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Lifshitz solutions in supergravity and string theory

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Abstract: We derive Lifshitz configurations in string theory for general dynamical exponents $z \geq 1$. We begin by obtaining simple $Li \times \Omega$ solutions to supergravities in diverse dimensions, with Ω a compact constant curvature manifold. Then we uplift the solutions to ten dimensions, providing configurations that correspond to warped compactifications in Type II string theory.

Keywords: ads/cft, Lifshitz scaling.

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1. Introduction

In the past few years, the adS/CFT, [1], more generally called the gauge/gravity correspondence, has been explored to yield insights into strongly coupled field theory. While the initial particle physics motivation may have been to glean a nonperturbative understanding of QCD, a more unexpected but extremely fruitful dialogue has emerged with condensed matter physics (see e.g. [2]), which also has many experimental systems believed to be governed by strongly coupled physics.

As suggested by the nomenclature adS/CFT, the higher dimensional bulk gravitational theory asymptotically tends to anti-de Sitter spacetime (adS), yielding a boundary theory which is typically relativistically invariant. However, while many systems do display such scale invariance, there is also interest in systems displaying a more general dynamical scaling:

$$t \to \lambda^z t\,, \quad x \to \lambda x\,, \tag{1.1}$$

where $z \neq 1$ is the dynamical critical exponent. This splitting of space and time can seem counter-intuitive for a relativistically invariant theory, however, it is always possible to single out particular directions by choosing an appropriate background. In [3], Kachru et al. showed how to construct Lifshitz geometries by switching on fluxes with nontrivial topological couplings. They constructed a four-dimensional (4D) spacetime with two gauge fields (a 1- and 2-form) coupled via a topological "$B_2 \wedge F_2$" term. The model was, of course, phenomenological, in that it was constructed to provide a holographic dual with the requisite dynamical scaling properties, but it was hoped that the model could be put on firmer footing by finding similar Lifshitz geometries as solutions to String/M-theory. In this way, genuine dual quantum field theories could be constructed which would hopefully encapsulate the physics of the condensed matter systems.

Explicit Lifshitz solutions in string theory however have proven surprisingly difficult to find. The expectation was that standard techniques, such as flux compactifications, would be sufficient. Disappointingly, the first attempts with reasonable and quite general Ansätze led only to various no-go statements [4, 5]. Several schemes which could support Lifshitz geometries were discussed in [6], and significant progress was made with [7] and, most recently, [8], where it was shown that Lifshitz solutions with the dynamical exponent $z = 2$ can be embedded into D=10 and D=11 supergravity, by compactifying on Einstein manifolds fibred with a circle. These are important steps forward, however they still are restricted to one particular value of the critical exponent, and it would certainly be most interesting to see if they can be generalized to z different from two, as well as to black hole backgrounds.

In the present paper, we make a first step to answer these questions, and provide a simple method to obtain explicit string constructions of Lifshitz geometries with general dynamical exponents, $z \geq 1$ ($z = 1$ being the adS_4 limit). Following a bottom-up approach, the idea is to start by looking for such solutions in d-dimensional supergravities, and then to uplift them to ten dimensions. This has been an often used trick in the adS/CFT literature (see e.g. [9]). Supergravities in diverse dimensions offer a host of simple $\mathrm{adS}_q \times \Omega_{d-q}$ backgrounds, with Ω_{d-q} a constant curvature spherical, flat or hyperbolic space (see [10] for a review). Some of these, it will be seen, can be straightforwardly generalized to simple $Li_q \times \Omega_{d-q}$ geometries.

Our search led us to two examples of supergravity theories. First, we considered Romans gauged, massive, $\mathcal{N} = 4$ supergravity in 6D, [11], which has a solution $Li_4 \times H_2$, with general $z \geq 1$ and H_2 a hyperboloid. The 2D hyperbolic space can then easily be rendered compact by modding out a non-compact discrete subgroup of the isometry group. This does not change the local geometry, only the topology, leading to some genus g Riemann surface, where g depends on the choice of discrete symmetry (for some useful references on compact hyperbolic spaces, see [9], [12] and [13]). Flux quantization on the compact manifold leads to a topological restriction on z in terms of the couplings of the 6D theory. The 6D solution can then be uplifted to massive Type IIA supergravity following the analysis of [14], and the resulting configuration can be interpreted in terms of intersecting D-branes of various dimensions. Second, we considered the gauged $\mathcal{N} = 4$ supergravity in 5D, also by Romans [15]. This has solutions $Li_3 \times H_2$, again with general $z \geq 1$ (up to quantization conditions). They can be uplifted to Type IIB supergravity using the results of [16], which leads to configurations that can be interpreted in terms of intersecting D3 branes, or further to 11D supergravity using [17, 18].

The paper is organized as follows. In Section 2 we review the 6D supergravity theory and present 4D Lifshitz solutions, and then uplift them to massive Type IIA supergravity. Then, in Section 3, we present an analogous discussion for 3D Lifshitz solutions in 5D supergravity, uplifting them to Type IIB string theory. Finally, we conclude in Section 4. Throughout the paper, our signature is mostly minus, and the curvature tensors are defined such that the scalar curvature of a sphere is negative. Moreover we choose conventions such that the d-dimensional Newton constant is $\kappa_d^2 = 2$.

2. Li_4 solutions in massive Type IIA via 6D

The first Lifshitz solution we present has four infinite (spacetime) dimensions, and arises from a compactification of $\mathcal{N}=4$, 6D gauged massive supergravity. It can provide a dual description for a $2+1$-dimensional strongly coupled field theory. We first review the action and equations of motion of Romans, [11], then show that a simple $Li_4 \times \Omega_2$ Ansatz reduces the equations of motion to a set of algebraic equations that can be solved straightforwardly. The result is a $Li_4 \times H_2$ geometry, where H_2 is a hyperbolic two dimensional geometry, which can be taken to be compact, (e.g. see [9]). We demonstrate that the solution breaks supersymmetry, and show how to uplift the configuration to massive Type IIA supergravity.

The action for $\mathcal{N}=4$ 6D supergravity was worked out by Romans in [11], and we use the conventions found there. The bosonic field content consists of the metric, $g_{\mu\nu}$, dilaton, ϕ, an anti-symmetric two-form gauge field, $B_{\mu\nu}$, and a set of gauge vectors, $(A^{(i)}_\mu, \mathcal{A}_\mu)$, for the gauge group $SU(2)\times U(1)$. Fermions fill out the supermultiplets, but they will not be of interest for our purposes. The bosonic part of the Lagrangian can be written as:

$$\begin{aligned} \mathrm{e}^{-1}\,\mathcal{L} = & -\frac{1}{4}R + \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{e^{-\sqrt{2}\phi}}{4}\left(\mathcal{H}^{\mu\nu}\mathcal{H}_{\mu\nu} + F^{(i)\,\mu\nu}F^{(i)}_{\mu\nu}\right) + \frac{e^{2\sqrt{2}\phi}}{12}G_{\mu\nu\rho}G^{\mu\nu\rho} \\ & -\frac{\mathrm{e}}{8}\,\epsilon^{\mu\nu\rho\lambda\sigma\tau}\,B_{\mu\nu}\times\left(\mathcal{F}_{\rho\lambda}\mathcal{F}_{\sigma\tau} + \mathrm{m}B_{\rho\lambda}\mathcal{F}_{\sigma\tau} + \frac{\mathrm{m}^2}{3}B_{\rho\lambda}B_{\sigma\tau} + F^{(i)}_{\rho\lambda}F^{(i)}_{\sigma\tau}\right) \\ & +\frac{1}{8}\left(g^2 e^{\sqrt{2}\phi} + 4g\mathrm{m}e^{-\sqrt{2}\phi} - \mathrm{m}^2 e^{-3\sqrt{2}\phi}\right)\,, \end{aligned} \tag{2.1}$$

where e is the determinant of the vielbein, the spacetime indices $\mu,\nu,\ldots$ run from $1,\ldots,6$, and the gauge indices, $(i),(j),\ldots$ run over $(1),(2),(3)$. The field strengths are given by:

$$\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu \tag{2.2}$$

$$F^{(i)}_{\mu\nu} = \partial_\mu A^{(i)}_\nu - \partial_\nu A^{(i)}_\mu + g\,\epsilon^{ijk}A^{(j)}_\mu A^{(k)}_\nu \tag{2.3}$$

$$G_{\mu\nu\rho} = 3\,\partial_{[\mu}\,B_{\nu\rho]}\,, \tag{2.4}$$

and we also define:

$$\mathcal{H}_{\mu\nu} = \mathcal{F}_{\mu\nu} + \mathrm{m}B_{\mu\nu}\,. \tag{2.5}$$

