Life Distribution Analysis Based on Lévy Subordinators for Degradation with Random Jumps

Yin Shu[1], Qianmei Feng [1,*], David W. Coit[2]

[1] *Department of Industrial Engineering, University of Houston, E206 Engineering Bldg.2, Houston, TX 77204, USA*

[2] *Department of Industrial and Systems Engineering, Rutgers University, Piscataway, NJ 08854, USA*

Abstract

For a component or a system subject to stochastic degradation with sporadic jumps that occur at random times and have random sizes, we propose to model the cumulative degradation with random jumps using a single stochastic process based on the characteristics of Lévy subordinators, the class of non-decreasing Lévy processes. Based on an inverse Fourier transform, we derive a new closed-form reliability function and probability density function for lifetime, represented by Lévy measures. The reliability function derived using the traditional convolution approach for common stochastic models such as gamma degradation process with random jumps, is revealed to be a special case of our general model. Numerical experiments are used to demonstrate that our model performs well for different applications, when compared with the traditional convolution method. More importantly, it is a general and useful tool for life distribution analysis of stochastic degradation with random jumps in multi-dimensional cases.

Keywords: Lévy processes, Subordinators, Degradation, Jumps, Reliability

*qfeng@central.uh.edu

1. Introduction

Engineering systems (e.g., mechanical devices, subsea pipelines) usually deteriorate and lose their intended functionality due to wear, fatigue, erosion, corrosion and aging. The continuous deteriorating process commonly experiences sporadic jumps due to discrete damages caused by random external shocks, e.g., sudden crack increase due to collision on pipelines. Stochastic processes are typically used to represent the inherent statistical uncertainty of a degradation process, e.g., compound Poisson process, gamma process, Wiener process. However, there is a lack of research on using a single stochastic process to describe degradation with random jumps. Degradation with random jumps is a process of stochastically continuous degradation with sporadic jumps that occur at random times and have random sizes. In this paper, we intend to model the overall change volume of degradation with random jumps using one stochastic process based on the characteristics of Lévy subordinators, the class of non-decreasing Lévy processes. Based on an inverse Fourier transform, we derive a new closed-form reliability function and probability density function for lifetime of a component or a system subject to a degradation process with random jumps. The reliability function is constructed and represented by a certain Lévy measure corresponding to a certain Lévy degradation process.

For systems subject to sporadic jump damages, a compound Poisson process, a stochastic process with independent and identically distributed (i.i.d) jumps that occur according to a Poisson process, is one of the appropriate candidates to model the cumulative damages. Mohamed [20] and Gottlieb [8] introduced the life distribution and its properties for systems subject to pure jump damage process. Due to the lack of failure time data for highly reliable systems, degradation data can be used to improve reliability analysis and failure prognosis. Different mathematical models have been studied for degradation-based reliability in the literature. Singpurwalla [27] provided a comprehensive survey to describe some stochastic failure models that can be applied to systems operated in dynamic environments, such as Wiener process, gamma process, and a deterministic diffusion process for system wear. Kharoufeh [10,11] used a Markov process to model the dynamic operating environment for wear-based reliability. Random coefficient regression models were first constructed by Lu and Meeker [13] to fit the degradation data from a population of units with normally-distributed measurement errors, and it was later extended in Lu, Park and Yang [14].

For degradation due to wear only, gamma process and Wiener process are good candidates to model wear processes. Gamma process is suitable for modeling degradation that progresses in one direction due to its property of independent and nonnegative increments. Mohamed [19] was the first study to use a gamma process for modelling wear process. Lawless and Crowder [15] later presented a gamma process model incorporating a random effect for degradation. Liao et al. [16] proposed a maintenance policy for gamma degrading systems. Detailed discussions were given in Noortwijk [22] that provided an overview and survey for applying gamma processes to model wear. Wiener process is appropriate for modeling degradation that changes non-monotonically because it can have non-negative and negative increments alternately. To derive the reliability function from a Wiener process, the failure time needs to be defined as the first passage time to the failure threshold [4,5,17]. Whitmore [35] used Wiener process to model wear process considering the normally distributed measurement errors. Si et al. [28,29] modeled the degradation process using Wiener process for remaining useful life (RUL) estimation. Tang and Su [33] presented a modified maximum likelihood estimator for failure time distribution derived based on Wiener process. Park and Padgett [24,25] proposed a generalized approach using stochastic processes to describe cumulative degradation volume, where gamma process and Wiener process are the two cases that were conveniently used. More details about choosing gamma or Wiener process are in Tsai [34]. Besides gamma and Wiener processes, Ebrahimi [7] suggested a non-stationary stochastic process to model wear and derived the reliability function, where the underlying process is gamma process.

In practice, however, few systems experience pure sporadic jump damage process or degradation only. Due to random environments, a degradation process is typically impacted by sporadic jump damages. By considering degradation with random jumps, a typical approach to calculate reliability is using convolution formula as in Peng, Feng, and Coit [26]. However, when the wear process has little common properties with the random jump damages, the calculation becomes complex. For example, when we use a Wiener process (or a gamma process) for wear and a compound Poisson process for sporadic jumps with normally-distributed jump sizes (or gamma-distributed jump sizes), it is straightforward to derive the reliability function by using convolution; however, the calculation becomes more complex when we consider a Wiener degradation process with gamma jumps. Noortwijk et al. [23] used a gamma process to model wear and a Poisson process with jump sizes following a peaks-over-threshold distribution to model random loads, and the computation of reliability is extensive. In addition, the traditional gamma-Wiener-based models may not be suitable enough to fit the general degradation data, especially when there are complex jump mechanisms that cannot be well described by gamma or normal distributions.

In order to overcome the aforementioned problems, we propose to use a single Lévy process to describe a degradation process with random jumps. Lévy process has been explored by researchers for degradation processes due to its properties such as independent and stationary increments. Mohamed [21] used Lévy processes to model wear and studied its life distribution properties where the threshold is assumed to be random. Yang and Klutke [37] used special cases of Lévy process to model degradation process and jump damages: gamma process for wear and compound Poisson process for random jump damages, respectively. They assumed that the threshold is exponentially distributed, which leads to a closed-form lifetime distribution.

In this paper, we assume the degradation process with random jumps is nondecreasing, and a single Lévy subordinator is proposed to construct our models. In our model, we can specify different Lévy measures to describe different jump mechanisms in degradation, which makes our methods general and can fit many different types of degradation data sets. By using inverse Fourier transform, we further derive the closed-form reliability function and probability density function of lifetime for a system or a component subject to a degradation process with random jumps, represented by the Lévy measure. The calculation for reliability is simple enough to be implemented in practice. More importantly, based on mathematical theories in multi-dimensional Lévy measures, our work in this paper provides a new framework to analyze multi-degradation processes in multi-component systems.

