence, so must the sign of the running coupling). One obtains

$$p(m) = \frac{4g^3}{27\alpha^2} + \frac{gm^2}{3\alpha} + \frac{m^4}{16g_R(\bar{\mu})} - \frac{\alpha m^6}{64g^3} + \frac{m^4}{64\pi^2}\ln\frac{\bar{\mu}^2 e^{\frac{3}{2}}}{m^2} + \frac{m^2T^2}{2\pi^2}\sum_{n=1}^{\infty}\frac{K_2(n\beta m)}{n^2} + \dots \quad (38)$$

Renormalizing the sextic coupling α as in (36), the square-root expansion in (29) is justified also for $\sigma_0^{(2)}$. Similar to (34), terms with positive powers of the bare coupling g in the numerator vanish, so that one finds

$$p(m) = \frac{M^4}{27\,648\,g^3} - \frac{m^6}{M^2} + \frac{m^4}{64\pi^2}\ln\frac{\Lambda_{LP}^2 e^{\frac{3}{2}}}{m^2} + \frac{m^2T^2}{2\pi^2}\sum_{n=1}^{\infty}\frac{K_2(n\beta m)}{n^2} \quad (39)$$

This expression still has a divergent term $\propto \frac{1}{g^3}$, but this term is independent from m. For this reason, this last divergence can be canceled by a vacuum pressure-counterterm in the Lagrangian, leading to

$$p(m) = -\frac{m^6}{M^2} + \frac{m^4}{64\pi^2}\ln\frac{\Lambda_{LP}^2 e^{\frac{3}{2}}}{m^2} + \frac{m^2T^2}{2\pi^2}\sum_{n=1}^{\infty}\frac{K_2(n\beta m)}{n^2}\,. \quad (40)$$

Inspecting (35) and (40), one finds that the two saddle point solutions for σ_0 give rise to the same form for the pressure function, except for the sign of the m^6 term, which can be attributed to the fact that the sign of $g_R(\bar{\mu})$ is flipped for the solution $\sigma_0^{(1)}$.

The final integral over ζ_0 is done by finding the saddle point solution

$$0 = \frac{dp}{dm^2} = \pm\frac{3m^4}{M^2} + \frac{m^2}{32\pi^2}\ln\frac{\Lambda_{LP}^2 e^1}{m^2} - \frac{mT}{4\pi^2}\sum_{n=1}^{\infty}\frac{K_1(n\beta m)}{n}\,, \quad (41)$$

where $\pm$ corresponds to the solutions $\sigma_0^{(1,2)}$, respectively. At zero temperature, where the contribution from the modified Bessel function vanishes, there is a different number of solutions depending on the sign in (41) and magnitude of M^2. For positive sign (corresponding to solution $\sigma_0^{(1)}$ above), and large M^2, there are three solutions: $\bar{m} = 0$, $\bar{m} \simeq \Lambda_{LP}\sqrt{e}$ and $\bar{m} \propto M$ up to logarithmic corrections. Of these, the saddle with the largest pressure (lowest free energy) is $\bar{m} \simeq \Lambda_{LP}\sqrt{e}$, hence this is the dominant saddle point at large N. One thus again recovers the solution (17). As M^2 decreased, the situation remains qualitatively the same until $M \lesssim 84\Lambda_{LP}$, at which point solutions (except for $\bar{m} = 0$) become complex-valued. However, close to the Landau pole where $g_R(\bar{\mu} = \Lambda_{LP}) \to \infty$, (36) suggests that $M^2 \to \infty$, so I will not consider small values of M in the following.

For the negative sign in (41) and large M^2, there are two solutions $\bar{m} = 0$ and $\bar{m} \simeq \Lambda_{LP}\sqrt{e}$, where again the second solution is thermodynamically preferred. Therefore, one also recovers the unmodified theory solution (17) for the second solution $\sigma_0^{(2)}$.

C. Conclusions

The O(N) model in 3+1 dimensions has a Landau pole at large N. Physical observables at the Landau pole remain finite and well-behaved. This feature does not change when adding either relevant operators (e.g. mass terms) or irrelevant operators (e.g. sextic interactions) to the theory. I therefore conclude that for the O(N) model in 3+1 dimensions at large N, the Landau pole is a harmless feature of the theory, and not a sign that the theory itself is 'sick'. Most importantly, using the analytic continuation provided by $\mathcal{PT}$-symmetric field theory, the O(N) model does not need to be treated as a cut-off theory. It is UV complete, and asymptotically free, despite (or perhaps because of) the negative coupling constant.