Notice that the theory has two parameters: the gauge coupling, g, and the mass parameter, m. Romans identified five distinct theories, labelled $\mathcal{N}=4^+$ (for $g>0,\mathrm{m}>0$), $\mathcal{N}=4^-$ (for $g<0,\mathrm{m}>0$), $\mathcal{N}=4^g$ (for $g>0,\mathrm{m}=0$), $\mathcal{N}=4^{\mathrm{m}}$ (for $g=0,\mathrm{m}>0$) and finally $\mathcal{N}=4^0$ (for $g=0,\mathrm{m}=0$).

The equations of motion that follow from the Lagrangian (2.1) read:

$$R_{\mu\nu} = 2\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}P(\phi) + e^{2\sqrt{2}\phi}\left(G_\mu{}^{\rho\lambda}G_{\nu\rho\lambda} - g_{\mu\nu}G^{\rho\lambda\sigma}G_{\rho\lambda\sigma}\right)$$

$$-e^{-\sqrt{2}\phi}\left(2\mathcal{H}_\mu^{\ \rho}\mathcal{H}_{\nu\rho} + 2F_\mu^{\ \rho\,(i)}F^{(i)}_{\nu\rho} - \frac{1}{4}g_{\mu\nu}\left(\mathcal{H}^{\rho\lambda}\mathcal{H}_{\rho\lambda} + F^{\rho\lambda\,(i)}F^{(i)}_{\rho\lambda}\right)\right) \tag{2.6}$$

$$\Box\phi = \frac{\partial P}{\partial\phi} + \frac{1}{3}\sqrt{\frac{1}{2}}e^{2\sqrt{2}\phi}G^{\mu\nu\rho}G_{\mu\nu\rho} + \frac{1}{2}\sqrt{\frac{1}{2}}e^{-\sqrt{2}\phi}\left(\mathcal{H}^{\mu\nu}\mathcal{H}_{\mu\nu} + F^{\mu\nu\,(i)}F^{(i)}_{\mu\nu}\right) \tag{2.7}$$

$$D_\nu\left(e^{-\sqrt{2}\phi}\mathcal{H}^{\nu\mu}\right) = \frac{1}{6}\mathrm{e}\,\epsilon^{\mu\nu\rho\lambda\sigma\tau}\mathcal{H}_{\nu\rho}G_{\lambda\sigma\tau} \tag{2.8}$$

$$D_\nu\left(e^{-\sqrt{2}\phi}F^{\nu\mu\,(i)}\right) = \frac{1}{6}\mathrm{e}\,\epsilon^{\mu\nu\rho\lambda\sigma\tau}F^{(i)}_{\nu\rho}G_{\lambda\sigma\tau} \tag{2.9}$$

$$D_\rho\left(e^{2\sqrt{2}\phi}G^{\rho\mu\nu}\right) = -\mathrm{m}e^{-\sqrt{2}\phi}\mathcal{H}^{\mu\nu} - \frac{1}{4}\mathrm{e}\,\epsilon^{\mu\nu\rho\lambda\sigma\tau}\left(\mathcal{H}_{\rho\lambda}\mathcal{H}_{\sigma\tau} + F^{(i)}_{\rho\lambda}F^{(i)}_{\sigma\tau}\right), \tag{2.10}$$

where we have defined the scalar potential function:

$$P(\phi) = \frac{1}{8}\left(g^2 e^{\sqrt{2}\phi} + 4g\mathrm{m}e^{-\sqrt{2}\phi} - \mathrm{m}^2 e^{-3\sqrt{2}\phi}\right). \tag{2.11}$$

In order to solve these equations, we make a simple Ansatz for the solution: that the metric consists of a direct product between a 4D Lifshitz geometry, and a constant curvature 2D internal space:

$$ds^2 = L^2\left(r^{2z}dt^2 - r^2dx_1^2 - r^2dx_2^2 - \frac{dr^2}{r^2}\right) - d\Omega_2^2\,, \tag{2.12}$$

where the 'internal' 2D part of the metric takes the form of flat space, or:

$$d\Omega_2^2 = a^2(d\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2) \qquad \text{or} \qquad d\Omega_2^2 = \frac{a^2}{y_2^2}\left(dy_1^2 + dy_2^2\right), \tag{2.13}$$

with a the radius of curvature of the sphere or hyperboloid respectively. In (2.12). the parameter z is the dynamical exponent, which measures the anisotropic scaling symmetry:

$$t \to \lambda^z t, \qquad x^i \to \lambda x^i, \qquad r \to \lambda^{-1}r\,. \tag{2.14}$$

Gauge field backgrounds that are invariant under the chosen symmetries are:

$$F^{(3)}_{tr} = \alpha L^2 r^{z-1} \quad , \quad F^{(3)}_{y_1y_2} = \gamma\,\mathrm{e}_2 \tag{2.15}$$

$$G_{x_1x_2r} = \beta L^3 r \quad \Rightarrow \quad B_{x_1x_2} = \frac{\beta}{2}L^3r^2\,, \tag{2.16}$$

with e_2 the determinant of the zweibein on Ω_2, and the scalar field is constant with value ϕ_0.

We take z to be some fixed number in the solution. When $z = 1$ the Ansatz reduces to the standard $\mathrm{adS}_4 \times \Omega_2$ one. These solutions were discussed in Romans' paper [11], where they were found to exist in the $\mathcal{N} = 4^+$ theory, with the internal space being H_2. For a special value of the parameters[1] these solutions are supersymmetric thanks to an embedding

[1] Note, however, that Romans' condition on the couplings $g = 2\mathrm{m}$ can be relaxed by allowing for a non-zero constant dilaton.

of the spin connection in the gauge connection, with half of the original supersymmetries surviving. Later, in [19], these solutions and their deformations were studied in the context of adS/CFT, along with similar $\mathrm{adS}_4 \times S^2$ geometries. Beware, however, that the fixed point S^2 configurations are not solutions to the supersymmetry conditions nor the equations of motion[2]. Therefore, in the following we focus on the hyperbolic internal manifold in the $\mathcal{N} = 4^+$ theory.

In the familiar case of an $\mathrm{adS}_4 \times H_2$ Ansatz, the equations of motion (2.6−2.10) reduce to a set of algebraic equations. The same is true for the more general Lifshitz Ansatz. For convenience, we make the following rescalings, which absorb the two quantities ϕ_0 and L into a redefinition of the other parameters:

$$\hat{\alpha} = L\alpha e^{-\phi_0/\sqrt{2}} \qquad \hat{\beta} = L\beta e^{\sqrt{2}\phi_0} \qquad \hat{\gamma} = L\gamma e^{-\phi_0/\sqrt{2}} \tag{2.17}$$

$$\hat{g} = Lg e^{\phi_0/\sqrt{2}} \qquad \hat{a} = a/L \qquad \hat{\mathrm{m}} = L\,\mathrm{m}\, e^{-3\phi_0/\sqrt{2}}\,. \tag{2.18}$$

In essence, we are interchanging the freedom to choose the parameters ϕ_0 and L with the freedom to choose the Lagrangian parameters $\hat{g}$ and $\hat{\mathrm{m}}$. It is of course straightforward to recover the values of ϕ_0 and L by inverting these relations. With our Ansatz and these redefinitions, the field equations (2.6−2.10) simplify to the following set of algebraic relations.