The organization of the paper is as follows. Section 2 begins with the key notions of the general Lévy process, and then introduces the special cases of Lévy processes typically used in the literature. In Section 3, we derive the reliability function and probability density function of lifetime for systems subject to degradation with random jumps described by the Lévy subordinators, based on the Fourier inversion theorem. Section 4 studies the reliability of temporally homogenous gamma degradation with different random jumps, a special case of Lévy subordinators. Numerical examples are developed in Section 5, and conclusions are given in Section 6.

Notation

- Euclidean space: $R^d, d \in N$
- The inner product on Euclidean space: $(x, y) = \sum_{i=1}^{d} x_i y_i$
- Euclidean norm: $|x| = (x, x)^{1/2} = \left(\sum_{i=1}^{d} x_i^2\right)^{1/2}$
- The set of all Borel probability measures on Euclidean space: $\mathrm{M}_1\left(R^d\right)$
- Indicator function: $I_A\left(x\right)$
- Borel σ algebra on Euclidean space: $\mathrm{B}\left(R^d\right)$
- The convolution of finite measures: $\mu_1 * \mu_2$
- Min $\{a, b\}$: $a \wedge b$
- Lévy processes: $X\left(t\right)$
- Lévy subordinators: $X_s\left(t\right)$

- Characteristic function: $\phi_X(u)$
- Lévy measure: ν
- Lévy symbol: $\eta(u)$
- Standard Brownian motion or Wiener process: $B_0(t)$
- Temporally homogeneous gamma process: $G(t)$
- Compound Poisson process: $C(t)$

2. Preliminaries of Lévy processes

Lévy processes are stochastic processes whose increments in nonoverlapping time intervals are independent and stationary in time. Their importance in modelling degradation processes stems from [30,1]: 1) they are analogues of random walks in continuous time; 2) they form special subclasses of Markov processes, for which the analysis is much simpler and provides a valuable guidance for the general case; 3) they are the simplest examples of random motion whose sample paths are right-continuous and have a number (at most countable) of random jump discontinuities occurring at random times, on each finite time interval; and 4) they include a number of important processes as special cases, such as Wiener process/Brownian Motion, compound Poisson process, gamma process and stable process. Therefore, Lévy process can serve as an important tool for the study of degradation-based reliability theory. In this section, we introduce Lévy processes along with their properties and characteristics on Euclidean space, where the increments can be positive or negative.

2.1. Characteristics

To make our model general, and provide a framework for multi-degradation processes, we introduce Lévy processes on Euclidean space.

Definition 1 **[1]** $\{X(t), t \geq 0\}$ is a Lévy process defined on a probability space $(\Omega, \mathfrak{F}, P), \Omega \in R^d$, $\mathfrak{F} \in \mathrm{B}\left(R^d\right)$, $P \in \mathrm{M}_1\left(R^d\right)$, if:

- $X(0) = 0$ with probability of 1;
- $X(t)$ has independent and stationary increments: for $n \in N$ and $0 \leq t_1 < t_2 < \cdots t_{n+1} < \infty$, the random variables $(X(t_{i+1}) - X(t_i), 1 \leq i \leq n)$ are independent and the distribution of $X(s+t) - X(s)$ does not depend on s;
- $X(t)$ is stochastically continuous: for all $\varepsilon > 0, s > 0$,

$$\lim_{t \to s} P\left(|X(t) - X(s)| > \varepsilon\right) = 0.$$

Characteristic functions are a primary tool for analysis when the distributions have no analytic forms, especially for Lévy processes. On Euclidean space, let $\phi_X(u) = \int_{R^d} e^{i(u,x)} P_X(dx) = E\left(e^{i(u,X)}\right)$ denote the characteristic function of a random variable X, where P_X is the distribution function of X, and $u \in R^d$. More generally, if $\mu \in \mathrm{M}_1\left(R^d\right)$, the set of all Borel probability measures on R^d, then

$$\phi_\mu(u) = \int_{R^d} e^{i(u,y)} \mu(dy). \tag{1}$$

Characteristic functions have many useful properties, and readers can refer to [18] for more details. One important property that can be used to analyze the sum of independent variables is described in Definition 2.

Definition 2 [1] The convolution μ of two finite measures μ_1 and μ_2 on R^d, denoted by $\mu = \mu_1 * \mu_2$ is a measure defined by

$$\mu\left(B\right) = \iint_{R^d \times R^d} I_B\left(x+y\right)\mu_1\left(dx\right)\mu_2\left(dy\right), \quad B \in B\left(R^d\right),$$

where B $\left(R^d\right)$ is Borel σ algebra on Euclidean space.

If $X \sim \mu_1, Y \sim \mu_2$, and X and Y are independent, then $X+Y \sim \mu$, and

$$\phi_{X+Y}\left(u\right) = \phi_X\left(u\right)\phi_Y\left(u\right), \tag{2}$$

which implies that the characteristic function of the sum of independent random variables is the product of the characteristic functions of individual random variables.

Characteristic functions of Lévy processes are characterized by Lévy measures or Lévy symbols. Next we give the definition of Lévy measure, and a Lévy symbol can be represented by a Lévy measure.

Definition 3 [1] A Borel probability measure on R^d, ν, is a Lévy measure if

$$\int_{R^d}\left(1 \wedge |x|^2\right)\nu\left(dx\right) < \infty, \nu\left(\{0\}\right) = 0.$$

Based on Lévy Khintchine formula, Lévy process $X\left(t\right)$ has a specific form for its characteristic function. More precisely, for all $t \geq 0, u \in R^d$,

$$\phi_{X(t)}(u) = E\left(e^{i(u,X(t))}\right) = e^{t\eta(u)}, \tag{3}$$

where

$$\eta(u) = i(b,u) - \frac{1}{2}(u,au) + \int_{R^d}\left(e^{i(u,x)} - 1 - i(u,x)\,I_{0<|x|<1}(x)\right)\nu(dx)$$

is Lévy symbol, in which ν is Lévy measure, b is a constant on R^d, and a is a positive definite symmetric $d \times d$ matrix.

Lévy measure is the most important element of a Lévy process, in a sense that if we specify a Lévy measure, we can get the corresponding Lévy process and its characteristic function.

Lévy subordinators [1] form the class of nondecreasing Lévy processes, taking values in $[0,\infty)$. Based on (3), a one-dimensional Lévy subordinator $X_s(t)$ has the characteristic function:

$$\phi_{X_s(t)}(u) = E\left(e^{iuX_s(t)}\right) = e^{t\eta_s(u)}, \tag{4}$$

where

$$\eta_s(u) = ib^*u + \int_{R^+}\left(e^{iux} - 1\right)\nu(dx),$$

is Lévy symbol, $b^* = b - \int_{0<x<1} x\nu(dx)$, ν is a Lévy measure satisfying an extra condition $\int_{R^+}(1 \wedge x)\,\nu(dx) < \infty$, and b^* is a constant on R^+.

2.2. *Special cases of Lévy processes*

Lévy processes are stochastic processes with independent and stationary increments over time. Some special Lévy processes have been widely used to model degradation processes in the literature, such as Wiener process and gamma process for wear, and compound Poisson process for pure jump damages. The Lévy measures and Lévy symbols for these common special cases are introduced in this section.

