LIFETIMES OF CONDITIONED DIFFUSIONS

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Abstract. We investigate when an upper bound on expected lifetimes of conditioned diffusions associated with elliptic operators in divergence and non-divergence form can be found. The critical value of the parameter is found for each of the following classes of domains: L^p-domains ($p = n-1$), uniformly regular twisted L^p-domains ($p = n-1$), and twisted Hölder domains ($\alpha = 1/3$). A related parabolic boundary Harnack principle is proved.

1. Introduction and main results. Suppose that D is a domain in $\mathbb{R}^n$, $n \geq 2$; let E_h^x denote the expectation corresponding to Brownian motion in D starting from x and conditioned by a positive harmonic function h in D (i.e., Doob's h-process); and let R be the lifetime of this process. Several authors have addressed the problem of characterizing those domains D for which there exists a constant $c(D) < \infty$ such that

$$E_h^x R < c(D) \tag{1.1}$$

for all positive harmonic h and all $x \in D$.

Cranston and McConnell (1983) proved that (1.1) is true for planar domains D with bounded area (see Chung (1984) for an alternative proof); they also gave an example of a bounded 3-dimensional domain where (1.1) fails. Cranston (1985) extended (1.1) to bounded Lipschitz domains in $\mathbb{R}^n$, $n \geq 2$. Bañuelos (1987) showed that (1.1) holds in uniform domains; he also generalized the result to some other diffusions besides Brownian motion. Some very recent results are discussed at the end of the introduction.

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It is clear from the known results that the domains where (1.1) fails should have long and thin canals and, on the other hand, a reasonably regular boundary for D assures validity of (1.1). In this paper, we will give a precise meaning to the idea of "long and thin canals" and use it to formulate theorems which give sharp sufficient conditions of a geometric nature for (1.1) to hold.

We discuss three families of domains. The definitions will be given in Sections 2 and 3. Here we content ourselves with an intuitive description.

The first family consists of L^p-domains. Roughly speaking, the boundary of an L^p-domain is given (locally) by the graph of an L^p-function. (Note, however, that Definition 2.1 excludes the half-space from this family.)

The second class of domains is the class of twisted L^p-domains. The boundary of a twisted L^p-domain does not have to be representable as the graph of a function anywhere. But we require, by definition, that if such a domain contains a canal of width r, then its length does not exceed $r^{(1-n)/p}$; this property is clearly true of L^p-domains.

Finally, we discuss twisted Hölder domains. A bounded domain D is called a Hölder domain of order α if every point $x \in \partial D$ has a neighborhood U such that $U \cap \partial D$ is the graph, in a suitable coordinate system depending on x, of a Hölder function with exponent α. The boundary of a twisted Hölder domain need not be locally representable as the graph of any function; the canals in a twisted Hölder domain of order α are, by definition, no longer and no thinner than those in a Hölder domain of order α; there is also a mild condition on the regularity of the boundary, less restrictive than uniform regularity. We would like to emphasize that although some Hölder domains are not regular (in the sense of the Dirichlet problem), every Hölder domain satisfies the aforementioned condition.

Recall that a domain D is called uniformly regular if for some $c > 0$ and all $x \in \partial D$, $r > 0$,

$$\mathrm{Cap}_\Delta^{B(x,2r)}(B(x,r) \cap D^c) > c\,\mathrm{Cap}_\Delta^{B(x,2r)}(B(x,r)) \tag{1.2}$$

where $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$ and $\mathrm{Cap}_\Delta^{B(x,2r)}$ is the capacity associated with the Laplacian Δ relative to $B(x,2r)$. We can replace $\mathrm{Cap}_\Delta^{B(x,2r)}$ by $\mathrm{Cap}_\Delta^{\mathbb{R}^n}$ in condition (1.2) for $n \geq 3$.

We will also use a "strong uniform regularity" condition, where (1.2) is replaced by

$$\mathrm{Vol}(B(x,r) \cap D^c) > c\,\mathrm{Vol}(B(x,r)) \tag{1.3}$$

for all $x \in \partial D$, $r > 0$.

Our results hold not only for Brownian motion which is, of course, associated with one half the Laplacian Δ, but for diffusions associated with some other operators L as well.

Recall that L is a uniformly elliptic operator in divergence form ($L \in \mathcal{D}$) if

$$Lf(x) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial f}{\partial x_j})(x)$$

where the a_{ij} are symmetric and for some $c_L < \infty$,

$$c_L^{-1} \sum_{j=1}^{n} (y_j)^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) y_i y_j \leq c_L \sum_{j=1}^{n} (y_j)^2 \tag{1.4}$$

for all $x, y \in \mathbb{R}^n$. Similarly, L is called a uniformly elliptic operator in non-divergence form ($L \in \mathcal{ND}$) if (1.4) holds and

$$Lf(x) = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x).$$

We will assume smoothness of the coefficients a_{ij} to ensure the existence of h-transforms, associated strong Markov processes, etc. Our estimates, however, depend only on c_L and do not depend on the smoothness of the coefficients.

¿From now on P^x and E^x will refer to the probabilities and expectations corresponding to the diffusion associated with the operator L and "harmonic" will mean "L-harmonic." See Stroock and Varadhan (1979) for the definition and discussion of such diffusions. Similarly, P_h^x and E_h^x will correspond to the diffusion conditioned by a positive harmonic function h (see Doob (1984), Section 2VI13 and Chapter 2X).

Theorem 1.1. *(i) Suppose that either*

(a) *L is a uniformly elliptic operator in divergence form or*

(b) *L is a uniformly elliptic operator in non-divergence form and D is strongly uniformly regular (i.e., D satisfies (1.3)).*

Now make one of the following assumptions about D:

(A) *D is an L^p-domain for some $p > n-1$; or*

(B) *D is a uniformly regular twisted L^p-domain for some $p > n-1$; or*

(C) *D is a twisted Hölder domain of order α for some $\alpha \in (1/3, 1]$.*

Then there exists $c(D) < \infty$ such that

$$E_h^x R < c(D)$$

for all $x \in D$ and all positive L-harmonic functions h in D.

(ii) For every $p < n-1$ and $\alpha \in (0, 1/3)$ there exist

(A) *an L^p-domain D_1,*

(B) *a uniformly regular twisted L^p-domain D_2, and*

(C) *a twisted Hölder domain D_3 of order α;*

and functions h_k that are positive and Δ-harmonic in D_k such that

$$R = \infty \quad P_{h_k}^x\text{-a.s.}$$

for all $x \in D_k$, $k = 1, 2, 3$, where $P_{h_k}^x$ stands for the distribution of Brownian motion conditioned by h_k.

Theorem 1.1 (i) holds, in particular, for every bounded domain which may be locally represented as the region above the graph of a function, with no assumptions on regularity in the sense of the Dirichlet problem.

If we consider a domain above the graph of a Hölder function, then Theorem 1.1(i)(b)(C) holds without the assumption of strong uniform regularity, in view of Remark 3.3 (i) below.

In this case $\alpha = 1/3$ is the critical exponent. (The counterexample to show this is too long and complicated to include in this paper; it may be constructed along the lines of Section 4.)

It is very likely that the techniques of this paper may be used to show that Theorem 1.1 (i) also holds for positive superharmonic functions h.

¿From Theorem 1.1 it follows immediately that

Corollary 1.1. *Under the hypotheses of Theorem 1.1 (i), there exists $c_1(D) > 0$ such that*

$$\liminf_{t\to\infty} \left(-\frac{1}{t} \log P_h^x(R > t) \right) > c_1(D)$$

for all x and all positive L-harmonic functions h.

A more precise version of Corollary 1.1 has been proved for Lipschitz domains by DeBlassie (1987, 1988) and Kenig and Pipher (1989) (see also Bañuelos (1991) for more general domains).

Xu (1991) and Davis (1991) have examples of simply connected planar domains with infinite area where (1.1) holds. In Section 2, we will describe some simple uniformly regular twisted L^p-domains with $p > n-1$ and infinite volume, hence providing a new class of examples of the same type.

The condition of uniform regularity in Theorem 1.1 (i) (B) is essential, as easy examples show.

The next result may be called a "parabolic boundary Harnack principle" for operators in divergence form. Let $p_t^D(x, y)$ denote the transition density for the L-diffusion killed on exiting D, $L \in \mathcal{D}$.

Theorem 1.2. *Suppose that $L \in \mathcal{D}$ and D satisfies one of the assumptions (A)-(C) of Theorem 1.1 (i). Then for each $u > 0$ there exists $c = c(D, L, u) > 0$ such that*

$$\frac{p_t^D(x, y)}{p_t^D(x, z)} \geq c\, \frac{p_s^D(v, y)}{p_s^D(v, z)}$$

for all $s, t \geq u$ *and all* $v, x, y, z \in D$.

Related theorems for the case D a Lipschitz domain can be found in Fabes et al. (1986).

We do not know what happens at the critical values $p = n - 1$ and $\alpha = 1/3$.

The proof of the parabolic boundary Harnack principle uses an idea which was also utilized to prove its elliptic counterpart. We have proved that the (elliptic) boundary Harnack principle holds in twisted Hölder domains of order α, $\alpha \in (1/2, 1]$, but counterexamples exist for $\alpha \in (0, 1/2)$ (Bass and Burdzy (1991a)). The elliptic boundary Harnck principle holds in every domain which lies above the graph of a Hölder function with exponent $\alpha \in (0, 1]$ provided $L \in \mathcal{D}$ (see Bañuelos, Bass and Burdzy (1991)), while the same is true for operators $L \in \mathcal{ND}$ if $\alpha \in (1/2, 1]$; here $\alpha = 1/2$ is the critical exponent (Bass and Burdzy (1991c)).

In a related paper, Bass and Burdzy (1991b), we address the question of equality of the Martin and the Euclidean boundaries, known to hold in bounded Lipschitz domains. The two boundaries coincide in domains whose Euclidean boundary can be represented locally by functions less regular than Lipschitz. The critical modulus of continuity lies between $cx \log\log(1/x)/\log\log\log(1/x)$ and $cx \log\log(1/x)$.

Very recently we have seen three papers related to lifetimes of conditioned diffusions. Xu (1991) has an example of a simply connected planar domain with infinite area for which (1.1) holds. We learned from R. Bañuelos and B. Davis about the concept of intrinsic ultracontractivity; see Davies and Simon (1984) for the original definition. Davis (1991) proves that intrinsic ultracontractivity is equivalent to what we call the parabolic boundary Harnack principle. He also proves intrinsic ultracontractivity for a family of planar domains of infinite area and for domains above the graph of a single bounded function; this last result inspired our Theorem 1.2. After writing this article we learned that Bañuelos (1991) had previously proved intrinsic ultracontractivity for a class of domains which he calls "uniformly Hölder domains;" these uniformly regular domains are very close to but slightly more general than our uniformly regular twisted L^p-domains. (Our proofs extend easily to his class of domains.) We remark that in our present paper we prove in Theorem 1.1 (ii) that our results are sharp.

It is perhaps worth discussing the relationship between intrinsic ultracontractivity and (1.1). Intrinsic ultracontractivity implies (1.1) (see Bañuelos (1991), Bañuelos and Davis (1989), Davis (1991) and Kenig and Pipher (1989)), and in fact is a strictly stronger property (Bañuelos and Davis (1989)). But proving intrinsic ultracontractivity is no more difficult than proving (1.1). Indeed, the only currently known widely applicable method of proving (1.1) is based on a method of Chung (1984). Our proof of Theorem 1.2, which is fairly simple, shows that whenever Chung's method works, then the parabolic boundary Harnack principle also holds.

Section 2 contains some estimates for L^p-domains and uniformly regular L^p-domains. Section 3 introduces twisted Hölder domains and also includes the proofs of Theorem 1.1 (i) and Corollary 1.1. Section 4 contains the proof of Theorem 1.1 (ii), while Theorem 1.2 is proved in Section 5.

The letters c_1, c_2, etc. denote constants whose values may change from one proof to another but do not change within a proof.

2. L^p-domains. We start with some general notation and a review of potential theoretic and probabilistic properties for operators $L \in \mathcal{D} \cup \mathcal{ND}$.

For $x \in \mathbb{R}^n$ we will write $x = (\widetilde{x}, x_n)$, i.e., $\widetilde{x} = (x_1, x_2, \dots, x_{n-1})$. Paths of stochastic processes will be denoted X and

$$T_A = T(A) \stackrel{\text{df}}{=} \inf\{t > 0 : X_t \in A\}.$$

If $L \in \mathcal{D}$ and K is a compact subset of a domain D then, by Littman et al. (1963),

$$c_1 G_D^{\Delta}(x, y) \le G_D^L(x, y) \le c_1^{-1} G_D^{\Delta}(x, y) \qquad \text{for all } x, y \in K, \tag{2.1}$$

where $c_1 > 0$ depends only on c_L, K and D and G_D^L is the Green function for L in the domain D. Their proof derives this from

$$c_2 < \mathrm{Cap}_L^D(A)/\mathrm{Cap}_{\Delta}^D(A) < c_2^{-1} \tag{2.2}$$

where $c_2 > 0$ depends only on c_L. Here Cap_L^D is the capacity in the domain D associated with L. (They only prove (2.1) and (2.2) for D a ball, but their proof goes through for arbitrary domains.) Recall that

$$\mathrm{Cap}_L^D(A) = \sup\{\mu(A) : \mu \text{ is a measure supported on } A \subset D \text{ with } G_L^D \mu \leq 1\}.$$

For $L \in \mathcal{D} \cup \mathcal{ND}$ we have the following Harnack principle (Moser (1961) for $\mathcal{D}$, Krylov and Safonov (1981) for $\mathcal{ND}$). If A is a compact subset of an open set D and h is positive and L-harmonic on D, then

$$h(x)/h(y) > c > 0$$

for all $x, y \in A$, where c depends only on D, A and c_L.

By a "chain of balls" connecting points x and y in D, we will mean a sequence of open balls contained in D, with centers $z^1 = x, z^2, z^3, \dots, z^k = y$ and radii $r_j \leq \mathrm{dist}(z^j, \partial D)$, such that

$$|z^j - z^{j+1}| < \min(r_j, r_{j+1})/2.$$

If x and y may be connected by a chain of balls of length k then, by the Harnack principle,

$$h(x)/h(y) > c^k$$

for every positive harmonic function h in D, where $c = c(c_L) > 0$.

We will often use "scaling" of L-diffusions analogous to the space-time scaling of Brownian motion. The resulting diffusion corresponds to a different operator than L, say $\widetilde{L}$, but the bound c_L in (1.4) remains valid for $\widetilde{L}$.

