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LIFTING RESULTS FOR SEQUENCES IN BANACH SPACES

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Several important classes of Banach spaces are characterized by means of convergence properties of sequences. For example, if X is a Banach space, then X belongs to Nl_1 : spaces without copies of l_1, R : reflexive spaces or F : finite dimensional spaces if and only if each bounded sequence has respectively a weakly Cauchy (w-Cauchy), weakly convergent (w-convergent) or convergent subsequence. Likewise X is in the class WSC : weakly sequentially complete spaces, or in SCH : spaces with the Schur property if and only if each w-Cauchy sequence is w-convergent or convergent, respectively; note that $X \in SCH$ is equivalent to each w-convergent sequence of X is convergent [12; p.47].

The weak* convergence (w^*-convergence) determines analogously another classes as Gr : Grothendieck spaces, where weak* and weak convergence of sequences in the duals coincide, and SW^*C : spaces with w^*-sequentially compact dual ball, in which the bounded sequences of the duals have w^*-convergent subsequences [2].

Using the characterization of Rosenthal-Dor of when a Banach space contains l_1 (see [12], [3]), Lohman proved in [8] a lifting result for w-Cauchy sequences from which he derived some structural statements.

In this note, by means of the Rosenthal-Dor theorem, we prove the following results:

2.1 The class of all Banach spaces such that each w^*-convergent sequence in the dual has a w-Cauchy subsequence coincides with the class NQc_0 of all Banach spaces without quotients isomorphic to c_0 : in a sense NQc_0 is dual of Nl_1.

2.2 Given a Banach space $X \in SW^*C$, the dual $X' \in Nl_1$ if and only if $X \in NQc_o$ (this extends a result of [6]).

2.3 For a Banach space X we have

$$X' \in Nl_1 \leftrightarrow X \in NQc_o \cap Nl_1 \leftrightarrow X \in NQc_o \cap SW^*C$$

Moreover we verify that no new class appears by considering all possible combinations of bounded, w^*-convergent, w-Cauchy, w-convergent and convergent sequences.

Next, for each of the above classes with the only exception of SW^*C, we obtain a lifting result of sequences analogous to that of Lohman for Nl_1:

2.7 Let M be a subspace of a Banach space E, and denote i the inclusion of M into E, p the quotient map onto E/M and $q = i'$ the quotient map onto E'/M°. Then:

(a) If M belongs to F, R or Nl_1 and (x_n) is a bounded sequence in E such that (px_n) is respectively convergent, w-convergent or w-Cauchy, then (x_n) has respectively a convergent, w-convergent or w-Cauchy subsequence.

(b) If M belongs to WSC or SCH and (x_n) is a w-Cauchy sequence such that (px_n) is respectively w-convergent or w-Cauchy, then (x_n) is w-convergent or w-Cauchy respectively.

(c) If E/M belongs to Gr or NQc_o and (f_n) is a w^*-convergent sequence of E' such that (qf_n) is w convergent or w-Cauchy respectively, then (f_n) has a w-convergent or w-Cauchy subsequence respectively.

Finally we show some consequences; in particular, we derive the three-space property for the corresponding classes, which is a new result for Gr and an alternative proof for the others. Since SW^*C has not the three-space property [2; p.237], a lifting result of sequences similar to the above cannot be true for this class. Also we prove the following result for operators:

2.10 Let $T \in L(X,Y)$ where X,Y are Banach spaces. Then either T' maps w^*-convergent sequences into sequences having a w-Cauchy subsequence or qT surjective for some $q \in L(Y,c_o)$.

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