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LIFTING OF THE ADDITIVE GROUP SCHEME ACTIONS

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Abstract. Let B be a normal affine $\boldsymbol{C}$-domain and let A be a $\boldsymbol{C}$-subalgebra of B such that B is a finite A-module. Let δ be a locally nilpotent derivation on A. Then δ lifts uniquely to the quotient field L of B, which we denote by Δ. We consider when Δ is a locally nilpotent derivation of B. This is a classical subject treated in [17, 19, 16]. We are interested in the case where A is the G-invariant subring of B when a finite group G acts on B. As a related topic, we treat in the last section the finite coverings of log affine pseudo-planes in terms of the liftings of the $\boldsymbol{A}^1$-fibrations associated with locally nilpotent derivations.

1. Introduction. An algebraic action of the additive group scheme G_a on an affine scheme Spec A over the complex number field $\boldsymbol{C}$ is described in terms of a locally nilpotent derivation on the $\boldsymbol{C}$-algebra A (see [3]). We have to consider often the liftability of the G_a-action (or the associated $\boldsymbol{A}^1$-fibration) on Spec A via a finite covering Spec $B \to$ Spec A. This is a special case of the classical problem of lifting derivations via finite extensions of algebras which are not necessarily locally nilpotent.

Let B be an integral domain defined over $\boldsymbol{C}$ and let A be its $\boldsymbol{C}$-subalgebra such that B is a finite A-module. Given a $\boldsymbol{C}$-derivation δ on A, δ extends to the quotient field K of A. Since the quotient field L of B is a simple extension of K, L is written as $L = K(\theta)$ for some $\theta \in L$. Let $F(X)$ be the minimal polynomial of θ over K. Then it is well-known that δ lifts uniquely to a $\boldsymbol{C}$-derivation Δ on L such that $\Delta(\theta) = -F^{\delta}(\theta)/F'(\theta)$, where $F^{\delta}(X)$ is the polynomial with all the coefficients of $F(X)$ replaced by their δ-images. The derivation Δ on L does not necessarily restrict to a derivation on B, i.e., $\Delta(B) \subset B$. By Vasconcelos [19], if $\Delta(B) \subseteq B$ is satisfied, then Δ is locally nilpotent provided so is δ.

Let $\mathfrak{R}$ be the radical of the annihilator $\mathrm{Ann}(\Omega_{B/A})$ and let $\mathfrak{b} = A \cap \mathfrak{R}$, which we call the *reduced ramification ideal* and the *reduced branch ideal* of B over A, respectively. Suppose that A and B are noetherian normal domains over $\boldsymbol{C}$. According to Scheja-Storch [16] where the assumption is a little more relaxed to the effect that B and A are Krull rings, $\Delta(B) \subseteq B$ if and only if $\delta(\mathfrak{p}) \subseteq \mathfrak{p}$ for every height 1 prime divisor $\mathfrak{p}$ of $\mathfrak{b}$. In particular, if $\Omega_{B/A} = (0)$, i.e., B is unramified over A, then Δ satisfies $\Delta(B) \subseteq B$, i.e., δ lifts to a derivation Δ of B.

Since we need the liftability criterion in more algebro-geometric settings, it is desirable to have a more geometric proof of the liftability criterion for locally nilpotent derivations without the normality of rings B and A if possible. This is our first objective. *Hereafter in the first two sections, we assume that δ is locally nilpotent.* We state the following results.

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THEOREM 1.1. *Suppose that B is an affine domain over $\boldsymbol{C}$ and that B is étale over A. Then $\Delta(B) \subset B$ and Δ is a locally nilpotent derivation on B.*

In the non-étale case, we can show the following two results. Theorem 1.2 is weaker than the result of Vasconcelos [19], though the proof is different.

THEOREM 1.2. *Suppose that B is an affine $\boldsymbol{C}$-domain and that $\Delta(B) \subset B$. Then Δ is locally nilpotent if and only if there exists an element a of A such that $\delta(a) = 0$ and $B[a^{-1}]$ is étale over $A[a^{-1}]$.*

THEOREM 1.3. *Suppose that B and A are normal affine domains over $\boldsymbol{C}$. Suppose further that there exists a nonzero ideal $\mathfrak{a}$ of A satisfying the conditions*:

(1) *The ideal $\mathfrak{a}$ has height at least two.*

(2) *The associated morphism* Spec $B \to$ Spec A *is étale outside $V(\mathfrak{a})$.*

Then $\Delta(B) \subset B$ and Δ is locally nilpotent.

These theorems are proved in the next section. In the third section, we elucidate the liftability of derivations and the local nilpotency of the lifted derivations by giving examples of G-invariant derivations which satisfy or dissatisfy the assumptions for a finite group G (see Theorems 3.2, 3.5, 3.6 and 3.7). In the last section, we give algebraic characterizations for an affine normal surface to be isomorphic to $\boldsymbol{A}^2/G$ for a finite cyclic group G (see Theorem 4.4). The existence of G_a-actions on such surfaces are also treated in detail in [2].

2. Proof of theorems.

2.1. Proof of Theorem 1.1. The proof is also outlined in [8]. Since B is étale over A, it follows that $\Omega_{A/\boldsymbol{C}} \otimes_A B \cong \Omega_{B/\boldsymbol{C}}$. Since the given derivation δ on A is locally nilpotent, there exist a nonzero element $a \in A$ and an element x in $A[a^{-1}]$ such that $\delta(x) = 1$ and hence $A[a^{-1}] \cong R[x]$, where R is the kernel of δ extended to $A[a^{-1}]$. The exact sequence of differential modules applied to the inclusions $R[x] \supset R \supset \boldsymbol{C}$ yields a direct sum decomposition

$$\Omega_{R[x]/\boldsymbol{C}} \cong (\Omega_{R/\boldsymbol{C}} \otimes_R R[x]) \oplus R[x]dx\,.$$

By tensoring it with $B[a^{-1}]$, we obtain a direct sum decomposition

$$\Omega_{R[x]/\boldsymbol{C}} \otimes_{R[x]} B[a^{-1}] \cong (\Omega_{R/\boldsymbol{C}} \otimes_R B[a^{-1}]) \oplus B[a^{-1}]dx\,.$$

Since $A[a^{-1}] \subset B[a^{-1}]$ is étale, we have

$$\Omega_{B[a^{-1}]/\boldsymbol{C}} \cong \Omega_{A[a^{-1}]/\boldsymbol{C}} \otimes_{A[a^{-1}]} B[a^{-1}] \cong \Omega_{R[x]/\boldsymbol{C}} \otimes_{R[x]} B[a^{-1}]\,.$$

Hence we have a direct sum decomposition

$$\Omega_{B[a^{-1}]/\boldsymbol{C}} \cong (\Omega_{R/\boldsymbol{C}} \otimes_R B[a^{-1}]) \oplus B[a^{-1}]dx\,.$$

The derivation Δ of the quotient field L, which is the extension of δ, is given as a $B[a^{-1}]$-module homomorphism α from $\Omega_{B[a^{-1}]/\boldsymbol{C}}$ to L. Here the restriction of α onto the direct summand $\Omega_{R/\boldsymbol{C}} \otimes_R B[a^{-1}]$ is zero because δ is zero on R, and $\alpha(dx) = \Delta(x) = \delta(x) = 1$. This implies that $\Delta(B[a^{-1}]) \subset B[a^{-1}]$. In fact, for any $z \in B[a^{-1}]$, we have $\Delta(z) = \alpha(dz)$

and $dz = \omega + f dx$, where $\omega \in \Omega_{R/\boldsymbol{C}} \otimes_R B[a^{-1}]$ and $f \in B[a^{-1}]$. Then $\Delta(z) = \alpha(\omega + f dx) = f\alpha(dx) = f \in B[a^{-1}]$.

We directly show that Δ is locally nilpotent on $B[a^{-1}]$. Let $C = \operatorname{Spec} R$, let $X_a = \operatorname{Spec} A[a^{-1}]$ and let $Y_a = \operatorname{Spec} B[a^{-1}]$. Then $X_a \cong C \times \boldsymbol{A}^1$. Hence X_a is topologically contractible to C. This implies that $\pi_1(X_a) \cong \pi_1(C)$ and that Y_a is a fiber product of an algebraic scheme D and $\boldsymbol{A}^1$, where D is a finite étale covering of C. Let S be the coordinate ring of D. Then $B[a^{-1}] \cong S[x]$, where S is étale and finite over R. Since Δ is trivial on R, it follows that Δ is trivial on S. Thus Δ is locally nilpotent on $B[a^{-1}]$.

We shall show that $\Delta(B) \subset B$. Since B is A-flat, we have isomorphisms of B-modules (see [9, Theorem 7.11]):

$$\operatorname{Der}_{\boldsymbol{C}}(B, B) \cong \operatorname{Hom}_A(\Omega_{A/\boldsymbol{C}}, B) \cong \operatorname{Der}_{\boldsymbol{C}}(A, A) \otimes_A B .$$

Hence there exists a unique element Δ' of $\operatorname{Der}_{\boldsymbol{C}}(B, B)$ which corresponds to $\delta \otimes 1_B$ of $\operatorname{Der}_{\boldsymbol{C}}(A, A) \otimes_A B$. Hence the restriction of Δ' on A is the given derivation δ. By the uniqueness of the extended derivation on L, we conclude that $\Delta' = \Delta$. Thereby, we conclude that $\Delta(B) \subset B$.

It is now easy to see that Δ itself is a locally nilpotent derivation since Δ restricted on $B[a^{-1}]$ is locally nilpotent.

2.2. Proof of Theorem 1.2. Suppose first that $\delta(a) = 0$ and $B[a^{-1}]$ is étale over $A[a^{-1}]$ for an element a of A. By Theorem 1.1, the derivation δ on $A[a^{-1}]$ lifts to a locally nilpotent derivation δ_a on $B[a^{-1}]$. Then δ_a coincides with Δ on $B[a^{-1}]$. Let b be any element of B. Then $\Delta^n(b) = \delta_a^n(b)$ which is zero if $n \gg 0$. Hence Δ is a locally nilpotent derivation of B.