2.2.1. *Linear process*

When $a = \nu = 0$, $b \neq 0$, Lévy symbol in (3) becomes $\eta(u) = i(b, u)$, and the characteristic function in (3) is $\phi_{X(t)}(u) = e^{it(b,u)}$, indicating that $X(t) = bt$, where b is a constant and usually called the drift. Therefore, $X(t)$ is a deterministic linear process, which is not suitable for modeling stochastic degradation process.

2.2.2. *Brownian motion/Wiener process*

When $a \neq 0, b \neq 0, \nu = 0$, Lévy symbol becomes $\eta(u) = i(b, u) - \frac{1}{2}(u, au)$ and $\phi_{X(t)}(u) = e^{t\left[i(b,u) - \frac{1}{2}(u,au)\right]}$, which is the characteristic function of Brownian motion with drift b. The case $a = I, b = 0, \nu = 0$ is usually called standard Brownian motion or Wiener process $B_0 = (B_0(t), t \geqslant 0)$, which has a Gaussian density

$$\rho_t(x) = \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{2t}}.$$

Wiener process and Brownian motion with drift are not suitable for modeling monotonically increasing/decreasing wear processes, because their increments are not always positive.

2.2.3. Temporally homogeneous gamma process

When $a = 0, b \neq 0, \nu \neq 0$, if ν is a finite measure, we have

$$\eta(u) = i(b^*, u) + \int_{R^d} \left(e^{i(u,x)} - 1\right)\nu(dx),$$

where $b^* = b - \int_{0<|x|<1} x\nu(dx)$. Next, We find the special form for ν to obtain the temporally homogeneous gamma process.

For a gamma process, if the shape parameter, $\alpha(t) = \alpha t$, $t \geqslant 0$ (i.e., the second condition in Definition 1 is satisfied), it is a temporally homogeneous gamma process, $G(t)$. On R^1, $G(t)$ has a density $f_{G(t)} = Ga(x|\alpha t, \beta) = \frac{\beta^{\alpha t} x^{\alpha t-1} e^{-\beta x}}{\Gamma(\alpha t)}, x > 0, t \geqslant 0$. Then the characteristic function of $G(t)$ can be expressed as

$$\phi_{G(t)}(u) = \left(\frac{\beta}{\beta - iu}\right)^{\alpha t} = \exp\left(\alpha t \ln \frac{\beta}{\beta - iu}\right) = \exp\left(\alpha t \int_0^\infty \left(e^{iux} - 1\right) \frac{e^{-\beta x}}{x} dx\right). \quad (5)$$

Therefore, the temporally homogeneous gamma process is a special case of Lévy process, and its Lévy measure is $\nu(dx) = \alpha x^{-1} e^{-\beta x} dx$, and Lévy symbol is $\eta(u) = \alpha \int_0^\infty (e^{iux} - 1) \frac{e^{-\beta x}}{x} dx$, with $a = 0$, $b^* = 0$. The temporally homogeneous gamma process is a Lévy process that is always positive and strictly increasing, and it is suitable for modeling strictly increasing wear processes with a linear mean path, $\alpha t/\beta$.

2.2.4. Compound Poisson process

For a Poisson process with parameter λ, $N(t) \sim Poisson(\lambda t)$, and $P(N(t) = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$, for $n = 0, 1, 2, \cdots$. Let $(J(n), n \in N(t))$ be the jump size described by a sequence of independent and identically distribution (i.i.d.) random variables taking values in $R^d, d \in N$ with distribution μ_J, and independent of $N(t)$. The compound Poisson process $C(t)$ is defined as follows:

$$C(t) = J(1) + \cdots + J(N(t)).$$

Based on Definition 2, we obtain the characteristic function of compound Poisson process:

$$\begin{aligned}\phi_{C(t)}(u) &= E\left(e^{iuC(t)}\right) = \sum_{n=0}^{\infty} P(N(t) = n) E\left(e^{i\left(u, \sum\limits_{k=1}^{n} J(k)\right)}\right) = \sum_{n=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} {\phi_J}^n(u) \\ &= \exp\left(\lambda t\left(\phi_J(u) - 1\right)\right) = \exp\left(\lambda t \int_{R^d} \left(e^{i(u,x)} - 1\right) \mu_J(dx)\right).\end{aligned} \tag{6}$$

Therefore, for a compound Poisson process $C(t)$, Lévy measure is $\nu(dx) = \lambda\mu_J(dx)$, and Lévy symbol is

$$\eta_C(u) = \int_{R^d} \left(e^{i(u,x)} - 1\right) \lambda\mu_J(dx).$$

The sample paths of $C(t)$ are piecewise constant on finite intervals with jump discontinuities at random times. It is suitable for modeling pure jump damages.

3. Life distribution analysis based on Lévy subordinators

We use the Lévy subordinator $X_s(t)$ to represent the monotonically non-decreasing volume of a degradation with random jumps up to time t. A component or a system fails when $X_s(t)$ exceeds a failure threshold x, assuming that it subjects to one degradation process that begins with $X_s(t) = 0$. To simplify the formula, we assume the failure threshold is a constant, and it is easy to extend the model when the failure threshold is a random variable.

The lifetime of the device is defined as

$$T_x = inf\{t : X_s(t) > x\}$$

Since $X_s(t)$ is nondecreasing, we have

$$\{T_x \geq t\} \equiv \{X_s(t) \leq x\}.$$

Then the reliability function can be defined as

$$R(t) = P(T_x \geqslant t) = P(X_s(t) \leqslant x) = F_{X_s(t)}(x). \tag{7}$$

In this section, we present a method based on inverse Fourier transform to derive the reliability function for systems subject to a degradation process with jumps that can be described by Lévy subordinator. Temporally homogeneous gamma process and compound

Poisson process are the special Lévy processes commonly used in degradation-based reliability analysis. For systems only subject to wear, temporally homogeneous gamma process can be used; for systems subject to pure jump damages, compound Poisson process can be used. It is straightforward to derive the reliability function for one of these special processes. For degradation processes exhibiting both wear and jump damages, convolution formula has been typically used to analyze reliability for these cases. However, when the wear process has little common properties with the jump process, it is difficult to calculate reliability in (7). A Lévy process can conveniently represent wear process with random jumps.

Although the probability density function of Lévy process is not readily available, we have the expression of its characteristic function. Since there is a one-to-one correspondence between the cumulative distribution function (cdf) and the characteristic function, we can obtain one of them if the other one is known. Based on Fourier inversion theorem, Shephard [31] provided the following remarkable theorem describing the cdf as the function of $\phi(u)$ for a random variable.

Lemma 1 [31] If the probability density function f and the characteristic function $\phi_X(u)$ are integrable in the Lebesgue sense, then under the assumption that the mean for the random variable of interest exists, the following equality holds:

$$F_X(x) = \frac{1}{2} - \frac{1}{2\pi}\int_0^{\infty} \underset{u}{\Delta}\left(\frac{e^{-iux}}{iu}\phi_X(u)\right)du,$$

where $\underset{u}{\Delta}\rho(u) = \rho(u) + \rho(-u)$.