Lemma 2.1. *(i) Suppose that $L \in \mathcal{D}$, A_1 is a compact subset of a domain D, $0 \in D \setminus A_1$, and for $r > 0$,*

$$A_1^r = \{x : x/r \in A_1\}$$
$$D^r = \{x : x/r \in D\}.$$

Then, for $A \subset A_1^r$, $r > 0$,

$$c_1 \frac{\mathrm{Cap}(A)}{\mathrm{Cap}(A_1^r)} \le P^0(T(A) < T(\partial D^r)) \le c_1^{-1} \frac{\mathrm{Cap}(A)}{\mathrm{Cap}(A_1^r)}, \tag{2.3}$$

where $\mathrm{Cap} = \mathrm{Cap}_\Delta^{D^r}$ *and* $c_1 > 0$ *depends only on* c_L, D *and* A_1.

(ii) Suppose that $L \in \mathcal{D} \cup \mathcal{ND}$ *and for some* $A \subset A_1^r$, $r > 0$,

$$\mathrm{Vol}(A)/\mathrm{Vol}(A_1^r) \ge c_2.$$

Then

$$P^0(T(A) < T(\partial D^r)) \ge c_3$$

where $c_3 > 0$ *depends only on* c_L, c_2, D *and* A_1.

Proof. We will give a proof only for $D = B(0,3)$ and $A_1 = B(0,2) \setminus B(0,1)$. Other cases may be treated analogously.

(i) We will follow Lemma 4.2 of Bass and Burdzy (1991a) closely. By scaling, we may assume that $r = 1$.

Let μ be the capacitory measure for A in $B(0,3)$. Then, using (2.1),

$$\begin{aligned} G^L_{B(0,3)}\mu(0) &= \int_A G^L_{B(0,3)}(0,y)\mu(dy) \\ &\ge \mu(A) \inf_{y \in A_1} G^L_{B(0,3)}(0,y) \\ &\ge \mu(A)\, c_4 \inf_{y \in A_1} G^\Delta_{B(0,3)}(0,y) \\ &\ge c_5 \mu(A) = c_5 \mathrm{Cap}_L^{B(0,3)}(A) \ge c_6 \mathrm{Cap}_\Delta^{B(0,3)}(A). \end{aligned}$$

By the strong Markov property,

$$\begin{aligned} G^L_{B(0,3)}\mu(0) &= \int_A G^L_{B(0,3)}\mu(y) P^0(T_A < T(\partial B(0,3)), X(T_A) \in dy) \\ &\le \sup_{y \in A} G^L_{B(0,3)}\mu(y) P^0(T(A) < T(\partial B(0,3))) \\ &\le P^0(T(A) < T(\partial B(0,3))). \end{aligned}$$

This and the previous inequality prove the first inequality in (2.3).

The function

$$x \to P^x(T(A) < T(\partial B(0,3)))$$

is a potential corresponding to a measure ν supported on A with mass less than or equal to $\mathrm{Cap}_L^{B(0,3)}(A)$, since the function is bounded by 1. Thus

$$\begin{aligned} P^0(T(A) < T(\partial B(0,3))) &= \int_A G^L_{B(0,3)}(0,y)\nu(dy) \\ &\le \sup_{y\in A} G^L_{B(0,3)}(0,y)\nu(A) \\ &\le c_7 \sup_{y\in A} G^\Delta_{B(0,3)}(0,y)\mathrm{Cap}_L^{B(0,3)}(A) \\ &\le c_8 \mathrm{Cap}_\Delta^{B(0,3)}(A) \end{aligned}$$

and (2.3) is proved.

(ii) For the case $L \in \mathcal{ND}$, see Krylov and Safonov (1979).

In the case $L \in \mathcal{D}$, the estimate follows from part (i) of the lemma and the following lower bound on capacity in terms of volume (see the remark following Lemma 4.2 in Bass and Burdzy (1991a)): $\mathrm{Cap}_\Delta(A) \ge c(\mathrm{Vol}(A))^\beta$. Sidney Port (private communication) pointed out that we can take $\beta = (n-2)/n$ for $n \ge 3$ and $\beta > 1/2$ for $n = 2$. $\square$

Lemma 2.2. *If $L \in \mathcal{D} \cup \mathcal{ND}$ then*

$$E^0 T(\partial B(0,r)) < c_1 r^2$$

for $r > 0$, where $c_1 > 0$ depends only on c_L.

Proof. If $L \in \mathcal{D}$ then

$$\begin{aligned} E^0 T(\partial B(0,r)) &\le \int_{B(0,r)} G^L_{B(0,2r)}(0,x)\,dx \\ &\le c_2 \int_{B(0,r)} G^\Delta_{B(0,2r)}(0,x)\,dx = c_3 r^2. \end{aligned}$$

The case $L \in \mathcal{ND}$ is discussed in Lemma 5.1 of Bass and Pardoux (1987). $\square$

Lemma 2.3. *Suppose that* $L \in \mathcal{D}$,

$$D = \{x \in \mathbb{R}^n : |\widetilde{x}| < 1, |x_n| < 1\},$$

$$M = \{x \in \partial D : |\widetilde{x}| < 1/2, x_n = 1\}.$$

Then, for $x \in B(0, 1/2)$,

$$P^x(T(M) \le T(\partial D)) > c_1 > 0$$

where c_1 *depends only on* c_L.

Proof. Let $M_1 = \{x \in M : |\widetilde{x}| < 1/4\}$, $D_1 = \{x \in \mathbb{R}^n : |\widetilde{x}| < 1/2, -1 < x_n < 2\}$. Since $\mathrm{Cap}_\Delta^{D_1}(M_1) > 0$, Lemma 2.1 shows that

$$P^0(T(M_1) < T(\partial D_1)) > c_2 > 0,$$

and, by the Harnack principle,

$$P^x(T(M_1) < T(\partial D_1)) > c_3 > 0$$

for all $x \in B(0, 1/2)$. To complete the proof, observe that the last probability is less than or equal to $P^x(T(M) < T(\partial D))$. $\square$

Now we introduce L^p-domains and twisted L^p-domains.

Definition 2.1. An open connected set $D \subset \mathbb{R}^n$ will be called an L^p-domain if there exist a constant $a > 0$, a finite family of orthonormal coordinate systems $CS_1, CS_2, \dots, CS_k$, reals $r_1, r_2, \dots, r_k$, and functions $f_1, f_2, \dots, f_k : \mathbb{R}^{n-1} \to (-\infty, 0]$ with the following three properties.

(i) $f_j \in L^p$ for every j;

(ii)

$$U_j \stackrel{\text{df}}{=} \{x \in D : |\widetilde{x}| < r_j, x_n < a \text{ in } CS_j\} = \{x \in \mathbb{R}^n : |\widetilde{x}| < r_j, f_j(\widetilde{x}) < x_n < a \text{ in } CS_j\};$$

(iii) $D = \bigcup_{j=1}^k U_j$.

The length of a rectifiable curve γ will be denoted $\ell(\gamma)$.

Definition 2.2. An open connected set $D \in \mathbb{R}^n$ will be called a twisted L^p-domain if for some base point $z \in D$, some constants $c_1, c_2 \in (0, \infty)$, and every $x \in D$ there exists a Jordan arc γ with endpoints x and z such that

(i) $\text{dist}(\gamma, \partial D) > c_1 \text{dist}(x, \partial D)$, and
(ii) $\ell(\gamma) < c_2 (\text{dist}(x, \partial D))^{(1-n)/p}$.

It is elementary to see that every L^p-domain is a twisted L^p-domain, although not, in general, a uniformly regular one.

Example 2.1. Let $m(k) = [k^{p/(n-1)} + 1]$ and consider the following domain.

$$D = \{x \in \mathbb{R}^n : |\widetilde{x}| < 1, x_n > 0\} \setminus \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{m(k)} \{x : |\widetilde{x}| \geq k^{p/(1-n)}, x_n = k + jk^{p/(1-n)}\}.$$

It is easy to check that D is a uniformly regular twisted L^p-domain. According to Theorem 1.1, if $L \in \mathcal{D}$ and we choose $p > n - 1$, the expected lifetime of conditioned L-diffusions in D is bounded by a finite constant, despite the fact that D has infinite volume.

We turn to probabilistic estimates of harmonic functions in L^p-domains.

Lemma 2.4. *Suppose that $L \in \mathcal{D}$, $f : \mathbb{R}^{n-1} \to \mathbb{R}$ is upper semi-continuous,*

$$D_k = \{x \in \mathbb{R}^n : |\widetilde{x}| < 1, \max(-1, f(\widetilde{x})) < x_n < k\}, \quad k \geq 1,$$

$$\widetilde{D}_k = \{x \in \mathbb{R}^n : |\widetilde{x}| < 1, -1 < x_n < k\}, \quad k \geq 1,$$

and

$$M_k = \{x \in \mathbb{R}^n : |\widetilde{x}| < 1/2, x_n = k\}, \quad k \geq 1.$$

There exists $p_0 < 1$ such that if $p \geq p_0$ and

$$P^0(T(\partial D_4) = T(\partial \widetilde{D}_4)) \geq p \tag{2.4}$$

then

$$P^0(T(\partial D_k) = T(M_k)) \geq e^{-ck}$$

for $k \geq 4$, where $c \in (0, \infty)$ depends on c_L and on p_0 but does not otherwise depend on f.

Proof. Let

$$\widehat{D}_k = \{x \in \mathbb{R}^n : |\widetilde{x}| < 3/4, -1 < x_n < k\}, \quad k \geq 0,$$

$$W_k = \{x \in \widetilde{D}_k : x_n \geq k-5\}, \quad k \geq 4,$$

$$V_k = \{x \in \widehat{D}_k \cap D_k^c : x_n \geq k-2\}, \quad k \geq 3.$$

By Lemma 2.1 we have

$$\begin{aligned}
\mathrm{Cap}_\Delta^{W_4}(V_3) &\leq c_1 \mathrm{Cap}_L^{W_4}(V_3) \\
&\leq c_1 c_2 P^0(T(V_3) < T(\partial W_4)) \\
&\leq c_1 c_2 P^0(T(\partial D_4) < T(\partial \widetilde{D}_4)) \\
&\leq c_3(1-p).
\end{aligned}$$

By our assumptions on D_k and translation invariance of Cap_Δ,

$$\mathrm{Cap}_\Delta^{W_{k+1}}(V_k) \leq c_3(1-p)$$

for all $k \geq 3$. Lemma 2.1 yields

$$P^x(T(V_k) < T(\partial W_{k+1})) \leq c_4 \mathrm{Cap}_\Delta^{W_{k+1}}(V_k) \leq c_5(1-p) \tag{2.5}$$

for $x = (0, \dots, 0, k-3)$. The inequality also holds for all $x \in M_{k-3}$, by the Harnack principle (we may have to change the constant c_5).

Let θ denote the usual shift operator for Markov processes.

$$A_k \stackrel{\text{df}}{=} \bigcap_{m=1}^{k} \{T(\partial D_m) = T(M_m), T(\partial \widehat{D}_{m-1}) \circ \theta_{T(M_m)} > T(\partial \widehat{D}_k)\}.$$

We will prove inductively that

$$P^0(A_{k+1}) > c_6 P^0(A_k)$$

for some $c_6 > 0$ and all $k \geq 2$, provided $p > p_0$.

We start with $k = 2$ and 3. By Lemma 2.3, and the strong Markov property applied at $T(M_m)$,

$$P^0\left(\bigcap_{m=1}^{4}\{T(\partial\widehat{D}_m) = T(M_m), T(\partial\widehat{D}_{m-1} \circ \theta_{T(M_m)} > T(\partial\widehat{D}_4)\}\right) > c_7.$$

It follows that if $1 - p < c_7/2$, then in view of (2.4),

$$P^0(A_{k+1}) > c_7/2 > (c_7/2)P^0(A_k)$$

for $k = 2, 3$.

By the strong Markov property applied at $T(M_{k+1})$ and Lemma 2.3,

$$P^0(A_{k+1} \cap \{T(\partial\widehat{D}_{k+2}) = T(M_{k+2})\} \cap \{T(\partial\widehat{D}_k) \circ \theta_{T(M_{k+1})} > T(\partial\widehat{D}_{k+2})\}) \geq c_8 P^0(A_{k+1}).$$

First choose $c_6 \in (0, c_7/2)$ small and then $p_0 < 1$ large so that for $p \geq p_0$ we have

$$c_8 - c_5(1-p)c_6^{-2} > c_6.$$

Now suppose that $k \geq 3$ and $P^0(A_{m+1}) > c_6 P^0(A_m)$ for all $m \leq k$. By (2.5),

$$\begin{aligned}
P^0(A_{k+2}) &\geq P^0(A_{k+1} \cap \{T(\partial\widehat{D}_{k+2}) = T(M_{k+2})\} \cap \{T(\partial\widehat{D}_k) \circ \theta_{T(M_{k+1})} > T(\partial\widehat{D}_{k+2})\}) \\
&\qquad - P^0(A_{k-1} \cap \{T(V_{k+2}) \circ \theta_{T(M_{k-1})} < T(W_{k+3}) \circ \theta_{T(M_{k-1})}\}) \\
&\geq c_8 P^0(A_{k+1}) - c_5(1-p)P^0(A_{k-1}) \\
&\geq c_8 P^0(A_{k+1}) - c_5(1-p)c_6^{-2} P^0(A_{k+1}) \\
&= P^0(A_{k+1})[c_8 - c_5(1-p)c_6^{-2}].
\end{aligned}$$

Then $P^0(A_{k+2}) > c_6 P^0(A_{k+1})$ which finishes the inductive argument. We conclude that

$$P^0(T(\partial D_k) = T(M_k)) \geq P^0(A_k) \geq c_6^{k-2} P^0(A_2)$$

and the proof is complete. $\square$

Lemma 2.5. *Suppose that $L \in \mathcal{D}$, $f : \mathbb{R}^{n-1} \to \mathbb{R}$ is upper semi-continuous,*

$$z^k = (0, 0, \ldots, 0, k - 1/2), \quad k \geq 1,$$

$$D_k = \{x \in \mathbb{R}^n : |\widetilde{x}| < 1, \max(-4, \min(f(\widetilde{x}), k - 1)) < x_n < k\}, \quad k \geq 1,$$

$$\widetilde{D}_k = \{x \in \mathbb{R}^n : |\widetilde{x}| < 1, -4 < x_n < k\}, \quad k \geq 1,$$

$$M_k = \{x \in \mathbb{R}^n : |\widetilde{x}| < 1/2, x_n = k\}, \quad k \geq 1,$$

and

$$N \subset M_1.$$

For each $q > 0$ there is $p_0 < 1$ such that if $p > p_0$,

$$P^0(T(\partial D_4) = T(\partial \widetilde{D}_4)) \geq p$$

and

$$P^0(T(N) = T(\partial D_1 \cup \partial D_{-1})) \geq q,$$

then

$$P^{z^k}(T(N) < T(\partial D_k)) \geq e^{-ck}$$

for all $k \geq 4$ where $c \in (0, \infty)$ depends only on c_L, q and p_0 but does not otherwise depend on f.