An important omission in the present study is the question about $\frac{1}{N}$ corrections – will they destroy the features of the large N limit? While this question is without doubt important, I cannot resist pointing out that (almost) the entirety of holography is built upon the strict large N limit of field theory, without systematic discussion of $\frac{1}{N}$ corrections [15]. Yet even without systematic understanding of $\frac{1}{N}$ terms, holography has indisputable been useful in building our understanding of quantum field theory, so maybe a similar attitude could be extended to the large N limit of the O(N) model.

III. SPECULATION – A LANDAU POLE IN QCD?

Let me conclude the discussion by speculating about a Landau pole in a seemingly unrelated theory – Quantum Chromodynamics. According to prevailing knowledge, QCD does not have a Landau pole, so it seems that it should not be discussed here. However, QCD does have asymptotic freedom, and since the $\mathcal{PT}$-symmetric O(N) model at large N shares

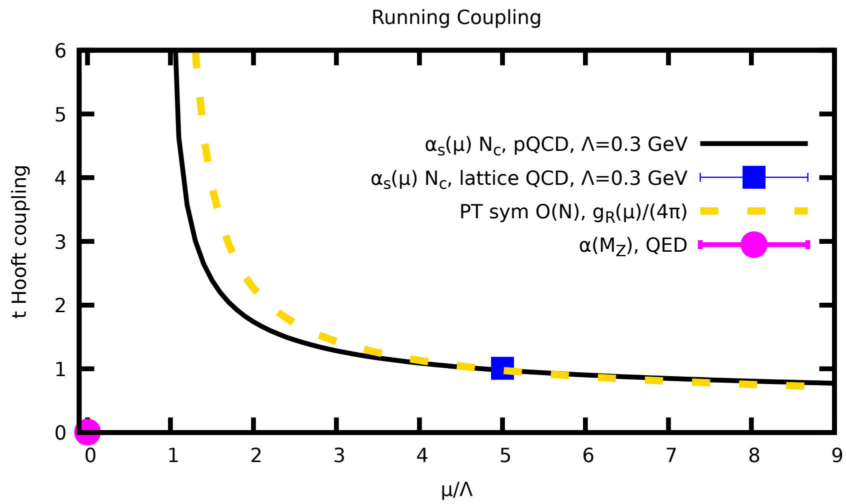

FIG. 3. Running coupling in QCD (with $\Lambda \equiv \Lambda_{\overline{\rm MS}} = 0.3$ GeV) and large N $\mathcal{PT}$-symmetric O(N) model. In order to have a fair comparison, I'm converting the QCD running coupling $\alpha_s(\bar\mu)$ to the 't Hooft coupling by multiplying with a factor of $N_c = 3$, and I am dividing the $\mathcal{PT}$-symmetric O(N) model coupling by a factor of 4π. Lattice QCD point at $\bar\mu = 1.5$ GeV is from Ref. [16], QED point at the Z-pole mass $\bar\mu = M_Z$ is from Ref. [17]. See text for details.

this property, one can nevertheless ask how different or similar these theories are.

It would seem more appropriate to attempt a comparison to QED rather than QCD, but the experimental determination of the QED fine-structure constant does not extend much beyond the Z-pole mass $\bar\mu = M_Z \simeq 91$ GeV. Since the QED Landau pole (2) is at vastly higher energy, the resulting QED information (shown in Fig. 3) is not particularly illuminating.

So instead of QED, in Fig. 3 I compare the running coupling from QCD to that in the $\mathcal{PT}$-symmetric O(N) model. For the QCD running coupling, I am using the 3-loop perturbative QCD expression resulting from numerically integrating

$$\frac{\partial a_s}{\partial \ln \bar\mu^2} = -\beta_0 a_s^2 - \beta_1 a_s^3 - \beta_2 a_s^3 \,, \tag{42}$$

where $a_s = \frac{\alpha_s(\bar\mu)}{4\pi}$ and $\beta_0 = 11 - \frac{2}{3}N_f$, $\beta_1 = 102 - \frac{38}{3}N_f$, $\beta_2 = \frac{2857}{2} - \frac{5033}{18}N_f + \frac{325}{54}N_f^2$ and I am taking $N_f = 5$ [17, 18]. As can be seen from Fig. 3, the QCD running coupling obtained from perturbation theory becomes very large at small scales $\bar\mu$. In fact, one finds that the perturbative solution for $\alpha_s(\bar\mu)$ thus obtain diverges for a particular value of $\bar\mu = \Lambda_{\overline{\rm MS}} \simeq 0.3$ GeV. This value is consistent with similar values reported by other methods [19], and indeed