Consider the differential module $\Omega_{B/A}$ which is a finite B-module. For a prime ideal $\mathfrak{P}$ of B and its contraction $\mathfrak{p} = \mathfrak{P} \cap A$, the extension $B_{\mathfrak{P}}$ is ramified over $A_{\mathfrak{p}}$ if and only if $\Omega_{B/A} \otimes_B B_{\mathfrak{P}} \cong \Omega_{B_{\mathfrak{P}}/A_{\mathfrak{p}}} \neq (0)$. This condition is equivalent to the condition that $\mathfrak{R} \subset \mathfrak{P}$, where $\mathfrak{R}$ is the reduced ramification ideal. Let $\mathfrak{b} = \mathfrak{R} \cap A$ be the reduced branch ideal. Suppose now that Δ is locally nilpotent. Then Δ defines a G_a-action $\tau : G_a \times Y \to Y$ on $Y = \operatorname{Spec} B$ which extends the G_a-action $\sigma : G_a \times X \to X$ on $X = \operatorname{Spec} A$, i.e., $p \cdot \tau = \sigma \cdot (\mathrm{id}_{G_a} \times p)$ holds on $G_a \times Y$, where $p : Y \to X$ is the natural finite morphism. For any $\lambda \in \boldsymbol{C}$, denote by $\lambda(\mathfrak{P})$ (resp. $\lambda(\mathfrak{p})$) the image $\tau(\lambda, \mathfrak{P})$ (resp. $\sigma(\lambda, \mathfrak{p})$). Then $\lambda(\mathfrak{p}) = \lambda(\mathfrak{P}) \cap A$, and $B_{\lambda(\mathfrak{P})}$ is unramified over $A_{\lambda(\mathfrak{p})}$ if and only if so is $B_{\mathfrak{P}}$ over $A_{\mathfrak{p}}$. In other words, $\lambda(\mathfrak{P}) \supset \mathfrak{R}$ if and only if $\mathfrak{P} \supset \mathfrak{R}$, and hence, $\lambda(\mathfrak{p}) \supset \mathfrak{b}$ if and only if $\mathfrak{p} \supset \mathfrak{b}$. This implies that $\lambda(\mathfrak{R}) = \mathfrak{R}$ (resp. $\lambda(\mathfrak{b}) = \mathfrak{b}$) for every $\lambda \in \boldsymbol{C}$. This can be said that $\mathfrak{R}$ (resp. $\mathfrak{b}$) is a Δ-ideal of B (resp. δ-ideal of A) in the sense that $\Delta(\mathfrak{R}) \subseteq \mathfrak{R}$ (resp. $\delta(\mathfrak{b}) \subseteq \mathfrak{b}$). If $\mathfrak{b} = A$, then $\mathfrak{R} = B$ and therefore $\Omega_{B/A} = (0)$, that is to say, B is étale over A (see Remark 2.1 below). In this case, we take $a = 1$. Suppose that $\mathfrak{b} \neq A$. For a general choice of $\mathfrak{P}$, we have $\Omega_{B/A} \otimes_B B_{\mathfrak{P}} = (0)$. Hence $\mathfrak{b} \neq (0)$. Since $\mathfrak{b}$ is a nonzero δ-ideal, we can choose a nonzero element a of $\mathfrak{b}$ such that $\delta(a) = 0$ and that $B[a^{-1}]$ is étale and finite over $A[a^{-1}]$. This completes a proof of Theorem 1.2.

REMARK 2.1. With the above notations, suppose that $\Omega_{B/A} = (0)$. Then, for any maximal ideal $\mathfrak{M}$ of B and $\mathfrak{m} = \mathfrak{M} \cap A$, it holds that $\mathfrak{M}B_{\mathfrak{M}} = \mathfrak{m}B_{\mathfrak{M}}$ and hence the completions $\widehat{B_{\mathfrak{M}}}$ and $\widehat{A_{\mathfrak{m}}}$ coincides with each other. Since $\widehat{B_{\mathfrak{M}}}$ (resp. $\widehat{A_{\mathfrak{m}}}$) is faithfully flat over $B_{\mathfrak{M}}$ (resp. $A_{\mathfrak{m}}$), it follows that $B_{\mathfrak{M}}$ is a flat $A_{\mathfrak{m}}$-module. Hence B is unramified and flat over A. So, B is étale over A.

2.3. Proof of Theorem1.3. Note that $\mathrm{ht}(\mathfrak{a}B) \geq 2$ because $p : Y \to X$ is a finite morphism. Let $\mathcal{D}er_{\boldsymbol{C}}(Y)$ be the coherent $\mathcal{O}_Y$-Module $\mathcal{H}om_{\mathcal{O}_Y}(\Omega_{Y/\boldsymbol{C}}, \mathcal{O}_Y)$. Then, by [4, Cor. 5.10.6], the canonical homomorphism

$$\Gamma(Y, \mathcal{D}er_{\boldsymbol{C}}(Y)) \to \Gamma(Y - V(\mathfrak{a}B), \mathcal{D}er_{\boldsymbol{C}}(Y))$$

is an isomorphism. Since $\Delta_{Y-V(\mathfrak{a}B)}$ is the lifting of $\delta_{X-V(\mathfrak{a})}$, we have $\Delta_{Y-V(\mathfrak{a}B)} \in \Gamma(Y - V(\mathfrak{a}B), \mathcal{D}er_{\boldsymbol{C}}(Y))$ by the condition (2) and Theorem 1.1. Hence $\Delta_{Y-V(\mathfrak{a}B)}$ extends to a $\boldsymbol{C}$-derivation $\Delta' \in \Gamma(Y, \mathcal{D}er_{\boldsymbol{C}}(Y))$. Since both Δ and Δ' restricted on the function field L is the extension of δ, it follows that $\Delta' = \Delta$. This implies that $\Delta(B) \subset B$. Then, by Vasconcelos [19], Δ is locally nilpotent. This completes a proof of Theorem 1.3.

REMARK 2.2. In the above proof of Theorem 1.3, we can show that Δ is locally nilpotent without using a result of Vasconcelos if the ideal $\mathfrak{a}$ is a δ-ideal, i.e., $\delta(\mathfrak{a}) \subset \mathfrak{a}$. In fact, since $\mathfrak{a}$ is a nonzero δ-ideal, there exists a nonzero element a of $\mathfrak{a}$ such that $\delta(a) = 0$. Then by the condition (2), $B[a^{-1}]$ is étale over $A[a^{-1}]$. Hence the restriction $\Delta|_{B[a^{-1}]}$ is locally nilpotent (see the proof of Theorem 1.1). It then follows that Δ is locally nilpotent.

If we assume that $\mathfrak{a}$ is generated by finitely many elements $a_1, \dots, a_m$ such that $\delta(a_i) = 0$ for every $1 \leq i \leq m$, then we can give another proof to Theorem 1.3. In fact, choose any a_i and denote it by a. Then the open set $D(a)$ is contained in $\operatorname{Spec} A \setminus V(\mathfrak{a})$. Hence $B[a^{-1}]$ is finite and étale over $A[a^{-1}]$ by the condition (2). Then Δ is a locally nilpotent derivation on $B[a^{-1}]$ by Theorem 1.1. In particular, $\Delta(B) \subset B[a^{-1}]$. Hence it follows that $\Delta(B) \subset \bigcap_{i=1}^{m} B[a_i^{-1}]$. Meanwhile, let $\mathfrak{p}$ be a prime ideal of A with $\mathrm{ht}(\mathfrak{p}) = 1$. Since $\mathrm{ht}(\mathfrak{a}) \geq 2$ by the hypothesis, $\mathfrak{p}$ does not contain a_i for some i. Let $B_{\mathfrak{p}} := B \otimes_A A_{\mathfrak{p}}$. Then $B_{\mathfrak{p}} \supset B[a_i^{-1}]$. Hence $\Delta(B) \subset B_{\mathfrak{p}}$. Since we can take $\mathfrak{p}$ as an arbitrary prime ideal of A with height one, we have $\Delta(B) \subset \bigcap_{\mathrm{ht}(\mathfrak{p})=1} B_{\mathfrak{p}} = B$ since B is a finite A-module and A is normal.

3. ***G*-invariant derivations.** Let G be a finite group and let G act faithfully on the affine domain B over $\boldsymbol{C}$. Let A be the ring of G-invariants of B. Then B is a finite A-module. With the same notations as in the previous sections, we let L and K be respectively the quotient fields of B and A. Then K is the G-invariant subfield of L. For a $\boldsymbol{C}$-algebra R, we denote by $\mathrm{Der}_{\boldsymbol{C}}(R, R)$ or simply $\mathrm{Der}_{\boldsymbol{C}}(R)$ the R-module of $\boldsymbol{C}$-derivations of R into R.

We shall begin with some elementary observations on the derivations.

LEMMA 3.1. *Suppose that G acts on a $\boldsymbol{C}$-algebra R. Then the following assertions hold.*

(1) *G acts on* $\mathrm{Der}_{\boldsymbol{C}}(R, R)$ *by $g(\Delta)(x) = g(\Delta(g^{-1}(x)))$, where $g \in G$, $\Delta \in \mathrm{Der}_{\boldsymbol{C}}(R, R)$ and $x \in R$.*

(2) *Taking the above L as R, $\Delta \in \mathrm{Der}_{\boldsymbol{C}}(L, L)$ is a lifting of an element $\delta \in \mathrm{Der}_{\boldsymbol{C}}(K, K)$ if and only if $g(\Delta) = \Delta$ for every $g \in G$.*

(3) *Taking the above B as R, $\Delta \in \mathrm{Der}_{\boldsymbol{C}}(B, B)$ is a lifting of an element $\delta \in \mathrm{Der}_{\boldsymbol{C}}(A, A)$ if and only if $g(\Delta) = \Delta$ for every $g \in G$.*

PROOF. (1) It is straightforward to verify that $g(\Delta) \in \mathrm{Der}_{\boldsymbol{C}}(R, R)$.