Proof. Let

$$W_k = \{x \in D_k : x_n \geq k - 5\}, \quad k \geq -1,$$

$$V_k = \{x \in \widetilde{D}_k \cap D_k^c : x_n \geq k-2\}, \quad k \geq -1.$$

We can show as in the previous proof that

$$\mathrm{Cap}_\Delta^{W_1}(V_{-1}) \leq \mathrm{Cap}_\Delta^{W_0}(V_{-1}) \leq c_1(1-p)$$

and, therefore,

$$\mathrm{Cap}_\Delta^{W_3}(V_1) \leq c_1(1-p) \tag{2.6}$$

and $\mathrm{Cap}_\Delta^{W_{k+1}}(V_k) \leq c_1(1-p)$ for $k \geq -1$. As in the proof of Lemma 2.4, we may show that

$$P^{z^k}(T(M_2) < T(M_0) < T(\partial D_k), T(\partial D_3) \circ \theta_{T(M_2)} > T(M_0)) \geq e^{-c_2 k}.$$

By the Harnack principle,

$$P^x(T(N) = T(\partial D_1 \cup \partial D_{-1})) \geq c_3 q$$

for all $x \in M_0$. Let

$$\begin{aligned} B_k = \{&T(M_2) < T(M_0) < T(N) = T(\partial D_1 \cup \partial D_{-1}) \circ \theta_{T(M_0)} < T(\partial D_k)\} \\ &\cap \{T(\partial D_3) \circ \theta_{T(M_2)} > T(M_0)\}. \end{aligned}$$

Then the strong Markov property applied at $T(M_0)$ yields

$$\begin{aligned} P^{z^k}(B_k) &\geq \int_{M_0} P^x(T(N) = T(\partial D_1 \cup \partial D_{-1}))\times \\ &\quad \times P^{z^k}(T(M_2) < T(M_0) < T(\partial D_k), T(\partial D_3) \circ \theta_{T(M_2)} > T(M_0), X(T(M_0)) \in dx) \\ &\geq \int_{M_0} c_3 q P^{z^k}(T(M_2) < T(M_0) < T(\partial D_k), T(\partial D_3) \circ \theta_{T(M_2)} > T(M_0), X(T(M_0)) \in dx) \\ &\geq e^{-c_2 k} c_3 q. \end{aligned}$$

Lemma 2.1 and (2.6) imply that

$$P^x(T(V_1) < T(\partial W_3)) \leq c_4(1-p)$$

for all $x \in M_2$. Then

$$P^{z^k}(T(N) < T(\partial D_k)) \geq P^{z^k}(B_k) - P^{z^k}(\{T(V_1) \circ \theta_{T(M_2)} < T(\partial W_3) \circ \theta_{T(M_2)}\} \cap B_k)$$
$$\geq e^{-c_2 k}(c_3 q - c_4(1-p)).$$

Now it remains to choose q, p_0 and c in an appropriate way. $\square$

Lemma 2.6. *Suppose that*

(i) $L \in \mathcal{D}$ *and* D *is an* L^p*-domain; or*

(ii) $L \in \mathcal{D}$ *and* D *is a uniformly regular twisted* L^p*-domain; or*

(iii) $L \in \mathcal{ND}$ *and* D *is a strongly uniformly regular twisted* L^p*-domain.*

Let h *be a positive harmonic function in* D *and*

$$U_k = \{x \in D : h(x) \in [2^k, 2^{k+1}]\}, \quad k \in \mathbb{Z}.$$

For some $\varepsilon > 0$ *let* $r = (1-\varepsilon)p/(n-1+p)$*. Then there exist* $c > 0$ *and* $k_0 > 0$ *such that*

$$P^x(T(\partial U_k) < T(B(x, |k|^{-r}))) > c \tag{2.7}$$

for all $|k| > k_0$ *and all* $x \in U_k$*.*

Proof. (i) Consider the following special L^p-domain. Suppose that f is an L^p-function, $f \leq 0$, and

$$D = \{x \in \mathbb{R}^n : |\widetilde{x}| < 1, f(\widetilde{x}) < x_n < 1\}.$$

Assume that $y \in D$, $y_n < 0$, and let

$$D_1 = \{x \in \mathbb{R}^n : |\widetilde{x} - \widetilde{y}| < |k|^{-r}/8, f(\widetilde{x}) < x_n < 1\},$$

$$D_2 = \{x \in \mathbb{R}^n : |\widetilde{x} - \widetilde{y}| < |k|^{-r}/8, |x_n - y_n| < |k|^{-r}/2\},$$

$$D_3 = D_1 \cap D_2,$$

$$M = \{x \in \mathbb{R}^n : |\widetilde{x} - \widetilde{y}| < |k|^{-r}/16, x_n = 1/2\}.$$

Suppose that

$$P^y(T(\partial D) < T(B(y, |k|^{-r}))) < 1 - p_0. \tag{2.8}$$

Then

$$P^y(T(\partial D_2) = T(\partial D_3)) > p_0 \tag{2.9}$$

and, by Lemma 2.4 and scaling,

$$P^y(T(M) < T(\partial D_1)) \geq \exp(-c_1(1 - y_n)/(|k|^{-r}/8)). \tag{2.10}$$

It follows from (2.9) and Lemma 2.1 (ii) that

$$\mathrm{Vol}(D^c \cap D_2) < c_2(|k|^{-r})^n,$$

where by taking p_0 sufficiently close to 1 we may suppose c_2 is small. Let us choose $p_0 \in (0, 1)$ large enough so that

$$\mathrm{Vol}(D^c \cap D_2) < c_3|k|^{-rn} \stackrel{\mathrm{df}}{=} \mathrm{Vol}(D_2)/2$$

and, therefore,

$$\mathrm{Vol}(D \cap D_2) > c_3|k|^{-rn}/2.$$

This and the fact that the lower boundary of D is the graph of a function imply that the $(n-1)$-dimensional volume of the set

$$\{x \in \partial D_2 : x_n = y_n + |k|^{-r}/2, f(\widetilde{x}) \leq y_n + |k|^{-r}/2\}$$

is greater than or equal to

$$(c_3|k|^{-rn}/2)/|k|^{-r}.$$

Since the function f belongs to L^p,

$$\int_{|\widetilde{x}-\widetilde{y}|<|k|^{-r}/8} |1-f(\widetilde{x})+|k|^{-r}/2|^p d\widetilde{x} < c_4 < \infty,$$

where c_4 does not depend on y. It follows that

$$(1-y_n)^p[(c_3|k|^{-rn}/2)/|k|^{-r}] < c_4,$$

so

$$1-y_n < c_5|k|^{r(n-1)/p},$$

and thus

$$(1-y_n)/|k|^{-r} \le c_5|k|^{r(n-1)/p}|k|^r = c_5|k|^{1-\varepsilon}. \tag{2.11}$$

A standard application of the Harnack principle shows that $h(x) \ge c_6|k|^{-c_7r}$ for all $x \in M$, since such points x may be connected with $(0,\dots,0,1/2)$ by a chain of balls in D of length $c_8 \log|k|^{-r}$.

The strong Markov property, (2.10) and (2.11) yield

$$\begin{aligned} h(y) &\ge c_6|k|^{-c_7r}P^y(T(M) < T(\partial D_1)) \\ &\ge c_6|k|^{-c_7r}\exp(-c_8|k|^{1-\varepsilon}) \\ &> 2^{-|k|+1} \end{aligned}$$

if $|k|$ is sufficiently large. It follows that $y \notin U_{-k}$, for large $k>0$. Thus, if we assume that $y \in U_{-k}$ for large $k>0$, then (2.8) must fail and (2.7) holds with $c=(1-p_0)/2$.

Now suppose that

$$P^y(T(\partial U_k) < T(B(y,|k|^{-r}))) < 1-p_0; \tag{2.12}$$

(this is a slight modification of (2.8)). Then again we have (2.9), i.e.,

$$P^y(T(\partial D_2) = T(\partial D_3)) > p_0.$$

By Lemma 2.3 and scaling, we have

$$P^y(T(M_1) = T(\partial D_2)) > c_9 > 0$$

where

$$M_1 = \{x \in \partial D_2 : |\widetilde{x} - \widetilde{y}| < |k|^{-r}/16, x_n = y_n + |k|^{-r}/2\},$$

and c_9 depends only on c_L. Given $p_0 \in (0,1)$ sufficiently large,

$$P^y(T(M_1) = T(\partial D_3)) > c_9/2 = c_{10}.$$

If we had

$$P^y(h(X(T(M_1))) < 2^{|k|}, T(M_1) = T(\partial D_3)) > c_{10}/2$$

then we would have

$$P^y(T(\partial U_k) < T(B(y, |k|^{-r}))) > c_{10}/2.$$

If p_0 is chosen so that $1 - p_0 < c_{10}/2$, then, by (2.12), the last inequality would be impossible. Therefore we must have had

$$P^y(h(X(T(M_1))) \geq 2^{|k|}, T(M_1) = T(\partial D_3)) > c_{10}/2.$$

In other words, if $M_2 = \{x \in M_1 : h(x) \geq 2^{|k|}\}$, then

$$P^y(T(M_2) = T(\partial D_3)) > c_{10}/2.$$

Let y^1 be defined by $\widetilde{y} = \widetilde{y}^1$, $y^1_n = 1/2$. By Lemma 2.5, scaling, and (2.11),

$$\begin{aligned} P^{y^1}(T(M_2) < T(\partial D_1)) &\geq \exp(-c_{11}(1 - y_n)/(|k|^{-r}/8)) \\ &\geq \exp(-c_{12}|k|^{1-\varepsilon}). \end{aligned}$$

We obtain

$$h(y^1) \geq 2^{|k|} P^y(T(M_2) < T(\partial D_1))$$
$$\geq 2^{|k|} \exp(-c_{12}|k|^{1-\varepsilon}). \tag{2.13}$$

However, $h(y^1) \leq c_{13}|k|^{-c_{14}r}$, by the chain of balls argument. This contradicts (2.13) for large $|k|$. Therefore, if $y \in U_k$ and $k > 0$ is large then (2.7) holds with $c = (1-p_0)/2$.

Points $y \in D$ with $y_n \geq 0$ may be treated analogously. The proof extends to general L^p-domains by a localization argument.

(ii) Now we turn our attention to uniformly regular twisted L^p-domains and $L \in \mathcal{D}$. We will consider two cases.

First, suppose that $x \in D$ and $\operatorname{dist}(x, \partial D) \leq |k|^{-r}/3$. Then there exists a point $y \in \partial D$ with $|x - y| = |k|^{-r}/2$ and we have, by uniform regularity,

$$\operatorname{Cap}_L^{B(y,|k|^{-r}/2)}(B(y, |k|^{-r}/6) \cap D^c) > c_{15} > 0.$$

It follows easily that

$$\operatorname{Cap}_L^{B(x,|k|^{-r})}(B(y, |k|^{-r}/6) \cap D^c) > c_{16} = c_{16}(c_{15}) > 0$$

and, by Lemma 2.1 (i),

$$P^x(T(\partial U_k) < T(B(x, |k|^{-r}))) \geq P^x(T(D^c) < T(B(x, |k|^{-r}))) > c_{17} > 0.$$

It remains to consider the case when $\operatorname{dist}(x, \partial D) \stackrel{\text{df}}{=} d > |k|^{-r}/3$. Let $z \in D$ be a base point. There exists a Jordan arc γ connecting x and z in D such that $\operatorname{dist}(\gamma, \partial D) > c_{18}d$ and $\ell(\gamma(x, z)) < c_{19}d^{(1-n)/p}$.

Let j be the largest integer not greater than $\ell(\gamma(x, z))/(c_{18}d)$. Then $j < c_{20}d^{(1-n)/p-1}$.

Let $y^0 = x, y^1, \ldots, y^j, y^{j+1} = z$ be the points on γ such that $\ell(\gamma(x, y^m)) = mc_{18}d/2$. The balls with centers y^m and radii $c_{18}d$ form a "chain of balls connecting x and z" and, therefore,

$$h(x) \leq h(z)c_{21}^j \leq h(z)c_{21}^{c_{22}|k|^{1-\varepsilon}} \leq 2^{|k|}$$

for large $|k|$. Thus, $x \notin U_k$ for large $k > 0$ and similarly, $x \notin U_{-k}$. This completes the proof of part (ii) of the lemma.

(iii) Part (iii) may be proved exactly like part (ii) except that we have to use volumes rather than capacities and Lemma 2.1 (ii) instead of Lemma 2.1 (i). $\square$

Remark 2.1. We will need in Section 5 the following extension of Lemma 2.6. Let $\widehat{U}_k = \{x \in D : h(x) \le 2^{k+1}\}$. Then the above proof shows that Lemma 2.6 holds for U_k replaced with $\widehat{U}_k$ and all $k < -k_0$.

3. Twisted Hölder domains. We start with a number of completely elementary results on twisted Hölder domains which are needed in this paper and its companion—Bass and Burdzy (1991a). We introduce the class of twisted Hölder domains as a natural generalization of Hölder domains.

Twisted Hölder domains have, by definition, canals no longer and no thinner than Hölder domains, but do not have to have their boundaries representable as graphs of functions.

A bounded domain $D \subset \mathbb{R}^n$ is called a Hölder domain of order α if every point $x \in \partial D$ has a neighborhood U such that $U \cap \partial D$ may be represented in some orthonormal coordinate system (depending on x) as the graph of a Hölder function with exponent α.

For a rectifiable Jordan arc γ and $x, y \in \gamma$, we denote the length of the piece of γ between x and y by $\ell(\gamma(x, y))$.

Definition 3.1. A bounded domain $D \subset \mathbb{R}^n$, $n \ge 2$, will be called a twisted Hölder domain of order α, $\alpha \in (0, 1]$, if there exist constants $c_1, \dots, c_5 \in (0, \infty)$, a point $z \in D$ and a continuous function $\delta : D \to (0, \infty)$ with the following properties.