The following Lemma 2 is the multivariate generalization of Lemma 1.

Lemma 2 [**31**] If the probability density function f and the characteristic function $\phi_X(u)$ are integrable in the Lebesgue sense, then under the assumption that the mean for the random multi-dimensional variable of interest exist, the following equality holds:

$$\frac{(-2)^d}{(2\pi)^d}\int_0^{\infty}\cdots\int_0^{\infty}\underset{u_1}{\Delta}\underset{u_2}{\Delta}\cdots\underset{u_d}{\Delta}\left(\frac{e^{-i(u,x)}}{iu_1iu_2\cdots iu_d}\phi_X(u)\right)du = F^*(x),$$

where

$$F^*(x) = 2^dF(x_1,\cdots x_d) - 2^{d-1}\left(F(x_2,x_3,\cdots,x_d)+\cdots+F(x_1,\cdots,x_{d-2},x_{d-1})\right)$$
$$+2^{d-2}\left(F(x_3,x_4,\cdots,x_d)+\cdots+F(x_1,\cdots,x_{d-3},x_{d-2})\right)+\cdots+(-1)^d.$$

Lemma 1 turns out to be a special case of Lemma 2 that deals with multi-dimensional variables. For an example of two-dimensional variables, if we know the characteristic function $\phi_X(u)$, we can get the expression of $F^*(x) = 4F(x_1,x_2) - 2F(x_1) - 2F(x_2) + 1$ based on Lemma 2. If we know the characteristic function of each variable, $\phi_{X_1}(u)$ and $\phi_{X_2}(u)$, we can have $F(x_1)$ and $F(x_2)$ based on Lemma 1. Finally we can solve for the joint distribution function $F(x_1,x_2)$ of X_1, X_2. Integration rules for the computation of the multivariate distribution function are derived in [32].

In this paper, we focus on the one-dimensional Lévy degradation process. When $d = 1$, for all $t \geq 0, u \in R^1$, the characteristic function of a Lévy subordinator $X_s(t)$ is expressed in (4). For $X_s(t)$, if the probability density function and the characteristic function are integrable in the Lebesgue sense, and the mean exists, then we derive the reliability function and pdf of lifetime in the following corollaries.

Corollary 1 For systems subject to stochastic degradation with random jumps that can be described by Lévy subordinators, assuming the failure threshold value is x, the reliability function represented by Lévy measure is

$$\begin{aligned} R(t) &= P(T \geqslant t) = P(X_s(t) \leqslant x) = F_{X_s(t)}(x) \\ &= \frac{1}{2} - \frac{1}{2\pi}\int_0^\infty \underset{u}{\Delta}\left(\frac{e^{-iux}}{iu}\exp\left(t\left(ib^*u + \int_{R^+}\left(e^{iux}-1\right)\nu(dx)\right)\right)\right)du, \end{aligned} \tag{8}$$

where $b^* = b - \int_{0<x<1} x\nu(dx)$, $a > 0$, and $\underset{u}{\Delta}\rho(u) = \rho(u) + \rho(-u)$.

Corollary 2 For systems subject to stochastic degradation with random jumps that can be described by Lévy subordinators, assuming the failure threshold value is x, the probability density function of lifetime represented by Lévy measure is

$$\begin{aligned} f(t) &= -\frac{\partial R(t)}{\partial t} \\ &= \frac{1}{2\pi}\int_0^\infty \underset{u}{\Delta}\left(\frac{e^{-iux}\exp\left(t\left(ib^*u + \int_{R^+}\left(e^{iux}-1\right)\nu(dx)\right)\right)}{iu\left(ib^*u + \int_{R^+}\left(e^{iux}-1\right)\nu(dx)\right)^{-1}}\right)du, \end{aligned} \tag{9}$$

where $b^* = b - \int_{0<x<1} x\nu(dx)$, $a > 0$, and $\underset{u}{\Delta}\rho(u) = \rho(u) + \rho(-u)$.

For systems subject to degradation with random jumps that can be described by Lévy subordinators, we can first specify a certain Lévy measure and then calculate the reliability function and pdf using Equations (8) and (9). Kharoufeh [10] gave explicit results for wear processes in Markovian environment, which requires to use multi-inverse algorithms to calculate. Although they are not explicit, our results in (8) and (9) can be computed comparatively cheap based on [31].

The advantages of our results in (8) and (9) are twofold: 1) They are general because we can specify different Lévy measures to fit different types of degradation data sets, while the models in the literature become special cases of our models; and 2) they provide a methodology to deal with complex random jumps in degradation processes, and our methods can solve the problems that the traditional convolution method cannot solve, i.e., when the distributions of jumps size are not additive.

4. Life distribution analysis for temporally homogeneous gamma process with random jumps

To demonstrate the advantages of our models, we present the reliability analysis by specifying a Lévy measure to model degradation with random jumps. As presented in Section 2.2, temporally homogeneous gamma process is a special Lévy subordinator that can be used to model strictly increasing degradation process, which is often the case in practice; and compound Poisson process can be used to model random jumps due to external or internal impacts. We present the life distribution analysis for temporally homogeneous gamma process with random jumps.

4.1. Reliability function using traditional convolution approach

We first present the reliability function derived from the traditional convolution approach for temporally homogeneous gamma process with random jumps. If a degradation process with sporadic jumps can be well described by the sum of a temporally homogeneous gamma process and a compound Poisson process, assuming these two processes are independent, the traditional convolution approach to derive reliability function is:

$$
\begin{aligned}
R(t) = P\left(X_s(t) \leqslant x\right) &= P\left(G(t) + \sum_{i=0}^{N(t)} J_i \leqslant x\right) \\
&= \sum_{n=0}^{\infty} P\left(G(t) + \sum_{i=0}^{N(t)} J_i \leqslant x \,|N(t) = n\right) P\left(N(t) = n\right) \\
&= \sum_{n=0}^{\infty} P\left(G(t) + \sum_{i=0}^{n} J_i \leqslant x\right) \frac{e^{-\lambda t}(\lambda t)^n}{n!}.
\end{aligned} \tag{10}
$$

If the jump size follows a gamma distribution, $J_i \sim Gamma(\alpha^*, \beta^*)$, then $\sum_{i=0}^{n} J_i \sim Gamma(n\alpha^*, \beta^*)$. If the scale parameter of $G(t)$ is the same as β^*, i.e., $\beta = \beta^*$, then $G(t) + \sum_{i=0}^{n} J_i \sim Gamma(\alpha t + n\alpha^*, \beta)$. The reliability function for this special case is

$$
\begin{aligned}
R(t) = P\left(X_s(t) \leqslant x\right) &= \sum_{n=0}^{\infty} P\left(G(t) + \sum_{i=0}^{n} J_i \leqslant x\right) \frac{e^{-\lambda t}(\lambda t)^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(1 - \frac{\Gamma\left(\alpha t + n\alpha^*, x\beta\right)}{\Gamma\left(\alpha t + n\alpha^*\right)}\right) \frac{e^{-\lambda t}(\lambda t)^n}{n!},
\end{aligned} \tag{11}
$$

where $\Gamma\left(\alpha t + n\alpha^*\right) = \int_{y=0}^{\infty} y^{\alpha t + n\alpha^* - 1} e^{-y} dy$, $\Gamma\left(\alpha t + n\alpha^*, x\beta\right) = \int_{y=x\beta}^{\infty} y^{\alpha t + n\alpha^* - 1} e^{-y} dy$, and x is the threshold value.