(2) Suppose that Δ is a lifting of $\delta \in \mathrm{Der}_{\boldsymbol{C}}(K, K)$. Then, for $z \in K$, we compute as $g(\Delta)(z) = g(\Delta(g^{-1}(z))) = g(\delta(z)) = \delta(z)$. This implies that $g(\Delta)$ is also a lifting of δ. Since the lifting of δ is unique, we have $g(\Delta) = \Delta$ for every $g \in G$. Conversely, if $g(\Delta) = \Delta$, then for $z \in K$, we have $g(\Delta(z)) = (g(\Delta))(g(z)) = \Delta(z)$, whence $\Delta(z) \in K$. So, Δ induces an element $\delta \in \mathrm{Der}_{\boldsymbol{C}}(K, K)$. Hence Δ is a lifting of δ.

(3) In the above proof, if $\Delta \in \mathrm{Der}_{\boldsymbol{C}}(B, B)$, then $g(\Delta) \in \mathrm{Der}_{\boldsymbol{C}}(B, B)$. Then the above proof applies to the present case. Q.E.D.

For an algebraic group G not necessarily finite, the action of G on $\mathrm{Der}_{\boldsymbol{C}}(R, R)$ is defined as above.

With the notations of the above assertion (3), if $\Delta \in \mathrm{Der}_{\boldsymbol{C}}(B, B)$ is locally nilpotent, so is the derivation $\delta \in \mathrm{Der}_{\boldsymbol{C}}(A, A)$. But the converse does not hold as shown in Proposition 3.4.

3.1. Symmetric derivations. We consider the case where G is the symmetric group S_n on n letters and G acts on the polynomial ring $B = \boldsymbol{C}[x_1, \dots, x_n]$ in the standard way such that $\sigma(x_i) = x_{\sigma(i)}$ for $\sigma \in S_n$. Then $A = B^G = \boldsymbol{C}[s_1, \dots, s_n] = \boldsymbol{C}[t_1, \dots, t_n]$, where s_i and t_i are the i-th elementary symmetric polynomials

$$s_i = \sum_{j_1 < \cdots < j_i} x_{j_1} x_{j_2} \cdots x_{j_i}\,, \quad t_i = \sum_{j=1}^{n} x_j^i\,.$$

The quotient field L of B is a minimal splitting field of $F(X)$ over the quotient field K of A, where

$$F(X) = X^n - s_1 X^{n-1} + \cdots + (-1)^i s_i X^{n-i} + \cdots + (-1)^n s_n\,.$$

Let $\Delta \in \mathrm{Der}_{\boldsymbol{C}}(L, L)$ be a lifting of $\delta \in \mathrm{Der}_{\boldsymbol{C}}(A, A)$. Since every x_i is a root of $F(X) = 0$, it follows that

$$\Delta(x_i) = -\frac{F^{\delta}(x_i)}{F'(x_i)}\,.$$

Furthermore, since Δ is G-invariant, Δ is in fact determined by $\Delta(x_1)$. We denote by $\mathrm{Der}_{\boldsymbol{C}}^G(B, B)$ the A-module of G-invariant derivations of B.

THEOREM 3.2. (1) *Let $\Delta \in \mathrm{Der}_{\boldsymbol{C}}(L, L)$ be a lifting of $\delta \in \mathrm{Der}_{\boldsymbol{C}}(A, A)$. Then $\Delta \in \mathrm{Der}_{\boldsymbol{C}}(B, B)$ if and only if $F'(x_1)$ divides $F^{\delta}(x_1)$ in B.*

(2) *The A-module $\mathrm{Der}_{\boldsymbol{C}}^G(B, B)$ is freely generated by*

$$\Delta_i = x_1^i \partial_{x_1} + \cdots + x_n^i \partial_{x_n}$$

for $0 \le i \le n-1$.

(3) *If $\Delta \in \mathrm{Der}_{\boldsymbol{C}}^G(B, B)$ is locally nilpotent, then $\Delta = f\Delta_0$ where f is an element of A such that $\Delta_0(f) = 0$.*

PROOF. (2) It is easily checked that Δ_i is a G-invariant homogeneous derivation for $0 \leq i \leq n-1$. We show that $\mathrm{Der}_{\boldsymbol{C}}^G(B, B)$ is generated by $\Delta_0, \dots, \Delta_{n-1}$ over A. Let $\Delta \in \mathrm{Der}_{\boldsymbol{C}}^G(B, B)$. Then Δ is written as

$$\Delta = f_1 \partial_{x_1} + \cdots + f_n \partial_{x_n}$$

where $f_i \in B$. Since Δ is G-invariant, it follows that $\sigma f_i = f_{\sigma(i)}$ for any $\sigma \in G$. In particular, $\sigma f_1 = f_1$ for any permutation σ of $2, 3, \dots, n$ and $f_i = \tau_{(1,i)} f_1$ for $i \geq 2$ where $\tau_{(1,i)}$ is the transposition of 1 and i. We may assume that f_1 is homogeneous. Let r be the degree of f_1 of Δ. If $r = 0$, then $\Delta = c\Delta_0$ for $c \in \boldsymbol{C}$. Let $r \geq 1$. Since $\sigma f_1 = f_1$ for any permutation σ of $2, 3, \dots, n$, it follows that

$$\begin{aligned} f_1 &= c_r x_1^r + c_{r-1} S_1(x_2, \dots, x_n) x_1^{r-1} \\ &\quad + \cdots + c_1 S_{r-1}(x_2, \dots, x_n) x_1 + c_0 S_r(x_2, \dots, x_n) \end{aligned}$$

where $S_k(x_2, \dots, x_n)$ is a symmetric polynomial of $x_2, \dots, x_n$ of degree k and $c_k \in \boldsymbol{C}$. Note that $S_1(x_2, \dots, x_n)$ is the first symmetric polynomial $s_1(x_2, \dots, x_n) = x_2 + \cdots + x_n$ of $x_2, \dots, x_n$. Suppose that $r \leq n-1$. It suffices to show that Δ is written as a sum of $\Delta_0, \dots, \Delta_r$ over A when $f_1 = S_k(x_2, \dots, x_n) x_1^{r-k}$ for $1 \leq k \leq r$. For $k \geq 2$, $S_k(x_2, \dots, x_n)$ is expressed by a linear combination of $s_k(x_2, \dots, x_n) = \sum_{2 \leq j_1 < \cdots < j_k} x_{j_1} x_{j_2} \cdots x_{j_k}$ and the products $s_{k_1}(x_2, \dots, x_n) \cdots s_{k_j}(x_2, \dots, x_n)$ such that $k_1 + \cdots + k_j = k$. Since

$$\begin{aligned} s_1(x_2, \dots, x_n) + x_1 &= s_1 \\ s_2(x_2, \dots, x_n) + s_1(x_2, \dots, x_n) x_1 &= s_2 \\ &\cdots \\ s_{n-1}(x_2, \dots, x_n) + s_{n-2}(x_2, \dots, x_n) x_1 &= s_{n-1}\,, \end{aligned}$$

it follows inductively on k that $s_k(x_2, \dots, x_n) x_1^{r-k}$ is written as a linear sum of $s_k x_1^{r-k}$, $s_{k-1} x_1^{r-k+1}, \dots, x_1^r$. Furthermore, by the above relations, the products $s_{k_1}(x_2, \dots, x_n) \cdots s_{k_j}(x_2, \dots, x_n)$ of degree k are reduced, inductively on k, to a sum of x_1^j for $0 \leq j \leq k$ over A. Hence it follows that Δ with $f_1 = S_k(x_2, \dots, x_n) x_1^{r-k}$ for $1 \leq k \leq r$ is written as a sum of $\Delta_0, \dots, \Delta_r$ over A when $r \leq n-1$. Suppose that $r \geq n$. Since $S_k(x_2, \dots, x_n)$ is a linear combination of the products of $s_1(x_2, \dots, x_n), \dots, s_{n-1}(x_2, \dots, x_n)$ of degree k, it follows by the argument as above that $S_k(x_2, \dots, x_n)$ is written by a sum of x_1^j for $0 \leq j \leq k$ over A. By the above relations and

$$x_1 s_{n-1}(x_2, \dots, x_n) = s_n\,,$$

it follows that

$$x_1^n = s_1 x_1^{n-1} - s_2 x_1^{n-2} + \cdots + (-1)^{n-2} s_{n-1} x_1 + (-1)^{n-1} s_n\,.$$

Hence x_1^r for $r \geq n$ is written as a sum of x_1^j for $0 \leq j \leq n-1$ over A. It follows that Δ with $f_1 = S_k(x_2, \dots, x_n) x_1^{r-k}$ for $1 \leq k \leq r$ and with $f_1 = x_1^r$ are written as a sum of $\Delta_0, \dots, \Delta_{n-1}$ over A when $r \geq n$.

We show that $\Delta_0, \dots, \Delta_{n-1}$ are linearly independent over A. Suppose that $a_0\Delta_0 + \cdots + a_{n-1}\Delta_{n-1} = 0$ for $a_0, \dots, a_{n-1} \in A$. Then we have $a_0 + a_1 x_i + \cdots + a_{n-1} x_i^{n-1} = 0$ for

$1 \le i \le n$. In a matrix form, ${}^tV^t(a_0, \dots, a_{n-1}) = {}^t(0, \dots, 0)$ where V is the van der Monde matrix. Multiplying the adjoint matrix of ${}^tV$, we have $d \cdot {}^t(a_0, \dots, a_{n-1}) = {}^t(0, \dots, 0)$ where $d = \prod_{i<j}(x_i - x_j)$. Hence it follows that $a_i = 0$ for all i, and the assertion follows.

(3) Write Δ as $\Delta = f_1\partial_{x_1} + \cdots + f_n\partial_{x_n}$ for $f_i \in B$. Since $f_i = \tau_{(1,i)}f_1$ for $i \ge 2$, it follows that $\Delta(x_i - x_j) = f_i - f_j \in (x_i - x_j)$. This implies that $\Delta(x_i - x_j) = 0$ since Δ is locally nilpotent (see [3, Cor. 1.20]). Hence $f_i = \Delta(x_i) = \Delta(x_j) = f_j$ for any i and j. Thus Δ is written as $\Delta = f\Delta_0$ for $f \in B$. Since Δ is G-invariant, it follows that $\Delta(s_1) = nf \in A$, i.e., $f \in A$. Furthermore, since $\Delta(f) = f\Delta_0(f)$ and Δ is locally nilpotent, we have $\Delta(f) = 0$ [3, *ibidem*]. Hence $\Delta_0(f) = 0$. Q.E.D.

