

Light-cone limits of large rectangular fishnets

I. Kostov

*Institut de physique théorique, Université Paris-Saclay,
CNRS, CEA, 91191 Gif-sur-Yvette, France*

*Department of Physics, Federal University of Espirito Santo,
29075-900, Vitória, Brazil*

E-mail: ivan.kostov@ipht.fr

ABSTRACT: Basso-Dixon integrals evaluate rectangular fishnets — Feynman graphs with massless scalar propagators which form a $m \times n$ rectangular grid — which arise in certain one-trace four-point correlators in the ‘fishnet’ limit of $\mathcal{N} = 4$ SYM. Recently, Basso et al. explored the thermodynamical limit $m \rightarrow \infty$ with fixed aspect ratio n/m of a rectangular fishnet and showed that in general the dependence on the coordinates of the four operators is erased, but it reappears in a scaling limit with two of the operators getting close in a controlled way. In this note I investigate the most general double scaling limit which describes the thermodynamics when one of two pairs of operators become nearly light-like. In this double scaling limit, the rectangular fishnet depends on both coordinate cross ratios. I show that all singular limits of the fishnet can be attained within the double scaling limit, including the null limit with the four points approaching the cusps of a null square. A direct evaluation of the fishnet in the null limit is presented any m and n .

KEYWORDS: Scale and Conformal Symmetries, Large-Order Behaviour of Perturbation Theory, Renormalons, Matrix Models, $1/N$ Expansion

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Contents

1	Introduction	1
1.1	BMN integral representation	4
1.2	The Fourier transformed integral	5
1.3	Determinant of ladders	5
2	Short-distance and light-cone limits for finite fishnets	6
2.1	Euclidean short-distance limit ($\sigma \rightarrow \infty$ with φ fixed)	6
2.2	Double light-cone, or null, limit ($\varphi \rightarrow \infty$ with σ fixed)	6
2.3	Single light-cone limit ($\varphi \rightarrow \infty$ with $\mu \equiv \varphi - \sigma$ fixed)	7
3	Double scaling limit	8
4	Solution of the saddle-point equations in the double scaling limit	11
4.1	Solution in regime I	13
4.2	Solution in regime II: square fishnet with $\sigma = 0$ and $\varphi \sim m$	14
4.3	Solution in the regime II: the general case	16
5	Discussion	18
A	Elliptic integrals	19
B	Fermionic representation of the $(m + \ell) \times m$ rectangular fishnet and derivation of the dual integral	20

1 Introduction

It has been known since decades that a series of Feynman graphs with massless propagators, the so called ladder graphs or simply ladders, can be evaluated explicitly [1]. In the last years it became clear that a vast family of planar Feynman graphs having a regular structure, the so called fishnet graphs, share this property. The term fishnet planar graphs was introduced by A. Zamolodchikov [2], who pointed out that they can be studied by two-dimensional integrability methods. The breakthrough was the invention of the ‘fishnet conformal field theory’ formulated as a certain projection of the integrable $\mathcal{N} = 4$ SYM [3–5], which opened the possibility to study fishnet Feynman integrals by adapting the integrability methods developed in $\mathcal{N} = 4$ SYM. Later the integrability of the fishnet CFT was established also in the spirit of the original paper [2] utilising the regular iterative structure of the fishnet graphs [6].

Since a fishnet Feynman graph can be interpreted as a (non-compact) lattice model with nearest neighbour interaction, there are hopes that its large fishnets is described by

some effective σ -model. In fact, the fishnet graphs with gaussian propagators have been first considered as a possible discretisation of the world sheet of a string, but such rigid discretisation does not respect the symmetry of the string path integral.

Remarkably, conformal fishnet integrals do have holographic interpretation in case of periodic boundary conditions [7–11]. There is strong evidence [7] that large periodic fishnet diagrams correspond to an integrable lattice regularisation of the two-dimensional AdS_5 non-linear sigma model.

It is believed that a holographic interpretation should exist also for large open fishnet graphs which are typically associated with correlation functions of products of matrix operators along a trace. The open fishnets possess nice integrability properties including Yangian symmetry [12–15], but for the moment it is not clear how to use them to explore their continuum limit in general. Gathering ‘experimental’ results concerning the thermodynamical limit of open fishnets might give us some intuition about its σ -model description.