$$z\,\hat{\beta} = \frac{\hat{\mathrm{m}}^2}{2}\,\hat{\beta} + 2\hat{\alpha}\hat{\gamma} \tag{2.19}$$

$$\hat{\alpha} = \hat{\gamma}\hat{\beta} \tag{2.20}$$

$$0 = \frac{1}{4}\left(\hat{g}^2 - 4\hat{g}\,\hat{\mathrm{m}} + 3\hat{\mathrm{m}}^2\right) - 2\hat{\beta}^2 + \left(\frac{\hat{\mathrm{m}}^2\hat{\beta}^2}{4} - \hat{\alpha}^2 + \hat{\gamma}^2\right) \tag{2.21}$$

$$z(2+z) = \mathcal{P} + \hat{\beta}^2 + \left(\frac{\hat{\mathrm{m}}^2\hat{\beta}^2}{8} + \frac{3\hat{\alpha}^2}{2} + \frac{\hat{\gamma}^2}{2}\right) \tag{2.22}$$

$$2+z = \mathcal{P} - \hat{\beta}^2 + \left(-\frac{3\hat{\mathrm{m}}^2\hat{\beta}^2}{8} - \frac{\hat{\alpha}^2}{2} + \frac{\hat{\gamma}^2}{2}\right) \tag{2.23}$$

$$2+z^2 = \mathcal{P} - \hat{\beta}^2 + \left(\frac{\hat{\mathrm{m}}^2\hat{\beta}^2}{8} + \frac{3\hat{\alpha}^2}{2} + \frac{\hat{\gamma}^2}{2}\right) \tag{2.24}$$

$$\frac{1}{\hat{a}^2} = \mathcal{P} + \hat{\beta}^2 + \left(\frac{\hat{\mathrm{m}}^2\hat{\beta}^2}{8} - \frac{\hat{\alpha}^2}{2} - \frac{3\hat{\gamma}^2}{2}\right) \tag{2.25}$$

where

$$\mathcal{P} = \frac{1}{8}\left(\hat{g}^2 + 4\hat{g}\,\hat{\mathrm{m}} - \hat{\mathrm{m}}^2\right)\,. \tag{2.26}$$

Although this would appear to be a system of seven equations in six independent variables, there is a Bianchi identity which relates a combination of (2.19) and (2.20) to a combination

[2]One of the components of the gravitino supersymmetry equation (component $\mu = 6$ in the case of an internal sphere, and component $\mu = 5$ for the hyperboloid) gives rise to an extra supersymmetry condition, $ag = 1$, not explicitly written in [19]. Together with the BPS conditions given in [19], this singles out H_2 as the only supersymmetric fixed-point solution. It is also straightforward to show that the S^2 geometry does not solve the second order Einstein equations. We thank the authors of [19] for discussions on this point.

of (2.22), (2.23) and (2.24). Thus a solution exists also for $z \neq 1$, and is given by:

$$\hat{\beta}^2 = z - 1 \tag{2.27}$$
$$\hat{\alpha}^2 = \hat{\gamma}^2(z-1) \tag{2.28}$$
$$\hat{\gamma}^2 = \frac{(2+z)(z-3) \pm 2\sqrt{2(z+4)}}{2z} \tag{2.29}$$
$$\hat{g}^2 = 2z(4+z) \tag{2.30}$$
$$\frac{\hat{m}^2}{2} = \frac{6+z \mp 2\sqrt{2(z+4)}}{z} \tag{2.31}$$
$$\frac{1}{\hat{a}^2} = 6 + 3z \mp 2\sqrt{2(z+4)}\,. \tag{2.32}$$

The internal hyperbolic space can be taken to be non-compact or compact. For our purposes, the compact case is most interesting, and we obtain this case as follows. The 2D hyperboloid can be written as a coset space $SO(1,2)/SO(2)$, and modding out by a freely acting, discrete non-compact subgroup of the isometry group, $SO(1,2)$, we arrive at a compact manifold. This manifold can be seen as a Riemann surface of some genus, which depends on the choice of subgroup (see e.g. [13]). Such spaces have been much discussed in the compactification literature, beginning with [12]. Notice now that, although classically solutions exist for all dynamical exponents $z \geq 1$, the quantization condition for the flux threading the compact internal manifold provides a relation between z and the parameters of the 6D theory, g, m.

It is easy to see that all the Lifshitz solutions ($z \neq 1$) break supersymmetry. The relevant supersymmetry transformations are the fermionic gravitini and dilatini ones. They take the form [11]:

$$\begin{aligned}\delta\psi_{\mu\, a} &= D_\mu \epsilon_a - \frac{1}{8\sqrt{2}}\left(g e^{\phi/\sqrt{2}} + \mathrm{m}\, \mathrm{e}^{-3\phi/\sqrt{2}}\right)\gamma_\mu \gamma_7 \epsilon_a - \frac{1}{24} e^{\sqrt{2}\phi}\gamma_7 \gamma^{\rho\lambda\sigma} G_{\rho\lambda\sigma}\gamma_\mu \epsilon_a \\ &\quad - \frac{1}{4\sqrt{2}} e^{-\phi/\sqrt{2}}\left(\gamma_\mu^{\nu\rho} - 6\delta_\mu^\nu \gamma^\rho\right)\left(\frac{1}{2}\mathcal{H}_{\mu\nu}\delta_a^b + \gamma_7 F_{\mu\nu}^{(i)} T_a^{(i)\ b}\right)\epsilon_b \end{aligned} \tag{2.33}$$

$$\begin{aligned}\delta\chi_a &= \frac{1}{\sqrt{2}}\gamma^\mu(\partial_\mu\phi)\epsilon_a + \frac{1}{4\sqrt{2}}\left(g e^{\phi/\sqrt{2}} - 3\mathrm{m}\, \mathrm{e}^{-3\phi/\sqrt{2}}\right)\gamma_7 \epsilon_a - \frac{1}{12} e^{\sqrt{2}\phi}\gamma_7 \gamma^{\rho\lambda\sigma} G_{\rho\lambda\sigma}\gamma_\mu \epsilon_a \\ &\quad \frac{1}{2\sqrt{2}}\gamma^{\mu\nu}\left(\frac{1}{2}\mathcal{H}_{\mu\nu}\delta_a^b + \gamma_7 F_{\mu\nu}^{(i)} T_a^{(i)\ b}\right)\epsilon_b\,. \end{aligned} \tag{2.34}$$

It is sufficient to consider the dilatini transformation. In this case, we can see that the non-trivial fluxes that support the Lifshitz solution would require us to impose four independent projection conditions for $\delta\chi_a = 0$, and so we cannot preserve supersymmetry. This fact can also be seen at the level of the ten dimensional, uplifted solution, as we show now.

Six dimensional Romans' gauge supergravity can be uplifted via an S^4 to massive Type IIA supergravity in ten dimensions [14]. Indeed, using the results of [14], it is trivial to uplift any solution of the 6D field equations to a solution of the 10D equations of motion. We now do so for the $Li_4 \times H_2$ configuration identified above. In order to go from the six dimensional action used in [14] to Romans' conventions used here, we make the following redefinitions[3] (a tilde denotes quantities in [14] notation):

$$\widetilde{g}_{\mu\nu} = -g_{\mu\nu}\,, \qquad\qquad \widetilde{\phi} - 2\widetilde{\phi}_0 = -2\phi\,, \tag{2.35}$$

$$e^{2\sqrt{2}\,\widetilde{\phi}_0} = \frac{3\mathrm{m}}{g}\,, \qquad\qquad \widetilde{g} = \frac{\left(3\mathrm{m}g^3\right)^{1/4}}{2}\,, \tag{2.36}$$

$$\frac{1}{2}e^{-\sqrt{2}\,\widetilde{\phi}_0}\,\widetilde{B}_2 = B_2\,, \qquad\qquad \frac{1}{2}e^{\sqrt{2}/2\,\widetilde{\phi}_0}\,\widetilde{F}_2^{(i)} = F_2^{(i)}\,. \tag{2.37}$$

