

Light-cone averaging in cosmology: formalism and applications

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Abstract

We present a general gauge invariant formalism for defining cosmological averages that are relevant for observations based on light-like signals. Such averages involve either null hypersurfaces corresponding to a family of past light-cones or compact surfaces given by their intersection with timelike hypersurfaces. Generalized Buchert-Ehlers commutation rules for derivatives of these light-cone averages are given. After introducing some adapted “geodesic light-cone” coordinates, we give explicit expressions for averaging the redshift to luminosity-distance relation and the so-called “redshift drift” in a generic inhomogeneous Universe.

1 Introduction

It is by now well-known (see, for example, [1]) that averaging solutions of the full inhomogeneous Einstein equations leads, in general, to different results from those obtained by solving the averaged (i.e. homogeneous) Einstein equations. In particular, the averaging procedure does not commute with the non-linear differential operators appearing in the Einstein equations and, as a result, the dynamics of the averaged geometry is affected by so-called “backreaction” terms, originating from the inhomogeneities present in the geometry and in the various components of the cosmic fluid.

Following the discovery of cosmic acceleration on large scales, interest in the possible effects of inhomogeneities for interpreting the data themselves has considerably risen (see [2, 3] for some recent review papers). It has even been suggested (see e.g. [4]) that the dynamical effects of the backreaction could replace, at least in part, the dark-energy sources as an explanation of such a cosmic acceleration, thereby providing an elegant solution to the well-known “coincidence problem”.

As a consequence, much work has been done in these last few years on trying to formulate a suitable “averaged” description of inhomogeneous cosmology. In most of these works, following Buchert’s seminal papers [5], the effective geometry emerging after the smoothing-out of local inhomogeneities has been determined by integrating over three-dimensional spacelike hypersurfaces.

However, as pointed out long ago [6, 7], a phenomenological reconstruction of the space-time metric and of its dynamic evolution on a cosmological scale is necessarily based on past light-cone observations, since most of the relevant signals travel with the speed of light. Hence, the averaging procedure should be possibly referred to a null hypersurface coinciding with a past light-cone rather than to some fixed-time spacelike hypersurface. Nonetheless, such a light-cone averaging procedure, whose importance has been repeatedly stressed in the specialized literature (see e.g. [2, 8]), has never been implemented in practice.

The aim of this paper is to introduce a general (covariant and gauge invariant) prescription for averaging scalar objects on null hypersurfaces, to apply it to the past light-cone of a generic observer in the context of an inhomogeneous cosmological metric, and to provide the analog of the Buchert-Ehlers commutation rules [9] for the derivatives of light-cone averaged quantities. We will also introduce an adapted system of coordinates (defining what we call a “geodesic light-cone frame”, which can be seen as a particular specification of the “observational coordinates” introduced in [6, 10]) where the averaging prescription greatly simplifies, while keeping all the required degrees of freedom for applications to general inhomogeneous metric backgrounds.

In order to illustrate our light-cone averaging procedure we will propose, in particular, two possible physical applications: the averaging of the luminosity-redshift relation and that of the so-called “redshift drift” parameter [11] (see [12, 13] for recent discussions), both evaluated for a generic inhomogeneous cosmological geometry. We will concentrate our attention on the realistic case in which the reference observer is geodesic, having in mind a model of light ray propagation based on the geometrical optics approximation (see e.g. [14, 15]).

Finally, we stress that our intent, in the present paper, is not to give a detailed physical discussion and interpretation of the many phenomenological effects of averaging, and to bring arguments supporting (or disproving) plausible theoretical alternatives to the “standard” dark-energy/cosmic acceleration scenario. We simply present a first step towards such an ambitious program, providing a formal procedure allowing an automatic implementation of light-cone averaging, a procedure that – to the best of our knowledge – was missing in the present literature, and that we believe to be of some utility because of its covariance and model independence. A discussion of the possible phenomenological consequences of its application to specific cosmological scenarios will be presented in a forthcoming paper.

The paper is organized as follows. In Sect.2 we recall the averaging prescription introduced in [16, 17] and we generalize it to different kinds of averaging on null hypersurfaces. In Sect.3 we give the covariant and gauge invariant version of the Buchert-Ehlers commutation rules for light-cone averages. In Sect.4 we introduce a coordinate system in which the light-cone averaging prescriptions and commutation rules take a much simpler form. In Sect.5 we discuss a possible approach to the average of the luminosity-redshift relation and of the redshift drift on the light-cone of a geodesic observer. Our conclusive remarks are presented in Sect.6.

2 Gauge invariant light-cone averaging

Let us first briefly recall the approach given in [16, 17] to gauge invariant averaging on a three-dimensional spacelike hypersurface $\Sigma(A)$, embedded in our four-dimensional space-time \mathcal{M}_4 . Assuming the hypersurface (or a spacelike foliation) to be defined by an equation involving a scalar field with timelike gradient $A(x)$:

$$A(x) - A_0 = 0, \tag{2.1}$$

the gauge (and hypersurface-parametrization) invariant definition of the integral of an arbitrary scalar $S(x)$ and of its average on such hypersurface was given in [16, 17] as¹:

$$\begin{aligned} I(S; A_0) &= \int_{\mathcal{M}_4} d^4x \sqrt{-g(x)} \delta(A(x) - A_0) \sqrt{-\partial_\mu A \partial^\mu A} S(x); \\ \langle S \rangle_{A_0} &= \frac{I(S; A_0)}{I(1; A_0)} \end{aligned} \quad (2.2)$$

(we are using the signature $(-, +, +, +)$). Here the spatial hypersurface has no boundary. However, as shown in [16, 17], a possible spatial boundary can be added through the following extension of the previous integral:

$$I(S; A_0; B_0) = \int_{\mathcal{M}_4} d^4x \sqrt{-g(x)} \delta(A(x) - A_0) \Theta(B_0 - B(x)) \sqrt{-\partial_\mu A \partial^\mu A} S(x), \quad (2.3)$$

and similarly for the corresponding average (Θ is the Heaviside step function, and B is a positive function of the coordinates, with spacelike gradient). As discussed in [16, 19], this is still a gauge invariant expression if $B(x)$ transforms as a scalar, while it gives violations of gauge invariance if B is not a scalar and keeps the same form in different coordinate systems. Even in that case, however, gauge invariance violations go to zero when we choose B_0 in such a way that the size of the spatial region goes to infinity [16, 19].

The above procedure unfortunately fails if $A(x) = A_0$ defines a null (lightlike) hypersurface, since in that case $\partial_\mu A \partial^\mu A = 0$. In order to circumvent this problem let us start with a spacetime integral where the four-dimensional integration region is bounded by two hypersurfaces, one spacelike and the other one null (corresponding e.g. to the past light-cone of some observer). Let us choose, in particular, the region inside the past light-cone of the observer bounded in the past by the hypersurface $A(x) = A_0$: clearly a gauge invariant definition of the integral of a scalar $S(x)$ over such a hypervolume can be written (in a useful notation generalizing the one used above) as

$$I(S; -; A_0, V_0) = \int_{\mathcal{M}_4} d^4x \sqrt{-g} \Theta(V_0 - V) \Theta(A - A_0) S(x), \quad (2.4)$$

where $V(x)$ is a (generalized advanced-time) scalar satisfying $\partial_\mu V \partial^\mu V = 0$, where V_0 specify the past light cone of a given observer, and where the “-” symbol on the l.h.s. denotes the absence of delta-like window functions.