We can see that Equation (11) is derived based on two assumptions: 1) the distribution of jump size is additive, i.e., gamma distribution is additive; and 2) the scale parameters are the same. However, if $\beta \neq \beta^*$, or if J_i follows a different distribution than gamma distribution (such as inverse Gaussian distribution, Lévy distribution, Pareto distribution, exponential distribution, or lognormal distribution), it becomes complex to calculate the reliability function in (11) based on Definition 2. Our approach in Corollary 1 is capable to deal with these cases by using Lévy measure.

4.2. Reliability function using Lévy measure

In this section, we present the reliability function and pdf derived from our new approach in Corollaries 1 and 2 using Lévy measure for temporally homogeneous gamma process with random jumps.

As given in Section 2.2.3, for a temporally homogeneous gamma process, Lévy measure is $\nu_1(dx) = \alpha x^{-1} e^{-\beta x} dx$. As given in Section 2.2.4, for a compound Poisson process, Lévy measure is $\nu_2(dx) = \lambda \mu_J(dx)$. Then for a Lévy subordinator called temporally homogeneous gamma process with compound Poisson jumps, the characteristic function is derived based on (2):

$$\phi_{X_s(t)}(u) = \phi_{G(t)+C(t)}(u) = \phi_{G(t)}(u)\,\phi_{C(t)}(u) = \exp\left(t\int_{R^+}\left(e^{iux}-1\right)\left(\frac{\alpha e^{-\beta x}}{x} + \lambda {\mu'}_J\right)dx\right),$$

where ${\mu'}_J$ is the probability density function of the jump size. Therefore, we can model the gamma degradation with random jumps by specifying a Lévy measure as $\nu = \nu_1 + \nu_2 = \alpha x^{-1} e^{-\beta x} dx + \lambda \mu_J(dx)$. Based on Corollary 1 and 2, the reliability function and pdf are derived.

$$\begin{aligned} R(t) &= P(T_x \geqslant t) = P(X_s(t) \leqslant x) = F_{X_s(t)}(x) \\ &= \frac{1}{2} - \int_0^\infty \underset{u}{\Delta}\left(\frac{e^{-iux}}{2\pi iu}\exp\left(t\left(\int_{R^+}\left(e^{iux}-1\right)\left(\frac{\alpha e^{-\beta x}}{x} + \lambda {\mu'}_J\right)dx\right)\right)\right)du. \end{aligned} \tag{12}$$

The probability density function of lifetime is

$$
\begin{aligned}
f(t) &= -\frac{\partial R(t)}{\partial t} \\
&= \int_0^{\infty} \underset{u}{\Delta} \left(\frac{e^{-iux} \exp\left(t \left(\int_{R^+} \left(e^{iux} - 1\right) \left(\frac{\alpha e^{-\beta x}}{x} + \lambda {\mu'}_J \right) dx \right) \right)}{2\pi iu \left(\int_{R^+} \left(e^{iux} - 1\right) \left(\frac{\alpha e^{-\beta x}}{x} + \lambda {\mu'}_J \right) dx \right)^{-1}} \right) du.
\end{aligned} \tag{13}
$$

The results in (12) and (13) can be applied to the jump size following a general distribution μ_J defined on $[0, \infty)$, as listed in Section 4.1. In the following, we derived the reliability function and pdf for three different jump types.

4.2.1. Gamma-distributed jump sizes

If the jump size follows a gamma distribution,

$$
{\mu'}_J = Ga\left(x|\alpha^*, \beta^*\right) = \frac{\beta^{\alpha^*} x^{\alpha^*-1} e^{-\beta^* x}}{\Gamma\left(\alpha^*\right)}, x > 0
$$

then the characteristic function for the compound Poisson process is

$$
\begin{aligned}
\phi_{C(t)}(u) &= \exp\left(\lambda t \left(\int_{R^+} \left(e^{iux} - 1\right) \frac{\beta^{\alpha^*} x^{\alpha^*-1} e^{-\beta^* x}}{\Gamma\left(\alpha^*\right)} dx \right) \right) \\
&= \exp\left(\lambda t \left(\int_{R^+} e^{iux} \frac{\beta^{\alpha^*} x^{\alpha^*-1} e^{-\beta^* x}}{\Gamma\left(\alpha^*\right)} dx - \int_{R^+} \frac{\beta^{\alpha^*} x^{\alpha^*-1} e^{-\beta^* x}}{\Gamma\left(\alpha^*\right)} dx \right) \right) \\
&= \exp\left(\lambda t \left(\int_{R^+} e^{iux} \frac{\beta^{\alpha^*} x^{\alpha^*-1} e^{-\beta^* x}}{\Gamma\left(\alpha^*\right)} dx - 1 \right) \right) \\
&= \exp\left(\lambda t \left(\left(\frac{\beta^*}{\beta^* - iu} \right)^{\alpha^*} - 1 \right) \right).
\end{aligned}
$$

Then the reliability function in (12) is

$$
R(t) = \frac{1}{2} - \int_0^{\infty} \underset{u}{\Delta} \left(\frac{e^{-iux}}{2\pi iu} \left(\frac{\beta}{\beta - iu} \right)^{\alpha t} \exp\left(\lambda t \left(\left(\frac{\beta^*}{\beta^* - iu} \right)^{\alpha^*} - 1 \right) \right) \right) du. \tag{14}
$$

Equation (14) is a general formula for reliability function for a gamma process with gamma-distributed jumps, regardless of $\beta = \beta^*$ or not, while Equation (11) is only for the case of $\beta = \beta^*$.