Using this dictionary, we can now write the 10D solution using the formulae in [14]. Defining

$$k_0 = e^{\phi_0/\sqrt{2}}\left(\frac{g}{3\mathrm{m}}\right)^{1/4} \tag{2.38}$$

$$C(\rho) = \cos\rho\,, \qquad S(\rho) = \sin\rho \tag{2.39}$$

$$\Delta(\rho) = k_0\,C^2 + k_0^{-3}\,S^2 \tag{2.40}$$

$$U(\rho) = k_0^{-6}\,S^2 - 3k_0^2\,C^2 + 4k_0^{-2}\,C^2 - 6k_0^{-2}\,, \tag{2.41}$$

as well as the constants

$$k_1 = \frac{8}{g^2}\frac{g}{3\mathrm{m}}e^{\sqrt{2}\,\phi_0}, \quad k_2 = \frac{2}{g^2}\left(\frac{g}{3\mathrm{m}}\right)^{1/4} e^{-\phi_0/\sqrt{2}}, \quad k_3 = -\frac{4\sqrt{2}}{3}\frac{1}{g^3}\left(\frac{g}{3\mathrm{m}}\right)^{3/4}, \tag{2.42}$$

$$k_4 = 3g^2 e^{2\sqrt{2}\phi_0} k_3\,, \quad k_5 = 3gk_3\,, \quad k_6 = -\frac{2\sqrt{2}\,e^{-3\phi_0/\sqrt{2}}}{g^2}, \quad k_7 = 2\left(\frac{3\mathrm{m}}{g}\right)^{1/2},$$

the ten dimensional, uplifted configuration that results is

$$\begin{aligned}
ds_{10}^2 &= S^{1/12}\,k_0^{1/8}\left[\Delta^{3/8}(Li_4 \times H_2) - k_1\Delta^{3/8}\,d\rho^2 - k_2\Delta^{-5/8}C^2\sum_i^3 (h^{(i)})^2\right],\\
\mathbf{F_4} &= k_3\,S^{1/3}\,C^3\,\Delta^{-2}\,U\,d\rho\wedge\epsilon_3 + k_4\,S^{1/3}C\,\star_6\,G_3\wedge d\rho\\
&\quad + k_5\,S^{1/3}\,C\,F_2^{(3)}\wedge h^{(3)}\wedge d\rho + k_6\,S^{4/3}\,C^2\Delta^{-1}\,F_2^{(3)}\wedge\sigma^{(1)}\wedge\sigma^{(2)}\,,\\
\mathbf{G_3} &= k_7\,S^{2/3}\,G_3\,, \quad \mathbf{F_2} = 0\,,\\
e^{\Phi} &= S^{-5/6}\,\Delta^{1/4}\,k_0^{-5/4}\,,
\end{aligned} \tag{2.43}$$

where

$$h^{(i)} = \sigma^{(i)} - g\,A_1^{(i)}\,, \tag{2.44}$$

with $\sigma^{(i)}$ the left-invariant 1-forms on S^3, and $\epsilon_3 = h^{(1)}\wedge h^{(2)}\wedge h^{(3)}$. The parameters of the 6D theory are related to the Type IIA mass parameter via $\mathbf{m} = \left(2\,\mathrm{m}\,g^3/27\right)^{1/4}$. Notice that the ten dimensional RR $\mathbf{F_2}$ field strength vanishes, while the RR $\mathbf{F_4}$ field and the NS $\mathbf{G_3}$ field are switched on. The uplifted solution (2.43) contains the six dimensional fields $F_2^{(3)}$, G_3 and $\star_6 G_3$, which we recall here:

$$\begin{aligned}
F_2^{(3)} &= \hat{\gamma}\,e^{\frac{\sqrt{2}\phi_0}{2}}\left[\sqrt{z-1}L\,r^{z-1}dt\wedge dr + \frac{a^2}{L\,y_2^2}\,dy_1\wedge dy_2\right]\\
G_3 &= e^{-\sqrt{2}\phi_0}\,L^2\,\sqrt{z-1}\,r\,dx_1\wedge dx_2\wedge dr\\
\star_6 G_3 &= e^{-\sqrt{2}\phi_0}\,\frac{a^2}{y_2^2}\,\sqrt{z-1}\,r^z\,dt\wedge dy_1\wedge dy_2\,.
\end{aligned} \tag{2.45}$$

[3] In addition, we take conventions where $\kappa_6^2 = 2$ rather than $\kappa_6^2 = 1/2$ as was taken in [14].

We can see the effects of the various charges in the 10D solution. The 3-form flux, G, lifts both directly to the 3-form $\mathbf{G}_3$, as well as contributing to the 4-form. $\mathbf{F}_4$ also contains a geometric term and contributions from the gauge field $F_2^{(3)}$, which now appears as a Kaluza-Klein (KK) field in the angular directions of the transverse 4D space[4]. The fluxes in 10D will give rise to further quantization conditions. At $\rho = \pi/2$, the metric has a coordinate singularity, as can be seen by considering the limit $k_2/\Delta k_1 \to 1/4$ as $\rho \to \pi/2$. Due to the overall factors $S^{1/12}$, the metric is also singular at $\rho = 0, \pi$.

The previous configuration can be interpreted as a system of D4-D8 branes, intersecting with D2 branes and an NS5 brane. We can understand this as follows. Take first $z = 1$, in this case the NS gauge field turns off, as well as the parts of the RR $\mathbf{F_4}$ field that contain a time component. Therefore, $\mathbf{F_4}$ is magnetically sourced by D4 branes, whereas $\mathbf{F_{10}}$ (associated with the IIA mass) is sourced electrically by D8 branes. One ends up with a D4-D8 system, which can preserve supersymmetry for certain configurations, see e.g. [19]. In contrast, when $z > 1$, the NS field, as well as the remaining components of the RR $\mathbf{F_4}$ field are turned on. These fields are, respectively, sourced magnetically by NS5 branes and electrically by D2 branes, which all intersect the previous D4-D8 system. Consequently, the four dimensional space-time symmetry is reduced from Lorentz to Lifshitz, and supersymmetry is broken completely.

3. Li_3 solutions in Type IIB via 5D

In this section we present three dimensional Lifshitz string solutions, which are dual to field theories in $1+1$ dimensions. We find the solutions via compactifications of $\mathcal{N} = 4$ 5D gauged supergravity, which can be uplifted to Type IIB supergravity in ten dimensions, or eleven dimensional supergravity. The steps are very similar to the 6D case of the previous section. After presenting the 5D action and field equations, we find the general $Li_3 \times \Omega_2$ solutions, where Ω_2 again turns out to be restricted to a hyperboloid. We show that supersymmetry is broken, and then we explain how to embed the solution in Type IIB string theory.

The $\mathcal{N} = 4$ 5D gauged supergravity was developed in [15], and our conventions are the same as the original reference. The field content consists of the metric, $g_{\mu\nu}$, dilaton, ϕ, gauge fields, $(A_\mu^{(i)}, \mathcal{A}_\mu)$, for an $SU(2) \times U(1)$ gauge group, two antisymmetric tensor fields, $B_{\mu\nu}^\alpha$ (α indicates the real two dimensional vector representation of $U(1)$), and the fermionic partners. The bosonic part of the Lagrangian is:

$$\begin{aligned} \mathrm{e}^{-1}\mathcal{L} = & -\frac{1}{4}R + \frac{1}{2}D_\mu\phi D^\mu\phi - \frac{1}{4}\xi^{-4}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} - \frac{1}{4}\xi^2\left(F_{\mu\nu}^{(i)}F^{\mu\nu\,(i)} + B^{\mu\nu\alpha}B_{\mu\nu}^\alpha\right) \\ & + \frac{1}{4}\,\mathrm{e}\,\epsilon^{\mu\nu\rho\sigma\lambda}\left(\frac{1}{g_1}\epsilon_{\alpha\beta}B_{\mu\nu}^\alpha D_\rho B_{\sigma\lambda}^\beta - F_{\mu\nu}^{(i)}F_{\rho\sigma}^{(i)}\mathcal{A}_\lambda\right) + P(\phi)\,, \end{aligned} \tag{3.1}$$

where we have defined $\xi \equiv e^{\sqrt{\frac{2}{3}}\phi}$, and the scalar field potential is

$$P(\phi) = \frac{g_2}{8}\left(g_2\,\xi^{-2} + 2\sqrt{2}\,g_1\,\xi\right)\,. \tag{3.2}$$

[4]The presence of a KK gauge field is similar to the Type IIB Lifshitz solution presented in [8].