Starting with this hypervolume integral we can construct gauge invariant hypersurface and surface integrals by applying to it appropriate differential operators – or, equivalently, by applying Gauss’s theorem to the volume integral of a covariant divergence. An example

¹In [17] the prescription introduced in [16] is used to give a covariant and gauge invariant generalization of the effective equations presented in [5]. Such a generalization has been recently used to deal with the backreaction of quantum fluctuations in an inflationary model [18].

of the latter, if we are interested into the variations of the volume averages along the flow lines normal to the reference hypersurface $\Sigma(A)$, is obtained by replacing the scalar S with the divergence of the unit normal to Σ ,

$$n_\mu = -\frac{\partial_\mu A}{\sqrt{-\partial_\nu A \partial^\nu A}}, \quad n_\mu n^\mu = -1, \quad (2.5)$$

and leads to the identity:

$$\begin{aligned} \int_{\mathcal{M}_4} d^4x \sqrt{-g} \Theta(V_0 - V) \Theta(A - A_0) \nabla^\mu n_\mu &= \\ &= - \int_{\mathcal{M}_4} d^4x \sqrt{-g} \Theta(V_0 - V) \delta(A - A_0) \sqrt{-\partial_\mu A \partial^\mu A} \\ &\quad + \int_{\mathcal{M}_4} d^4x \sqrt{-g} \delta(V_0 - V) \Theta(A - A_0) \frac{-\partial_\mu V \partial^\mu A}{\sqrt{-\partial_\nu A \partial^\nu A}}. \end{aligned} \quad (2.6)$$

Hence, if we start from Eq. (2.4), and we consider the variation of the average integral by shifting the light-cone $V = V_0$ along the flow lines defined by n_μ , we are led to define the hypersurface integral (with positive measure):

$$I(1; V_0; A_0) = \int_{\mathcal{M}_4} d^4x \sqrt{-g} \delta(V_0 - V) \Theta(A - A_0) \frac{|\partial_\mu V \partial^\mu A|}{\sqrt{-\partial_\nu A \partial^\nu A}}. \quad (2.7)$$

Similarly, if we consider the variation of the average integral by shifting the hypersurface $A = A_0$ (along the same flow lines defined by n_μ), we are led to another hypersurface integral:

$$I(1; A_0; V_0) = \int_{\mathcal{M}_4} d^4x \sqrt{-g} \Theta(V_0 - V) \delta(A - A_0) \sqrt{-\partial_\mu A \partial^\mu A}. \quad (2.8)$$

In the first case, Eq. (2.7), the integration region is on the light-cone itself, and it is spanned by the variation of $\Sigma(A_0)$ along its normal, at fixed light-cone V_0 (see Fig. 1, (a)). In the second case of Eq. (2.8) – which gives exactly the same integral as in Eq. (2.3) with V replacing B – the hypersurface $\Sigma(A_0)$ is kept fixed, and the integration region describes the causally connected section of Σ spanned by the variation of the light-cone hypersurface (see Fig. 1, (b)).

Further differentiation also lead to the following invariant surface integral

$$I(1; V_0, A_0; -) = \int_{\mathcal{M}_4} d^4x \sqrt{-g} \delta(V_0 - V) \delta(A - A_0) |\partial_\mu V \partial^\mu A|, \quad (2.9)$$

with a compact, two-dimensional integration region defined by the intersection of $\Sigma(A_0)$ with the light-cone V_0 (Fig. 1, (c)). This integral, as well as the integrals of Eqs. (2.7), (2.8), is not only covariant and gauge invariant but also invariant under separate reparametrizations of the scalar fields $A \rightarrow \tilde{A}(A)$ and $V \rightarrow \tilde{V}(V)$. Eq. (2.9), in addition, is a particular case of an invariant integration over an arbitrary codimension-2 hypersurface defined by the

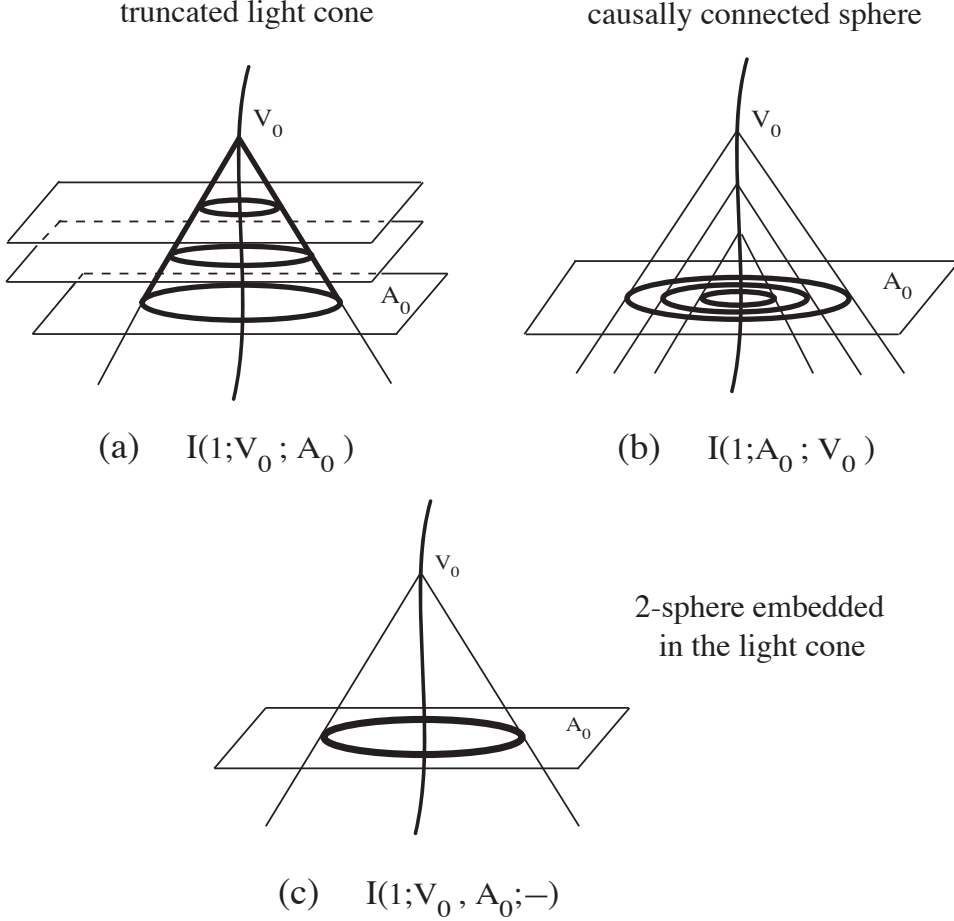


Figure 1: Three different light-cone averaging prescriptions. (a): the average of Eq. (2.7) on the light-cone itself starting from a given hypersurface in the past; (b): the average of Eq. (2.8) on the section of the hypersurface $A(x) = A_0$ which is causally connected with us; (c): the average of Eq. (2.9) on a 2-sphere embedded in the light-cone.

conditions $A^{(n)}(x) = 0$, $n = 1, 2$. In general, and in D spacetime dimensions, such an integral can be written as

$$\int_{\mathcal{M}_D} d^D x \sqrt{-g} \prod_n \delta(A^{(n)}(x)) \sqrt{|\det \bar{g}^{pm}|}; \quad \bar{g}^{pm} \equiv \partial_\mu A^{(p)} \partial_\nu A^{(m)} g^{\mu\nu}, \quad (2.10)$$

(as can be shown by considering the induced metric on the $(D - 2)$ -hypersurface), and is invariant under the more general reparametrizations $A^{(1)} \rightarrow \tilde{A}^{(1)}(A^{(1)}, A^{(2)})$ and $A^{(2)} \rightarrow \tilde{A}^{(2)}(A^{(1)}, A^{(2)})$. It can be easily checked that Eq. (2.10) reduces to (2.9) if $D = 4$ and if $A^{(1)} = A - A_0$ and $A^{(2)} = V - V_0$ are scalar functions with timelike and null gradient, respectively.