REMARK 3.3. Let $\delta_0 = \Delta_0|_A$. Then the locally nilpotent derivation δ_0 on $A = \boldsymbol{C}[t_1, \dots, t_n]$ is triangular, i.e., $\delta_0(t_i) \in \boldsymbol{C}[t_1, \dots, t_{i-1}]$ and has a slice $s = t_1/n$. Hence the kernel of δ_0 is a polynomial ring $\boldsymbol{C}[\pi_s(t_2), \dots, \pi_s(t_n)]$ where $\pi_s(a) = \sum_{i\ge0}((-1)^i/i!)\delta_0^i(a)s^i$ for $a \in A$.

By Theorem 3.2 together with a result of Vasconcelos, a lifting $\Delta \in \mathrm{Der}_{\boldsymbol{C}}(L, L)$ of a locally nilpotent derivation δ on A satisfies $\Delta(B) \subset B$ if and only if $\delta = f\delta_0$ where $\delta_0 = \Delta_0|_A$ and f is an element of A such that $\delta_0(f) = 0$. Let $d = \prod_{i<j}(x_i - x_j)$ (resp. $D = d^2$) be the discriminant (resp. the determinant) of B over A. Since $\delta_0(D) = 0$, our result accords with the criterion of Scheja-Storch [16]. We give an example of a lifting $\Delta \in \mathrm{Der}_{\boldsymbol{C}}(L, L)$ which is not a derivation of B.

PROPOSITION 3.4. *Let δ be a locally nilpotent derivation on A such that $\delta(s_i) = 0$ for $1 \le i < n$ and $\delta(s_n) = 1$. Then $\Delta(x_1), \dots, \Delta(x_n)$ are determined as*

$$
{}^t(\Delta(x_1), \dots, \Delta(x_n)) = \frac{1}{d}V^{*t}(0, \dots, (-1)^{n+1}),
$$

where V^* is the adjoint matrix of the van der Monde matrix

$$
V = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ & & \cdots & \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}.
$$

Hence $\Delta(B) \not\subset B$.

PROOF. Note that $t_i \in \boldsymbol{C}[s_1, \dots, s_i]$ for $1 \le i \le n-1$ and that $t_n + (-1)^n ns_n \in \boldsymbol{C}[s_1, \dots, s_{n-1}]$. Hence we have $\Delta(t_i) = 0$ for $0 \le i \le n-1$ and $\Delta(t_n) = (-1)^{n+1}n$, i.e., $\sum_{j=1}^n x_j^i \Delta(x_j) = 0$ for $0 \le i < n-1$ and $\sum_{j=1}^n x_j^{n-1}\Delta(x_j) = (-1)^{n+1}$. Namely we have

$$
\begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ & & \cdots & \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}
\begin{pmatrix} \Delta(x_1) \\ \Delta(x_2) \\ \vdots \\ \Delta(x_n) \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ \vdots \\ (-1)^{n+1} \end{pmatrix}.
$$

Thence follows the assertion. Q.E.D.

Let $\mathfrak{sl}_n$ be the Lie algebra with the adjoint action of SL_n. Then the algebraic quotient $\mathfrak{sl}_n//\mathrm{SL}_n$ is isomorphic to $\mathfrak{t}/W$ where $\mathfrak{t}$ is the Lie subalgebra of a maximal torus T of SL_n and W is the Weyl group which is isomorphic to S_n. Let R (resp. B) be the coordinate ring of $\mathfrak{sl}_n$ (resp. $\mathfrak{t}$). Then $B \cong \boldsymbol{C}[x_1, \dots, x_n]/(x_1 + \cdots + x_n)$ and $W = S_n$ acts on B by permutation of the coordinates $\bar{x}_i$'s where $\bar{x}_i \in B$ denotes the residue class of x_i. As remarked above, the SL_n-invariant subring R^{SL_n} is isomorphic to B^W. As an application of Theorem 3.2, we show the following.

THEOREM 3.5. *There exists no non-trivial*, SL_n*-invariant, locally nilpotent derivation on R. Hence there is no non-trivial G_a-action on $\mathfrak{sl}_n$ which commutes with the adjoint* SL_n*-action.*

PROOF. Let Δ be an SL_n-invariant locally nilpotent derivation on R. Then since the corresponding G_a-action on $\mathfrak{sl}_n$ commutes with the SL_n-action, it induces a G_a-action on $\mathfrak{sl}_n^T = \mathfrak{t}$ commuting with the action of $W = NT/T$, where $\mathfrak{sl}_n^T$ is the T-fixed locus of $\mathfrak{sl}_n$ and NT is the normalizer of T in SL_n. Let Δ' be the corresponding $W(= S_n)$-invariant locally nilpotent derivation on $B = \boldsymbol{C}[x_1, \dots, x_n]/(x_1 + \cdots + x_n)$. Via an isomorphism $\alpha : B \to \boldsymbol{C}[x_1, \dots, x_{n-1}]$ defined by $\alpha(\bar{x}_i) = x_i$ for $1 \le i \le n-1$ and $\alpha(\bar{x}_n) = -(x_1 + \cdots + x_{n-1})$, Δ' induces a locally nilpotent derivation $\bar{\Delta}'$ on $\boldsymbol{C}[x_1, \dots, x_{n-1}]$. Since Δ' is W-invariant, $\bar{\Delta}'$ is invariant under the action of S_{n-1} which permutes the coordinates $x_1, \dots, x_{n-1}$. Hence by Theorem 3.2, $\bar{\Delta}'$ is of a form $f\Delta_0$ where $\Delta_0 = \partial_{x_1} + \cdots + \partial_{x_{n-1}}$ and $f \in \boldsymbol{C}[x_1, \dots, x_{n-1}]^{S_{n-1}}$ satisfies $\Delta_0(f) = 0$. Since $\alpha \circ \Delta' = \bar{\Delta}' \circ \alpha$, we have $\alpha(\Delta'(\bar{x}_i)) = f$ for $1 \le i \le n-1$ and $\alpha(\Delta'(\bar{x}_n)) = -(n-1)f$. It holds that $\tau_{(1n)}\Delta'(\bar{x}_1) = \Delta'(\tau_{(1n)}\bar{x}_1)$ since Δ' is W-invariant. Applying α on both sides, we obtain $f(-(x_1 + \cdots + x_{n-1}), x_2, \dots, x_{n-1}) = -(n-1)f(x_1, \dots, x_{n-1})$. Further, by applying Δ_0 to the above equation, we induce $\partial_{x_1} f = 0$. Hence it follows that $f = 0$, i.e., $\Delta' = 0$, and the G_a-action on $\mathfrak{t}$ is trivial. Since the G_a-action commutes with the SL_n-action, it is trivial on $\mathrm{SL}_n \cdot \mathfrak{t}$, which is open in $\mathfrak{sl}_n$. Hence the assertion follows. Q.E.D.

3.2. D_d-invariant derivations. Consider the case G is a dihedral group $D_d = \boldsymbol{Z}/d\boldsymbol{Z} \rtimes \boldsymbol{Z}/2\boldsymbol{Z}$ for an odd prime integer d. Let $B = \boldsymbol{C}[x, y]$ and G acts on B by

$$\sigma(x, y) = (\zeta x, \zeta^{-1} y), \qquad \tau(x, y) = (y, x)$$

where σ is a generator of $\boldsymbol{Z}/d\boldsymbol{Z}$, τ the generator of $\boldsymbol{Z}/2\boldsymbol{Z}$, and ζ is a d-th primitive root of unity. Then $A = B^G = \boldsymbol{C}[s, t]$ where $s = x^d + y^d$ and $t = xy$. The minimal polynomial $F(X)$ of L over K is

$$F(X) = X^{2d} - sX^d + t^d .$$

THEOREM 3.6. (1) *Let $\Delta \in \mathrm{Der}_{\boldsymbol{C}}(L, L)$ be a lifting of $\delta \in \mathrm{Der}_{\boldsymbol{C}}(A, A)$. Then $\Delta \in \mathrm{Der}_{\boldsymbol{C}}(B, B)$ if and only if $x^d - y^d$ divides $\delta(s)x - dy^{d-1}\delta(t)$ in B.*

(2) *If* $\Delta \in \mathrm{Der}^G_{\boldsymbol{C}}(B, B)$, *then*

$$\Delta = f_1(x\partial_x + y\partial_y) + \sum_{i=1}^{l} f_{2i}(y^{id-1}\partial_x + x^{id-1}\partial_y)$$

where $f_1, f_{2i} \in A$ *and* $l \geq 1$.

(3) *There is no non-trivial derivation* $\Delta \in \mathrm{Der}^G_{\boldsymbol{C}}(B, B)$ *which is locally nilpotent.*

PROOF. (1) The assertion follows from $F^\delta(x) = -(\delta(s)x - dy^{d-1}\delta(t))x^{d-1}$ and $F'(x) = d(x^d - y^d)x^{d-1}$. Here we note that $\Delta(x) \in B$ if and only if $\Delta(y) \in B$ since Δ is G-invariant.

(2) Write Δ as $\Delta = f\partial_x + g\partial_y$ for $f, g \in B$. Since Δ is G-invariant, it follows that $f(\zeta x, \zeta^{-1}y) = \zeta f(x, y)$ and $g(x, y) = f(y, x)$. Hence $f = f_1 x + \sum_{i=1}^{l} f_{2i} y^{id-1}$ and $g = f_1 y + \sum_{i=1}^{l} f_{2i} x^{id-1}$ for $f_1, f_{2i} \in \boldsymbol{C}[s, t]$. Now the derivation is written as in the statement.

(3) With the above notations, suppose that Δ is locally nilpotent. Since $\Delta(x^d - y^d)$ is in $(x^d - y^d)$, we induce $\Delta(x^d - y^d) = 0$ by [3, 1.4]. Hence we obtain $s\Delta(s) = 2dt^{d-1}\Delta(t)$ by

$$\Delta((x^d - y^d)^2) = \Delta(s^2 - 4t^d) = 0\,.$$

Since $\Delta|_A$ is locally nilpotent, it is trivial (cf. *ibid.*). Hence $\Delta(t) = 0$. Since $\Delta(x)y = -x\Delta(y)$, it follows that $\Delta = 0$ (cf. *ibid.*) Q.E.D.