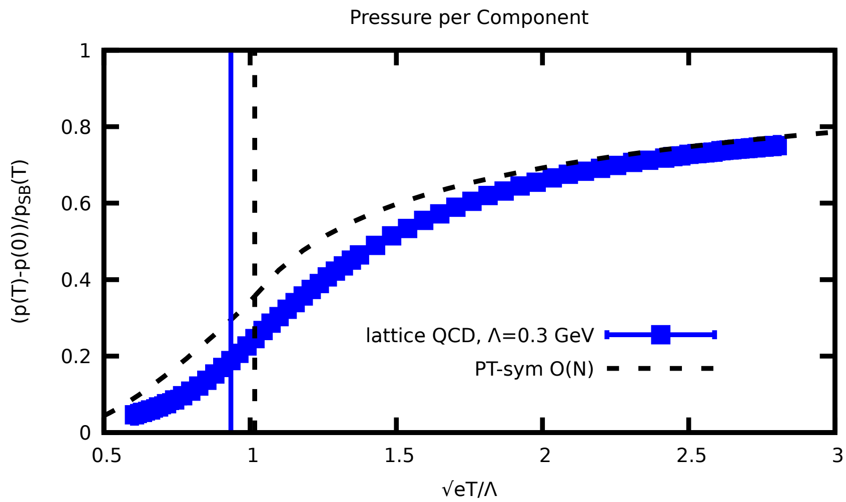

FIG. 4. Pressure as a function of temperature for QCD (with $\Lambda \equiv \Lambda_{\overline{\rm MS}} = 0.3$ GeV) and large N $\mathcal{PT}$-symmetric O(N) model. The QCD pressure is from lattice QCD with $N_f = 2+1$ flavors of quarks with physical masses [20], with a cross-over temperature reported as $T_c = 170(4)(3)$ MeV [21] (full vertical line). Massless O(N) model results with a second-order phase transition located at $T_c \simeq 0.616\Lambda_{LP}$ (dashed vertical line) are adapted from [5]. See text for details.

the running coupling $\alpha_s(\bar\mu)$ is consistent with calculations from lattice QCD [16] at values as low as $\bar\mu \simeq 5\Lambda_{\overline{\rm MS}}$ (also shown in Fig. 3).

In my opinion, Fig. 3 indicates a certain qualitative similarity between QCD and the O(N) model. Pushing the similarity further, this would lead to the interpretation that $\alpha_s(\bar\mu)$ actually does diverge at a finite momentum scale $\Lambda_{\overline{\rm MS}}$, and that deep in the infrared α_s should be analytically continued to negative (or complex) values.

It is well-known that QCD becomes confining in the infrared, and that physical observables are well-behaved and finite for all momentum scales. In particular, thermodynamic quantities such as the pressure are continuous as a function of temperature for QCD, with a broad analytic crossover from confined to quark-gluon plasma phase around $T_c \simeq 170$ MeV [21]. Normalizing the pressure by the Stefan-Boltzmann pressure $p_{SB}(T) = \frac{\pi^2 T^4}{90}\left(2(N_c^2-1)+\frac{7}{2}N_c N_f\right)$ with $N_c = N_f = 3$ for QCD and $p_{SB}(T) = \frac{\pi^2 T^4 N}{90}$ for the O(N) model, a comparison is shown in Fig. 4.

Similar to the case of the running coupling shown in Fig. 3, thermodynamic properties for QCD and the O(N) model shown in Fig. 4 seem to have a certain qualitative similarity when expressed in units of $\Lambda_{LP}, \Lambda_{\overline{\rm MS}}$.

IV. SUMMARY

In this work, I have considered the O(N) model in 3+1 dimensions at large N, which has a Landau pole. Using technology borrowed from $\mathcal{PT}$-symmetric field theories, I have extended the theory beyond the Landau pole, and I have found that adding relevant and irrelevant operators do not qualitatively change the behavior of the theory close to the Landau pole. Physical observables in the O(N) model are finite and well-behaved at and close to the Landau pole.

I take this to constitute evidence that at least at large N, the Landau pole in the O(N) model is harmless, and the theory does constitute a UV-complete interacting and asymptotically free theory.

Moreover, I compared results from the large N O(N) model for the running coupling and the finite temperature pressure to QCD, finding qualitative similarities between these two theories.

Based on these observations, my interpretation is as follows: Landau poles are common features for quantum field theories in 3+1 dimensions, since the O(N) model at large N, QED, and even QCD possess diverging coupling constants at a finite momentum scale $\bar{\mu} = \Lambda$. For two of these theories (O(N) model and QCD), we know that nothing 'bad' happens at this scale. Instead, $\bar{\mu} = \Lambda$ merely marks the scale at which the O(N) model and QCD seem to transition from a low temperature phase to a high temperature phase.

Perhaps it would be time to critically reassess the current dogma that Landau poles constitute fatal flaws of interacting quantum field theories.

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