In order to make contact with Eqs. (2.7), (2.8), it may be useful to remark that the

integral (2.9) can be also obtained starting from the hypervolume integral (2.4) by considering the variation of the volume average along the flow lines normal to $\Sigma(A)$ for both $\Theta(A)$ and $\Theta(V)$, namely by using the following window function:

$$\begin{aligned} -n^\mu \nabla_\mu \Theta(A(x) - A_0) n^\mu \nabla_\mu \Theta(V_0 - V(x)) &= \\ &= \sqrt{-\partial_\mu A \partial^\mu A} \delta(A(x) - A_0) \frac{-\partial_\mu V \partial^\mu A}{\sqrt{-\partial_\mu A \partial^\mu A}} \delta(V_0 - V(x)). \end{aligned} \quad (2.11)$$

We note, finally, that averages of a scalar S over different (hyper)surfaces are trivially defined – with self explanatory notation – by:

$$\langle S \rangle_{V_0, A_0} = \frac{I(S; V_0, A_0; -)}{I(1; V_0, A_0; -)}; \quad (2.12)$$

$$\langle S \rangle_{V_0}^{A_0} = \frac{I(S; V_0; A_0)}{I(1; V_0; A_0)}; \quad (2.13)$$

$$\langle S \rangle_{A_0}^{V_0} = \frac{I(S; A_0; V_0)}{I(1; A_0; V_0)}. \quad (2.14)$$

3 Buchert-Ehlers commutation rules on the light-cone

For the phenomenological applications of the following sections we are interested, in particular, in the derivatives of the averages defined in (2.12) and (2.13). To this purpose, let us first consider the derivatives of $I(S; V_0, A_0; -)$ and $I(S; V_0; A_0)$ with respect to A_0 and V_0 (quantities that, like the starting integrals themselves, are covariant and gauge invariant).

We will use the identities (here $k_\mu \equiv \partial_\mu V$):

$$\delta'(A - A_0) = \frac{k^\mu \partial_\mu \delta(A - A_0)}{k^\nu \partial_\nu A}, \quad \delta(A - A_0) = \frac{k^\mu \partial_\mu \Theta(A - A_0)}{k^\nu \partial_\nu A}, \quad (3.1)$$

$$\delta'(V - V_0) = \frac{\partial^\mu A \partial_\mu \delta(V - V_0)}{k^\nu \partial_\nu A}, \quad \delta(V - V_0) = \frac{\partial^\mu A \partial_\mu \Theta(V - V_0)}{k^\nu \partial_\nu A}, \quad (3.2)$$

and the relation $k^\mu \partial_\mu \delta(V - V_0) = k^\mu k_\mu \delta'(V - V_0) = 0$. Integrating by parts, we then obtain

$$\begin{aligned} \frac{\partial}{\partial A_0} I(S; V_0, A_0; -) &= I\left(\frac{k \cdot \partial S}{k \cdot \partial A}; V_0, A_0; -\right) + I\left(\frac{\nabla \cdot k}{k \cdot \partial A} S; V_0, A_0; -\right), \\ \frac{\partial}{\partial A_0} I(S; V_0; A_0) &= I\left(\frac{k \cdot \partial S}{k \cdot \partial A}; V_0; A_0\right) + I\left(\left[\nabla \cdot k - \frac{1}{2} \frac{k^\mu \partial_\mu ((\partial A)^2)}{(\partial A)^2}\right] \frac{S}{k \cdot \partial A}; V_0; A_0\right), \end{aligned} \quad (3.3)$$

where, to simplify notations, we have introduced the following definitions: $k^\mu \partial_\mu S = k \cdot \partial S$, $k^\mu \partial_\mu A = k \cdot \partial A$, $\partial_\mu A \partial^\mu S = \partial A \cdot \partial S$, $\partial_\mu A \partial^\mu A = (\partial A)^2$, $\nabla_\mu k^\mu = \nabla \cdot k$ and $\square = \nabla^\mu \nabla_\mu$.

Using these results we can write a generalized version of the Buchert-Ehlers commutation rule [9] for light-cone average. In the case of averages defined over the integration domain (2.9) we find

$$\frac{\partial}{\partial A_0} \langle S \rangle_{V_0, A_0} = \left\langle \frac{k \cdot \partial S}{k \cdot \partial A} \right\rangle_{V_0, A_0} + \left\langle \frac{\nabla \cdot k}{k \cdot \partial A} S \right\rangle_{V_0, A_0} - \left\langle \frac{\nabla \cdot k}{k \cdot \partial A} \right\rangle_{V_0, A_0} \langle S \rangle_{V_0, A_0}, \quad (3.4)$$

while for 3-dimensional averages over the domain (2.7) we obtain:

$$\begin{aligned} \frac{\partial}{\partial A_0} \langle S \rangle_{V_0}^{A_0} &= \left\langle \frac{k \cdot \partial S}{k \cdot \partial A} \right\rangle_{V_0}^{A_0} + \left\langle \left[\nabla \cdot k - \frac{1}{2} \frac{k^\mu \partial_\mu ((\partial A)^2)}{(\partial A)^2} \right] \frac{S}{k \cdot \partial A} \right\rangle_{V_0}^{A_0} \\ &\quad - \left\langle \left[\nabla \cdot k - \frac{1}{2} \frac{k^\mu \partial_\mu ((\partial A)^2)}{(\partial A)^2} \right] \frac{1}{k \cdot \partial A} \right\rangle_{V_0}^{A_0} \langle S \rangle_{V_0}^{A_0}. \end{aligned} \quad (3.5)$$

Following a similar procedure we can also evaluate the generalization of the Buchert-Ehlers commutation rule for the derivative of $I(S; V_0, A_0; -)$ and $I(S; V_0; A_0)$ with respect to V_0 . As we shall see in Sect. 5, such derivatives, together with the previous ones, may be useful in evaluating an averaged version of the ‘‘redshift drift’’ parameter. After some calculations we are led to

$$\begin{aligned} \frac{\partial}{\partial V_0} \langle S \rangle_{V_0, A_0} &= \left\langle \frac{\partial A \cdot \partial S}{k \cdot \partial A} \right\rangle_{V_0, A_0} - \left\langle k \cdot \partial S \frac{(\partial A)^2}{(k \cdot \partial A)^2} \right\rangle_{V_0, A_0} \\ &\quad + \left\langle \left[\square A - \nabla_\mu \left(k^\mu \frac{(\partial A)^2}{k \cdot \partial A} \right) \right] \frac{S}{k \cdot \partial A} \right\rangle_{V_0, A_0} \\ &\quad - \left\langle \left[\square A - \nabla_\mu \left(k^\mu \frac{(\partial A)^2}{k \cdot \partial A} \right) \right] \frac{1}{k \cdot \partial A} \right\rangle_{V_0, A_0} \langle S \rangle_{V_0, A_0}, \end{aligned} \quad (3.6)$$