3.3. $\boldsymbol{Z}/n\boldsymbol{Z}$-invariant derivations. Let $G = \boldsymbol{Z}/n\boldsymbol{Z}$ be a cyclic group of order n which acts linearly on $B = \boldsymbol{C}[x, y]$. Suppose that the isotropy group of every closed point of Spec B except the origin is trivial. Then by choosing an appropriate generator σ of G, we may assume that the G-action on B is given by $\sigma(x, y) = (\zeta x, \zeta^d y)$, where ζ is a primitive n-th root of unity and d is an integer such that $0 < d < n$ and $(d, n) = 1$. Then the G-invariant subring $A = B^G$ is generated by monomials $x^i y^j$ such that $i + dj \equiv 0 \pmod{n}$. The quotient field L of B is a minimal splitting field of $F(X) = X^{2n} - (x^n + y^n)X^n + x^n y^n$ over the quotient field K of A.

THEOREM 3.7. (1) *Let* $\Delta \in \mathrm{Der}_{\boldsymbol{C}}(L, L)$ *be a lifting of* $\delta \in \mathrm{Der}_{\boldsymbol{C}}(A, A)$. *Then* Δ *is in* $\mathrm{Der}_{\boldsymbol{C}}(B, B)$ *if and only if* $\delta(x^n)$ *is divided by* x^{n-1} *and* $\delta(y^n)$ *is divided by* y^{n-1} *in* B.

(2) *If* $\Delta \in \mathrm{Der}^G_{\boldsymbol{C}}(B, B)$, *then*

$$\Delta = (a_1 x + a_2 y^{d'})\partial_x + (b_1 x^d + b_2 y)\partial_y$$

where $a_1, a_2, b_1, b_2 \in A$ *and* d' *is an integer such that* $0 < d' < n$ *and* $dd' \equiv 1 \pmod{n}$.

PROOF. (1) Since x is a root of $F(X) = 0$, it follows that $\Delta(x) = -F^\delta(x)/F'(x)$. Similarly as for y, we obtain $\Delta(y) = -F^\delta(y)/F'(y)$, and the assertion follows.

(2) We write Δ as $\Delta = f\partial_x + g\partial_y$ for $f, g \in B$. Since Δ is G-invariant, it follows that $\sigma f = \zeta f$ and $\sigma g = \zeta^d g$. Since the A-module $B_1 = \{b \in B\,;\ \sigma b = \zeta b\}$ is generated by x and $y^{d'}$, f is written as $f = a_1 x + a_2 y^{d'}$ for $a_1, a_2 \in A$. As for g, it follows that $g = b_1 x^d + b_2 y$

for $b_1, b_2 \in A$ since $B_d = \{b \in B\,;\ \sigma b = \zeta^d b\}$ is generated by x^d and y over A. Hence the assertion follows. Q.E.D.

It is easily checked that $\Delta = bx^d\partial_y$ with $b \in \boldsymbol{C}[x^n]$ and $\Delta' = ay^{d'}\partial_x$ with $a \in \boldsymbol{C}[y^n]$ are G-invariant locally nilpotent derivations on B, hence restrict to locally nilpotent derivations on A. We show in a geometric way that any G-invariant locally nilpotent derivation on B is of the form Δ or Δ' (Theorem 4.5).

REMARK 3.8. Let G be a finite group and let $\rho : G \to \mathrm{GL}(n, \boldsymbol{C})$ be a non-trivial representation. We consider the G-action on the polynomial ring $B = \boldsymbol{C}[x_1, \dots, x_n]$ induced by ρ. Let $\Delta = c_1\partial_{x_1} + \cdots + c_n\partial_{x_n}$ be a linear derivation of B with $c_i \in \boldsymbol{C}$. Then it is easy to see that Δ is G-invariant if and only if the column vector ${}^t(c_1, \dots, c_n)$ is an eigenvector with value 1 of $\rho(g)$ for all $g \in G$. Hence $\Delta = 0$ if ρ is irreducible.

4. Algebraic characterizations of $\boldsymbol{A}^2/G$ with G cyclic. A normal affine surface X is called a *log affine pseudo-plane* if it has an $\boldsymbol{A}^1$-fibration over $\boldsymbol{A}^1$ such that all fibers are isomorphic to $\boldsymbol{A}^1$ when reduced and there is at most one multiple fiber [15, 14]. If a log affine pseudo-plane is smooth, it is simply called an affine pseudo-plane. The significance of affine pseudo-planes is clear from the following fact [15, Theorem 1.2]:

Let X be a $\boldsymbol{Q}$-factorial smooth affine surface. Then X is an affine pseudo-plane if and only if there exists a dominant morphism $p : \boldsymbol{A}^2 \to X$.

The quotient surface $\boldsymbol{A}^2/G$ of the affine plane $\boldsymbol{A}^2 = \mathrm{Spec}\, B$ by a linear action of a finite cyclic group $G = \boldsymbol{Z}/n\boldsymbol{Z}$ described in the previous subsection 3.3 is one of log affine pseudo-planes (see Theorem 4.3). To fix the notation, let σ be a generator of G and define a G-action on $B = \boldsymbol{C}[x, y]$ by $\sigma(x, y) = (\zeta x, \zeta^d y)$, where ζ is a primitive n-th root of unity and d is a positive integer with $(d, n) = 1$. As remarked in 3.3, the G-invariant subring A of B is given as $A = \boldsymbol{C}[x^n, y^n, x^i y^j\,;\ i + dj \equiv 0 \pmod n]$. The quotient surface $\boldsymbol{A}^2/G$ by this G-action is Spec A. We then say that $\boldsymbol{A}^2/G$ has a cyclic quotient singularity of *type* (n, d). In the sequel, we mean by $\boldsymbol{A}^2/G$ the quotient surface Spec A. The objective of the present section is to characterize $\boldsymbol{A}^2/G$ in terms of the liftings of $\boldsymbol{A}^1$-fibrations on $\boldsymbol{A}^2/G$.

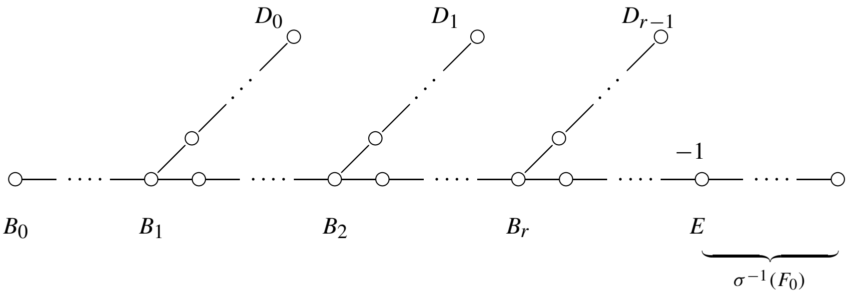
We shall first collect some known results on an $\boldsymbol{A}^1$-fibration on a normal affine surface. Given a fibration $\rho : X \to C$ and a point $p \in C$, we denote by $\rho^*(p)$ (resp. $\rho^{-1}(p)$) the scheme-theoretic (resp. set-theoretic) fiber of ρ over p.

LEMMA 4.1. *Let X be a normal affine surface and let $\rho : X \to C$ be an $\boldsymbol{A}^1$-fibration with a smooth curve C. Let P_0 be a singular point on X and let $F_0 := \rho^*(p_0)$ with $p_0 = \rho(P_0)$. Let $\sigma : Y \to X$ be the minimal resolution of singularities of X and let $\tau : Y \to C$ be the $\boldsymbol{A}^1$-fibration which is the extension of ρ. Then the following assertions hold.*

(1) *The point P_0 is a cyclic quotient singular point, say, of type (n, d) with $0 < d < n$ and $(n, d) = 1$ and the fiber $\rho^{-1}(p_0)$ is a disjoint union of the affine lines, each of which carries at most one singular point.*

We assume below that the fiber F_0 is irreducible.

(2) *There exist a smooth projective surface V and a $\boldsymbol{P}^1$-fibration $\varphi : V \to \bar{C}$ such that Y is an open set of V, $\bar{C}$ is the smooth completion of C, the $\boldsymbol{P}^1$-fibration φ induces the $\boldsymbol{A}^1$-fibration τ and the $\sigma^*(F_0)$ is a part of the degenerate fiber $\Sigma_0 := \varphi^*(p_0)$. We may assume that the weighted dual graph of Σ_0 is the following slanted tree and the exceptional locus $\sigma^{-1}(P_0)$ is the rightmost, horizontal linear twig sprouting from E, where E is the proper transform of $\rho^{-1}(p_0)$ and a unique (-1) curve of Σ_0:*



The contraction of Σ_0 to a smooth fiber starts with the contraction of E followed by a successive contractions of all components which lie on the right side of the component B_r. After these contractions, the component B_r becomes a (-1) curve, and B_r as well as all components lying on the right side of B_{r-1} are contracted. Continuing the contractions of this kind, we can contract all components except for the leftmost component B_0 which becomes finally a smooth fiber.