Also, the field strengths are as usual:

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu \tag{3.3}$$

$$F^{(i)}_{\mu\nu} = \partial_\mu A^{(i)}_\nu - \partial_\nu A^{(i)}_\mu + g_2\, \epsilon^{ijk} A^{(j)}_\mu A^{(k)}_\nu \,, \tag{3.4}$$

and g_2, g_1 are the gauge couplings for $SU(2) \times U(1)$, respectively. The 5D gauged supergravity thus has two independent parameters, g_1, g_2, which give rise to three physically distinct theories. Following Romans [15], when $g_1 g_2 > 0$, we call the theory $\mathcal{N} = 4^+$, when $g_2 = 0$ we call it $\mathcal{N} = 4^0$, and when $g_1 g_2 < 0$ we have $\mathcal{N} = 4^-$.

The equations of motion that result from the Lagrangian (3.1):

$$R_{\mu\nu} = 2\partial_\mu\phi\partial_\nu\phi + \frac{4}{3} g_{\mu\nu} P - \xi^{-4}\left(2\mathcal{F}_{\mu\rho}\mathcal{F}^{\rho}_{\nu} - \frac{1}{3} g_{\mu\nu}\mathcal{F}_{\rho\sigma}\mathcal{F}^{\rho\sigma}\right) - \xi^{2}\left(2F^{(i)}_{\mu\rho}F^{\rho\ (i)}_{\nu} - \frac{1}{3} g_{\mu\nu}\, F^{(i)}_{\rho\sigma}F^{\rho\sigma\ (i)}\right) \tag{3.5}$$

$$\Box\phi = \frac{\partial P}{\partial\phi} + \sqrt{\frac{2}{3}}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \sqrt{\frac{1}{6}}\xi^2\, F^{(i)}_{\rho\sigma}F^{\rho\sigma\ (i)} \tag{3.6}$$

$$D_\nu\left(\xi^{-4}\mathcal{F}^{\nu\mu}\right) = \frac{1}{4} e^{-1}\epsilon^{\mu\nu\rho\sigma\tau}\, F^{(i)}_{\nu\rho}F^{(i)}_{\sigma\tau} \tag{3.7}$$

$$D_\nu\left(\xi^{2}F^{\nu\mu\,(i)}\right) = \frac{1}{2} e^{-1}\epsilon^{\mu\nu\rho\sigma\tau}\, F^{(i)}_{\nu\rho}\mathcal{F}_{\sigma\tau} \,, \tag{3.8}$$

where we have set the antisymmetric tensor fields $B^{\alpha}_{\mu\nu}$ to zero, in view of our Ansatz below. The maximally symmetric solutions to these equations were studied in [15]. The $\mathcal{N} = 4^+$ theory was found to have vacua of the form $\mathrm{adS}_3 \times \Omega_2$ for Ω_2 a sphere, hyperboloid or flat space. The $\mathcal{N} = 4^0$ and $\mathcal{N} = 4^-$ theories admit similar vacua, but only for Ω_2 a hyperboloid.

Given the above, we assume an Ansatz of the form $Li_3 \times \Omega_2$, with any constant curvature internal 2D space. The metric is then:

$$ds^2 = L^2\left(r^{2z}dt^2 - r^2dx^2 - \frac{dr^2}{r^2}\right) - L^2 d\Omega_2^2 \,, \tag{3.9}$$

where, $d\Omega_2^2$ is given by flat space or (2.13) for the sphere or hyperboloid, as before. Similarly, our Ansatz for the gauge fields, motivated by the symmetries, is:

$$\mathcal{F}_{rt} = \frac{\xi_0^2\,\alpha_1}{L} r^{z-1} \;;\; \mathcal{F}_{rx} = \frac{\xi_0^2\,\beta_1}{L} \;;\; \mathcal{F}_{y_1y_2} = \frac{\xi_0^2\,\gamma_1}{L}\mathrm{e}_2 \tag{3.10}$$

$$F^{(3)}_{rt} = \frac{\xi_0^{-1}\,\alpha_2}{L} r^{z-1} \;;\; F^{(3)}_{rx} = \frac{\xi_0^{-1}\,\beta_2}{L} \;;\; F^{(3)}_{y_1y_2} = \frac{\xi_0^{-1}\,\gamma_2}{L}\mathrm{e}_2 \,, \tag{3.11}$$

and the scalar field is constant, $\phi = \phi_0$. Here, e_2 is the square-root of the determinant of the metric $d\Omega_2^2$. Finally, it again proves useful to rescale the quantities g_1 and g_2 in such a way as to absorb into their values the free parameters L and ϕ_0:

$$g_1 = \frac{\hat{g}_1\xi_0^{-2}}{L} \tag{3.12}$$

$$g_2 = \frac{\hat{g}_2\xi_0}{L} \,. \tag{3.13}$$

Thus in the following we will consider $\hat{g}_1$ and $\hat{g}_2$ as free parameters, to be fixed by the equations of motion.

As before, this Lifshitz Ansatz reduces the field equations to a set of (ten) algebraic equations[5]:

$$\begin{aligned}
\alpha_2 &= 2\gamma_2\beta_1 + 2\gamma_1\beta_2 && (3.14)\\
z\beta_2 &= 2\alpha_2\gamma_1 + 2\gamma_2\alpha_1 && (3.15)\\
\alpha_1 &= 2\gamma_2\beta_2 && (3.16)\\
z\beta_1 &= 2\alpha_2\gamma_2 && (3.17)\\
0 &= z(z+1) - \frac{4}{3}P - \frac{4}{3}(\alpha_1^2+\alpha_2^2) - \frac{2}{3}(\beta_1^2+\beta_2^2) - \frac{2}{3}(\gamma_1^2+\gamma_2^2) && (3.18)\\
0 &= z+1 - \frac{4}{3}P + \frac{2}{3}(\alpha_1^2+\alpha_2^2) + \frac{4}{3}(\beta_1^2+\beta_2^2) - \frac{2}{3}(\gamma_1^2+\gamma_2^2) && (3.19)\\
0 &= z^2+1 - \frac{4}{3}P - \frac{4}{3}(\alpha_1^2+\alpha_2^2) + \frac{4}{3}(\beta_1^2+\beta_2^2) - \frac{2}{3}(\gamma_1^2+\gamma_2^2) && (3.20)\\
0 &= \alpha_1\beta_1 + \alpha_2\beta_2 && (3.21)\\
0 &= -\frac{\lambda}{a^2} - \frac{4}{3}P + \frac{2}{3}(\alpha_1^2+\alpha_2^2) - \frac{2}{3}(\beta_1^2+\beta_2^2) + \frac{4}{3}(\gamma_1^2+\gamma_2^2) && (3.22)\\
0 &= \sqrt{\frac{3}{2}}\frac{dP}{d\phi} + 2(\beta_1^2+\gamma_1^2-\alpha_1^2) - (\beta_2^2+\gamma_2^2-\alpha_2^2)\,, && (3.23)
\end{aligned}$$

with

$$\begin{aligned}
P &= \frac{\hat{g}_2}{8}\left(\hat{g}_2 + 2\sqrt{2}\hat{g}_1\right) && (3.24)\\
\frac{dP}{d\phi} &= \sqrt{\frac{2}{3}}\frac{\hat{g}_2}{4}\left(-\hat{g}_2 + \sqrt{2}\hat{g}_1\right)\,. && (3.25)
\end{aligned}$$

Here, we have introduced the parameter $\lambda = 1, 0, -1$, to include the cases of the 2D sphere, flat space and hyperboloid, respectively, at once.