(i) $\delta(x) \le c_1(\mathrm{dist}(x, \partial D))^\alpha$ for all $x \in D$;

(ii) for every $x \in D$ there exists a rectifiable Jordan arc γ connecting x and z in D and such that

$$\delta(y) \ge c_2(\ell(\gamma(x, y)) + \delta(x))$$

for all $y \in \gamma$;

(iii) $\mathrm{Cap}(B(x,c_3a)\cap F_a^c)/\mathrm{Cap}(B(x,c_3a)) \ge c_4$ for all $x\in F_a$ and $a\le c_5$, where $F_a=\{y\in D:\delta(y)\le a\}$ and $\mathrm{Cap}=\mathrm{Cap}_\Delta^{B(x,2c_3a)}$.

Remarks 3.1. (i) The term "Hölder domains" has been used to denote related but different classes of domains (Smith and Stegenga (1990), Bañuelos (1991)).

(ii) Condition (iii) of Definition 3.1 is a very mild version of uniform regularity. The main theorems on twisted Hölder domains of this article and Bass and Burdzy (1991a) seem to be false without this assumption. The counterexamples are complicated and will be omitted.

Our first result is a rigorous counterpart of the heuristic idea that "twisted Hölder domains have canals no longer and no thinner than Hölder domains."

Proposition 3.1. *Suppose that $D\subset\mathbb{R}^n$, $n\ge 2$, is a bounded domain and there exist $\alpha\in(0,1]$, $c_1,c_2,c_3,c_4\in(0,\infty)$ and $z\in D$ with the following properties.*

(i) For each $x\in D$ there exist $b>0$ and a rectifiable Jordan arc γ connecting x and z in D and such that for all $y\in\gamma$

$$\mathrm{dist}(y,\partial D)\ge c_1(b+\ell(\gamma(x,y)))^{1/\alpha}. \tag{3.1}$$

Let $\delta(x)$ be the supremum of b's which satisfy (3.1) and let $F_a=\{y\in D:\delta(y)\le a\}$.

(ii)

$$\mathrm{Cap}(B(x,c_2a)\cap F_a^c)/\mathrm{Cap}(B(x,c_2a))\ge c_3$$

for all $x\subset F_a$, $a\le c_4$, where $\mathrm{Cap}=\mathrm{Cap}_\Delta^{B(x,2c_2a)}$.

Then the domain D is a twisted Hölder domain of order α and δ satisfies Definition 3.1.

Remark 3.2. If (3.1) is satisfied only by $b=0$ for some $x\in D$, then replace c_1 by $c_1/2$. As a result, the corresponding δ will be always strictly positive.

Proof. It will suffice to show that δ is a continuous function satisfying conditions (i) and (ii) of Definition 3.1. Condition (iii) holds by assumption.

For a fixed $x \in D$, the lengths of γ's satisfying (3.1) are bounded away from 0 and ∞ and, therefore, by compactness, there is a Jordan arc γ_0 connecting x and z in D and such that

$$\operatorname{dist}(y, \partial D) \geq c_1(\delta(x) + \ell(\gamma_0(x, y)))^{1/\alpha} \tag{3.2}$$

for all $y \in \gamma_0$. Now let $y \in \gamma_0$ and let v be a point on γ_0 between y and z. By (3.2),

$$\begin{aligned}\operatorname{dist}(v, \partial D) &\geq c_1(\delta(x) + \ell(\gamma_0(x, v)))^{1/\alpha} \\ &= c_1(\delta(x) + \ell(\gamma_0(x, y)) + \ell(\gamma_0(y, v)))^{1/\alpha}.\end{aligned}$$

It follows that (3.1) is satisfied for y in place of x if $\gamma = \gamma_0(y, z)$ and $b = \delta(x) + \ell(\gamma_0(x, y))$. Hence,

$$\delta(y) \geq \delta(x) + \ell(\gamma_0(x, y)) \tag{3.3}$$

and condition (ii) of Definition 3.1 is verified.

By taking $y = x$ in (3.2) we have

$$\operatorname{dist}(x, \partial D) \geq c_1(\delta(x))^{1/\alpha}$$

which implies condition (i) of Definition 3.1.

It remains to show that δ is continuous.

Fix some $x \in D$ and let γ_0 be the curve satisfying (3.2) for x. For $y \in D$ with $|x - y| < \operatorname{dist}(x, \partial D)$ let

$$b_1 = (\delta(x)^{1/\alpha} - |x - y|/c_1)^\alpha - |x - y|.$$

Let γ_1 consist of γ_0 and γ_2, the latter being the line segment joining y and x. Since $\delta(x) \geq b_1 + |x - y|$, we have for $v \in \gamma_0$, by (3.2),

$$\begin{aligned}\operatorname{dist}(v, \partial D) &\geq c_1(\delta(x) + \ell(\gamma_0(x, v)))^{1/\alpha} \\ &\geq c_1(b_1 + |x - y| + \ell(\gamma_0(x, v)))^{1/\alpha} \\ &\geq c_1(b_1 + \ell(\gamma_1(y, v)))^{1/\alpha}.\end{aligned} \tag{3.4}$$

Since $\mathrm{dist}(x,\partial D) \geq c_1(\delta(x))^{1/\alpha}$ and by our choice of b_1, we have for $v \in \gamma_2$,

$$\begin{aligned}
\mathrm{dist}(v,\partial D) &\geq \mathrm{dist}(x,\partial D) - |x-y| \\
&\geq c_1(\delta(x))^{1/\alpha} - |x-y| \\
&= c_1(b_1 + |x-y|)^{1/\alpha} \\
&\geq c_1(b_1 + \ell(\gamma_1(y,v)))^{1/\alpha}.
\end{aligned}$$

This and (3.4) show that (3.1) is satisfied for y if we take $\gamma = \gamma_1$ and $b = b_1$. Thus

$$\delta(y) \geq b_1 = (\delta(x)^{1/\alpha} - |x-y|/c_1)^\alpha - |x-y|. \tag{3.5}$$

As a result we have

$$\liminf_{\substack{|x-y|\to 0 \\ |x-y|<\mathrm{dist}(x,\partial D)}} (\delta(y) - \delta(x)) \geq 0$$

which clearly implies the continuity of δ. $\square$

There are several types of domains in the literature which are candidates for the name of twisted Lipschitz domain. We recall now their names and definitions, following Bañuelos (1987) and Smith and Stegenga (1990).

A bounded domain D is called a *John domain* provided there exist $z \in D$ and $c > 0$ such that for every $x \in D$ there is an arc γ connecting x and z in D and satisfying

$$\mathrm{dist}(y,\partial D) \geq c\,\ell(\gamma(x,y)) \qquad \text{for all } y \in \gamma.$$

A bounded domain D is called a *uniform domain* if there exist $c_1, c_2 < \infty$ such that every pair of points $x, y \in D$ may be joined by an arc γ in D with

$$\ell(\gamma(x,y)) \leq c_1|x-y|,$$

$$\min(\ell(\gamma(x,z)), \ell(\gamma(z,y))) \leq c_2 \mathrm{dist}(z,\partial D) \qquad \text{for all } \ z \in \gamma.$$

A bounded domain is called a *non-tangentially accessible (NTA) domain* if there exist $M > 1$ and $r_0 > 0$ such that

(i) for every $x \in \partial D$ and $r < r_0$ there is $y \in D$ such that $|x-y| < Mr$ and $B(x, r/M) \subset D$;

(ii) property (i) holds for D^c in place of D;

(iii) for every $c > 0$ there is an N such that if $0 < \varepsilon < r_0$, $x^1, x^2 \in D$ with $\text{dist}(x^k, \partial D) > \varepsilon$ for $k = 1, 2$ and $|x^1 - x^2| < c\varepsilon$, then there exists a sequence of N points $z^1 = x^1, z^2, z^3, \dots, z^N = x^2$ such that $|z^j - z^{j+1}| < \varepsilon/M$ and $B(z^j, 2\varepsilon/M) \subset D$ for all j.

It is well known (and quite elementary to prove) that every NTA domain and every uniform domain is a John domain. We are going to show that John domains are the same as twisted Hölder domains of order 1. This means that all results on twisted Hölder domains of order 1, e.g., Theorems 1.1 (i) (C) and 1.2 automatically hold for uniform, NTA and John domains.

Proposition 3.2. *The classes of John domains and twisted Hölder domains of order 1 are identical.*

Proof. If D is a twisted Hölder domain of order 1 then by Definition 3.1, for each $x \in D$ we have an arc γ connecting x and z with

$$\begin{aligned}\text{dist}(y, \partial D) &\geq c_1^{-1}\delta(y)\\ &\geq c_1^{-1}c_2(\ell(\gamma(x,y)) + \delta(x))\\ &\geq c_1^{-1}c_2\ell(\gamma(x,y))\end{aligned}$$

for $y \in \gamma$, which shows that D is a John domain.

Now assume that D is a John domain. Then there is a constant $c_3 > 0$ and for each $x \in D$ there is an arc γ connecting x and z with

$$\text{dist}(y, \partial D) \geq c_3\ell(\gamma(x,y))$$

for all $y \in \gamma$. This implies that

$$\text{dist}(y, \partial D) \geq (c_3/2)(b + \ell(\gamma(x,y))) \qquad \text{for all } \ y \in \gamma,$$

for some $b > 0$ and, therefore, condition (i) of Proposition 3.1 is satisfied. It remains to verify hypothesis (ii) of Proposition 3.1.

Fix some $x \in D$, $a > 0$, and let γ_0 be defined as in the proof of the last proposition. Let $y \in \gamma_0 \cap \partial B(x, 2a)$. Then, by (3.3),

$$\delta(y) \geq \delta(x) + \ell(\gamma_0(x, y)) \geq \delta(x) + 2a.$$

For $|v - y| < \operatorname{dist}(y, \partial D)$ we have by (3.5)

$$\delta(v) \geq \delta(y) - c_2|v - y|$$

for some $c_2 > 0$. It follows that

$$\delta(v) \geq 2a - c_2|v - y|$$

and $B(y, a/2c_2) \cap F_a = \emptyset$. Hence

$$\operatorname{Cap}(B(x, 2a) \cap F_a^c)/\operatorname{Cap}(B(x, 2a)) \geq \operatorname{Cap}(B(x, 2a) \cap B(y, a/2c_2))/\operatorname{Cap}(B(x, 2a)).$$

It is easy to see that the last expression is greater than some $c_3 > 0$ and this completes the proof. $\square$

Twisted Hölder domains have to satisfy condition (iii) of Definition 3.1, which does not have a counterpart in the definition of a Hölder domain. For this reason, the next result is not completely obvious.

Proposition 3.3. *Every Hölder domain of order α is a twisted Hölder domain of order α.*

Proof. We will leave some of the elementary details of this proof to the reader.

Suppose that D is a Hölder domain of order α. By compactness, ∂D may be covered by a finite number of open cylinders such that ∂D can be represented as the graph of a

Hölder function in each of them. It will suffice to consider only one of these cylinders, say U. Assume without loss of generality that in some orthonormal coordinate system

$$U \cap D = \{x \in \mathbb{R}^n : |\widetilde{x}| < c_1, f(\widetilde{x}) < x_n < c_2\}$$

and $f(\widetilde{x}) < c_3 < c_2$ for $|\widetilde{x}| < c_1$ for some Hölder function f with exponent α.

For $x \in U \cap D$, let $\widehat{\delta}(x) = x_n - f(\widetilde{x})$. Fix some $z \in D \backslash U$ and for each $x \in U \cap D$ let γ be a curve connecting x and z in D such that the portion of γ lying in U consists of a vertical line segment and such that $\operatorname{dist}(\gamma \cap U^c, \partial D) > c_4 > 0$ for every $x \in U \cap D$. With such a choice of γ, (3.1) is satisfied for every $x \in U \cap D$ provided we take $b = \widehat{\delta}(x)$ and the constant c_1 in (3.1) is sufficiently small. This is, of course, a consequence of the Hölder character of the function f.

Now let δ be defined as in Proposition 3.1. We will show that condition (ii) of that proposition is satisfied. Note that $\delta(x) \geq \widehat{\delta}(x)$ for $x \in U \cap D$. Let

$$\widehat{F}_a = \{x \in U \cap D : \widehat{\delta}(x) \leq a\},$$

$$S(x) = \{y \in U \cap D : |\widetilde{y} - \widetilde{x}| < 5a/4, |y_n - x_n| < 5a/4\}.$$

Then $S(x) \subset B(x, 2a)$. Let $\lambda(\widetilde{y})$ be the 1-dimensional Lebesgue measure of $\widehat{F}_a^c \cap S(x) \cap (\widetilde{y} \times \mathbb{R})$. Note that, for a less than some $c_5 > 0$, $\lambda(\widetilde{y}) \geq 3a/2$. Hence

$$\begin{aligned}
\operatorname{Vol}(B(x, 2a) \cap \widehat{F}_a^c) &\geq \operatorname{Vol}(S(x) \cap \widehat{F}_a^c) \\
&\geq \int\limits_{\{y \in U \cap D \cap S(x)\}} \lambda(\widetilde{y}) d\widetilde{y} \\
&\geq \int\limits_{\{y \in U \cap D \cap S(x)\}} (3a/2) d\widetilde{y} \geq c_5 a^n.
\end{aligned}$$

This implies

$$\operatorname{Vol}(B(x, 2a) \cap \widehat{F}_a^c) / \operatorname{Vol}(B(x, 2a)) \geq c_6 > 0$$

and, consequently,

$$\operatorname{Cap}(B(x,2a)\cap \widehat{F}_a^c)/\operatorname{Cap}(B(x,2a)) \geq c_7 > 0. \tag{3.6}$$

Since $\widehat{\delta}(x) \leq \delta(x)$ and $\widehat{F}_a$ is defined in terms of $\widehat{\delta}$ in the same way as F_a is defined in terms of δ, (3.6) implies condition (ii) of Proposition 3.1. According to the proposition, D is a twisted Hölder domain of order α. $\square$

Lemma 3.1. *Suppose that D is a twisted Hölder domain of order α, $\alpha \in (0,1]$, and δ satisfies Definition 3.1. Fix some $z \in D$ and $a > 0$. Then there exists $c_1 = c_1(D,\delta,z,a) < \infty$ such that for every $x \in D$ there is a "chain of balls" connecting x and z (see Section 2) of length $k \leq c_1\delta(x)^{1-1/\alpha}$.*

Proof. Recall the definition of "chain of balls" given in Section 2. Suppose that γ is an arc connecting x and z and satisfying Definition 3.1. Find an integer r such that $\delta(x) \in [2^{-r}, 2^{-r+1})$. Let $y^1 = x$ and define $y^2, y^3, \dots$ inductively. Given y^{m-1} pick j so that

$$\ell(\gamma(x, y^{m-1})) + \delta(x) \in [2^{-j}, 2^{-j+1}) \tag{3.7}$$

and then pick the point y^m lying on γ between y^{m-1} and z so that

$$\ell(\gamma(y^m, y^{m-1})) = \frac{1}{2}\min(a, 2^{-j/\alpha}(c_2/c_1)^{1/\alpha}).$$

Here c_1 and c_2 are the constants in Definition 3.1. At some point the inductive procedure will have to stop because γ has a finite length (a consequence of Definition 3.1). More specifically, for some y^{m-1} we have

$$\ell(\gamma(z, y^{m-1})) \leq \frac{1}{2}\min(a, 2^{-j/\alpha}(c_2/c_1)^{1/\alpha}).$$

Then let $y^m = z, k = m$.