In a recent work [16], the continuum limit was computed in for the simplest open fishnet graphs, introduced previously by Basso and Dixon [17], which are described briefly below. Basso-Dixon fishnets correspond to the four-point correlators

$$G_{m,n}(x_1, x_2, x_3, x_4) = \langle \text{Tr}\{\phi_2^n(x_1)\phi_1^m(x_2)\phi_2^\dagger(x_3)\phi_1^\dagger(x_4)\} \rangle, \quad (1.1)$$

in a theory of two $N_c \times N_c$ complex matrix fields ϕ_1 and ϕ_2 with chiral quartic interaction $\sim g^2 \text{Tr}[\phi_1\phi_2\phi_1^\dagger\phi_2^\dagger]$. The perturbative series for the correlator $G_{m,n}$ consists of a single Feynman graph representing regular square lattices of size $m \times n$ with the external legs on each side attached to four distinct points in the Minkowski space. Up to a standard factor, $G_{m,n}$ depends on the positions of the operators through the two conformal cross ratios,

$$G_{m,n}(x_1, x_2, x_3, x_4) = \begin{array}{c} x_2 \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ x_4 \end{array} x_1 \quad x_3 = \frac{g^{2mn}}{(x_{13}^2)^n (x_{24}^2)^m} \times I_{m,n}^{\text{BD}}(z, \bar{z}), \quad (1.2)$$

where $x_{ij}^2 = (x_i - x_j)^2$ and z, \bar{z} are defined by

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = \frac{z\bar{z}}{(1-z)(1-\bar{z})}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = \frac{1}{(1-z)(1-\bar{z})}. \quad (1.3)$$

A canonical choice for the positions of the four operators is

$$x_1 = (0, 0), \quad x_2 = (z, \bar{z}), \quad x_3 = (\infty, \infty), \quad x_4 = (1, 1). \quad (1.4)$$

It is convenient to use (for Minkowski kinematics) the exponential parametrisation

$$z = -e^{-\sigma-\varphi}, \quad \bar{z} = -e^{-\sigma+\varphi}. \quad (1.5)$$

The Euclidean kinematics is described by the analytic continuation of φ to the imaginary axis such that $\bar{z} = z^*$. Due to the symmetries $z \leftrightarrow \bar{z}$ and $z \leftrightarrow 1/\bar{z}$ one can consider only

the fundamental domain $z\bar{z} \leq 1, \bar{z} \leq 1$, or equivalently $\sigma > 0$ and $\varphi > 0$. Basso and Dixon [17] conjectured, using the integrability properties inherited from $\mathcal{N} = 4$ SYM, two different integral representations for $G_{m,n}$ which they named BMN (Berenstein-Maldacena-Nastase [18]) and FT (flux-tube) representations. The BMN integral representation was obtained by the adapting the ‘hexagonalisation’ [19–23] while the FT representation is obtained by the ‘pentagon OPE’ [24, 25]. A rigorous proof of the former was subsequently given using the method of separation of variables for conformal integrable spin chains [11, 26–28], generalising the 2d calculation of [29].

The BMN integral representation of $G_{m,n}$ looks as the radial part of an $m \times m$ hermitian matrix integral. In general, such an m -fold integral can be expressed as an $m \times m$ determinant of single integrals. These were identified in [16] as the ladder integrals for the case in question. In particular, the expression for $G_{1,n}$ is identical to the integral representation of the ladders found in [30].

The thermodynamical limit $m \rightarrow \infty$ with fixed cross ratios u, v and aspect ratio n/m was computed in [16] using the saddle-point approximation. The open fishnet, considered as a statistical model defined on the $m \times n$ rectangular grid, is characterised by an extensive free energy $\mathcal{F} = \log I_{m,n}^{\text{BD}} \sim mn$. In [16] it was observed that the bulk free energy per site,

$$\hat{\mathcal{F}} = \lim_{m \rightarrow \infty} \frac{\mathcal{F}}{mn}, \quad \mathcal{F} \equiv \log I_{m,n}^{\text{BD}}, \quad (1.6)$$

is different than the bulk free energy in case of periodic boundary conditions computed in [2]. Moreover, it is a non-trivial function of the aspect ratio n/m . A physical interpretation of this unusual thermodynamical limit due to the boundary conditions is still missing.

A similar phenomenon has been observed in the six-vertex model with domain-wall boundary conditions [31]. In the case of the six-vertex model, the dynamics in the continuum limit is described by a massless bosonic field which in case of domain-wall boundary conditions has diverging variation on the boundary. One can speculate that the rectangular fishnets correspond to singular boundary conditions for the ADS_5 σ -model.