The probability density function of lifetime in (13) is derived to be

$$f(t) = -\frac{\partial R(t)}{\partial t}$$
$$= \int_0^\infty \underset{u}{\Delta}\left(\frac{e^{-iux}}{2\pi iu}\left(\frac{\beta}{\beta - iu}\right)^{\alpha t} e^{\lambda t\left(\left(\frac{\beta^*}{\beta^* - iu}\right)^{\alpha^*} - 1\right)}\left(\ln\left(\frac{\beta}{\beta - iu}\right)^{\alpha} + \lambda\left(\left(\frac{\beta^*}{\beta^* - iu}\right)^{\alpha^*} - 1\right)\right)\right) du$$

4.2.2. Lévy-distributed jump sizes

If the jump size follows a different distribution than gamma distribution, we can also derive the reliability function and pdf using (12) and (13). When the jump size follows a Lévy distribution, the probability density function is given as [2]

$$\mu_J'(x; \omega, \xi) = \begin{cases} \dfrac{\sqrt{\frac{\xi}{2\pi}}\exp(-\frac{\xi}{2(x-\omega)})}{(x-\omega)^{\frac{3}{2}}} & for \quad x \geqslant \omega \\ 0 & otherwise \end{cases}$$

Lévy distribution is a continuous probability distribution of a non-negative random variable when $\omega \geq 0$. It has little common properties with gamma distribution, leading to complex calculation in the convolution approach. Since the characteristic function of Lévy distributed variable is $e^{iu\omega - \sqrt{-2iu\xi}}$ [9], the reliability function of gamma degradation with Lévy-distributed jumps is derived from (12) to be

$$R(t) = \frac{1}{2} - \int_0^\infty \underset{u}{\Delta}\left(\frac{e^{-iux}}{2\pi iu}\left(\frac{\beta}{\beta - iu}\right)^{\alpha t}\exp\left(\lambda t\left(e^{iu\omega - \sqrt{-2iu\xi}} - 1\right)\right)\right) du. \qquad (15)$$

The probability density function of lifetime in (13) is

$$
\begin{aligned}
f(t) &= -\frac{\partial R(t)}{\partial t} \\
&= \int_0^\infty \underset{u}{\Delta}\left(\frac{e^{-iux}}{2\pi iu}\left(\frac{\beta}{\beta-iu}\right)^{\alpha t} e^{\lambda t\left(e^{iu\omega-\sqrt{-2iu\xi}}-1\right)}\left(\ln\left(\frac{\beta}{\beta-iu}\right)^{\alpha}+\lambda\left(e^{iu\omega-\sqrt{-2iu\xi}}-1\right)\right)\right)du.
\end{aligned}
$$

4.2.3. Inverse Gaussian-distributed jump sizes

The inverse Gaussian distribution is used to describe positive continuous random variables. Its probability density function is

$$
\mu_J'(x;\varsigma,\vartheta) = \begin{cases} \sqrt{\dfrac{\vartheta}{2\pi x^3}}\exp\{-\dfrac{\vartheta(x-\varsigma)^2}{2\varsigma^2 x}\} & for \quad x>0 \\ 0 & otherwise \end{cases}
$$

where $\varsigma > 0$ is the mean, and $\vartheta > 0$ is the shape parameter. It also has little common properties with gamma distribution. Since the characteristic function of an inverse Gaussian-distributed variable is $e^{\frac{\vartheta}{\varsigma}\left(1-\sqrt{1-\frac{2iu\varsigma^2}{\vartheta}}\right)}$, the reliability function of gamma degradation with inverse Gaussian-distributed jumps in (12) is

$$
R(t) = \frac{1}{2} - \int_0^\infty \underset{u}{\Delta}\frac{e^{-iux}}{2\pi iu}\left(\frac{\beta}{\beta-iu}\right)^{\alpha t}\exp\left(\lambda t\left(e^{\frac{\vartheta}{\varsigma}\left(1-\sqrt{1-\frac{2iu\varsigma^2}{\vartheta}}\right)}-1\right)\right)du. \tag{16}
$$

The probability density function of lifetime in (13) is

$$
\begin{aligned}
f(t) &= -\frac{\partial R(t)}{\partial t} \\
&= \int_0^\infty \underset{u}{\Delta}\left(\frac{e^{-iux}}{2\pi iu}\left(\frac{\beta}{\beta-iu}\right)^{\alpha t} e^{\lambda t\left(e^{\frac{\vartheta}{\varsigma}\left(1-\sqrt{1-\frac{2iu\varsigma^2}{\vartheta}}\right)}-1\right)}\left(\ln\left(\frac{\beta}{\beta-iu}\right)^{\alpha}+\lambda\left(e^{\frac{\vartheta}{\varsigma}\left(1-\sqrt{1-\frac{2iu\varsigma^2}{\vartheta}}\right)}-1\right)\right)\right)du.
\end{aligned}
$$

Besides the Lévy measures used in this section, we can specify other Lévy measures for model construction in order to fit the corresponding degradation data. Some interesting Lévy measures have been studied in [3], such as $\nu(dx) = \frac{\delta\gamma^{-2\kappa}\kappa x^{-\kappa-1}\exp(-\frac{1}{2}\gamma^2 x)}{\Gamma(\kappa)\Gamma(1-\kappa)}dx, x, \delta > 0, 0 < \kappa < 1, \gamma \geq 0$ for the positive tempered stable process $PTS(\kappa, \delta, \gamma)$.

5. Numerical examples

We consider the crack growth process in a device, which is subject to degradation due to fatigue and a variety of overloads that can occur in manufacturing, deployment, and operation phases. We use a Lévy subordinator $X_s(t)$ to represent the growth of a crack at time t, specifically, a temporally homogeneous gamma process with random jumps. Then the Lévy measure is $\nu = \alpha x^{-1}e^{-\beta x}dx + \lambda\mu_J(dx)$. In particular, we consider three different distributions to model the jump size: gamma, Lévy and inverse Gaussian. The specific values for the parameters are given in Table 1. A device fails when the crack length exceeds the threshold x.

Table 1: Values for parameters

Parameters	**Values**	**Parameters**	**Values**
α	5	ω	1
β	0.8	ξ	0.002
λ	3	ς	1
α^*	10	ϑ	1
β^*	15	x	50

Figure 1 shows the reliability over time of devices subject to a gamma degradation with gamma-distributed jumps. When the parameter $\beta^* = \beta = 0.8$, both traditional convolution