and

$$\begin{aligned} \frac{\partial}{\partial V_0} \langle S \rangle_{V_0}^{A_0} &= \left\langle \frac{\partial A \cdot \partial S}{k \cdot \partial A} \right\rangle_{V_0}^{A_0} - \left\langle k \cdot \partial S \frac{(\partial A)^2}{(k \cdot \partial A)^2} \right\rangle_{V_0}^{A_0} \\ &\quad + \left\langle \left[\square A - \partial^\mu A \partial_\mu \ln((\partial A)^2) \right] \frac{S}{k \cdot \partial A} \right\rangle_{V_0}^{A_0} \\ &\quad - \left\langle \left[\nabla \cdot k \frac{(\partial A)^2}{(k \cdot \partial A)^2} + \frac{1}{2} k^\mu \partial_\mu \left(\frac{(\partial A)^2}{(k \cdot \partial A)^2} \right) \right] S \right\rangle_{V_0}^{A_0} \\ &\quad - \left\langle \left[\square A - \partial^\mu A \partial_\mu \ln((\partial A)^2) \right] \frac{1}{k \cdot \partial A} \right\rangle_{V_0}^{A_0} \langle S \rangle_{V_0}^{A_0} \\ &\quad + \left\langle \left[\nabla \cdot k \frac{(\partial A)^2}{(k \cdot \partial A)^2} + \frac{1}{2} k^\mu \partial_\mu \left(\frac{(\partial A)^2}{(k \cdot \partial A)^2} \right) \right] \right\rangle_{V_0}^{A_0} \langle S \rangle_{V_0}^{A_0}. \end{aligned} \quad (3.7)$$

In a similar way one could also derive equations for the derivatives of the average $\langle S \rangle_{A_0}^{V_0}$. However, since we will not discuss applications of those, we will not give their explicit form here.

4 Geodesic light-cone coordinates

4.1 Definition of geodesic light-cone gauge

We now turn to a special (adapted) coordinate system, $x^\mu = (w, \tau, \theta^a)$, $a = 1, 2$, in which the previous equations take a simpler form. In this sense they will play a role similar to the one played by synchronous gauge coordinates for spatial averages with respect to geodesic observers [19]. We are interested in coordinates such that the level sets of one of them define the past light-cones, while those of another coordinate define a set of geodesic observers. We claim that such coordinates, that we will call geodesic light-cone (GLC) coordinates, are defined by the metric:

$$ds^2 = \Upsilon^2 dw^2 - 2\Upsilon dw d\tau + \gamma_{ab}(d\theta^a - U^a dw)(d\theta^b - U^b dw); \quad a, b = 1, 2. \quad (4.1)$$

This metric depends on six arbitrary functions (Υ , the two-dimensional vector U^a and the symmetric tensor γ_{ab}) and corresponds to a complete gauge fixing (modulo residual transformations involving non-generic functions of all the coordinates) of the so-called observational coordinates² discussed in detail in [6, 10]. In matrix form, the metric and its inverse are given by:

$$g_{\mu\nu} = \begin{pmatrix} \Upsilon^2 + U^2 & -\Upsilon & -U_b \\ -\Upsilon & 0 & \vec{0} \\ -U_a & \vec{0} & \gamma_{ab} \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} 0 & -1/\Upsilon & \vec{0} \\ -1/\Upsilon & -1 & -U^b/\Upsilon \\ \vec{0} & -U^a/\Upsilon & \gamma^{ab} \end{pmatrix}, \quad (4.2)$$

where γ^{ab} is the inverse of γ_{ab} . Clearly w is a null coordinate (i.e. $\partial_\mu w \partial^\mu w = 0$). More interestingly, one can check that $\partial_\mu \tau$ defines a geodesic flow, i.e. that

$$(\partial^\nu \tau) \nabla_\nu (\partial_\mu \tau) \equiv 0, \quad (4.3)$$

as a consequence of $g^{\tau\tau} = -1$. Thus an observer defined by constant τ spacelike hypersurfaces is in geodesic motion. Also note that $\sqrt{-g} = \Upsilon \sqrt{|\gamma|}$, with $\gamma = \det \gamma_{ab}$.

In order to understand the geometric meaning of these variables, it is useful to consider the limiting case of a spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) Universe, in the cosmic time gauge, with scale factor $a(t)$. Such a limit is easily reproduced by Eq. (4.1) by setting

$$\begin{aligned} w &= r + \eta, & \tau &= t, & \Upsilon &= a(t), & U^a &= 0, \\ \gamma_{ab} d\theta^a d\theta^b &= a^2(t) r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \end{aligned} \quad (4.4)$$

²Note that our coordinates θ^a are not necessarily ‘‘observational’’, in general, but they can be reduced to such form (e.g. to standard spherical coordinates, parallelly propagated along the observer world-line) through an appropriate relabelling of null generators [10].

where η is the conformal time of the homogeneous metric: $d\eta = dt/a$.

In these coordinates a past light-cone hypersurface is specified by the condition $w = w_0$, and the momentum of a photon traveling on it, being proportional to $k_\mu = \partial_\mu w$, is orthogonal both to itself ($k_\mu k^\mu = 0$) and to the 2-surface generated by ∂_{θ^a} ($\partial_\mu \theta^a k^\mu = 0$). The velocity of a generic observer defined by a scalar A as in Eq. (2.5) satisfies:

$$k_\mu n^\mu = \frac{1}{\Upsilon} \frac{\partial_\tau A}{\sqrt{-(\partial A)^2}}. \quad (4.5)$$

For a geodesic observer A depends only on τ and we can always set $A = \tau$, so the above relation simplifies to:

$$k_\mu n^\mu = \Upsilon^{-1}. \quad (4.6)$$

These equations will be used in Sect. 5 in connection with the redshift and luminosity distance. We also note for later use that, in these coordinates, the covariant divergence of k^μ takes the simple form:

$$\nabla_\mu k^\mu = -\frac{1}{\Upsilon} \partial_\tau \left(\ln \sqrt{|\gamma|} \right). \quad (4.7)$$

4.2 Average equations in the GLC gauge

In the GLC gauge the averaging integrals introduced in Sect. 2 greatly simplify, especially in the case where the reference hypersurface $\Sigma(A)$ defines a geodesic observer. We will concentrate on the integrals (2.9) and (2.7) which, setting $V = w$, $V_0 = w_0$, are now given by:

$$I(S; w_0, A_0; -) = \int d^2\theta dw d\tau \sqrt{|\gamma|} |\partial_\tau A| \delta(w - w_0) \delta(A - A_0) S, \quad (4.8)$$

$$I(S; w_0; A_0) = \int d^2\theta dw d\tau \sqrt{|\gamma|} \frac{|\partial_\tau A|}{\sqrt{-(\partial A)^2}} \delta(w - w_0) \Theta(A - A_0) S. \quad (4.9)$$

For a geodesic reference observer, with $A = \tau$ (and $A_0 = \tau_0$), we obtain:

$$I(S; w_0, \tau_0; -) = \int d^2\theta \sqrt{|\gamma(w_0, \tau_0, \theta^a)|} S(w_0, \tau_0, \theta^a), \quad (4.10)$$

$$I(S; w_0; \tau_0) = \int d^2\theta d\tau \sqrt{|\gamma(w_0, \tau, \theta^a)|} \Theta(\tau - \tau_0) S(w_0, \tau, \theta^a). \quad (4.11)$$

The generalized Buchert-Elhers commutation rules introduced in Sect. 3 also simplify as illustrated below.