Since the ADS_5 σ -model has more degrees of freedom than one massless boson, it is expected that the bulk free energy can be influenced by the boundary in different ways. Interestingly, the authors of [16] found that in the *scaling limit* combining the thermodynamical limit $m \rightarrow \infty$ and the short-distance limit $x_2 \rightarrow x_1$, the bulk free energy depends both on the aspect ratio n/m and on the hyperbolic angle $\sigma = -\log \sqrt{u/v}$. The relevant scaling parameter is $\hat{\sigma} \sim \sigma/m$. The scaling variable $\hat{\sigma}$ parametrises the flow between the ‘bulk’ thermodynamical limit ($\hat{\sigma} \rightarrow 0$) and the Euclidean short-distance limit ($\hat{\sigma} \rightarrow \infty$). The saddle-point equation in the scaling limit turns out to be the same as the finite-gap equation for a classical folded string rotating in $\text{AdS}_3 \times S^1$ solved in [32–34]. The leading term in the expansion at $\hat{\sigma} \rightarrow \infty$,

$$\hat{\mathcal{F}} = \log |\hat{\sigma}| + C_0 + \frac{C_1}{|\hat{\sigma}|} + \dots, \quad (1.7)$$

describes, in the string theory interpretation, a short folded string rotating in the flat space, while the subleading terms are corrections to the flat-space regime coming from

the curvature of AdS [35]. The analogy with the classical folded string is intriguing but purely formal.

More generally, one can consider the *double scaling limit* $m, \sigma, \varphi \rightarrow \infty$ where the bulk free energy depends on all three boundary parameters. It happens that this problem is exactly solvable and its solution is the main subject of this paper. The double scaling limit is characterised by the aspect ratio n/m and the two scaling parameters

$$\hat{\sigma} \equiv \frac{\sigma}{2\pi m}, \quad \hat{\varphi} \equiv \frac{\varphi}{2\pi m}. \tag{1.8}$$

The double scaling limit analytically connects the bulk thermodynamical limit ($\hat{\sigma} = \hat{\varphi} = 0$) with the Euclidean short-distance limit $\hat{\sigma} \rightarrow \infty$ studied in [16] as well as with the light-cone limit $\hat{\varphi} \rightarrow \infty$ to be considered here. The light-cone limit appears to be more subtle since the result depends on the ray in the $\{\hat{\varphi}, \hat{\sigma}\}$ plane along which the infinity is reached.

The computation of the free energy is based on the integral representations obtained in [17] and [16] which are reminded below to make the presentation self-consistent. The so called dual integral, related to the BMN integral by a Fourier transformation, represents a variant of the $O(-2)$ matrix model for which resolution techniques have been established in the last century [36]. For generic values of the two scaling parameters explicit expression is found not for the free energy itself but for its derivative with respect to m . This is in principle sufficient to compute to all orders the corrections to the leading log asymptotics in the short-distance and light-cone limits, which is however beyond the scope of this short paper.

The paper is organised as follows. Section 2 presents the derivation of the leading log asymptotics in the Euclidean short-distance limit ($\sigma \rightarrow \infty, \varphi = 0$) following [16], as well as in the double light-cone limit ($\sigma = 0, \varphi \rightarrow \infty$). The two limits are particular cases of the single light-cone limit which is characterised by a continuous parameter $\mu = \sigma - \varphi$. Section 3 is devoted to the double scaling limit which explore the whole $\{\hat{\sigma}, \hat{\varphi}\}$ plane. Since the matrix-model representation is not symmetric under exchanging m and n , it is more convenient to use as independent variables m , which is also the number of the ‘eigenvalues’, and $\ell \equiv n - m \geq 0$, which enters the external potential. The saddle-point equations for a general potential are reformulated in section 3 as a Riemann-Hilbert problem. In section 4 I give the solution of the R-H problem in terms of incomplete elliptic integrals. Finally I will show how the leading log asymptotics in the short-distance and light-like limits are extracted from the general solution.