(3) *Let m be the multiplicity of the fiber F_0. Let m_i be the multiplicity of the component B_i in the fiber Σ_0 for $0 \le i \le r$. Then $m_0 = 1$, $m_i \mid m_{i+1}$ for $1 \le i < r$ and $m_r \mid m$.*

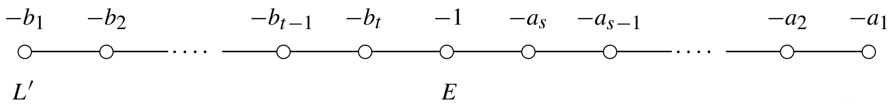
(4) *The right side part of the component B_r is produced by blowing up a point on B_r and its infinitely near point, i.e., reversing the contracting process in* (2) *above, with B_r viewed as a smooth fiber L of a $\boldsymbol{P}^1$-fibration on a smooth projective surface. This blowing-up process is determined by the pair (n, d) or (n, d'), where d' is an integer such that $0 < d' < n$ and $dd' \equiv 1 \pmod{n}$. More precisely, define a sequence of positive integers $[a_1, a_2, \ldots, a_s]$ with $a_i \ge 2$ by expanding n/d into a continued fraction*

$$\frac{n}{d} := [a_1, a_2, \ldots, a_s] = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots\, a_{s-1} - \cfrac{1}{a_s}}}$$

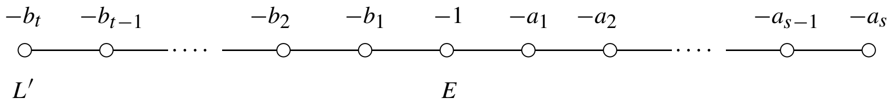
and also a sequence $[b_1, b_2, \ldots, b_t]$ by

$$\frac{n}{n-d} = [b_1, b_2, \ldots, b_t]$$

Then there is a sequence of blowing-ups which starts from a point P on L of the smooth projective surface such that the total transform of L containing the proper transform L' of L has one of the following linear chains consisting of rational curves:



$(Case\ 1)$



$(Case\ 2)$

Let A be the part consisting of vertices with self-intersection numbers $-a_s, -a_{s-1}, \dots, -a_2, -a_1$. Then the fiber F_0 is obtained from the above fiber Σ_0 by contracting the part A to the singular point P_0 and removing all other components except for E. The self-intersection number of B_r in the fiber Σ_0 is $-(b_1+1)$ or $-(b_t+1)$ in the case 1 or 2, respectively.

(5) *We have $m = nm_r$.*

(6) *For an irreducible fiber $F = m'\ell'$ of ρ with multiplicity m', there is no singular point on F if $m' = 1$.*

PROOF. (1) See [12]. It is also shown that the proper transform of each irreducible component of $\rho^{-1}(p_0)$ meets one of the end components of the linear chain which constitute the exceptional locus of the minimal resolution of the singular point.

(2) We may assume that the component E, which corresponds to the fiber $\rho^{-1}(p_0)$, is the unique (-1) component in the fiber Σ_0. Then we obtain the above slanted tree as the dual graph of Σ_0.

(3) Reversing the contraction process, one can obtain the fiber Σ_0 by blowing up a point on B_0 and its infinitely near points. When we blow up a point on B_1, the exceptional curve has the same multiplicity m_1, and the exceptional curves appearing by further blowing-ups have multiples of m_1 as multiplicities. Hence $m_1 \mid m_2$. By a similar reason, we have further divisions $m_i \mid m_{i+1}$ for $2 \le i < r$ and $m_r \mid m$.

(4) The integer d' is obtained from the pair (n, d) as the denominator of

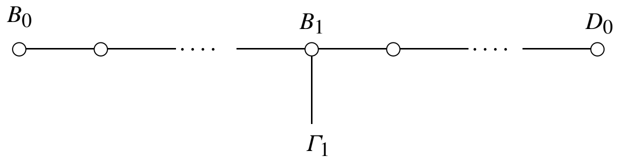
$$\frac{n}{d'} = [a_s, a_{s-1}, \dots, a_1]\,.$$

Hence the pairs (n, d) and (n, d') yield a cyclic singular point with the same resolution graph. In order to show the assertion, we may assume that the dual graph of Σ_0 is a linear chain, i.e., B_0 coincides with B_r. By making use of [6, Lemma 6.1] and by induction on n, one can readily show that $m = n$. We refer to the arguments in [10] for the assertion that the linear chain in Case 2 is also obtained by blowing-ups from a smooth fiber of a $\boldsymbol{P}^1$-fibration.

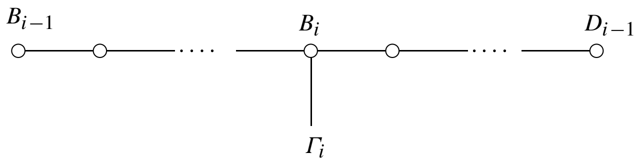
(5) This follows from the assertions (3) and (4).

(6) This is a corollary of the assertion (5). Q.E.D.

Concerning the slanted tree given in the assertion (2) of Lemma 4.1, we write it as



where Γ_1 is the side tree sprouting out from the component B_1. The horizontal chain is called the *first linear chain* of the fiber Σ_0. Similarly, for $1 \le i \le r$, we call the following horizontal chain the *i-th linear chain* of Σ_0, where Γ_i is the connected component of $\Sigma_{0,\mathrm{red}} \setminus B_i$ that contains the component E.



Suppose, in Lemma 4.1, that the base curve C of the $\boldsymbol{A}^1$-fibration ρ is the affine line. Let p_∞ be the point at infinity of C. Let $\mu_1 : C_1 \to C$ be a cyclic covering of degree m_1 which ramifies at the points p_0 and p_∞. Let X_1 be the normalization of the fiber product $X \times_C C_1$ and let $\nu_1 : X_1 \to X$ be the normalization morphism composed with the projection onto X. The projection $\rho_1 : X_1 \to C_1$ is an $\boldsymbol{A}^1$-fibration with the fiber $\rho_1^*(p_1)$, where p_1 is the unique point of C_1 lying over p_0. For $1 \le i \le r$, we define inductively a cyclic covering $\mu_i : C_i \to C_{i-1}$ of degree m_i/m_{i-1}, a point p_i of C_i and an affine normal surface X_i with an $\boldsymbol{A}^1$-fibration $\rho_i : X_i \to C_i$ such that

(i) the covering $\mu_i : C_i \to C_{i-1}$ ramifies totally at the points p_{i-1} and the point at infinity of C_{i-1}, where $C_{i-1} \cong \boldsymbol{A}^1$ and $C_0 = C$, and

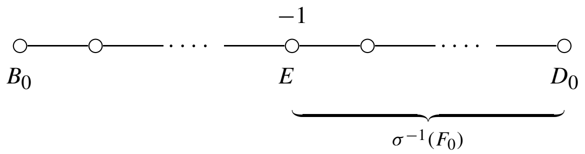
(ii) X_i is the normalization of $X_{i-1} \times_{C_{i-1}} C_i$ and ρ_i is the projection from X_i to C_i.

Finally, let $\mu : \tilde{C} \to C_r$ be a cyclic covering of degree n ramifying at the point p_r and the point at infinity of C_r, and let $\tilde{X}$ be the normalization of $X_r \times_{C_r} \tilde{C}$. The projection $\tilde{\rho} : \tilde{X} \to \tilde{C}$ is an $\boldsymbol{A}^1$-fibration. The composite $\tilde{\mu} := \mu_1 \cdots \mu_r \cdot \mu : \tilde{C} \to C$ is a cyclic covering of degree m which ramifies over the point p_0 and the point at infinity p_∞ of C, and $\tilde{X}$ is the normalization of $X \times_C \tilde{C}$. Let $\tilde{p}_0$ be the unique point lying over p_0 (and hence over the point p_r of C_r).

The following result gives a description of the fibers $\rho_i^*(p_i)$.

LEMMA 4.2. *The following assertions hold.*

(1) *Suppose that $r = 0$ and the graph of Σ_0 is reduced to the chain*



Then $\tilde{\rho}^*(\tilde{p}_0)$ is a smooth $\boldsymbol{A}^1$-fiber.

(2) *Suppose $r > 0$. The fiber $\rho_1^*(p_1)$ splits into a disjoint sum of m_1 irreducible components, each of which has multiplicity m/m_1 and carries a cyclic quotient singularity of the same type (n, d) as in Lemma* 4.1, (4).

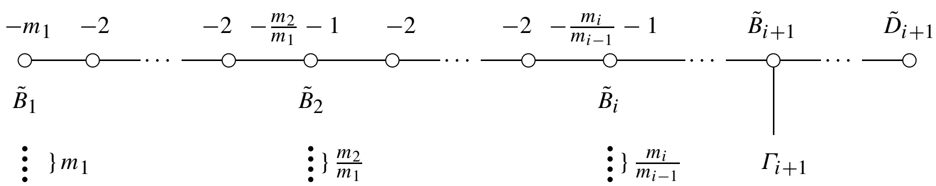
(3) *Suppose $r > 0$ and $1 \leq i \leq r$. Then the fiber $\rho_i^*(p_i)$ is a disjoint union of m_i copies of irreducible components, each of which has multiplicity m/m_i and carries a cyclic quotient singularity of the same type (n, d) as in Lemma* 4.1, (4).

(4) *Finally, the fiber $\tilde{\rho}^*(\tilde{p}_0)$ is a disjoint union of m_r smooth reduced components.*

PROOF. (1) In the fiber Σ_0, the component E (resp. the components B_0 and D_0) has multiplicity n (resp. 1). Let $\tilde{V}$ be the normalization of $V \times_{\bar{C}} D$, where D is the smooth completion of $\tilde{C}$ and hence $D \cong \boldsymbol{P}^1$. The projection $\tilde{\varphi} : \tilde{V} \to D$ is a $\boldsymbol{P}^1$-fibration. Let $\tilde{\nu} : \tilde{V} \to V$ be the normalization morphism composed with the projection onto V. The fiber $\tilde{\varphi}^{-1}(\tilde{p}_0)$ carries, in general, cyclic quotient singularities which lie over the intersection points of the components of Σ_0. Since the components B_0 and D_0 of Σ_0 have multiplicity 1, $\tilde{\nu}$ ramifies totally over B_0 and D_0. Then the inverse image by $\tilde{\nu}$ of the components adjacent to B_0 or D_0 is irreducible since the degenerate $\boldsymbol{P}^1$-fiber cannot contain a loop. By the same reason, all irreducible components of Σ_0 are irreducible on $\tilde{V}$. Note that $\tilde{\nu}$ is unramified over the component E. Since $\tilde{\nu}^*(\Sigma_0) = n\tilde{\varphi}^*(\tilde{p}_0)$, it follows that the respective inverse images $\tilde{B}_0$, $\tilde{D}_0$ and $\tilde{E}$ of B_0, D_0 and E by $\tilde{\nu}$ have multiplicities 1 in the fiber $\tilde{\varphi}^*(\tilde{p}_0)$. This implies that, after resolving minimally all singular points lying over $\tilde{\varphi}^*(\tilde{p}_0)$, all the components except for the proper transform of $\tilde{E}$ are contractible to smooth points. The assertion (1) follows readily from this observation.