Average on the 2-sphere embedded in the light-cone

Let us consider separately, as before, derivatives with respect to A_0 and to $V_0 = w_0$. Starting with A_0 , Eq. (3.4) can now be written as

$$\partial_{A_0} \langle S \rangle_{w_0, A_0} = \left\langle \frac{\partial_\tau S}{\partial_\tau A} \right\rangle_{w_0, A_0} + \left\langle S \frac{\partial_\tau \ln \sqrt{|\gamma|}}{\partial_\tau A} \right\rangle_{w_0, A_0} - \left\langle \frac{\partial_\tau \ln \sqrt{|\gamma|}}{\partial_\tau A} \right\rangle_{w_0, A_0} \langle S \rangle_{w_0, A_0}, \quad (4.12)$$

and for the special case where $\Sigma(A)$ defines a geodesic observer (with $A = \tau$ and $A_0 = \tau_0$), it reduces to

$$\partial_{\tau_0} \langle S \rangle_{w_0, \tau_0} = \langle \partial_\tau S \rangle_{w_0, \tau_0} + \left\langle S \partial_\tau \ln \sqrt{|\gamma|} \right\rangle_{w_0, \tau_0} - \left\langle \partial_\tau \ln \sqrt{|\gamma|} \right\rangle_{w_0, \tau_0} \langle S \rangle_{w_0, \tau_0}. \quad (4.13)$$

Note that, in this last case with $A = \tau$, the partial derivative with respect to A_0 reduces to a standard derivative with respect to the proper (cosmic) time if we consider the limit of a homogeneous FLRW Universe (see Eq. (4.4)).

In the geodesic case we have also the following simplification for the derivative with respect to w_0 as given in Eq. (3.6):

$$\begin{aligned} \partial_{w_0} \langle S \rangle_{w_0, \tau_0} &= \langle \partial_w S + U^a \partial_a S \rangle_{w_0, \tau_0} + \left\langle S \partial_w \ln \sqrt{|\gamma|} \right\rangle_{w_0, \tau_0} \\ &+ \left\langle S \left[\partial_a U^a + U^a \partial_a \ln \sqrt{|\gamma|} \right] \right\rangle_{w_0, \tau_0} - \left\langle \partial_w \ln \sqrt{|\gamma|} \right\rangle_{w_0, \tau_0} \langle S \rangle_{w_0, \tau_0} \\ &- \left\langle \partial_a U^a + U^a \partial_a \ln \sqrt{|\gamma|} \right\rangle_{w_0, \tau_0} \langle S \rangle_{w_0, \tau_0}. \end{aligned} \quad (4.14)$$

Average on the truncated light-cone

For a generic hypersurface we cannot specify $(\partial A)^2$, and Eq. (3.5) just simplifies as

$$\begin{aligned} \partial_{A_0} \langle S \rangle_{w_0}^{A_0} &= \left\langle \frac{\partial_\tau S}{\partial_\tau A} \right\rangle_{w_0}^{A_0} + \left\langle S \frac{\partial_\tau \ln \sqrt{|\gamma|} - \partial_w (\partial A \cdot \partial A)}{\partial_\tau A} \right\rangle_{w_0}^{A_0} \\ &- \left\langle \frac{\partial_\tau \ln \sqrt{|\gamma|} - \partial_w (\partial A \cdot \partial A)}{\partial_\tau A} \right\rangle_{w_0}^{A_0} \langle S \rangle_{w_0}^{A_0}. \end{aligned} \quad (4.15)$$

However, in the case where $\Sigma(A)$ defines a geodesic observer (with $A = \tau$ and $A_0 = \tau_0$) one obtains:

$$\partial_{\tau_0} \langle S \rangle_{w_0}^{\tau_0} = \langle \partial_\tau S \rangle_{w_0}^{\tau_0} + \left\langle S \partial_\tau \ln \sqrt{|\gamma|} \right\rangle_{w_0}^{\tau_0} - \left\langle \partial_\tau \ln \sqrt{|\gamma|} \right\rangle_{w_0}^{\tau_0} \langle S \rangle_{w_0}^{\tau_0}. \quad (4.16)$$

Comparing with Eq. (4.13) we find, for this last case, the same commutation rule independently of the averaging prescription used.

Finally, for a hypersurface $\Sigma(A)$ associated to a geodesic observer, we also have the following simplification of Eq. (3.7):

$$\partial_{w_0} \langle S \rangle_{w_0}^{\tau_0} = \langle \partial_w S + U^a \partial_a S \rangle_{w_0}^{\tau_0} + \left\langle S \partial_w \ln \sqrt{|\gamma|} \right\rangle_{w_0}^{\tau_0}$$

$$\begin{aligned}
& + \left\langle S \left[\partial_a U^a + U^a \partial_a \ln \sqrt{|\gamma|} \right] \right\rangle_{w_0}^{\tau_0} - \left\langle \partial_w \ln \sqrt{|\gamma|} \right\rangle_{w_0}^{\tau_0} \langle S \rangle_{w_0}^{\tau_0} \\
& - \left\langle \partial_a U^a + U^a \partial_a \ln \sqrt{|\gamma|} \right\rangle_{w_0}^{\tau_0} \langle S \rangle_{w_0}^{\tau_0}.
\end{aligned} \tag{4.17}$$

Hence, we obtain the same commutation rule (see Eq. (4.14)) even for derivatives with respect to w_0 .

5 Some physical applications

Information about the large scale structure of our Universe reaches us travelling along the null geodesics of a possibly inhomogeneous spacetime. Hence, an averaged description of the cosmological geometry should unavoidably make use of some light-cone averaging procedure, like those illustrated in the previous sections. In this section we will suggest two possible applications of the previous formalism: we will consider, in particular, the averaging of the luminosity-distance redshift relation and the averaging of the redshift drift parameter for a generic inhomogeneous cosmology. Note that in this section we will be dealing with the τ coordinate both at the source and at the observer's position. They play the same role as $A_0 \equiv \tau_0$ of the previous sections, and will be denoted by τ_s and τ_0 respectively.

5.1 The redshift to luminosity-distance relation

It is well known that the scalar redshift parameter z , for a photon with momentum k_μ emitted by a source S and received by the observer O , depends on the scalar product between k_μ and the four-velocity n_μ of a local reference observer, and is given in general (with self-explanatory notation) by:

$$1 + z = \frac{(k_\mu n^\mu)_s}{(k_\nu n^\nu)_0}. \tag{5.1}$$

Note that this object may be regarded as a bi-scalar, as it depends on the ratio of the same scalar quantity calculated in two different spacetime points. In the GLC gauge of Sect. 4, and in the physically interesting case of a geodesic observer (that we will consider hereafter), we can use Eq. (4.6) and we thus obtain:

$$1 + z = \frac{\Upsilon_0}{\Upsilon_s}, \tag{5.2}$$

where $\Upsilon_0 = \Upsilon(w_0, \tau_0, \theta_0^a)$ and $\Upsilon_s = \Upsilon(w_0, \tau_s, \theta_s^a)$.

In a cosmological context we can define at least three different distances of physical interest (see e.g. [20]): d_s , namely the angular distance of the source as seen from the

observer (in the FLRW limit of Eq. (4.4) we have $d_s = ra(t_s)$, where r is the comoving radial distance of the source); d_o , namely the angular distance of the observer “as seen from” the source (in the FLRW limit we have $d_o = ra(t_0)$); and d_L , the so-called luminosity distance. These three observational distances are always related to each other, independently of the given cosmological model, by redshift factors [15, 21]:

$$d_L = (1 + z)d_o = (1 + z)^2 d_s, \quad (5.3)$$

as a consequence of the Etherington reciprocity law [22].