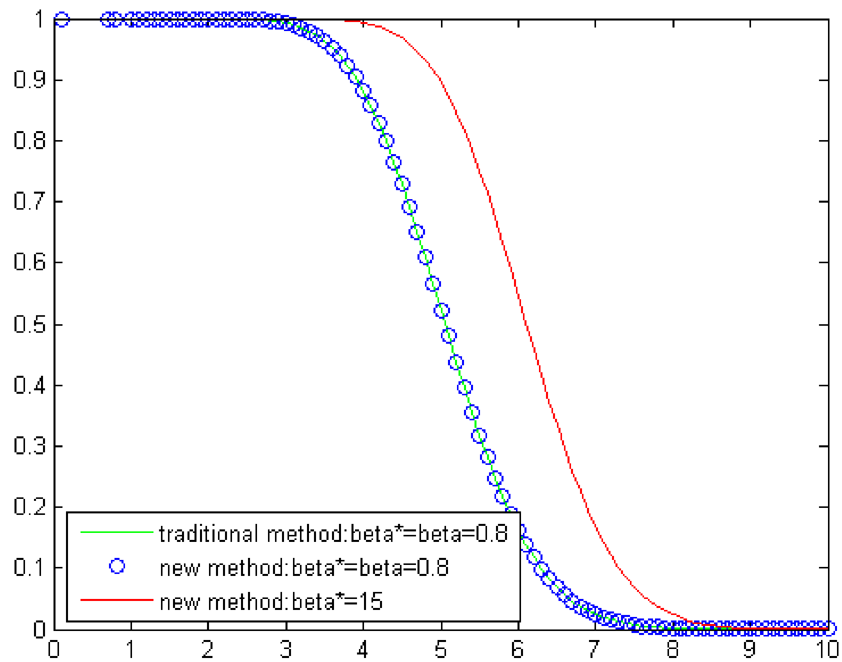

Figure 1: Reliability for gamma degradation with gamma-distributed jumps

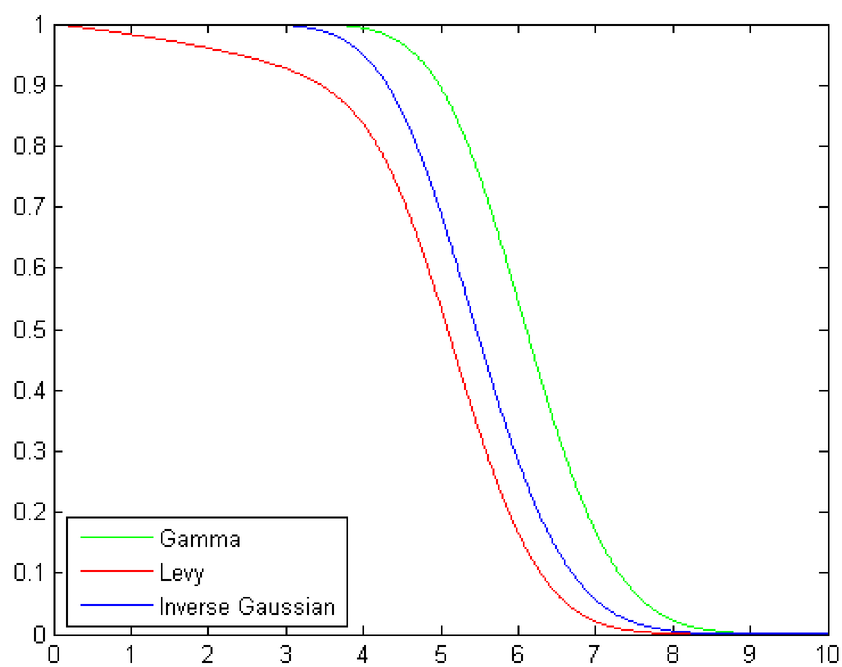

Figure 2: Reliability for gamma degradation with three jump types

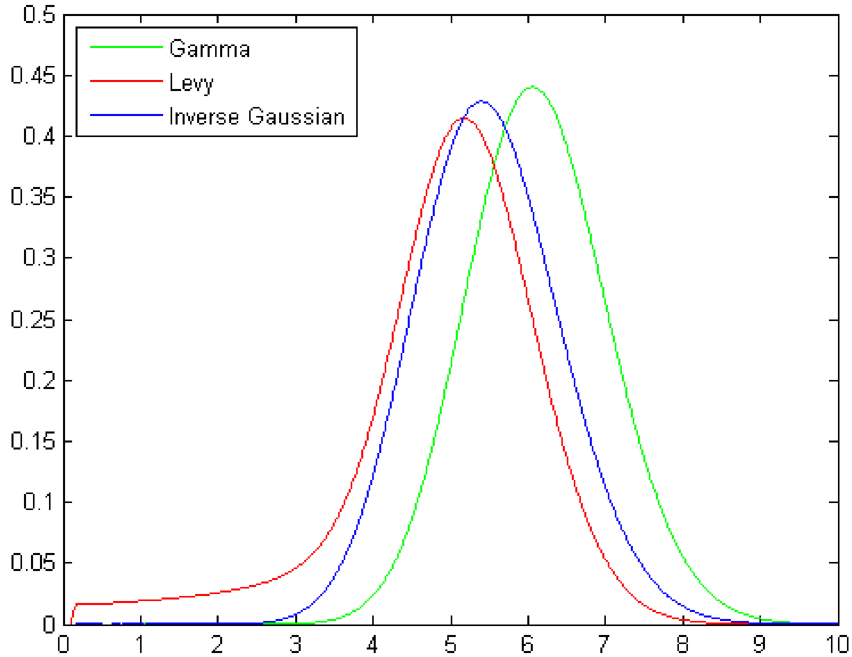

Figure 3: Pdf of lifetime for gamma degradation with three jump types

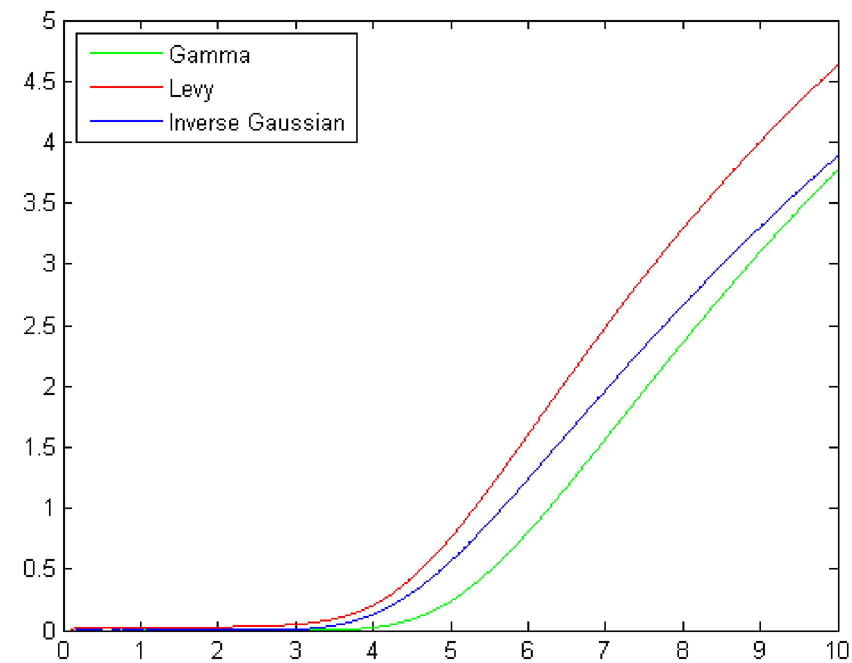

Figure 4: Hazard rate for gamma degradation with three jump types

(11) and our proposed method (14) can solve the problem, showing the same curve of $R(t)$. When $\beta^* \neq \beta$, the reliability curve is provided by our model in (14), and the convolution approach becomes complex in this case.

Figure 2 shows the reliability over time of devices subject to a gamma degradation with three different jump distributions. It demonstrates that for Lévy and Inverse Gaussian distributed jump sizes, whose probability density functions have little common properties with gamma, we can readily calculate the reliability by using our proposed model in (14). Figure 3 and Figure 4 illustrate the probability density function and hazard rate for the lifetime of the devices. We can see that the hazard rates increase over time for all three cases.