By Definition 3.1, for $y \in \gamma$,

$$\begin{aligned}\operatorname{dist}(y, \partial D) &\geq (\delta(y)/c_1)^{1/\alpha} \\ &\geq ((\ell(\gamma(x,y)) + \delta(x))c_2/c_1)^{1/\alpha}.\end{aligned}$$

So, using (3.7),

$$\begin{aligned}|y^m - y^{m-1}| \leq \ell(\gamma(y^m, y^{m-1})) &\leq \frac{1}{2}(2^{-j}c_2/c_1)^{1/\alpha} \\ &\leq \frac{1}{2}(\ell(\gamma(x, y^{m-1})) + \delta(x))c_2/c_1)^{1/\alpha} \\ &\leq \frac{1}{2}\operatorname{dist}(y^{m-1}, \partial D).\end{aligned}$$

A similar inequality holds with y^{m-1} replaced by y^m on the right hand side. Thus, if we choose the balls to have centers y^m and radii $\operatorname{dist}(y^m, \partial D)$, then they will satisfy the definition of a "chain of balls."

Now we will estimate k. It follows from Definition 3.1 that the length of γ is bounded by $(\operatorname{diam} D)^\alpha c_1/c_2$, so the number of m's with $\ell(\gamma(y^m, y^{m-1})) = a/2$ is not greater than $(\operatorname{diam} D)^\alpha 2c_1/ac_2 \overset{\mathrm{df}}{=} k_1$.

There are no more than

$$2 \cdot 2^{-j}/(2^{-j/\alpha}(c_2/c_1)^{1/\alpha})$$

points y^{m-1} with

$$\ell(\gamma(x, y^{m-1})) + \delta(x) \in [2^{-j}, 2^{-j+1})$$

and

$$\ell(\gamma(y^m, y^{m-1})) = \frac{1}{2}2^{-j/\alpha}(c_2/c_1)^{1/\alpha}. \tag{3.8}$$

Find an integer i such that $\delta(z) < 2^{-i}$ and recall the definition of r. The total number k_2 of points y^{m-1} satisfying (3.8) is less or equal to

$$\sum_{j=i}^{r} 2 \cdot 2^{-j}/(2^{-j/\alpha}(c_2/c_1)^{1/\alpha}) = \sum_{j=i}^{r} c_3 2^{(1/\alpha - 1)j} \leq c_4(2^{-r})^{1-1/\alpha} \leq c_4\delta(x)^{1-1/\alpha}.$$

Since δ is bounded on D,

$$k \leq k_1 + k_2 \leq k_1 + c_4 \delta(x)^{1-1/\alpha} \leq c_5 \delta(x)^{1-1/\alpha}. \quad \square$$

Lemma 3.2. *Suppose that D is a twisted Hölder domain of order α, $\alpha \in (0,1]$, and h is a positive harmonic function in D. Assume that*

(i) *$L \in \mathcal{D}$, or*

(ii) *$L \in \mathcal{ND}$ and D is strongly uniformly regular.*

For some $\varepsilon > 0$ let $r = (1-\varepsilon)/(1-1/\alpha)$ and

$$U_k = \{x \in D : h(x) \in [2^k, 2^{k+1}]\}, \quad k \in \mathbb{Z}.$$

Then there exists $k_0 > 0$ such that

$$P^x(T(\partial U_k) < T(\partial B(x, |k|^r))) > c_1 > 0$$

for all $|k| > k_0$ and all x.

Proof. Find a chain of points $y^1 = x, y^2, \ldots, y^k = z$ as in Lemma 3.1. By the Harnack principle, for some $c_1 \in (0,1)$,

$$\begin{aligned} h(x) &\geq h(z) c_1^k \\ &= h(z) \exp(k \log c_1) \\ &\geq h(z) \exp(c_2 \delta(x)^{1-1/\alpha} \log c_1) \\ &= h(z) \exp(-c_3 \delta(x)^{1-1/\alpha}) \\ &\geq \exp(-c_4 \delta(x)^{1-1/\alpha}). \end{aligned}$$

The following inequality holds for similar reasons:

$$h(x) \leq \exp(c_5 \delta(x)^{1-1/\alpha}).$$

If $x \in U_k$ and $k > 0$ then,

$$2^k \le h(x) \le \exp(c_5 \delta(x)^{1-1/\alpha}),$$

and so

$$\delta(x) \le (k \log 2/c_5)^{1/(1-1/\alpha)} \le c_6 k^{1/(1-1/\alpha)}.$$

For $k < 0$,

$$2^{k+1} \ge h(x) \ge \exp(-c_3 \delta(x)^{1-1/\alpha}),$$

and

$$\delta(x) \le (-(k+1) \log 2/c_3)^{1/(1-1/\alpha)} \le c_7 |k|^{1/(1-1/\alpha)}.$$

It follows that $U_k \subset F_a$ with

$$a \le c_8 |k|^{1/(1-1/\alpha)}.$$

Condition (iii) of Definition 3.1 and Lemma 2.1 (i) imply, for $L \in \mathcal{D}$, that

$$P^x(T(\partial F_a) < T(\partial B(x, c_9 a))) > c_{10} > 0$$

for all x. Thus, for $a_0 = c_8 |k|^{1/(1-1/\alpha)}$ and large $|k|$,

$$\begin{aligned} P^x(T(\partial U_k) < T(\partial B(x, |k|^r))) &\ge P^x(T(\partial U_k) < T(\partial B(x, c_9 a_0))) \\ &\ge P^x(T(\partial F_{a_0}) < T(\partial B(x, c_9 a_0))) > c_{10} > 0. \end{aligned}$$

The case $L \in \mathcal{ND}$ may be treated in an analogous way, using Lemma 2.1 (ii). □

Remarks 3.3. (i) By the proof of Proposition 3.3, we may omit "D is strongly uniformly regular" for $L \in \mathcal{ND}$ if D is a Hölder domain.

(ii) As in the case of Lemma 2.6 we have the following variation of Lemma 3.2. Suppose that $\widehat{U}_k = \{x \in D : h(x) \le 2^{k+1}\}$. Lemma 3.2 then holds with U_k replaced by $\widehat{U}_k$ and $k < -k_0$. The proof does not require any changes.

Lemma 3.3. *Suppose that $L \in \mathcal{D} \cup \mathcal{ND}$ and for some set U and all x we have*

$$P^x(T(U^c) < T(\partial B(x,r))) > c_1 > 0.$$

Then

$$E^x(T(U^c)) \leq c_2 r^2$$

for all x.

Proof. We have $E^x(T(\partial B(x,r))) \leq c_3 r^2$ by Lemma 2.2. Suppose we had that

$$P^x(T(U^c) > c_4 r^2) > c_5, \tag{3.9}$$

where $c_5 = 1 - c_1/2$. Then we would have

$$P^x(T(\partial B(x,r)) > T(U^c) > c_4 r^2) \geq c_1 - (1 - c_5)$$

and, therefore,

$$c_3 r^2 \geq E^x(T(\partial B(x,r))) \geq c_4 r^2 (c_1 + c_5 - 1),$$

where $c_4 = 4c_3/c_1$, a contradiction. Therefore (3.9) must be false, i.e.,

$$P^x(T(U^c) > c_4 r^2) \leq c_5 < 1.$$

By the Markov property applied at $c_4 r^2$ we have $P^x(T(U^c) > 2c_4 r^2) \leq c_5^2$ and, by induction, $P^x(T(U^c) > kc_4 r^2) \leq c_5^k$. This clearly implies $E^x(T(U^c)) \leq c_2 r^2$. $\square$

Proof of Theorem 1.1 (i). Recall that

$$U_k = \{x \in D : h(x) \in [2^k, 2^{k+1}]\}, \quad k \in \mathbb{Z},$$

for a positive harmonic function h in D.

Chung (1984) (see also Cranston (1985) and Bañuelos (1987)) showed that

$$E_h^x R \le c_1 \sum_{k=-\infty}^{\infty} \sup_{x \in U_k} E^x T(U_k^c).$$

If D is an L^p-domain or a uniformly regular L^p-domain and $p > n-1$ then let

$$\begin{aligned}
\beta &= 2p/(n-1+p) - 1, \\
\varepsilon &= 1 - (1+\beta/2)(n-1+p)/2p, \\
r &= (1-\varepsilon)p/(n-1+p).
\end{aligned}$$

Note that $\beta, \varepsilon > 0$. By Lemmas 2.6 and 3.3 we have for $L \in \mathcal{D} \cup \mathcal{ND}$,

$$E^x(T(U_k^c)) \le c_2 |k|^{-2r} = c_2 |k|^{-1-\beta/2}, \qquad |k| > k_0.$$

If D is a twisted Hölder domain of order $\alpha \in (1/3, 1]$ then let

$$\begin{aligned}
\beta &= -1 - 2/(1-1/\alpha), \\
\varepsilon &= 1 + (1+\beta/2)(1-1/\alpha)/2, \\
r &= (1-\varepsilon)/(1-1/\alpha).
\end{aligned}$$

In this case we also have $\beta, \varepsilon > 0$. Lemmas 3.2 and 3.3 imply that

$$E^x(T(U_k^c)) \le c_3 |k|^{2r} = c_3 |k|^{-1-\beta/2}, \qquad |k| > k_0.$$

Thus, under each of the assumptions (a)-(b), (A)-(C) of Theorem 1.1 (i), we have

$$E^x(T(U_k^c)) \le c_4 |k|^{-1-\beta/2}, \qquad |k| > k_0,$$

for some $\beta > 0$.

We need a similar estimate for $|k| \le k_0$. First assume that D is an L^p-domain or a twisted Hölder domain of order α. It follows easily from the definitions that D has a finite volume for $p > 1$ and any α, say $\mathrm{Vol}(D) < c_5$. Choose $c_6 < \infty$ so that $\mathrm{Vol}\, B(x, c_6) > 2c_5$. Then

$$\mathrm{Vol}(D^c \cap B(x, c_6)) > c_5$$

for all x and, according to Lemma 2.1 (ii),

$$P^x(T(D^c) < T(\partial B(x, 2c_6))) > c_7 > 0.$$

Lemma 3.3 implies that, for all k and x,

$$E^x(T(U_k^c)) \le E^x(T(D^c)) \le c_8 c_6^2 = c_9 < \infty.$$

Now suppose that D is a uniformly regular twisted L^p-domain. Then there is $c_{10} < \infty$ such that $\mathrm{dist}(x, D^c) < c_{10}$ for all x. In other words, for each $x \in D$, there is $y \in \partial D$ with $|x - y| < c_{10}$. By uniform regularity, we have for $L \in \mathcal{D}$,

$$\mathrm{Cap}_L^{B(y,2c_{10})}(B(y, c_{10}) \cap D^c) \ge c_{11} \mathrm{Cap}_\Delta^{B(y,2c_{10})}(B(y, c_{10}) \cap D^c) \ge c_{12} > 0.$$

This easily implies that, for $L \in \mathcal{D}$ and $x \in D$,

$$\mathrm{Cap}_L^{B(x,3c_{10})}(B(x, 2c_{10}) \cap D^c) \ge c_{13} > 0.$$

An application of Lemmas 2.1 (i) and 3.3 gives for $L \in \mathcal{D}$, all k and all x,

$$E^x(T(U_k^c)) \le E^x(T(D^c)) \le c_{14} < \infty.$$

The case of a strongly uniformly regular twisted L^p-domain and $L \in \mathcal{ND}$ may be handled in a similar manner using Lemmas 2.1 (ii) and 3.3.

In each case we have

$$E^x(T(U_k^c)) \le c_{15} \le c_{16}|k|^{-1-\beta/2}$$

for some $c_{16} < \infty$ and all x, $|k| \le k_0$.

It follows that

$$\begin{aligned} E_h^x R &\le c_1 \sum_{k=-\infty}^{\infty} \sup_{x \in U_k} E^x T(U_k^c) \\ &\le c_1 \sum_{k=-\infty}^{\infty} c_{17}|k|^{-1-\beta/2} < \infty. \quad \square \end{aligned}$$

Remark 3.4. Let $\widehat{U}$ be defined as in Remarks 2.1 and 3.3. These two remarks and the argument of the last proof show that

$$\sum_{k=-\infty}^{k_0} \sup_{x \in \widehat{U}_k} E^x T(\widehat{U}_k^c) < \infty.$$

Proof of Corollary 1.1. Let $\mathcal{F}_t = \sigma\{X_s, s \le t\}$. Theorem 1.1 (i) and the Markov property imply that, for all x,

$$E_h^x(\max(0, R-t) \mid \mathcal{F}_t) = E_h^{X_t} R < c_2 < \infty \qquad P_h^x\text{-a.s.}$$

Then by Dellacherie and Meyer (1980), page 193, there are $c_3 > 0$ and $c_4 < \infty$ such that

$$E_h^x \exp(c_3 R) < c_4$$

for all x. Chebyshev's inequality yields

$$P_h^x(R > t) = P_h^x(\exp(c_3 R) > \exp(c_3 t)) \le c_4 \exp(-c_3 t),$$

so

$$-\frac{1}{t} \log P_h^x(R > t) \ge -(\log c_4)/t + c_3,$$

which completes the proof. $\square$

4. Counterexamples. The counterexamples for Theorem 1.1 (ii) (A)-(B) are trivial. Let D be the interior of

$$\bigcup_{k=1}^{\infty} \{x \in \mathbb{R}^n : |\widetilde{x}| \le 1/k, -k \le x_n \le -k+1\}.$$

It is evident that D is a strongly uniformly regular L^p-domain for every $p < n-1$. Let h be the positive harmonic function in D with boundary values 0 everywhere on the Euclidean boundary ∂D and such that $h((0,0,\dots,0,a)) \to \infty$ when $a \to -\infty$. Brownian motion conditioned by h escapes to minus infinity along the thin canal constituting D. The lifetime of this process is infinite a.s., which may be proved as in Step 4 below.