On the other hand, in a general spacetime, we can also identify (point-by-point on the 2-sphere) the square of the angular distance of the source with the ratio of the intrinsic cross-sectional area element to the subtended solid angle element, according to $d_s^2 = dA/d\Omega$ (see e.g. [14]). In our GLC gauge we have $k^\mu = (0, -1/\Upsilon, 0, 0)$, and, as already mentioned, the 2-sphere embedded in the light-cone (corresponding to the integration region of Eq. (2.9)) is orthogonal to the photon momentum, so that the cross-sectional area element is proportional to $\sqrt{|\gamma|}d\theta^1 d\theta^2$. Therefore, d_s^2 can be written as

$$d_s^2(w = w_0, \tau_s, \theta^a) = \lim_{\rho \rightarrow 0} \rho^2 \frac{\sqrt{|\gamma_s|}}{\sqrt{|\gamma(\rho)|}}, \quad (5.4)$$

where ρ is the proper radius of an infinitesimal sphere centered around the observer, $w = w_0$ defines the past light-cone connecting source and observer, and $\tau = \tau_s$ defines the spacelike hypersurface normal to n^μ at the source position. Eq. (5.4) easily reduces to $d_s = ra(t_s)$ in the FLRW case of Eq. (4.4).

It is appropriate to mention that in the present discussion we are neglecting the possible occurrence of caustics in the past light cone of our observer, which would affect the assumed relation between area distance and angular size distance. The possible relevance of this interesting effect, and its dependence on gravitational lensing and on the distribution of inhomogeneities has been addressed in detail in [23].

The relevant variables related to supernovae observations are the luminosity distance d_L and the redshift z . Their relation has been studied within a gauge invariant approach, for a linearly perturbed FLRW metric, in [24]. Averaging their values on the two-sphere embedded in the light-cone, and using the above results, we obtain, for a general inhomogeneous metric background,

$$\langle d_L \rangle_{w_0, \tau_s} = \frac{\int d^2\theta \sqrt{|\gamma(w_0, \tau_s, \theta^a)|} [\Upsilon^2(w_0, \tau_0, \theta^a) / \Upsilon^2(w_0, \tau_s, \theta^a)] d_s(w_0, \tau_s, \theta^a)}{\int d^2\theta \sqrt{|\gamma(w_0, \tau_s, \theta^a)|}}, \quad (5.5)$$

where d_s is given by Eq.(5.4), and

$$\langle 1 + z \rangle_{w_0, \tau_s} = \frac{\int d^2\theta \sqrt{|\gamma(w_0, \tau_s, \theta^a)|} [\Upsilon(w_0, \tau_0, \theta^a) / \Upsilon(w_0, \tau_s, \theta^a)]}{\int d^2\theta \sqrt{|\gamma(w_0, \tau_s, \theta^a)|}}. \quad (5.6)$$

Since our coordinates should not be pathological near $w = w_0$ and $\tau = \tau_0$ we should have $\lim_{\tau \rightarrow \tau_0} \Upsilon(w = w_0, \tau, \theta^a) = \Upsilon(w = w_0, \tau_0) \equiv \Upsilon_0$ independent of θ^a . Hence, in Eq. (5.5) and (5.6), the factor $\Upsilon(w_0, \tau_0, \theta^a) = \Upsilon_0$ behaves like a constant, leaving a well defined integral of a scalar object over the 2-sphere.

The above expressions can be used to see how the usual redshift to luminosity-distance relation gets affected by the inhomogeneities of the cosmological geometry. To this purpose it is also possible (and probably more convenient) to consider averages of the luminosity distance on constant redshift surfaces³. In that case the geodesic observers measuring the redshift do not coincide any longer with the “observers” associated to the flow lines of the reference hypersurface $\Sigma(A)$, chosen to specify the averaging region. Even in such a case, however, our general formalism provides a rather simple averaging prescription by identifying the scalar A in Eq. (2.12) with $k_\mu u^\mu$ (where now u^μ is the velocity of a geodesic observer), and the quantity to be averaged with d_L (or with some convenient observable related to it).

The straightforward calculation simplifies considerably in our gauge giving, for instance,

$$\langle d_L \rangle_{w_0, z} = (1+z)^2 \frac{\int d^2\theta \sqrt{|\gamma(w_0, \tau(z, w_0, \theta^a), \theta^a)|} d_s(w_0, \tau(z, w_0, \theta^a), \theta^a)}{\int d^2\theta \sqrt{|\gamma(w_0, \tau(z, w_0, \theta^a), \theta^a)|}}, \quad (5.7)$$

where $\tau(z, w_0, \theta^a)$ is the solution of:

$$\frac{\Upsilon(w_0, \tau, \theta^a)}{\Upsilon_0} = \frac{1}{1+z}. \quad (5.8)$$

We plan to come back to an explicit application of our formalism to this problem in the near future.

5.2 Redshift drift

The redshift drift (RSD) is defined as the rate of change of the redshift of a given source with respect to the observer’s proper time. Since both the observer and the source simultaneously evolve in time, the relevant hypersurfaces for the RSD effect will consist of two light-cones with two different bases (see Fig. 2). Assuming that source and observer are in geodesic motion and have negligible peculiar velocity, the RSD in a FLRW Universe, and in the proper-time interval Δt_0 , can be simply expressed as (see e.g. [13]):

$$\frac{\Delta z}{\Delta t_0} = (1+z)H_0 - H_s = \frac{\dot{a}_0 - \dot{a}_s}{a_s}, \quad (5.9)$$

where H_0 is the present value of the Hubble parameter while H_s is its value at the emission time. Clearly the RSD effect is a good indicator of cosmic acceleration as a function of z which *does not* use hypothetical standard candles.

³We would like to thank Misao Sasaki for stressing this point with us.

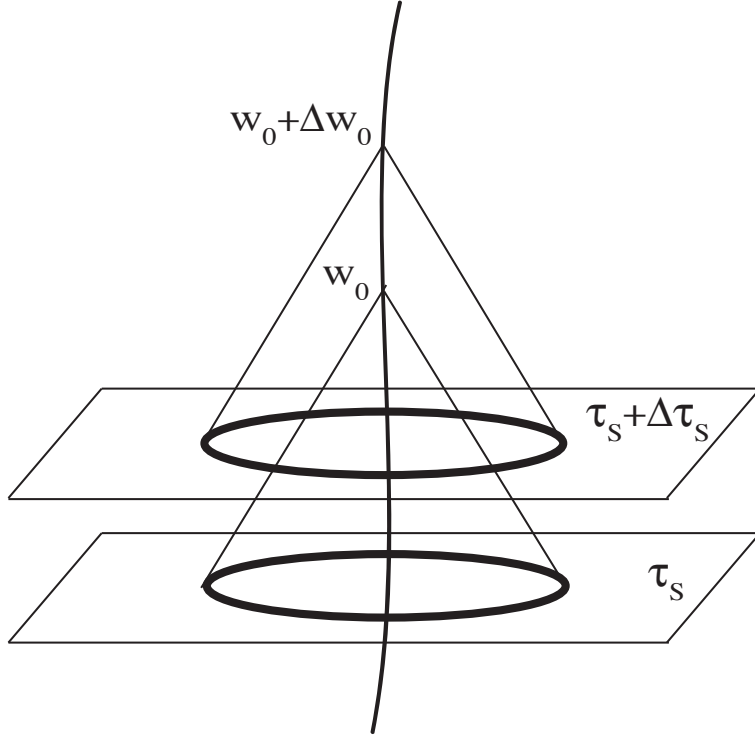


Figure 2: A graphic illustration of the redshift drift effect. A possible variation of the cosmological expansion rate is detected by comparing observations performed on two different past light-cones.