In total we have nine free parameters, α_k, β_k, γ_k, a, $\hat{g}_1$, $\hat{g}_2$, with $k = 1, 2$, to fix by means of the equations. Although we have more equations than unknowns, as before it turns out that not all the equations are independent and solutions can exist for every value of $z \geq 1$. Indeed, it is straightforward to solve the previous system of equations completely, as we demonstrate in detail in Appendix A. There are two sets of solutions, which are qualitatively similar:

- $\alpha_1 = 0 = \beta_2$

[5]We are grateful to Jerome Gauntlett and Aristomenis Donos for pointing out that Equation (3.21) was missing in the original version of this paper.

The solution takes the form

$$
\begin{aligned}
\alpha_2^2 &= \frac{z(z-1)}{2} && (3.26)\\
\beta_1^2 &= \frac{(z-1)}{2} && (3.27)\\
\gamma_1^2 &= 0 && (3.28)\\
\gamma_2^2 &= \frac{z}{4} && (3.29)\\
\frac{2\lambda}{a^2} &= -3z && (3.30)\\
\hat{g}_2^2 &= 2z^2+3z-2 && (3.31)\\
\hat{g}_1^2 &= \sqrt{2}\,(1+z)\ . && (3.32)
\end{aligned}
$$

We must have $z \geq 1$.

- $\alpha_2 = 0 = \beta_1$

The solution takes the form

$$
\begin{aligned}
\alpha_1^2 &= \frac{z(z-1)}{2} && (3.33)\\
\beta_2^2 &= \frac{(z-1)}{2} && (3.34)\\
\gamma_1^2 &= 0 && (3.35)\\
\gamma_2^2 &= \frac{z}{4} && (3.36)\\
\frac{2\lambda}{a^2} &= -3z && (3.37)\\
\hat{g}_2^2 &= -2z^2+3z+2 && (3.38)\\
\hat{g}_1^2 &= \frac{1}{\sqrt{2}}\left(2z^2+z+1\right)\ . && (3.39)
\end{aligned}
$$

Here, to ensure that the gauge couplings are real, we must restrict $1 \leq z \leq 2$.

Notice that, since $\lambda < 0$ in both cases, the geometry turns out to be restricted to $Li_3 \times H_2$. Also, similar to the 6D case, quantization of the internal fluxes implies a topological relation between z and the 5D gauge couplings, g_1, g_2.

Although some of the adS$_3 \times \Omega_2$ solutions partially preserve supersymmetry, breaking $\mathcal{N} = 4$ to $\mathcal{N} = 1$ [15], it is again easy to see that the Lifshitz fluxes prevent any supersymmetric Lifshitz configurations. The spinorial supersymmetry transformations are (setting

$B^{\alpha}_{\mu\nu}=0$):

$$\begin{aligned}\delta\psi_{\mu a} &= D_\mu\epsilon_a+\gamma_\mu\left(\frac{1}{6}\sqrt{\frac{1}{2}}g_2\xi^{-1}+\frac{1}{12}g_1\xi^2\right)T_{ab}\epsilon^b\\ &\quad -\frac{1}{6}\sqrt{\frac{1}{6}}\left(\gamma_\mu{}^{\nu\rho}-4\delta^\nu_\mu\gamma^\rho\right)\left(\xi F^{(i)}_{\nu\rho}T_{(i)\,ab}-\sqrt{\frac{1}{2}}\xi^{-2}\mathcal{F}_{\nu\rho}\Omega_{ab}\right)\epsilon^b \qquad (3.40)\\ \delta\chi_a &= \sqrt{\frac{1}{2}}\gamma^\mu(\partial_\mu\phi)\epsilon_a+\left(\frac{1}{2}\sqrt{\frac{1}{6}}g_2\xi^{-1}-\frac{1}{2}\sqrt{\frac{1}{3}}g_1\xi^2\right)T_{ab}\epsilon^b\\ &\quad -\frac{1}{2}\sqrt{\frac{1}{6}}\gamma^{\mu\nu}\left(\xi F^{(i)}_{\mu\nu}T_{(i)\,ab}-\sqrt{2}\xi^{-2}\mathcal{F}_{\mu\nu}\Omega_{ab}\right)\epsilon^b\,, \qquad (3.41)\end{aligned}$$

where $T^{(i)}_{ab}, T_{ab}$ are generators of the $SU(2)\times U(1)$ gauge group, and Ω_{ab} is a metric used to raise and lower any spinor index. Plugging our Lifshitz solutions into the dilatino supersymmetry transformation, we see that three independent projection conditions would be required to make it vanish, showing that no supersymmetry can survive.

The 5D Romans' theory has been lifted to Type IIB supergravity in ten dimensions in [16] by means of an S^5 reduction. Building on this result, 11D interpretations of the 5D theory were given in [17] and [18]. Here, we use the results of [16] to uplift our 5D Lifshitz solutions to solutions of the Type IIB supergravity equations of motion.

First, we write down the dictionary to go from the conventions in [16] to Romans' conventions used above. This requires the following redefinitions of the fields and parameters[6] (a tilde denotes quantities in [16] notation):

$$\widetilde{g}_{\mu\nu}=-g_{\mu\nu}\,, \qquad \widetilde{\phi}+2\widetilde{\phi}_0=2\phi \qquad (3.42)$$

$$\sqrt{2}\,e^{3\sqrt{2/3}\,\widetilde{\phi}_0}=\frac{g_2}{g_1}\,, \qquad \widetilde{g}=\left(\frac{g_1\,g_2^2}{16}\right)^{1/3} \qquad (3.43)$$

$$\frac{1}{2}e^{2\sqrt{2/3}\,\widetilde{\phi}_0}\,\widetilde{\mathcal{A}}_1=\mathcal{A}_1\,, \qquad \frac{1}{2}e^{-\sqrt{2/3}\,\widetilde{\phi}_0}\,\widetilde{F}^{(i)}_2=F^{(i)}_2\,. \qquad (3.44)$$

Using this dictionary, we can immediately write down our ten dimensional solution using the formulae in [16]. It is convenient to define:

$$k_0=\xi_0^{-1}\left(\frac{g_2}{g_1\sqrt{2}}\right)^{1/3} \qquad (3.45)$$

$$C(\rho)=\cos\rho\,, \qquad S=\sin\rho \qquad (3.46)$$

$$\Delta(\rho)=k_0^{-2}\,S^2+k_0\,C^2 \qquad (3.47)$$

$$U(\rho)=k_0^{-1}\,S^2+k_0^2\,C^2+k_0^{-1}\,, \qquad (3.48)$$

along with the constants:

$$k_1=\frac{4\sqrt{2}}{\xi_0\,g_1g_2},\quad k_2=\left(\frac{g_1\,g_2^2}{2}\right)^{1/3},\quad k_3=\frac{2\xi_0^2}{g_2k_2},\quad k_4=-\frac{8k_0^2}{k_2^2\xi_0^2},$$
$$k_5=-\frac{4}{k_2^4},\quad k_6=-\frac{4\xi_0^2}{g_2k_2^2},\quad k_7=\frac{\sqrt{2}\,\xi_0k_1}{k_2},\quad k_8=-\frac{2}{\xi_0^2k_0k_2^3}. \qquad (3.49)$$

[6] We should also take into account the convention $\kappa_5^2=1/2$ taken in [16]. Note also that in the final 10D expressions in [16] they absorbed g_1, g_2 in a single $\tilde{g}$.