One can also verify that the domains constructed in the next proof are uniformly regular twisted L^p-domains and p takes values arbitrarily close to $n-1$ when $\alpha \to 1/3$.

The counterexample announced in Theorem 1.1 (ii) (C) is fairly complicated and the rest of this section is devoted to it. For a given $\alpha \in (0, 1/3)$, we will construct a twisted Hölder domain of order α, a positive harmonic function h in D and $x \in D$ such that $R = \infty$, P_h^x-a.s. (in fact, x is irrelevant—the lifetime is infinite either for all $x \in D$ or no $x \in D$). Our example is based on an idea similar to that of Cranston and McConnell (1983) but requires a more refined construction and careful estimates. For simplicity, we will discuss the 3-dimensional case only. It is routine to extend the result to higher dimensions.

Step 1. First, we construct D. We will have to define several objects, starting with a planar curve $\widetilde{\Gamma}$. We will apply a method of Koch (see Mandelbrot (1982)).

Take the line segment joining $(0,0)$ and $(2,0)$, remove the piece between $(1,0)$ and $(1+1/k, 0)$, and replace it with a polygonal line with consecutive vertices $(1,0)$, $(1,1/k)$, $(1+1/k, 1/k)$ and $(1+1/k, 0)$. Here k is a (large) integer which will be specified later. The resulting line—which we call Γ_1—may be written as the union of $2k+2$ line segments J_m of length $1/k$ and endpoints in the lattice $\mathbb{Z}^2/k$.

Now we will construct Γ_2, Γ_3, etc. inductively. In order to obtain Γ_2, replace each of the $k+1$ line segments J_m closest to $(2,0)$ with a copy Γ_1^2 of Γ_1 shrunk k times; Γ_1^2 is translated, and rotated by the angle $\pi/2$ if necessary, so that its endpoints coincide with the endpoints of the replaced line segment. Note that Γ_2 consists of $k+1$ line segments of length $1/k$ and $2(k+1)^2$ line segments of length $1/k^2$.

Suppose that Γ_m has been constructed; it contains, among others, $2(k+1)^m$ line segments of length $1/k^m$ with endpoints in $\mathbb{Z}^2/k^m$. To obtain Γ_{m+1}, replace the half of them (i.e., the $(k+1)^m$ line segments) closest to (2, 0), each with a copy of Γ_1 shrunk k^m times, translated and possibly rotated.

The sequence $\{\Gamma_m\}_{m\geq 1}$ of curves converges to a set $\widetilde{\Gamma}$. It is easy to see that $\widetilde{\Gamma}$ is a Jordan arc connecting $(0,0)$ and $(2,0)$ and lying above $\{x : x_2 = -1/2\}$. It is the union in order, starting from $(0,0)$, of $k+1$ line segments of length $1/k$, $(k+1)^2$ line segments of length $1/k^2$, etc. These constituent line segments will be called $I_1, I_2, \dots$ and the length of I_m will be denoted $d_1(m)$.

For $d > 0$, let $\widetilde{A}(d)$ be a planar set defined by

$$\widetilde{A}(d) = ([0,3d) \times (0,d)) \setminus \Big([d,2d] \times [(100d)^{1/\alpha}, d)\Big).$$

Let $\widetilde{C}$ be the open bounded set enclosed by $\widetilde{\Gamma}$ and the polygonal line with consecutive vertices $(0,0)$, $(0,-1)$, $(2,-1)$, and $(2,0)$.

$$C \stackrel{\text{df}}{=} \widetilde{C} \times (0,1).$$

For a line segment I_m in $\widetilde{\Gamma}$, let I_m^1 be its middle part of length $d_1(m)/8 \stackrel{\text{df}}{=} d = d(m)$. Let φ_m be a composition of translation and rotation which maps $\widetilde{A}(d(m))$ onto a set $\widetilde{B}_m$ so that $\{x \in \partial\widetilde{A}(d(m)) : x_1 = 0\}$ is mapped onto I_m^1 and, moreover, $\widetilde{B}_m$ lies outside $\widetilde{C}$. It is easy to see that such a mapping exists and that the $\widetilde{B}_m$'s are disjoint for distinct m's. Let

$$\begin{aligned}
B_m &= \widetilde{B}_m \times \bigcup_{\substack{j\in\mathbb{Z}\\ j\geq 1\\ 2jd\leq 1}} ((2j-1)d(m), 2jd(m));\\
F^1(a) &= \{x \in \widetilde{A}(a) : x_1 > 2a\},\\
\widetilde{F}_m &= \varphi_m(F^1(d(m))),\\
F_m &= \widetilde{F}_m \times (-2d(m), 1+2d(m)),\\
F_m^{-1} &= \widetilde{F}_m \times (-d(m), 0),\\
F_m^{-2} &= \widetilde{F}_m \times (-2d(m), -d(m)),\\
F_m^{+1} &= \widetilde{F}_m \times (1, 1+d(m)),\\
F_m^{+2} &= \widetilde{F}_m \times (1+d(m), 1+2d(m)).
\end{aligned}$$

Let K_m be the convex hull of $F_m^{-2} \cup F_{m+1}^{-2}$ ($F_m^{+2} \cup F_{m+1}^{+2}$) for m odd (even). Finally, let

$$D = C \cup \bigcup_{m\geq 1} (B_m \cup F_m \cup K_m).$$

The set D consists in part of an infinite winding canal which is composed of tubes F_m whose ends are connected by relatively short K_m's.

Step 2. Clearly, D is an open bounded connected set. We start analyzing it by sketching an argument showing that it is a twisted Hölder domain of order α.

Consider a point $x \in F_m \cup K_m$. The set

$$D \cap \Big[\varphi_m(\{x \in \widetilde{A}(d(m)) : d(m) < x_1 < 2d(m)\}) \times (0,1)\Big]$$

consists of thin parallelepipeds. The one closest to x will be called Q.

Let $z = (1, -1/2, 1/2) \in D$ be our base point. We will connect x with z by a curve γ consisting of three parts: γ_1, γ_2 and γ_3.

The middle part γ_2 sits inside Q at an equal distance from its sides. The arc γ_1 joins x with an endpoint x^1 of γ_2. Since there is plenty of room inside $F_m \cup K_m$ (as compared to Q), γ_1 may be chosen so that $\ell(\gamma_1(x, x^1)) < 100d$ and $\mathrm{dist}(y, \partial D) \geq c_1 \ell(\gamma_1(x, y))$ for $y \in \gamma_1$. The width of Q, as a result of the definition of $\widetilde{A}(d)$, is such that it is possible to have

$$\mathrm{dist}(y, \partial D) \geq \frac{1}{2} \ell(\gamma_4(x, y))^{1/\alpha} \qquad \text{for } y \in \gamma_4 \stackrel{\mathrm{df}}{=} \gamma_1 \cup \gamma_2.$$

It is elementary to show that C is a uniform domain. In particular, each point $v \in C$ may be connected with z by a curve γ_5 such that

$$\mathrm{dist}(y, \partial D) \geq c_2 \ell(\gamma_5(v, y)) \qquad \text{for } y \in \gamma_5.$$

One can find a curve γ_3 connecting the other endpoint x^2 of γ_2 with z with properties similar to those of γ_5. It is now clear that for some $c_3 > 0$,

$$\mathrm{dist}(y, \partial D) \geq c_3 \ell(\gamma(y, x))^{1/\alpha} \qquad \text{for } y \in \gamma \stackrel{\mathrm{df}}{=} \gamma_1 \cup \gamma_2 \cup \gamma_3.$$

Other points $x \in D$ may be treated in a similar way. Thus, assumption (i) of Proposition 3.1 is satisfied.

As for the second assumption of Proposition 3.1, it is not hard to show that it holds for $\widetilde{\delta}(x) = c_4 \mathrm{dist}(x, \partial D)$ in place of $\delta(x)$. Then one uses the fact that $\delta(x) \geq \widetilde{\delta}(x)$ for some c_4, to prove that the assumption holds for δ as well. We leave the details to the reader. According to Proposition 3.1, this completes the proof that D is a twisted Hölder domain of order α.

Step 3. In this step, we will define a harmonic function in D and prove that it is bounded on C.

For $m \geq 1$, let x^m be the center of the cube F_m^{-1}. A subsequence $\{x^{m_k}\}_{k \geq 1}$ converges in the Martin topology to x^∞, which corresponds to a positive harmonic function h in D. In other words, the sequence of functions $G_D(\,\cdot\,, x^{m_k})/G_D(z, x^{m_k})$ converges to $h(\,\cdot\,)$ uniformly on compact subsets of D as $k \to \infty$.

Let

$$\widetilde{V}_m = \{x \in \widetilde{A}(d(m)) : x_1 = d(m)/2\},$$

$$V_m = [\varphi_m(\widetilde{V}_m) \times (0, 1)] \cap D.$$

The event that the path hits F_m and goes through B_m will be denoted H_m. More precisely,

$$H_m = \{T(V_m) < T(\bigcup_{j\geq 1}(F_j \cup K_j)) < T(\partial B_m) \circ \theta_{T(V_m)}\}$$

where θ denotes the usual shift operator.

Let $\widetilde{W}_1, \widetilde{W}_2, \ldots, \widetilde{W}_s$ be the vertical line segments of length $(100d)^{1/\alpha}$, each one dividing $\widetilde{A}(d)$ into two subdomains, lying in order on the lines $\{x_1 = d\}$, $\{x_1 = d + (100d)^{1/\alpha}\}$, $\{x_1 = d + 2(100d)^{1/\alpha}\}$, etc. We have

$$s \geq d(100d)^{-1/\alpha}/2. \tag{4.1}$$

$$W_k \stackrel{\mathrm{df}}{=} [\varphi_m(\widetilde{W}_k) \times (0,1)] \cap D.$$

By scaling, the chance of hitting W_{k+1} or W_{k-1} before hitting ∂D for Brownian motion starting from $y \in W_k$ is less than $p < 1$, where p does not depend on m or k. Repeated applications of the strong Markov property at the $T(W_k)$'s give for $x \in V_m$,

$$P^x(H_m) \leq p^{s-1}.$$

Hence, in view of (4.1), there exists $c_4 > 0$ independent of d such that

$$P^x(H_m) \leq \exp(-c_4 d(m)^{(\alpha-1)/\alpha}), \tag{4.2}$$

for $x \in V_m$ and small d (i.e. large m).

Let Z_m consist of the three squares obtained by intersecting F_m with the planes $\{x_3 = -d/4\}$, $\{x_3 = -3d/4\}$ and $\{x_3 = 1 + d/4\}$. Next we will estimate $P^x(T(Z_m) < T(\partial D))$ for $x \in V_m$. Suppose that x is the center of one of the squares which constitute V_m. Then x may be linked with our base point $z = (1, -1/2, 1/2) \in C$ by a chain of balls of length less than $c_5 \log d(m), c_5 < 0$. This follows from the fact that C is a uniform domain.

Recall that x^m is the center of F_m^{-1} and, therefore, belongs to F_m. We have $d(m) = k^{-t}/8$ for some integer $t = t(m)$. Let us find a chain of balls connecting z and x^m and going through the F_r's and K_r's for all $r \le m$. Choose a so that $d(r) = k^{-a}/8$. Note that we need $c_6 k^a$ balls in $F_r \cup K_r$ and there are $(k+1)^a$ sets $F_r \cup K_r$ corresponding to a given a, so the total number of balls needed to connect z and x^m is less than

$$\begin{aligned}\sum_{j=1}^{t} c_6 k^j (k+1)^j &= c_6[(k(k+1))^{t+1} - 1]/[k(k+1) - 1] \\ &\le c_7 (k(k+1))^t,\end{aligned}$$

for large t. Hence, there is a chain of balls connecting x and x^m of length less than

$$c_5 \log d + c_7(k(k+1))^t \le c_8(k(k+1))^t$$

for large t. The function $y \to P^y(T(Z_m) < T(\partial D))$ is harmonic in $D \backslash Z_m$ so the Harnack principle yields

$$P^x(T(Z_m) < T(\partial D)) \ge \exp(-c_9(k(k+1))^t) \tag{4.3}$$

where x is the center of a square in V_m, provided we choose $c_9 > 0$ sufficiently large.

Now we will find a large j_0 so that the P^x-probability that the process hits Z_m before hitting ∂D and goes through one of the B_j's, $j > j_0$, is relatively small when compared to (4.3).

Suppose that k is large enough so that

$$(k(k+1)) < k^{2+2\beta} < k^{(1-\alpha)/\alpha}$$

for some $\beta > 0$. This is possible since we have assumed that $\alpha < 1/3$ and, consequently, $(1-\alpha)/\alpha > 2$. Recall that $d(m) = k^{-t}/8$, and use (4.2) and (4.3) to see that

$$\begin{aligned}P^x(H_m)/P^x(T(Z_m) < T(\partial D)) &\le \exp(-c_{10} k^{-t(\alpha-1)/\alpha})/\exp(-c_9(k(k+1))^t) \\ &\le \exp(-c_{11} k^{\beta t}),\end{aligned} \tag{4.4}$$

where x is the center of a square in V_m, and $c_{11} > 0$ is large. In fact, (4.4) holds for all $x \in V_m$, by the boundary Harnack principle (the constant c_{11} may need to be changed).

The Green function $G_D(x^m, \cdot)$ is bounded below and above by q and $c_{12}q$ on $U_m \stackrel{\text{df}}{=} \partial B(x^m, d/8)$, by the Harnack principle. It follows that

$$G_D(x^m, x) \geq q P^x(T(U_m) < T(\partial D)) \tag{4.5}$$

and

$$G_D(x^m, x) \leq c_{12} q P^x(T(U_m) < T(\partial D)) \tag{4.6}$$

for all $x \in C$.

Note that the sphere U_m is cut off from C by Z_m.