In order to generalize this expression to a general inhomogeneous Universe (and attempt some averaging of it) let us consider again the geodesic observer of the GLC gauge, with coordinates $x^\mu = (w, \tau, \theta^a)$. As shown in Eq. (5.2), z is in principle a function of seven independent variables, namely of w_0 , τ_0 , θ_0^a , τ_s and θ_s^a (note that, since we are assuming that source and observer are on the same light-cone $w = w_0$ at $\tau = \tau_0$, they will be both on the light-cone $w = w_0 + \Delta w_0$ at $\tau = \tau_0 + \Delta\tau_0$, see Fig. 2). We can also note that Υ_0 is independent of θ_0^a (see the comment after Eq.(5.6)). As a consequence we have only five independent contributions to the variation of $1 + z$, and we can write, in general:

$$\Delta(1 + z) = \frac{\partial(1 + z)}{\partial w_0} \Delta w_0 + \frac{\partial(1 + z)}{\partial \tau_0} \Delta \tau_0 + \frac{\partial(1 + z)}{\partial \tau_s} \Delta \tau_s + \frac{\partial(1 + z)}{\partial \theta_s^a} \Delta \theta_s^a. \quad (5.10)$$

As shown in Eq.(4.4), in the homogeneous limit the coordinate τ goes to the proper (cosmic) time t of the synchronous gauge. So, locally around our geodesic observer, we choose to evaluate the redshift drift Δz with respect to his/her time parameter τ , and we need, to this purpose, an explicit relation between $\Delta\tau_0$ and the variation of the other coordinates

involved. For a geodesic observer with $n_\mu = -\partial_\mu\tau$ we have $\dot{x}^\mu \sim n^\mu = (1/\Upsilon, 1, U^a/\Upsilon)$, so that we can express Δx^μ in terms of Δw , at all times, as:

$$\Delta\tau = \Upsilon\Delta w \quad , \quad \Delta\theta^a = U^a\Delta w. \quad (5.11)$$

Using Eq. (5.11) we find $\Delta w_0 = \Delta\tau_0/\Upsilon_0 = \Delta\tau_s/\Upsilon_s$, and the final result for the RSD can be written in terms of z and its derivatives as:

$$\frac{\Delta z}{\Delta\tau_0} = (1+z)\tilde{H}_0 + \frac{1}{\Upsilon_0}\frac{\partial}{\partial w_0}(1+z) + \frac{\partial}{\partial\tau_s}\ln(1+z) + \frac{U_s^a}{\Upsilon_0}\frac{\partial}{\partial\theta_s^a}(1+z), \quad (5.12)$$

where:

$$\tilde{H}_0 \equiv \frac{1}{\Upsilon_0}\frac{\partial\Upsilon_0}{\partial\tau_0} \quad (5.13)$$

(let us note again that ∂_{w_0} acts on both metric factors Υ contained in $(1+z)$). This result is valid for a general inhomogeneous metric, and can be compared, as a useful consistency check, with a similar result previously obtained in the particular case of a spherically symmetric geometry [12]. If we move from our coordinates to the adapted coordinates used in [12] we find that our expression (5.12) exactly reduces to the expression for the RSD reported in Eq. (5) of Ref. [12].

The quantities appearing in Eq. (5.12) can now be averaged over the 2-sphere embedded, at $\tau = \tau_s$, in the light-cone $w = w_0$. Using our prescription based on Eq. (4.10) we find that both Υ_0 and \tilde{H}_0 can be taken out from the averaging integrals (which are performed over the variables θ^a), and we obtain:

$$\frac{\langle\Delta z\rangle_{w_0,\tau_s}}{\Delta\tau_0} = \langle 1+z \rangle_{w_0,\tau_s}\tilde{H}_0 + \frac{1}{\Upsilon_0}\langle\partial_w(1+z)\rangle_{w_0,\tau_s} + \langle\partial_\tau\ln(1+z)\rangle_{w_0,\tau_s} + \frac{1}{\Upsilon_0}\langle U^a\partial_a(1+z)\rangle_{w_0,\tau_s}. \quad (5.14)$$

For the derivatives performed with respect to the hypersurface parameters w and τ we can now apply the Buchert-Ehlers commutation rules in the simplified form of Sect. 4.2. Using in particular Eqs. (4.13) and (4.14) we are lead to:

$$\begin{aligned} \frac{\langle\Delta z\rangle_{w_0,\tau_s}}{\Delta\tau_0} &= \langle 1+z \rangle_{w_0,\tau_s}\tilde{H}_0 + \frac{1}{\Upsilon_0}\partial_{w_0}\langle 1+z \rangle_{w_0,\tau_s} + \partial_{\tau_s}\langle\ln(1+z)\rangle_{w_0,\tau_s} \\ &\quad + Q_w(z) + Q_\tau(z), \end{aligned} \quad (5.15)$$

where

$$Q_\tau(z) = -\left\langle\ln(1+z)\partial_\tau\ln\sqrt{|\gamma|}\right\rangle_{w_0,\tau_s} + \langle\ln(1+z)\rangle_{w_0,\tau_s}\left\langle\partial_\tau\ln\sqrt{|\gamma|}\right\rangle_{w_0,\tau_s}, \quad (5.16)$$

$$\begin{aligned} Q_w(z) &= \frac{1}{\Upsilon_0}\left\{-\left\langle(1+z)\partial_w\ln\sqrt{|\gamma|}\right\rangle_{w_0,\tau_s} - \left\langle(1+z)\left[\partial_a U^a + U^a\partial_a\ln\sqrt{|\gamma|}\right]\right\rangle_{w_0,\tau_s}\right. \\ &\quad \left. + \langle 1+z \rangle_{w_0,\tau_s}\left[\left\langle\partial_w\ln\sqrt{|\gamma|}\right\rangle_{w_0,\tau_s} + \left\langle\partial_a U^a + U^a\partial_a\ln\sqrt{|\gamma|}\right\rangle_{w_0,\tau_s}\right]\right\}. \end{aligned} \quad (5.17)$$

These last terms, together with the other terms of Eq. (5.15) control the difference between the *averaged redshift drift* and the *drift* for the *averaged redshift* and represent a light-cone analog of the backreaction terms due to the geometric inhomogeneities, first computed by Buchert [5] in the context of three-dimensional spacelike averages. Unlike in that case, however, our backreaction is physically controlled by the geometric dynamics of a two-dimensional surface, with metric γ_{ab} . Also, the possible physical meaning of these backreaction terms is in principle strongly model-dependent, and their physical effects are to be explicitly computed for any given model chosen to parametrize deviations from FLRW geometry.

For a homogeneous FLRW metric the averages disappear, and the backreaction terms Q_τ and Q_w are identically vanishing. In that case $\tilde{H}_0 = H_0$ and, using the homogeneous limit of Eq.(4.4), one also finds $\partial_w(1+z) = 0$ and $\partial_\tau \ln(1+z)_s = -H_s$, so that Eq. (5.15) gives back the result (5.9). For a general inhomogeneous model, however, several new contributions appear. It would be very interesting to see whether such backreaction effects are relevant for determining the kinematics of the Universe (and the equation of state of its energy components as a function of z) from the forthcoming RSD experiments (see e.g. [25]). We plan to come back to this question elsewhere [26], at least within specific inhomogeneous models such as the Lemaître-Tolman-Bondi (LTB) Universe [27].