The solution then reads:

$$
\begin{aligned}
ds^2_{10} &= \Delta^{1/2}(Li_3 \times d\Omega_2^2) - k_1 \Delta^{-1/2}\left[\Delta d\rho^2 + k_0 S^2 (d\eta - g_1 \mathcal{A}_1)^2 + \frac{C^2}{4k_0^2}\sum_i^3 (h^{(i)})^2\right] \\
\mathbf{F_5} &= k_2 U \epsilon_5 + k_3\, C^2 \star_5 F_2^{(3)} \wedge \sigma^{(1)} \wedge \sigma^{(2)} - 2k_3\, S\, C\, \star_5 F_2^{(3)} \wedge h^{(3)} \wedge d\rho \\
&\qquad + k_4 \star_5 \mathcal{F}_2 \wedge d\rho \wedge (d\eta - g_1 \mathcal{A}_1) \\
\mathbf{F_3} &= 0, \quad \mathbf{G_3} = 0, \quad \Phi = 0, \quad \chi = 0\,,
\end{aligned} \tag{3.50}
$$

and we may also write down the ten dimensional Hodge dual of the RR five-form as:

$$
\begin{aligned}
\star\mathbf{F_5} \;=\; & k_5\, S\, C^3\, U\, \Delta^{-2} d\rho \wedge (d\eta - g_1 \mathcal{A}_1) \wedge \sigma^{(1)} \wedge \sigma^{(2)} \wedge h^{(3)} \\
& + k_6\, S^2\, C^2\, \Delta^{-1} F_2^{(3)} \wedge \sigma^{(1)} \wedge \sigma^{(2)} \wedge (d\eta - g_1 \mathcal{A}_1) \\
& + k_7\, S\, C\, F_2^{(3)} \wedge h^{(3)} \wedge d\rho \wedge (d\eta - g_1 \mathcal{A}_1) + k_8\, C^4\, \Delta^{-1} \mathcal{F}_2 \wedge \sigma^{(1)} \wedge \sigma^{(2)} \wedge h^{(3)}\,.
\end{aligned} \tag{3.51}
$$

Here, the 1-forms $h^{(i)}$ are now given in terms of the left-invariant 1-forms on S^3 as:

$$
h^{(i)} = \sigma^{(i)} - g_2\, A_1^{(i)}\,, \tag{3.52}
$$

and ϵ_5 is the volume form in the five dimensional $Li_3 \times \Omega_2$ space. We also recall the 5D fields $\mathcal{F}_2$, $\star_5\mathcal{F}_2$, $F_2^{(3)}$ and $\star_5 F_2^{(3)}$:

$$
\begin{aligned}
\mathcal{F}_2 &= \frac{\xi_0^2}{L}\left[\alpha_1 r^{z-1}\, dr \wedge dt \,+\, \beta_1\, dr \wedge dx\right], \\
\star_5 \mathcal{F}_2 &= \xi_0^2\, r^z\, \mathrm{e}_2 \left[-\alpha_1 r^{1-z}\, dx \wedge dy_1 \wedge dy_2 \,-\, \beta_1\, dt \wedge dy_1 \wedge dy_2\right], \\
F_2^{(3)} &= \frac{\xi_0^{-1}}{L}\left[\alpha_2 r^{z-1}\, dr \wedge dt \,+\, \beta_2\, dr \wedge dx \,+\, \gamma_2\, \mathrm{e}_2\, dy_1 \wedge dy_2\right], \\
\star_5 F_2^{(3)} &= \xi_0^{-1}\, r^z\, \mathrm{e}_2 \left[-\alpha_2 r^{1-z}\, dx \wedge dy_1 \wedge dy_2 \,-\, \beta_2\, dt \wedge dy_1 \wedge dy_2 \,+\, \gamma_2\, \mathrm{e}_2^{-1}\, dt \wedge dx \wedge dr\right],
\end{aligned} \tag{3.53}
$$

where α_k, β_k $(k = 1, 2)$, γ_2, ξ_0 and L are z dependent constants to be read off from the 5D solution, and in particular α_k, β_k are vanishing when $z = 1$.

As with the previous Lifshitz example, we see the presence of KK gauge fields when uplifting the 5D solutions, due to the non-trivial backgrounds for $A_1^{(3)}$ and $\mathcal{A}_1$. However in this case, our ten dimensional metric is everywhere regular (apart from the usual coordinate singularities). Again, flux quantization conditions in the 10D system, will lead to further constraints on z, g_1 and g_2.

We can geometrically interpret the ten dimensional uplifted configuration as follows. When $z = 1$, the parameters α_k and β_k vanish, so that the 5D $\mathcal{F}_2$ vanishes and $F_2^{(3)}$ has components only in the internal directions. The ten dimensional dual of $\mathbf{F_5}$, (3.51), is then sourced magnetically by various intersecting D3 branes, and if the D3 brane configuration satisfies certain conditions, the system can be supersymmetric. Meanwhile, when $z > 1$, additional components of $\star\mathbf{F_5}$ are turned on, which are sourced both magnetically and electrically by further D3 branes. The overall effect is to reduce the symmetry of the three infinite dimensions from Lorentz to Lifshitz, and to break supersymmetry completely.

4. Discussion

In this paper, we provided a simple method that allowed us to obtain explicit string constructions of Lifshitz geometries for general dynamical exponents, $z \geq 1$. Following a bottom-up approach, our starting point was to look for Lifshitz solutions in d-dimensional supergravities, appropriately deforming $\mathrm{adS}_q \times \Omega_{d-q}$ solutions already known in the literature. Then we uplifted them to ten dimensional configurations. First, we considered the gauged, massive $\mathcal{N} = 4$ six dimensional supergravity, and showed that it admits a solution of the form $Li_4 \times H_2$, with Li_4 characterized by dynamical exponents z larger than one, which may be subject to quantization conditions, and H_2 a hyperboloid that can be compact. Then we discussed the uplifting of this geometry to massive Type IIA string theory, giving a basic interpretation of the resulting configuration in terms of intersecting branes of various dimensions. Second, we considered gauged $\mathcal{N} = 4$ five dimensional supergravity. We found that this admits solutions of the form $Li_3 \times H_2$. The resulting geometry can be uplifted to IIB string theory, and can be interpreted as a system of intersecting D3 branes. It would be interesting to study more deeply the brane interpretations of our 10D configurations.

Our results indeed suggest various issues that deserve further investigation. In Ref. [8], the authors argue that their 10/11D Lifshitz compactifications can be supersymmetric when they are based on Sasaki-Einstein manifolds. In our case in contrast, it is easy to see that supersymmetry is broken. For example, looking at the dilatino supersymmetry transformation in 6D supergravity, the 4D fluxes which we use to support the Lifshitz geometry lead to a proliferation of projection conditions. Since we do not have supersymmetry, there is no reason to believe that our solutions are stable, and it would be important to investigate this issue. Along these lines, it is intriguing to recall a parallel discussion in the literature on non-relativistic Schrödinger solutions. Also there, both supersymmetric and non-supersymmetric solutions have been found in string theory. Among these are the supersymmetric solutions of [20] (albeit with kinematical supersymmetry only [21]), which were surprisingly found to be unstable. Moreover it was argued in [20] that turning on supersymmetry breaking fluxes can actually help to restore stability.

We should also note that not all known $\mathrm{adS}_q \times \Omega_{d-q}$ supergravity solutions can be generalized to Lifshitz solutions. For example, $\mathcal{N} = 2$ 8D gauged supergravity has an $\mathrm{adS}_4 \times S^4$ background [22], but it turns out that the simple extension to a Lifshitz Ansatz is inconsistent with the equations of motion. The same can be said of the $\mathrm{adS}_3 \times S^3$ solution to $\mathcal{N} = 4$ gauged, massless, 6D supergravity [11]. It would be interesting to understand what makes our working examples special. One characteristic that seems to distinguish them is that the internal space is a negative curvature hyperboloid, although we do not know yet whether or not this is a coincidence.

Lastly, the main advantage of our approach is its simplicity and the solutions presented may be useful in developing the Li/CMP correspondence further. A possible next step is to find Lifshitz black hole solutions and study their properties, which we leave for future work.