(2) We apply the arguments in the assertion (1) to the first linear chain of Σ_0, where the components B_0, D_0 and B_1 have multiplicities 1, 1 and m_1, respectively. In fact, instead of $\tilde{V}$ and $\tilde{\nu}$ above, we consider the normalization V_1 of $V \times_{\bar{C}} \bar{C}_1$, where $\bar{C}_1$ is the smooth completion of C_1, and a finite covering $\bar{\nu}_1 : V_1 \to V$ of degree m_1 which is the normalization morphism composed with the projection to V. Then the morphism $\bar{\nu}_1$ is unramified over B_1 and the inverse image $\bar{\nu}^{-1}(B_1)$ is irreducible and has multiplicity 1 in the fiber $\varphi_1^*(p_1)$, where $\varphi_1 : V_1 \to \bar{C}_1$ is the induced $\boldsymbol{P}^1$-fibration. Furthermore, the inverse image of the side tree Γ_1 splits into a disjoint union of m_1 copies of Γ_1. After resolving the cyclic quotient singularities on the inverse image by $\bar{\nu}_1$ of the first linear chain, we can contract all the components to smooth points except for the proper transform $\tilde{B}_1$ of B_1 and $\bar{\nu}_1^{-1}(\Gamma_1)$. Since the multiplicity of $\tilde{B}_1$ is 1, all the components of $\bar{\nu}_1^{-1}(\Gamma_1)$ have multiplicities equal to the corresponding multiplicities in Σ_0 divided by m_1. Thence follows the assertion (2).

(3) Suppose $i \geq 2$. Let $\bar{C}_i$ be the smooth completion of C_i. We construct a projective normal surface V_i and a $\boldsymbol{P}^1$-fibration $\varphi_i : V_i \to \bar{C}_i$ inductively as follows. The surface V_i is the normalization of $V_{i-1} \times_{\bar{C}_{i-1}} \bar{C}_i$ and φ_i is the projection to $\bar{C}_i$. Let $\bar{\nu}_i : V_i \to V_{i-1}$ be the composite of the normalization morphism and the projection to V_{i-1}. It is a finite covering of degree m_i/m_{i-1}. Let $\tau_i = \bar{\nu}_1 \cdots \bar{\nu}_i : V_i \to V$. In the fiber $\varphi_{i-1}^*(p_{i-1})$, the inverse image $\tau_{i-1}^{-1}(B_{i-1})$ is a disjoint union of the irreducible components $B_{i-1}^{(j)}$ $(1 \leq j \leq m_{i-2})$, each of which has multiplicity 1 and meets a disjoint union of m_{i-1}/m_{i-2} copies of the $(i-1)$-st linear chain with the side tree Γ_{i-1} added and B_{i-2} subtracted, where all the multiplicities are one m_{i-1}-th of those in Σ_0. Here we understand $m_0 = 1$. Now $\bar{\nu}_i$ is a finite covering of degree m_i/m_{i-1} which are totally ramifying over the $B_{i-1}^{(j)}$ and the opposite end components of (the copies of) the $(i-1)$-st linear chain and unramified over the m_{i-1} copies of B_i. Hence, in the fiber $\varphi_i^*(p_i)$, the inverse image $\tau_i^{-1}(B_i)$ consists of the irreducible components $B_i^{(j)}$ $(1 \leq j \leq m_{i-1})$, each of which has multiplicity 1 and meets a disjoint union of m_i/m_{i-1} copies of the i-th linear chain with the side tree Γ_i added and B_{i-1} subtracted. In V_i, there appear cyclic quotient singularities lying over the intersection points of the components of $\varphi_{i-1}^*(p_{i-1})$. After resolving the singularities and contracting the possible components, the fiber $\varphi_i^*(p_i)$ is modified to a fiber with the following dual graph, where the vertical dots below the vertex $\tilde{B}_i$ mean that the graph lying on the right side of the vertex is copied m_i/m_{i-1} times and attached to the vertex $\tilde{B}_i$. We call this operation the *completion* of the graph at the vertex $\tilde{B}_i$. As for the vertex $\tilde{B}_{i-1}$, the graph lying on the right side of the vertex is copied m_{i-1}/m_{i-2} times after the completion at $\tilde{B}_i$ and attached to the vertex $\tilde{B}_{i-1}$. We continue the operations of completing the right subgraph, copying and attaching them at the vertices $\tilde{B}_{i-2}, \ldots, \tilde{B}_2, \tilde{B}_1$. All components between $\tilde{B}_1$ and $\tilde{B}_i$ have multiplicity 1, and all the unnamed components in between $\tilde{B}_1$ and $\tilde{B}_i$ have self-intersection number -2. The assertion (3) follows easily from this observation.



(4) If $i = r$, there are m_r-copies of $\tilde{B}_r$ in the fiber $\varphi_r^*(p_r)$, each of which has multiplicity 1 and the same linear chain as the one in the assertion (1) where the component B_0 is replaced by $\tilde{B}_r$. Then the assertion is proved by the same argument as in the proof of the assertion (1). Q.E.D.

Let X be a normal algebraic variety and let X° be the smooth locus of X. Suppose that $\pi_1(X^\circ)$ is a finite group. Let Z be the universal covering of X°, which is an algebraic variety. The normalization $\tilde{X}$ of X in the function field of Z is a normal algebraic variety containing X as an open set. We call $\tilde{X}$ the *quasi-universal* covering of X. A normal affine surface X is called a *log affine surface* if it has at worst quotient singularities. The Makar-Limanov invariant $\mathrm{ML}(X)$ is defined for a normal affine surface as in the case X is smooth.

THEOREM 4.3. *Let X be a normal affine surface. Then the following assertions hold.*

(1) *X is isomorphic to $\boldsymbol{A}^2/G$ with a finite cyclic group G if and only if X is a log affine pseudo-plane and the quasi-universal covering $\tilde{X}$ is isomorphic to $\boldsymbol{A}^2$.*

(2) *Suppose that X is a log affine pseudo-plane with a cyclic quotient singular point P_0 of type (n, d). Let $\rho : X \to C$ be an $\boldsymbol{A}^1$-fibration and let F_0 be the fiber through P_0. Then* $\mathrm{ML}(X) = \boldsymbol{C}$ *if and only if either $r = 0$ or $r = 1$ and $d = n - 1$ with the notations in Lemma* 4.1.

(3) *With the hypothesis in* (2) *above, X is isomorphic to $\boldsymbol{A}^2/G$ if and only if $r = 0$.*

(4) *Let X be a log affine pseudo-plane. Then the quasi-universal covering space of X is a Danielewski surface which is, by definition, a smooth affine surface with an $\boldsymbol{A}^1$-fibration over $\boldsymbol{A}^1$ such that all fibers are smooth but possibly only one reduced, reducible fiber.*

PROOF. (1) *Only if part.* With the notations before Lemma 4.1, define a G-invariant locally nilpotent derivation Δ on B by $\Delta(x) = 0$ and $\Delta(y) = x^d$. Then Δ induces a locally nilpotent derivation δ on A, hence an $\boldsymbol{A}^1$-fibration $\rho : X \to C$, where $X = \operatorname{Spec} A$ and $C = \operatorname{Spec} \boldsymbol{C}[u]$ with $u = x^n$. Let $\mathfrak{M} = (x, y)$ be the maximal ideal of B and let $\mathfrak{m} = \mathfrak{M} \cap A$. Then it is known that the quotient morphism $q : \operatorname{Spec} B \to \operatorname{Spec} A$ is étale outside $V(\mathfrak{m})$ and $\mathfrak{M} = \sqrt{\mathfrak{m}B}$. Hence $q : \boldsymbol{A}^2 \to \boldsymbol{A}^2/G$ is the quasi-universal covering. The linear pencil $\{u = c\,;\ c \in \boldsymbol{C}\}$ defines the $\boldsymbol{A}^1$-fibration ρ, and the $\boldsymbol{A}^1$-fibration ρ lifted onto $\boldsymbol{A}^2 = \operatorname{Spec} B$ is defined by the linear pencil $\{x = c\,;\ c \in \boldsymbol{C}\}$. Since the fiber $x = 0$ is G-stable, the fiber F_0 of ρ passing through the singular point is irreducible. This implies that X is a log affine pseudo-plane.

If part. Let $q : \boldsymbol{A}^2 \to X$ be the quasi-universal covering with group G. Then X is isomorphic to $\boldsymbol{A}^2/G$. Since X has an $\boldsymbol{A}^1$-fibration, G is a cyclic group.

(2) *Only if part.* The minimal normal compactification, say V, of X has the boudary divisor $\Delta = L_\infty + S + \bar{\Sigma}_0$ such that

(i) L_∞ (resp. S) is the smooth fiber (resp. a cross-section) of a $\boldsymbol{P}^1$-fibration $\varphi : V \to \bar{C}$ lying outside X, where we may assume that $({L_\infty}^2) = (S^2) = 0$, and

(ii) $\bar{\Sigma}_0$ is the fiber Σ_0 given in Lemma 4.1 with $\sigma^{-1}(F_0)$ contracted to the fiber F_0 with the singular point P_0.

Then, by [5, Theorems 2.9, 2.10], $\mathrm{ML}(X) = \boldsymbol{C}$ if and only if Δ is a linear chain. The last condition is equivalent to saying that either $r = 0$ or $r = 1$ and $\sigma^{-1}(F_0) - E$ is a linear (-2)-chain, i.e., every vertex has weight -2. Hence $d = n - 1$.

(3) If $r = 1$, then B_1 has multiplicity ≥ 2. Hence the quasi-universal covering of X has m_1 affine lines mapped onto F_0. Hence we have the assertion.

(4) The assertion follows from Lemma 4.2, (4). Q.E.D.

Given a cyclic quotient singularity P_0 of type (n, d) on a normal algebraic surface, we call the integer n the *order* of P_0. It is, in fact, the order of the local fundamental group at P_0. With this terminology, the condition that $r = 0$ in the assertion (3) above is equivalent to saying that the multiplicity m of the fiber F_0 is equal to n. This is the case for any $\boldsymbol{A}^1$-fibration on X and its fiber F_0 passing through P_0.