6 Conclusions

In this paper we have made a first step towards defining suitable covariant and gauge invariant light-cone averages that should be relevant to analyze the effects of inhomogeneities on astrophysical observables related to light-like (massless) signals. We were able to define such averages both on light-like hypersurfaces and on ordinary 2-surfaces embedded on a specific light-cone (e.g. our past one). We have also written down the generalized version of the Buchert-Ehlers commutation rules (between averaging and differentiation) that hold for all such light-cone averages.

One obvious problem to which we would like to apply our technique is that of cosmic acceleration. This is actually nothing but a measured relation between redshift and luminosity distance which can only be explained in a FLRW cosmology if the Universe underwent a late-time accelerated expansion. Within General Relativity it implies, in turn, the existence of a large dark energy component in the cosmic fluid. Our formalism allows in principle to study directly the luminosity-redshift relation for a given non homogeneous Universe (e.g. of the LTB type [27]) or, even more interestingly, for a Universe containing a stochastic spectrum of inhomogeneities like those that originate from inflationary cosmology.

In order to prepare the ground for such investigations we have introduced some adapted “geodesic null” coordinates that allow to express in a simple way redshift and luminosity distance in terms of local metric components (i.e. calculated at the source’s position). Furthermore, our expressions for both the averages and for their derivatives take a considerably simpler form in these coordinates.

We have also applied our formalism to the case of the so-called redshift drift, a quantity which, without any need of using standard candles, should be sensitive to a possible acceleration of the scale factor at different redshift values, and thus could distinguish among different forms of dark energy. Once more, however, one should address the question of how inhomogeneities could modify the FLRW relation between redshift drift and expansion rate. By using the geodesic light-cone gauge we were able to find the formal modification of the relation and show that it contains, among others, Buchert-Ehlers-like commutator terms. The physical interpretation of such terms is, however, model-dependent, and their possible effect are to be extracted for any given model with an explicit computation.

Unlike in the case of spacelike hypersurfaces we have found that the most useful averages for physical applications are related to surface averages. However, it is easy to see that the quantities to be averaged are themselves sensitive to inhomogeneities over the whole past light-cone hypersurface even when they can be expressed in terms of quantities living on the surface. So far we have not found physical examples in which a true average over the full light-cone is involved, and we strongly suspect that there isn’t one.

In order to find out whether the formalism we have introduced is actually useful one will have to perform detailed calculations within realistic and explicit models for the relevant inhomogeneities. These can be either abstract mathematical ones, like LTB, swiss-cheese, or fractal models, or realistic ones based on our present knowledge of the large scale structure.

Another interesting exercise would be to compute the magnitude of several effects to second order in a perturbed FLRW Universe. Even if, as most people suspect, the effect of inhomogeneities will turn out to be small and not to change the conclusions that follow from simple FLRW consideration, the result of such an investigation will be important in that it will sharpen considerably the conclusion that dark energy is indeed unavoidable.

Acknowledgements

We wish to thank Ido Ben-Dayan for useful discussions during the early stages of this investigation. Useful correspondence with Luca Amendola, Roy Maartens and Misao Sasaki is also gratefully acknowledged.

References

- [1] G. F. R. Ellis, in “*10th International Conference on General Relativity and Gravitation*”, ed. by B. Bertotti, De Felice and A. Pascolini (Reidel, Dordrecht, 1984), p. 215.
- [2] T. Buchert, arXiv:1103.2016 [gr-qc].
- [3] G. F. R. Ellis, arXiv:1103.2335 [astro-ph.CO].
- [4] T. Buchert, Jap. Gen. Rel. Grav. **9** (2000) 306; T. Buchert and M. Carfora, Phys. Rev. Lett. **90** (2003) 031101; S. Rasanen, JCAP **02** (2004) 003; E. W. Kolb, S. Matarrese, A. Notari and A. Riotto, Phys. Rev. **D 71** (2005) 023524; E. W. Kolb, S. Matarrese and A. Riotto, New. J. Phys. **8** (2006) 322.
- [5] T. Buchert, Gen. Rel. Grav. **32** (2000) 105; **33** (2001) 1381.
- [6] R. Maartens, PhD Thesis, University of Cape Town, South Africa (1980).
- [7] R. Maartens and D. Matravvers, Class. Quant. Grav. **11** (1994) 2693.
- [8] V. Marra, E. W. Kolb and S. Matarrese, Phys. Rev. **D77** (2008) 023003; N. Li, M. Seikel and D. J. Schwarz, Fortsch. Phys. **56** (2008) 465; S. Rasanen, JCAP **02** (2009) 011; S. Rasanen, JCAP **03** (2010) 018; E. W. Kolb and C. R. Lamb, arXiv:0911.3852.
- [9] T. Buchert and J. Ehlers, Astron. Astrophys. **320** (1997) 1.
- [10] G. F. R. Ellis, S. D. Nel, R. Maartens, W. R. Stoeger and A. P. Whitman, Phys. Rep. **124** (1985) 315.
- [11] A. Sandage, Ap. J. **136** (1962) 319; A. Loeb, Ap. J. **499** (1998) L11.
- [12] J. P. Uzan, C. Clarkson and G. F. R. Ellis, Phys. Rev. Lett. **100** (2008), 191303.
- [13] C. Quercellini, L. Amendola, A. Balbi, P. Cabella and M. Quartin, arXiv:1011.2646 [astro-ph.CO].
- [14] J. Kristian and R. K. Sachs, Ap. J. **143** (1966) 379.
- [15] G. F. R. Ellis, in *General Relativity and Cosmology*, Proc. of the International School of Physics Enrico Fermi, ed. by R. K. Sachs (Academic Press, New York, 1971), p. 104. Reprinted as Gen. Rel. Grav. **39** (2007) 1055.
- [16] M. Gasperini, G. Marozzi and G. Veneziano, JCAP **03** (2009) 011.

- [17] M. Gasperini, G. Marozzi and G. Veneziano, JCAP **02** (2010) 009.
- [18] F. Finelli, G. Marozzi, G. P. Vacca and G. Venturi, Phys. Rev. Lett. **106** (2011) 121304.
- [19] G. Marozzi, JCAP **01** (2011) 012.
- [20] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
- [21] P. Schneider, J. Ehlers and E. E. Falco, *Gravitational Lenses* (Springer, Berlin, 1992).
- [22] I. M. H. Etherington, Phil. Mag. **15** (1933) 761.
- [23] G. F. R. Ellis, B. A. C. C. Bassett and P. K. S. Dunsby, Class. Quantum Grav. **15** (1998) 2345.
- [24] M. Sasaki, Mon. Not. Roy. Astron. Soc. **228** (1987) 653; M. Kasai and M. Sasaki, Mod. Phys. Lett. **A2** (1987) 727.
- [25] L. Pasquini et al., The Messenger **122** (2005) 10.
- [26] I. Ben-Dayan, M. Gasperini, G. Marozzi, F. Nugier and G. Veneziano, in preparation.
- [27] G. Lemaître, Ann. Soc. Sci. Brux. **A 53** (1933) 51. Reprinted in G. Lemaître, Gen. Rel. Grav. **29** (1997) 5; R.C. Tolman, Proc. Natl. Acad. Sci. **20** (1934) 169; H. Bondi, Mon. Not. R. Astron. Soc. **107** (1947) 410.