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A. Appendix: Details of the five dimensional solution

In this appendix, we analyse in detail the system of the ten equations (3.14)−(3.23) associated with Lifshitz configurations in five dimensional gauged supergravity. We determine the general solutions to these equations, providing the values of the nine free parameters α_k, β_k, γ_k, a, $\hat{g}_1$, $\hat{g}_2$ as a function of z.

There are combinations of the equations that provide simple relations among the previous free parameters. Taking the differences between (3.18) and (3.20), and between (3.18) and (3.19), we obtain

$$\frac{z-1}{2} = \beta_1^2 + \beta_2^2 \tag{A.1}$$

$$\frac{z(z-1)}{2} = \alpha_1^2 + \alpha_2^2\,. \tag{A.2}$$

The first of the previous equations show that $z \geq 1$. These two equations also imply

$$\alpha_1^2 - z\beta_1^2 \;=\; z\beta_2^2 - \alpha_2^2\,. \tag{A.3}$$

Meanwhile, multiplying together Eqs. (3.16) and (3.17), and imposing (3.21), leads to the condition:

$$\alpha_1\beta_1 \;=\; 0 \;=\; \alpha_2\,\beta_2\,. \tag{A.4}$$

Thus, we may take either $\alpha_1 = 0 = \beta_2$ or $\alpha_2 = 0 = \beta_1$. In both cases, Eqs. (3.14)-(3.17) then imply $\gamma_1 = 0$ and $\gamma_2^2 = z/4$. Now, combining Eqs. (3.19), (3.22) and (3.23) one finds

$$\frac{\lambda}{a^2} = -2\beta_1^2 + 2\gamma_2^2 - 2\beta_2^2 - z - 1 \tag{A.5}$$

$$\frac{\hat{g}_2^2}{4} = \frac{z+1}{2} + 2\beta_1^2 - \alpha_1^2 + \alpha_2^2 - \gamma_2^2 \tag{A.6}$$

$$\frac{\hat{g}_1\hat{g}_2}{\sqrt{2}} = z + 1 + 2\beta_2^2 + 2\alpha_1^2\,. \tag{A.7}$$

There are two sets of solutions, which are qualitatively similar:

• for $\alpha_1 = 0 = \beta_2$, then $2\beta_1^2 = z-1$, $2\alpha_2^2 = z(z-1)$, $\gamma_1^2 = 0$, $4\gamma_2^2 = z$, and Eqs. (A.5)-(A.7) imply

$$\frac{\lambda}{a^2} = -\frac{3}{2}z \tag{A.8}$$

$$\hat{g}_2^2 = 2z^2 + 3z - 2 \tag{A.9}$$

$$\hat{g}_1^2 = \sqrt{2}(1+z) \tag{A.10}$$

This solution is valid for all $z \geq 1$. Notice that the internal space corresponds to a hyperboloid, since λ has to be negative.

• for $\alpha_2 = 0 = \beta_1$, then $2\beta_2^2 = z-1$, $2\alpha_1^2 = z(z-1)$, $\gamma_1^2 = 0$, $4\gamma_2^2 = z$, and Eqs. (A.5)-(A.7) imply

$$\frac{\lambda}{a^2} = -\frac{3}{2}z \tag{A.11}$$

$$\hat{g}_2^2 = -2z^2 + 3z + 2 \tag{A.12}$$

$$\hat{g}_1^2 = \frac{1}{\sqrt{2}}\left(2z^2 + z + 1\right) \tag{A.13}$$

This solution is physical only for $1 \leq z \leq 2$. The internal space is again a hyperboloid.

References

[1] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[2] S. A. Hartnoll, "Lectures on holographic methods for condensed matter physics," Class. Quant. Grav. **26**, 224002 (2009) [arXiv:0903.3246 [hep-th]].
S. Sachdev, "Condensed matter and AdS/CFT," arXiv:1002.2947 [hep-th].

[3] S. Kachru, X. Liu and M. Mulligan, "Gravity Duals of Lifshitz-like Fixed Points," Phys. Rev. D **78** (2008) 106005 [arXiv:0808.1725 [hep-th]].

[4] W. Li, T. Nishioka and T. Takayanagi, "Some No-go Theorems for String Duals of Non-relativistic Lifshitz-like Theories," JHEP **0910**, 015 (2009) [arXiv:0908.0363 [hep-th]].

[5] J. Blaback, U. H. Danielsson and T. Van Riet, "Lifshitz backgrounds from 10d supergravity," JHEP **1002**, 095 (2010) [arXiv:1001.4945 [hep-th]].

[6] S. A. Hartnoll, J. Polchinski, E. Silverstein and D. Tong, "Towards strange metallic holography," JHEP **1004**, 120 (2010) [arXiv:0912.1061 [hep-th]].

[7] K. Balasubramanian and K. Narayan, "Lifshitz spacetimes from AdS null and cosmological solutions," JHEP **1008**, 014 (2010) [arXiv:1005.3291 [hep-th]].

[8] A. Donos and J. P. Gauntlett, "Lifshitz Solutions of D=10 and D=11 supergravity," arXiv:1008.2062 [hep-th].

[9] J. M. Maldacena and C. Núñez, "Supergravity description of field theories on curved manifolds and a no go theorem," Int. J. Mod. Phys. A **16** (2001) 822 [arXiv:hep-th/0007018].

[10] A. Salam and E. Sezgin, "Supergravities in Diverse Dimensions, Vol. 1, 2,", *Amsterdam, Netherlands: North-Holland (1989) 1499 p. Singapore, Singapore: World Scientific (1989)*

[11] L. J. Romans, "The F(4) Gauged Supergravity In Six-Dimensions," Nuc. Phys. B **269** (1986) 691.

[12] N. Kaloper, J. March-Russell, G. D. Starkman and M. Trodden, "Compact hyperbolic extra dimensions: Branes, Kaluza-Klein modes and cosmology," Phys. Rev. Lett. **85**, 928 (2000) [arXiv:hep-ph/0002001].

[13] D. Orlando and S. C. Park, "Compact hyperbolic extra dimensions: a M-theory solution and its implications for the LHC," JHEP **1008**, 006 (2010) [arXiv:1006.1901 [hep-th]].

[14] M. Cvetic, H. Lu and C. N. Pope, "Gauged six-dimensional supergravity from massive type IIA," Phys. Rev. Lett. **83** (1999) 5226 [arXiv:hep-th/9906221].

[15] L. J. Romans, "Gauged N=4 Supergravities In Five-Dimensions And Their Magnetovac Backgrounds," Nucl. Phys. B **267**, 433 (1986).

[16] H. Lu, C. N. Pope and T. A. Tran, "Five-dimensional N = 4, SU(2) x U(1) gauged supergravity from type IIB," Phys. Lett. B **475** (2000) 261 [arXiv:hep-th/9909203].

[17] M. Cvetic, H. Lu and C. N. Pope, "Consistent warped-space Kaluza-Klein reductions, half-maximal gauged supergravities and CP(n) constructions," Nucl. Phys. B **597** (2001) 172 [arXiv:hep-th/0007109].

[18] J. P. Gauntlett and O. Varela, "D=5 SU(2)xU(1) Gauged Supergravity from D=11 Supergravity," JHEP **0802** (2008) 083 [arXiv:0712.3560 [hep-th]].

[19] C. Núñez, I. Y. Park, M. Schvellinger and T. A. Tran, "Supergravity duals of gauge theories from F(4) gauged supergravity in six dimensions," JHEP **0104** (2001) 025 [arXiv:hep-th/0103080].

[20] S. A. Hartnoll and K. Yoshida, "Families of IIB duals for nonrelativistic CFTs," JHEP **0812** (2008) 071 [arXiv:0810.0298 [hep-th]].

[21] A. Donos and J. P. Gauntlett, "Schrodinger invariant solutions of type IIB with enhanced supersymmetry," JHEP **0910** (2009) 073 [arXiv:0907.1761 [hep-th]].

[22] A. Salam and E. Sezgin, "D = 8 Supergravity," Nucl. Phys. B **258** (1985) 284.