6. Conclusions and discussion

One of the challenging aspects in reliability analysis is how to formulate the reliability function from the degradation process that a system or a component experiences. In this paper, we presented a novel model concerning the stochastic mechanism of a complex degradation process that also subjects to random jumps. Based on inverse Fourier transforms, the reliability function and pdf of lifetime were derived. Our model is general because we can specify many different Lévy measures to fit many different types of degradation data sets, and the models in the literature become special cases of our model. In addition, by providing a methodology to deal with complex random jumps in degradation processes, our method can solve the problems that the traditional convolution method cannot solve, i.e., when the distribution of jumps size is not additive. Our new method provides a convenient and general way to evaluate the system reliability. The calculation for reliability is simple enough to implement in practice.

More importantly, the model provides a framework for reliability analysis of multi-degradation processes in multi-component systems. To derive the reliability function for multi-Lévy degradation processes on $R^d, d \in N$, we need to construct multi-dimensional Lévy measures $\nu \in \mathrm{M}_1\left(R^d\right)$. If the multi-degradation processes are dependent, the construction of the multi-dimensional Lévy measures can refer to Lévy copula theory [12,6]. In order to apply the model to degradation data analysis, statistical inference on Lévy measures is another potential research topic.

7. Acknowledgements

This paper is based upon work supported by the Texas Norman Hackerman Advanced Research Program under Grant no. 003652-0122-2009, and by USA National Science Foundation (NSF) under Grant no. 0970140 and 0969423.

References

[1] D. Applebaum, Lévy processes and stochastic calculus, Cambridge University Press, Cambridge, 2009.

[2] M.M. Ali and J. Woo, Inference on reliability $P(Y < X)$ in the Lévy distribution, Mathematical and Computer Modelling 41 (2005), 965-971.

[3] O.E. Barndorff-Nielsen and N. Shephard, Basics of Lévy processes, Unpublished chapter of the authors' book on Lévy Driven Volatility Models, 2012.

[4] D.R. Cox and H.D. Miller, The theory of stochastic processes, Methuen and Company, London, 1965.

[5] R.S. Chhikara and J.L. Folks, The inverse Gaussian distribution: Theory, methodology, and applications, Marcel Dekker, New York, 1989.

[6] R. Cont and P. Tankov, Financial modelling with jump processes, Chapman & Hall, New York, 2004.

[7] N. Ebrahimi, System reliability based on system wear, Stochastic Models 22 (2006), 21-36.

[8] G. Gottlieb, Failure distributions of shock models, J. Appl. Prob. 17 (1980), 745-752.

[9] K. Gorska and K. A. Penson, Lévy stable distributions via associated integral transform, J. Math. Phys. 53 (2012), 1-10.

[10] J. Kharoufeh, Explicit results for wear processes in a Markovian environment, Operations Research Letters 31 (2003), 237-244.

[11] J. Kharoufeh and D.G. Mixon, On a Markov-modulated shock and wear process, Naval Research Logistics 56 (2009), 563-576.

[12] J. Kallsen and P. Tankov, Characterization of dependence of multidimensional Lévy processes using Lévy copulas, Journal of Multivariate Analysis 97 (2006), 1551-1572.

[13] J.C. Lu and W.Q. Meeker, Using degradation measures to estimate a time-to-failure distribution. Technometrics 35 (1993), 161-174.

[14] J.C. Lu, J. Park, and Q. Yang, Statistical inference of a time-to-failure distribution derived from linear degradation data, Technometrics 39 (1997), 391-400.

[15] J. Lawless and M. Crowder, Covariates and random effects in a gamma process model with application to degradation and failure, Lifetime Data Analysis 10 (2004), 213-227.

[16] H. Liao, E. Elsayed, and L. Chan, Maintenance of continuously monitored degrading systems, European Journal of Operational Research 175 (2006), 821-835.

[17] M.T. Lee and G.A. Whitmore, Threshold regression for survival analysis: Modeling event times by a stochastic process reaching a boundary, Statistical Science 21 (2006), 501-513.

[18] E. Lukacs, Characteristic functions, Griffin, London, 1970.

[19] A.H. Mohamed. A gamma wear process, IEEE Trans Reliab 24 (1975), 152-153.

[20] A.H. Mohamed, Life distribution properties of devices subject to a pure jump damage process, J. Appl. Prob. 21 (1984), 816-825.

[21] A.H. Mohamed, Life distribution properties of devices subject to a Lévy wear process, Mathematics of Operations Research 9 (1984), 606-614.

[22] J.M. Van Noortwijk, A survey of the application of gamma processes in maintenance, Reliability Engineering and System Safety 94 (2009), 2-21.

[23] J.M. Van Noortwijk, J.A.M. van der Weide, M.J. Kallen, and M.D. Pandey, Gamma processes and peaks-over-threshold distributions for time-dependent reliability, Reliability Engineering and System Safety 92 (2007), 1651-1658.

[24] C. Park and W. Padgett, Accelerated degradation models for failure based on geometric Brownian motion and gamma processes, Lifetime Data Analysis 11 (2005), 511-527.

[25] C. Park and W. Padgett, Stochastic degradation models with several accelerating variables, IEEE Transactions on Reliability 55 (2006), 379-390

[26] H. Peng, Q.M. Feng, and D.W. Coit, Reliability and maintenance modeling for systems subject to multiple dependent competing failure processes, IIE Transactions 43 (2011), 12-22.

[27] N.D. Singpurwalla, Survival in dynamic environment, Statis Sci 10 (1995), 86-103.

[28] X.S. Si, W.B. Wang, C.H. Hu, and D.H. Zhou, Remaining useful life estimation - A review on the statistical data driven approaches, European Journal of Operational Research 213 (2011), 1-14.

[29] X.S. Si, W.B. Wang, C.H. Hu, M.Y. Chen, and D.H. Zhou, A Wiener-process-based degradation model with a recursive filter algorithm for remaining useful life estimation, Mechanical Systems and Signal Processing 35 (2013) 219-237.

[30] K. Sato, Lévy processes and infinitely divisible distributions, Cambridge University Press, Cambridge, 1999.

[31] N.G. Shephard, From characteristic function to distribution function: A simple framework for the theory, Econometric Theory 7 (1991), 519-529.

[32] N.G. Shephard, Numerical integration rules for multivariate inversions, J. Statist. Comput. Simul. 39 (1991), 37-46.

[33] J. Tang and T. Su, Estimating failure time distribution and its parameters based on intermediate data from a Wiener degradation model, Naval Research Logistics 55 (2008), 265-276.

[34] C. Tsai, S. Tseng, and N. Balakrishnan, Mis-specification analyses of gamma and Wiener degradation processes, Journal of Statistical Planning and Inference 141 (2011), 3725-3735.

[35] G.A. Whitmore, Estimating degradation by a Wiener diffusion process subject to measurement error, Lifetime Data Analysis 1 (1995), 307-319.

[36] L.A. Waller, B.W. Turnbull, and J.M. Hardin, Obtaining distribution functions by numerical inversion of characteristic functions with applications, The American Statistician 49 (1995), 346-350.

[37] Y. Yang and G. Klutke, Lifetime-characteristics and inspection-schemes for Lévy degradation processes, IEEE Transactions on Reliability 49 (2000), 377-382.