The following result is a restatement of Theorem 4.3.

THEOREM 4.4. *Let $X = \operatorname{Spec} A$ be a singular normal affine surface. Then X is isomorphic to the quotient surface $\boldsymbol{A}^2/G$ with a finite cyclic group G if and only if the following three conditions satisfied.*

(1) $|\pi_1(X^\circ)| < \infty$, *where X° is the smooth part of X.*

(2) *The divisor class group of X is a torsion group.*

(3) *X has a non-trivial G_a-action such that the fiber of the $\boldsymbol{A}^1$-fibration associated with a G_a-action has multiplicity equal to the order of the singular point.*

If X is isomorpic to $\boldsymbol{A}^2/G$ with a finite cyclic group G, then the divisor class group and $\pi_1(X^\circ)$ are isomorphic to $\mathbf{Z}/n\mathbf{Z}$, where $n = |G|$.

PROOF. *Only if part.* If $X \cong \boldsymbol{A}^2/G$ then $\pi_1(X^\circ)$ is a homomorphic image of G. Furthermore, if $q : \boldsymbol{A}^2 \to X$ is the quotient morphism, then $(\deg q)D = q_*q^*(D) \sim 0$ for any divisor on X. So, the divisor class group of X is a torsion group. The condition (3) follows from the above remark.

If part. Let $\rho : X \to C$ be the $\boldsymbol{A}^1$-fibration as given by the condition (3). Since $\pi_1(X^\circ)$ is a finite group, the base curve C is a rational curve. Hence C is an open set of $\boldsymbol{A}^1$. Furthermore, since $\mathrm{Cl}(X)$ is a torsion group, the fibration ρ has only irreducible fibers, and C is not a complete curve. The fibers of ρ passing through the singular points of X must be multiple fibers because all singular points are cyclic quotient singular points (see Lemma 4.1, (6)). If C is not isomorphic to $\boldsymbol{A}^1$, let $\boldsymbol{A}^1 - C = \{p_1, \ldots, p_s\}$ and let p_∞ be the point at infinity of $\boldsymbol{A}^1$. By [1], there exists a Galois ramified covering $\alpha : D \to \boldsymbol{P}^1$ with branch locus $p_1, \ldots, p_s, p_\infty$ with arbitrarily assigned multiplicities $\mu_1, \ldots, \mu_s, \mu_\infty$. If $s = 1$, we must have $\mu_1 = \mu_\infty$. Let $C' = \alpha^{-1}(C)$ and let $\nu = \alpha|_{C'}$. Then the normalization X' of the fiber product $X \times_C C'$ yields a finite étale covering of X°. Hence the condition (1) implies that $C \cong \boldsymbol{A}^1$. If a fiber of ρ not passing through a singular point is a multiple fiber, we can argue exactly in the same way as above. Hence only multiple fibers are those passing through singular points. Note that each fiber of ρ has at most one singular point (see Lemma 4.1). If there are two singular points, we can argue as above to find out that $\pi_1(X^\circ)$ is an infinite group. So, X has only one singular point. Let m be the multiplicity of the fiber F_0 of ρ passing through the singular point P_0. Let $p_0 = \rho(P_0)$ and let $\tau : C' \to C$ be a cyclic covering of order m ramifying totally at the point p_0. Let X' be the normalization of the fiber product $X \times_C C'$. Then the projection $\rho' : X' \to C'$ is an $\boldsymbol{A}^1$-fibration over $C' \cong \boldsymbol{A}^1$ such that every fiber is irreducible and reduced. Hence X' is isomorphic to $\boldsymbol{A}^2$ and X is isomorphic to the quotient surface X'/G, where $G \cong \mathbf{Z}/m\mathbf{Z}$.

The rest of the assertion is easy to show. Q.E.D.

Let $X = \boldsymbol{A}^2/G$ be a quotient surface of $\boldsymbol{A}^2 = \operatorname{Spec} B$ by $G = \boldsymbol{Z}/n\boldsymbol{Z}$. Then as observed in 3.3, there exists a G-invariant locally nilpotent derivation $\Delta = x^d \partial_y$ on $B = \boldsymbol{C}[x, y]$, which defines a locally nilpotent derivation on A. Let δ' be a locally nilpotent derivation on A. Since the singular point of X is fixed by the G_a-action induced by δ', it follows that $\delta'(\mathfrak{m}) \subset \mathfrak{m}$ where $\mathfrak{m} = \mathfrak{M} \cap A$ and $\mathfrak{M} = (x, y)$ is the maximal ideal of B. We can determine a locally nilpotent derivation of A.

THEOREM 4.5. *Let δ' be an arbitrarily chosen, locally nilpotent derivation on A such that $\mathfrak{m}$ is a δ'-ideal and let Δ' be the locally nilpotent derivation on B which lifts δ'. Then, after a suitable change of coordinates x, y of B, Δ' is given by $\Delta' = f(x^n)\Delta$ with $f(x^n) \in \boldsymbol{C}[x^n]$.*

PROOF. Let $Y = \operatorname{Spec} B$, $X = \operatorname{Spec} A$ and $q : Y \to X$ the associated morphism. The derivation δ' gives rise to a G_a-action on X and hence an $\boldsymbol{A}^1$-fibration $\rho : X \to C$. Note that the base curve C is a smooth rational curve with only constants as units, whence it is isomorphic to $\boldsymbol{A}^1$. Let $\tilde{\rho} : Y \to \tilde{C}$ be the $\boldsymbol{A}^1$-fibration which is a lifting of ρ, where $\tilde{C} \cong \boldsymbol{A}^1$. Let x be a coordinate of $\tilde{C}$. Then the G-action maps the fibers of $\tilde{\rho}$ to the fibers. This implies that $\sigma(x) = \zeta^i x + c$ with $c \in \boldsymbol{C}$ and $0 < i < n$. Since the inverse image $q^{-1}(F)$ of a general fiber F of ρ is a disjoint union of n distinct fibers of $\tilde{\rho}$, we have $(n, i) = 1$. We may take $i = 1$. By the change of coordinates $x \mapsto x + c/(\zeta - 1)$, we may assume that $\sigma(x) = \zeta x$. Now the generic fiber of $\tilde{\rho}$ has the coordinate ring $\boldsymbol{C}(x)[y]$ and σ induces an automorphism on it. Hence we may write $\sigma(y) = f(x)y + g(x)$ with $f(x), g(x) \in \boldsymbol{C}[x]$. If we write $\sigma^n(y) = A(x)y + B(x)$, then $A(x) = \prod_{i=0}^{n-1} f(\zeta^i x)$, which must be 1 as σ^n is the identity automorphism. This implies that $f(x)$ is a constant and an n-th root of unity. Write $f(x) = \zeta^e$ with $0 \le e < n$. Write $g(x) = \sum_{j=0}^{m} k_j x^j$. Then we can compute

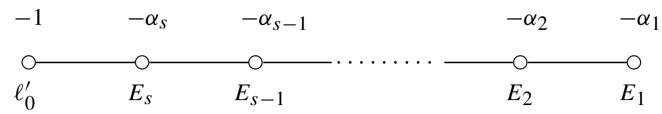
$$B(x) = \sum_{j=1}^{m} k_j(\zeta^{(n-1)e} + \zeta^{(n-2)e+j} + \cdots + \zeta^{(n-1)j})x^j\,.$$

If $\zeta^j \neq \zeta^e$, a coordinate change $y \mapsto y + k_j x^j/(\zeta^e - \zeta^j)$ allows us to assume $k_j = 0$. We make this change of coordinates for every j with $\zeta^j \neq \zeta^e$. If $\zeta^j = \zeta^e$, then we have

$$\zeta^{(n-1)e} + \zeta^{(n-2)e+j} + \cdots + \zeta^{(n-1)j} = n\zeta^{(n-1)e} \neq 0\,.$$

Since $B(x) = 0$, we must have $k_j = 0$ if $\zeta^j = \zeta^e$. Thus we may assume that $\sigma(y) = \zeta^e y$. Now the quotient surface of Y under this action of G coincides with X, and hence has the cyclic singularity of type (n, d). Write $n/d = [a_1, a_2, \ldots, a_s]$ as a continued fraction with $a_i \geq 2$. Then the fiber $F_0 = m\ell_0$ of ρ passing through the singular point P_0 after a minimal resolution of the singular point is a linear chain with the dual graph

where m is the multiplicity of the fiber, ℓ'_0 is the proper transform of ℓ_0 and $[\alpha_1, \alpha_2, \ldots, \alpha_s]$ is equal to either $[a_1, a_2, \ldots, a_s]$ or $[a_s, a_{s-1}, \ldots, a_1]$. By Theorem 4.4, we have $m = n$.



Hence $e = d$ or $e = d'$ with $dd' \equiv 1 \pmod{n}$. In the case $e = d'$, the G-action is written as $\sigma(x, y) = (\zeta x, \zeta^{d'} y) = (\zeta_1^d x, \zeta_1 y)$, where $\zeta_1 = \zeta^{d'}$. Hence the change of coordinates $(x, y) \mapsto (y, x)$ will give the desired G-action. Now consider the derivation Δ'. Since $\Delta'(x) = 0$, it induces a locally nilpotent derivation on the coordinate ring $\boldsymbol{C}(x)[y]$ of the generic fiber of $\tilde{\rho}$. Then $\Delta'(y)$ is equal to an element $h(x)$ in $\boldsymbol{C}[x]$. Since Δ' is G-invariant, it follows that $\sigma(\Delta'(\sigma^{-1} y)) = \zeta^{-d} h(\zeta x) = h(x)$. This implies that $h(x) = x^d f(x^n)$ with $f(x^n) \in \boldsymbol{C}[x^n]$. So, $\Delta' = f(x^n)\Delta$. Q.E.D.

It should be noted that a G-invariant locally nilpotent derivation on $\boldsymbol{C}[x, y]$ is not essentially unique. It is unique up to a change of coordinates and the multiplication factor $f(x^n)$. In fact, the derivation Δ_1 determined by $\Delta_1(y) = 0$ and $\Delta_1(x) = y^{d'}$ is also G-invariant and locally nilpotent. In fact, the Makar-Limanov invariant of A is equal to $\boldsymbol{C}$.

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