Let Z_m^1 consist of 6 squares in D obtained by translation up or down by $d/8$ from the 3 squares comprising Z_m. By the boundary Harnack principle,

$$\frac{P^u(X(T(Z_m)) \in dy, T(Z_m) < T(\partial D))}{P^u(X(T(Z_m)) \in dv, T(Z_m) < T(\partial D))} \frac{P^w(X(T(Z_m)) \in dv, T(Z_m) < T(\partial D))}{P^w(X(T(Z_m)) \in dy, T(Z_m) < T(\partial D))}$$

is bounded away from 0 and ∞ for all $u, w \in Z_m^1$, $y, v \in Z_m$, and, by scaling, the bounds do not depend on m. Then

$$\begin{aligned}
&\frac{P^x(X(T(Z_m)) \in dy, T(Z_m) < T(\partial D), H_m)}{P^x(X(T(Z_m)) \in dy, T(Z_m) < T(\partial D))} \\
&= \frac{\int_{Z_m^1} P^x(X(T(Z_m^1)) \in du, T(Z_m^1) < T(\partial D), H_m) P^u(X(T(Z_m)) \in dy, T(Z_m) < T(\partial D))}{\int_{Z_m^1} P^x(X(T(Z_m)) \in dw, T(Z_m^1) < T(\partial D)) P^w(X(T(Z_m)) \in dy, T(Z_m) < T(\partial D))} \\
&\leq \frac{P^u(X(T(Z_m)) \in dy, T(Z_m) < T(\partial D))}{P^u(X(T(Z_m)) \in dv, T(Z_m) < T(\partial D))} \frac{P^w(X(T(Z_m)) \in dv, T(Z_m) < T(\partial D))}{P^w(X(T(Z_m)) \in dy, T(Z_m) < T(\partial D))} \times \\
&\times \frac{\int_{Z_m^1} P^x(X(T(Z_m^1)) \in du, T(Z_m^1) < T(\partial D), H_m) P^u(X(T(Z_m)) \in dv, T(Z_m) < T(\partial D))}{\int_{Z_m^1} P^x(X(T(Z_m)) \in dw, T(Z_m^1) < T(\partial D)) P^w(X(T(Z_m)) \in dv, T(Z_m) < T(\partial D))} \\
&\leq c_{13} \frac{P^x(X(T(Z_m)) \in dv, T(Z_m) < T(\partial D), H_m)}{P^x(X(T(Z_m)) \in dv, T(Z_m) < T(\partial D))}
\end{aligned}$$

for all $x \in C$, $y, v \in Z_m$. This is equivalent to

$$\frac{P^x(X(T(Z_m)) \in dv, T(Z_m) < T(\partial D))}{P^x(X(T(Z_m)) \in dy, T(Z_m) < T(\partial D))} \leq c_{13} \frac{P^x(X(T(Z_m)) \in dv, T(Z_m) < T(\partial D), H_m)}{P^x(X(T(Z_m)) \in dy, T(Z_m) < T(\partial D), H_m)}.$$

By integrating both sides with respect to dv we obtain

$$\begin{aligned} &\frac{P^x(T(Z_m) < T(\partial D))}{P^x(X(T(Z_m)) \in dy, T(Z_m) < T(\partial D))} \\ &\qquad \leq c_{13} \frac{P^x(T(Z_m) < T(\partial D), H_m)}{P^x(X(T(Z_m)) \in dy, T(Z_m) < T(\partial D), H_m)}. \end{aligned} \tag{4.7}$$

It follows that, for $x \in C$,

$$\begin{aligned} \frac{P^x(X(T(Z_m)) \in dy, T(Z_m) < T(\partial D), H_m)}{P^x(X(T(Z_m)) \in dy, T(Z_m) < T(\partial D))} &\leq c_{13} \frac{P^x(T(Z_m) < T(\partial D), H_m)}{P^x(T(Z_m) < T(\partial D))} \\ &\leq c_{13} \frac{P^x(H_m)}{P^x(T(Z_m) < T(\partial D))}. \end{aligned}$$

By the strong Markov property applied at $T(Z_j)$,

$$\begin{aligned} &P^x(T(U_m) < T(\partial D), H_j) \\ &\qquad = \int_{Z_j} P^y(T(U_m) < T(\partial D)) P^x(X(T(Z_j)) \in dy, T(Z_j) < T(\partial D), H_j) \\ &\qquad \leq \int_{Z_j} P^y(T(U_m) < T(\partial D)) P^x(X(T(Z_j)) \in dy, T(Z_j) < T(\partial D)) \times \\ &\qquad\qquad \times c_{13} P^x(H_j) / P^x(T(Z_j) < T(\partial D)) \\ &\qquad \leq c_{13} P^x(T(U_m) < T(\partial D)) P^x(H_j) / P^x(T(Z_j) < T(\partial D)) \end{aligned}$$

for $m, j \geq 1$ and $x \in C$. Let $k^{-t}/8 = d(j)$. The strong Markov property applied at V_j and (4.4) show that

$$P^x(T(U_m) < T(\partial D), H_j) \leq P^x(T(U_m) < T(\partial D)) c_{14} \exp(-c_{11} k^{\beta t})$$

for $x \in C$. Since there are $(k+1)^t$ indices j with $d(j) = k^{-t}$,

$$
\begin{aligned}
&P^x(T(U_m) < T(\partial D), H_j \text{ for some } j \geq j_0) \\
&\qquad \leq \sum_{j=j_0}^{\infty} P^x(T(U_m) < T(\partial D), H_j) \\
&\qquad \leq \sum_{j=j_0}^{\infty} P^x(T(U_m) < T(\partial D)) c_{14} \exp(-c_{11} d(j)^{-\beta}) \\
&\qquad \leq P^x(T(U_m) < T(\partial D)) \times c_{14} \sum_{t=t_0}^{\infty} \exp(-c_{11} k^{t\beta})(k+1)^t
\end{aligned}
$$

where $k^{-t_0}/8 = d(j_0)$, $x \in C$. Let j_0 and t_0 be sufficiently large so that

$$
P^x(T(U_m) < T(\partial D), H_j \text{ for some } j \geq j_0) \leq P^x(T(U_m) < T(\partial D))/2,
$$

and, therefore,

$$
P^x(T(U_m) < T(\partial D)) \leq 2P^x(T(U_m) < T(\partial D), H_j^c \text{ for all } j \geq j_0)
$$

for $x \in C$.

An argument similar to the one that leads to (4.7) gives

$$
\frac{P^x(X(T(Z_{j_0})) \in dy, T(Z_{j_0}) < T(\partial D))}{P^x(T(Z_{j_0}) < T(\partial D))} \leq c_{15} \frac{P^z(X(T(Z_{j_0})) \in dy, T(Z_{j_0}) < T(\partial D))}{P^z(T(Z_{j_0}) < T(\partial D))}
$$

for our base point z and all $x \in C$. The probability $P^z(T(Z_{j_0}) < T(\partial D))$ is a constant and $P^x(T(Z_{j_0}) < T(\partial D)) \leq 1$, so

$$
P^x(X(T(Z_{j_0})) \in dy, T(Z_{j_0}) < T(\partial D)) \leq c_{15} P^z(X(T(Z_{j_0})) \in dy, T(Z_{j_0}) < T(\partial D))
$$

(we may have to change c_{15}). This implies, for $x \in C$ and $m > j_0$,

$$
\begin{aligned}
&P^x(T(U_m) < T(\partial D)) \\
&\qquad \leq 2P^x(T(U_m) < T(\partial D), H_j^c \text{ for all } j \geq j_0) \\
&\qquad \leq 2 \int_{Z_{j_0}} P^y(T(U_m) < T(\partial D)) P^x(X(T(Z_{j_0})) \in dy, T(Z_{j_0}) < T(\partial D)) \\
&\qquad \leq 2c_{15} \int_{Z_{j_0}} P^y(T(U_m) < T(\partial D)) P^z(X(T(Z_{j_0})) \in dy, T(Z_{j_0}) < T(\partial D)) \\
&\qquad \leq 2c_{15} P^z(T(U_m) < T(\partial D)).
\end{aligned}
$$

The last formula, (4.5) and (4.6) imply that

$$G_D(x^m, x) \le c_{16} G_D(x^m, z)$$

for all $x \in C$ and large m. Hence, the function $h(\,\cdot\,)$, being the limit of $G_D(x^{m_k}, \cdot\,)/G_D(x^{m_k}, z)$, is bounded by $c_{16} < \infty$ on C.

Step 4. We will prove that every h-process has infinite lifetime a.s. We start with a few remarks on the function h and h-processes. The remarks are standard but we could not find a reference.

First we will show that the function h has boundary values 0 except at

$$\partial_* D \stackrel{\text{df}}{=} \{x \in \partial D : x_1 = 2, x_2 = 0\}.$$

To see this, take any $x \in \partial D \backslash \partial_* D$ and let $r > 0$ be such that $B(x, 3r) \cap \partial_* D = \emptyset$. Fix some $y^0 \in B(x, r)$ and let N be a compact subset of D containing y^0 and z. For large m, say $m \ge m_1$, $x^m \notin N \cup B(x, 2r)$. Use the Harnack principle in N and then use the boundary Harnack principle in $B(x, r)$ to obtain

$$\frac{G_D(x^m, y)}{G_D(x^m, z)} \le c_{17} \frac{G_D(x^m, y)}{G_D(x^m, y^0)} \le c_{18} \frac{G_D(x^{m_1}, y)}{G_D(x^{m_1}, y^0)}$$

for $m \ge m_1$, $y \in B(x, r) \cap D$. The right hand side has zero limit when $y \to x$. The same is true for $h(y)$ since it is the limit of the left hand side when $m \to \infty$ through a subsequence $\{m_k\}$.

Since $\partial_* D$ is a polar set, the function h has 0 boundary values almost everywhere on the boundary with respect to the harmonic measure. A bounded harmonic function with this property would have to be identically zero, so h takes arbitrarily large values.

Note that $h(z) = 1$.

The process $1/h(X_t)$ is a positive supermartingale under P_h^z with the convention that for t larger than the lifetime R we let $h(X_t) = \lim_{s \to R} h(X_s)$ (see Doob (1984), Section

2X8). This process converges P_h^z-a.s. as $t \to \infty$, possibly to ∞. It follows that $h(X_t)$ converges P_h^z-a.s. as $t \to \infty$, and we will show that the limit is infinite P_h^z-a.s. Let

$$\begin{aligned} L_1^\varepsilon &= \{x \in D : h(x) \le \varepsilon\} \qquad \text{for } \varepsilon < 1, \\ L_2^m &= \{x \in D : h(x) \ge m\}, \\ T_1 &= T(L_1^\varepsilon \cup L_2^m). \end{aligned}$$

Since h has 0 boundary values almost everywhere,

$$T_1 \le T(L_1^\varepsilon) < \infty \quad P^z\text{-a.s.}$$

Then

$$\begin{aligned} P_h^z(X(T_1) \in L_1^\varepsilon) &= \int_{L_1^\varepsilon} [h(x)/h(z)] P^z(X(T_1) \in dx) \\ &= \varepsilon P^z(X(T_1) \in L_1^\varepsilon) \le \varepsilon. \end{aligned}$$

As $\varepsilon \to 0$ we have $P_h^z(X(T_1) \in L_2^m) \to 1$ and it follows that $P_h^z(T(L_2^m) < \infty) = 1$. Since m is arbitrary, $h(X_t) \to \infty$, P_h^z-a.s.

The last observation has two consequences. The first one is that, since h has 0 boundary values away from $\partial_* D$, the process X_t converges P_h^z-a.s. to $\partial_* D$,

$$\lim_{t \to R} \operatorname{dist}(X_t, \partial_* D) = 0 \quad P_h^z\text{-a.s.}$$

The second one is that the last visit to C will occur strictly before the lifetime R as the function h is bounded on C. If $L(C)$ is the last exit time from C then $\{X(L(C)+t), t > 0\}$ under P_h^z is an h_1-process in $D \backslash C$, converging to $\partial_* D$. It will suffice to show that such a process must have infinite lifetime.

The set

$$\left[\bigcup_{m \ge 1} F_m \cap \{x : 0 < x_3 < 1\} \right] \setminus \bigcup_{m \ge 1} B_m$$

consists of a sequence of cubes $Q_1, Q_2, \dots$ arranged in order along $\bigcup_{m \ge 1} (F_m \cup K_m)$. The h_1-process will have to pass through all cubes $Q_j, j \ge j_1$, where j_1 depends on the starting point of the h_1-process.

We make a digression concerning the lifetime of a conditioned Brownian motion in a cube. First consider a Brownian motion starting from the center of a sphere and conditioned to hit a fixed point x on the sphere at the time of exiting it. By symmetry, the lifetime of this process has the same distribution for each point x. Every Brownian motion conditioned by a harmonic function in the sphere and starting from its center is a mixture of such processes, so its lifetime has the same distribution. Let $\mathcal{L}$ denote this distribution in the case when the sphere has radius 1/8.

Now suppose that $Q = [-1,1]^3$, $\overline{Q} = \{x \in Q : x_1 = 0\}$ and g is a positive harmonic function in Q vanishing on $\{x \in \partial Q : x_1 \in (-1,1)\}$. By the boundary Harnack principle,

$$P^x(T(B(0,1/8)) < T(\partial Q))/g(x) > c_{19} > 0$$

for $x \in \overline{Q}$ with $|x| > 1/2$. Note that $g(y) > c_{20} > 0$ for $y \in B(0,1/8)$. Thus

$$\begin{aligned}
&P^x_g(T(B(0,1/8)) < T(\partial Q)) \\
&\qquad = \int_{B(0,1/8)} \frac{g(y)}{g(x)} P^x(T(B(0,1/8)) < T(\partial Q), X(T(B(0,1/8))) \in dy) \\
&\qquad \geq \frac{c_{20}}{g(x)} P^x(T(B(0,1/8)) < T(\partial Q)) \geq c_{20}c_{19} > 0,
\end{aligned}$$

for all $x \in \overline{Q}$ with $|x| > 1/2$. It is easy to see that a similar inequality holds for all $x \in \overline{Q}$. The time spent between $T(B(0,1/8))$ (assuming it is finite) and the hitting time of $B(X(T(B(0,1/8))),1/8)$ is independent of $x \in \overline{Q}$ and g and has distribution $\mathcal{L}$ under P^x_g. As a result, we can find a bounded random variable Y such that $EY > 0$ and the time spent in Q by the g-process starting from x is stochastically larger than Y, for every $x \in \overline{Q}$ and g. If $\widetilde{Q}$ is a cube with side length b then the analogous statement is true with Y replaced by $(b/2)^2 Y$.

Let us go back to our h_1-process. Define squares $\overline{Q}_j$ relative to Q_j in the same way as $\overline{Q}$ was defined relative to Q; moreover, orient them so that the h_1-process has to pass through each of the $\overline{Q}_j$'s.

Let S_j be the time elapsed between the first hit of $\overline{Q}_j$ and the first exit from Q_j afterwards. Suppose that $\rho(Q_j)$ is the side length of Q_j and that Y_j is a sequence of independent copies of Y. The distribution of S_j is stochastically larger than $(\rho(Q_j)/2)^2\, Y_j$ and, by the strong Markov property applied at the hitting times of $\overline{Q}_j$'s, the distribution of

$$S_{j_1} + S_{j_1+1} + \ldots + S_m$$

is stochastically greater than the distribution of

$$(\rho(Q_{j_1})/2)^2 Y_{j_1} + (\rho(Q_{j_1+1})/2)^2 Y_{j_1+1} + \ldots + (\rho(Q_m)/2)^2 Y_m. \tag{4.8}$$

Note that there are at least $k^m(k+1)^m/4$ cubes Q_j with $\rho(Q_j) = k^{-m}$. It follows that the sum of expectations of the terms in (4.8) is divergent:

$$\sum_{j=j_1}^{\infty} E(\rho(Q_j)/2)^2 Y_j = EY/4 \sum_{j=j_1}^{\infty} \rho(Q_j)^2$$

$$\geq EY/4 \sum_{m=m_1}^{\infty} (k^{-m})^2 k^m (k+1)^m/4 = \infty.$$

Since the Y_j's are independent and bounded, the three series theorem shows that the series in (4.8) converges a.s. to infinity as $m \to \infty$. Since (4.8) is stochastically smaller than the sum of the S_j's, we have $\sum_{j=j_1}^{\infty} S_j = \infty$ a.s. Of course, the lifetime of the h_1-process is larger than $\sum S_j$, so it is also infinite a.s. This completes the proof that h-processes in D have infinite lifetime and finishes the proof of Theorem 1.1 (ii) (C). $\square$

5. A parabolic boundary Harnack principle.

Lemma 5.1. *Under the assumptions of Theorem 1.2, for every $u > 0$ there exist a non-degenerate closed ball $M \subset D$ and $c > 0$ such that for all $x \in D$,*

$$P^x(X_u \in M, T(D^c) > u) \geq cP^x(T(D^c) > u).$$

Proof. Let

$$A = A(\beta) = \{x \in D : \operatorname{dist}(x, \partial D) \geq \beta\}.$$

It is easy to see that A is a bounded and closed set, hence, a compact set.

Fix some $z \in D$ and find $\beta_0 > 0$ and a closed ball M such that $M \subset A(\beta_0)$. The domain $D_1 \stackrel{\text{df}}{=} D \setminus M$ satisfies the same assumptions (A)-(C) as D. Let

$$\begin{aligned}
h(x) &\stackrel{\text{df}}{=} G_D^L(x,z),\\
D_2 = D_2(\beta) &\stackrel{\text{df}}{=} D \setminus A(\beta),\\
U_k &\stackrel{\text{df}}{=} \{x \in D_1 : h(x) \in [2^k, 2^{k+1}]\},\\
\widehat{U}_k &\stackrel{\text{df}}{=} \{x \in D_1 : h(x) \le 2^{k+1}\},\\
\widetilde{U}_k &\stackrel{\text{df}}{=} \{x \in D_2 : h(x) \in [2^k, 2^{k+1}]\}.
\end{aligned}$$

We have $\widetilde{U}_k = U_k \setminus A(\beta)$ for $\beta < \beta_0$. Note that U_k is bounded, so it has finite volume and, therefore, $\mathrm{Vol}(\widetilde{U}_k) \to 0$ as $\beta \to 0$. For any open set N and any x we have, by Lemma 1 of Bañuelos (1987),

$$E^x(T(N^c)) \le c_1(\mathrm{Vol}(N))^{1/n}.$$

It follows that

$$\lim_{\beta\to 0} E^x(T(\widetilde{U}_k^c)) = 0. \tag{5.1}$$

The function h is bounded in D_1 by 2^{k_0+1} for some $k_0 < \infty$. According to the proof of Theorem 1.1 (i), we have

$$\sum_{k=-\infty}^{k_0} \sup_{x\in U_k} E^x(T(U_k^c)) < \infty. \tag{5.2}$$

Since $\widetilde{U}_k \subset U_k$,

$$E^x(T(\widetilde{U}_k^c)) \le E^x(T(U_k^c)).$$

This, (5.1) and (5.2) show that for any constant $c_2 < \infty$ there is $\beta > 0$ with

$$c_2 \sum_{k=-\infty}^{k_0} \sup_{x\in \widetilde{U}_k} E^x(T(\widetilde{U}_k^c)) < u/8.$$

For suitable c_2, the expression on the left hand side is an upper bound for $E_h^x(T(D_2^c))$. It follows that

$$P_h^x(T(D_2^c)) < u/4) > 1/2. \tag{5.3}$$

Before we proceed with the proof, we introduce some notation. Let the $\mathbb{R}^n$-valued process be denoted as usual by X and let Y stand for the space-time process. More precisely, if X has law P^x, then the law of the space-time diffusion

$$\{Y(t) \stackrel{\text{df}}{=} (X(t), s-t), t \geq 0\}$$

will be denoted $P^{x,s}$. The distribution of space-time diffusion conditioned by a parabolic function g will be denoted $P_g^{x,s}$. See Doob (1984) for the discussion of these processes and their properties in the case $L = \Delta$. By abuse of notation, $T(A)$ will denote the first hitting time of A for Y as well as for X. The function

$$(x,t) \mapsto g(x,t) \stackrel{\text{df}}{=} P^x(T(\partial D) > t)$$

is parabolic in $D \times [0,\infty)$ with boundary values 1 on $D \times \{0\}$ and 0 otherwise; more precisely, it is zero at (y,t) provided $t > 0$ and y is a regular point of ∂D.

Let g_1 be a parabolic function in $D \times [0,\infty)$ which has the same boundary values as g except that $g_1(x,0) = \varepsilon$ for $x \in D \backslash M$, where $\varepsilon \in (0,1)$ will be chosen later. Now we will estimate g_1 on $D \times [u/2, u]$.

Lemma 5.1 of Fabes and Stroock (1986) implies that $g_1(x,s) > c_3$ for all $x \in M$ and $s \in [u/4, u]$. We also have $h(y) < c_4$ for all $y \in \partial D_2$. Let $h(x,s) \stackrel{\text{df}}{=} h(x)$. For $x \in D_2$ and

$s \geq 1/2$ we have, by (5.3),

$$
\begin{aligned}
g_1(x,s) &\geq \int\limits_{\substack{t\in[u/4,u]\\ y\in\partial D_2}} g_1(y,t) P^{x,s}(T(D_2^c)\in dt, X(T(D_2^c))\in dy) \\
&= \int\limits_{\substack{t\in[u/4,u]\\ y\in\partial D_2}} \frac{h(x,s)}{h(y,t)}\frac{h(y,t)}{h(x,s)} g_1(y,t) P^{x,s}(T(D_2^c)\in dt, X(T(D_2^c))\in dy) \\
&= \int\limits_{\substack{t\in[u/4,u]\\ y\in\partial D_2}} \frac{h(x,s)}{h(y,t)} g_1(y,t) P_h^{x,s}(T(D_2^c)\in dt, X(T(D_2^c))\in dy) \\
&\geq \int\limits_{\substack{t\in[u/4,u]\\ y\in\partial D_2}} h(x,s) c_4^{-1} c_3 P_h^{x,s}(T(D_2^c)\in dt, X(T(D_2^c))\in dy) \\
&= h(x,s) c_4^{-1} c_3 P_h^{x,s}(T(D_2^c)\in [u/4,s]) \\
&\geq h(x,s) c_4^{-1} c_3/2 \\
&= c_5 h(x,s) = c_5 h(x).
\end{aligned}
$$

Let

$$W_k = \{(x,s) : g_1(x,s) \in [2^k, 2^{k+1}], s\in[u/2,u]\},$$

$$W = \bigcup_{k=-\infty}^{k_1} W_k,$$

where $k_1 < 0$ will be chosen later. If $2^{-m} < c_5$ then $W_k \subset \widehat{U}_{k+m}\times[u/2,u]$. Using the estimate of Chung (1984) and Remark 3.4 we obtain for small k_1

$$
\begin{aligned}
E_{g_1}^{x,u}(T(W^c)) &\leq c_6 \sum_{k=-\infty}^{k_1} \sup_{(y,s)\in W_k} E^{y,s} T(W_k^c) \\
&\leq c_6 \sum_{k=-\infty}^{k_1} \sup_{(y,s)\in \widehat{U}_{k+m}} E^{y,s} T(\widehat{U}_{k+m}^c) < \infty.
\end{aligned}
$$

Choose k_1 so small that

$$E_{g_1}^{x,u} T(W^c) < u/8. \tag{5.4}$$

Let

$$V = \{(x,s) : g_1(x,s) \geq 2^{k_1}, s \in [u/2, u]\}.$$

Since the g_1-process cannot exit $D \times [0,\infty)$ through $\partial D \times [0,\infty)$, (5.4) implies

$$P^{x,u}_{g_1}(T(V) > u/4) < 1/2. \tag{5.5}$$

Now let $\varepsilon = 2^{k_1-1}$. Since $0 \leq g_1 \leq 1$, the process $g_1(Y_t)$ is a martingale under $P^{x,s}$, and $g_1(x,s) \geq 2^{k_1}$ for $(x,s) \in V$, we see that there is at least $2^{k_1-1}/2$ chance that Y under $P^{x,s}$ will hit $M \times \{0\}$ before hitting any other part of $\partial(D \times [0,\infty))$. Thus we have for $(x,s) \in V$,

$$\begin{aligned} P^{x,s}_{g_1}(Y_s \in M \times \{0\}) &= \int_M (g_1(y,0)/g_1(x,s)) P^{x,s}(Y_s \in dy, T(\partial(D \times [0,\infty))) = s) \\ &\geq \int_M P^{x,s}(Y_s \in dy, T(\partial(D \times [0,\infty))) = s) \\ &\geq 2^{k_1-1}/2. \end{aligned}$$

This and (5.5) yield, by the strong Markov property, for all $x \in D$,

$$P^{x,u}_{g_1}(Y_u \in M \times \{0\}) \geq c_{10} > 0.$$

The ratio of g and g_1 is bounded away from 0 and ∞ on the boundary of $D \times [0,\infty)$, so

$$P^{x,u}_{g}(Y_u \in M \times \{0\}) \geq c_{11} > 0$$

for all $x \in D$. This is equivalent to the statement in the lemma. $\square$

Proof of Theorem 1.2. First we will show that $p^D_u(x,y)$ is comparable to $\psi(x)\psi(y)$ where $\psi(x) \stackrel{\text{df}}{=} P^x(T(D^c) > u/3)$. To simplify the notation, let us take $u = 3$.

Note that $p^D_1(\,\cdot\,,\,\cdot\,) < c$ by Fabes and Stroock (1986) and $p^D_1(v,z) = p^D_1(z,v)$ for all $v,z \in D$ (see Fukushima (1980)). We have

$$\begin{aligned} p^D_2(z,y) &= \int_D p^D_1(z,v) p^D_1(v,y)\, dv \\ &\leq \int_D c p^D_1(v,y)\, dv = \int_D c p^D_1(y,v)\, dv \\ &= c\psi(y). \end{aligned}$$

It follows that

$$
\begin{aligned}
p_3^D(x,y) &= \int_D p_1^D(x,z)p_2^D(z,y)\,dz \\
&\le \int_D p_1^D(x,z)c\psi(y)\,dz \\
&= c\psi(x)\psi(y).
\end{aligned}
$$

In order to obtain the opposite inequality, first observe that Lemma 5.1 of Fabes and Stroock (1986) implies immediately that $p_1^D(z,v) > c_1$ for all $z, v \in M$, where M is a compact ball in D. For $z \in M$, we obtain, using our Lemma 5.1,

$$
\begin{aligned}
p_2^D(z,y) &\ge \int_M p_1^D(z,v)p_1^D(v,y)\,dv \\
&\ge \int_M c_1 p_1^D(v,y)\,dv \\
&= \int_M c_1 p_1^D(y,v)\,dv \\
&= c_1 P^y(X_1 \in M, T(D^c) > 1) \\
&\ge c_1 c_2 P^y(T(D^c) > 1) = c_1 c_2 \psi(y).
\end{aligned}
$$

Hence, for all $x, y \in D$,

$$
\begin{aligned}
p_3^D(x,y) &\ge \int_M p_1^D(x,z)p_2^D(z,y)\,dz \\
&\ge \int_M p_1^D(x,z)c_1c_2\psi(y)\,dz \\
&= c_1c_2 P^x(X_1 \in M, T(D^c) > 1)\psi(y) \\
&\ge c_1 c_2^2 P^x(T(D^c) > 1)\psi(y) \\
&= c_1 c_2^2 \psi(x)\psi(y).
\end{aligned}
$$

Thus, for some $c_3 > 0$ and all $x, y \in D$,

$$
c_3 < p_u^D(x,y)/\psi(x)\psi(y) < c_3^{-1}.
$$

This implies that

$$\text{(5.6)} \qquad \frac{p_u^D(x,y)}{p_u^D(x,z)}\frac{p_u^D(v,z)}{p_u^D(v,y)} \ge \frac{c_3\psi(x)\psi(y)}{c_3^{-1}\psi(x)\psi(z)}\frac{c_3\psi(v)\psi(z)}{c_3^{-1}\psi(v)\psi(y)} = c_3^4$$

which is Theorem 1.2 for $s = t = u$.

In order to extend the last formula to times greater than u we use the Markov property as follows. Let $a = c_3^4 p_u^D(v,y)/p_u^D(v,z)$. Then, according to (5.6),

$$p_u^D(w,y) \ge a p_u^D(w,z)$$

for all $w, y, z \in D$. Then, for $s > u$, $x, y, z \in D$,

$$\begin{aligned} p_s^D(x,y) &= \int_D p_{s-u}^D(x,w) p_u^D(w,y) dv \\ &\ge a \int_D p_{s-u}^D(x,w) p_u^D(w,z) dv \\ &= a p_s^D(x,z) \\ &= c_3^4 (p_u^D(v,y)/p_u^D(v,z)) p_s^D(x,z), \end{aligned}$$

and so

$$\frac{p_s^D(x,y)}{p_s^D(x,z)} \ge c_3^4 \frac{p_u^D(v,y)}{p_u^D(v,z)}.$$

An analogous argument may be used to replace u in the right hand side with an arbitrary $t > u$ and we obtain

$$\frac{p_s^D(x,y)}{p_s^D(x,z)} \ge c_3^4 \frac{p_t^D(v,y)}{p_t^D(v,z)}$$

for all $v, x, y, z \in D$, $s, t > u$. $\square$

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