1.1 BMN integral representation

The BMN integral representation, conjectured in [17] and proved in [26, 37] reads

$$I_{m,n}^{\text{BD}} = (uv)^{-m/2} \sum_{a_1, \dots, a_m=1}^{\infty} \prod_{j=1}^m \frac{\sinh(a_j \varphi)}{\sinh \varphi} (-1)^{a_j-1} \int \prod_{j=1}^m \frac{du_j}{2\pi} e^{2i\sigma u_j} \times \frac{\prod_{i=1}^m a_i \prod_{i<j} [(u_i - u_j)^2 + \frac{(a_i+a_j)^2}{4}]}{(u_j^2 + a_j^2/4)^{m+n}}, \tag{1.9}$$

where $(uv)^{-1/2} = 2 \cosh \sigma + 2 \cosh \varphi$ in the exponential parametrisation (1.5). The Euclidean kinematics is attained by the analytic continuation of φ to the imaginary axis such that $\bar{z} = z^*$.

The BMN integral is very similar to the expansion of the octagon [38] in a series of multiple integrals at weak coupling. More precisely, the BMN integral is obtained by retaining the term with m virtual particles and taking the weak coupling limit of the weights. The representation of the octagon with bridge $\ell = n - m$ in terms of free fermions [39] implies a similar representation for the fishnet, which is spelt out in appendix B.

1.2 The Fourier transformed integral

In [16], the BMN integral (1.9) was given, using the determinant representation in terms of ladders obtained in [17], a dual form

$$I_{m,n}^{\text{BD}} = \mathcal{Z}_m(\ell, \sigma, \varphi), \quad \ell \equiv n - m. \tag{1.10}$$

Up to a normalisation factor, the dual integral takes the form of the partition function of the $O(-2)$ matrix model [36]

$$\mathcal{Z}_m(\ell, \sigma, \varphi) = \frac{1}{\mathcal{N}} \frac{1}{m!} \int_{|\sigma|}^{\infty} \prod_{j=1}^m dt_j e^{-V(t_j)} \prod_{j,k=1}^m (t_j + t_k) \prod_{j < k}^m (t_j - t_k)^2 \tag{1.11}$$

with particular interaction potential

$$V(t) = \log \frac{\cosh t + \cosh \varphi}{\cosh \sigma + \cosh \varphi} - \ell \log(t^2 - \sigma^2). \tag{1.12}$$

In the matrix-model interpretation, the integration variables t_1, \dots, t_m the eigenvalues of a Hermitian $m \times m$ matrix. The normalisation factor reads

$$\mathcal{N} = \prod_{i=0}^{m-1} (2i + \ell)!(2i + 1 + \ell)! = \frac{G(2m + \ell + 1)}{G(\ell + 1)}, \tag{1.13}$$

where $G(m) \equiv \text{BarnesG}[m] = 1!2! \dots (m - 2)!$ is Barnes' G-function. In appendix B I show that the dual integral is obtained from the BMN integral (1.9) by a Fourier transformation. The spectral variable t is therefore the 'momentum' conjugated to the rapidity u .

1.3 Determinant of ladders

The r.h.s. of (1.11) gives, for $m = 1$, the integral representation of the ladder integrals obtained earlier by Broadhurst and Davydychev [30],

$$\begin{aligned} f_k(z, \bar{z}) &\equiv \frac{(1-z)(1-\bar{z})}{z-\bar{z}} k!(k-1)! L_k(z, \bar{z}) = \int_{|\sigma|}^{\infty} \frac{\cosh \sigma + \cosh \varphi}{\cosh t + \cosh \varphi} (t^2 - \sigma^2)^{k-1} 2t dt \\ &= (1-z)(1-\bar{z}) \sum_{j=k}^{2k} \frac{(k-1)! j!}{(j-k)!(2k-j)!} (-\log z\bar{z})^{2k-j} \frac{\text{Li}_j(z) - \text{Li}_j(\bar{z})}{z-\bar{z}}. \end{aligned} \tag{1.14}$$

The general m , the integral (1.11) is equivalent to the original Basso-Dixon determinant representation of the fishnet [17]. The latter is obtained by writing the product in the integrand in (1.11) as

$$\begin{aligned} \prod_{i=1}^m (t_j^2 - \sigma^2)^\ell \prod_{j < k}^m \left((t_j^2 - \sigma^2) - (t_k^2 - \sigma^2) \right)^2 &= (t_j^2 - \sigma^2)^\ell \left(\det_{j,k=1}^m \left[(t_j^2 - \sigma^2)^{k-1} \right] \right)^2 \\ &= \det_{j,k=1}^m \left[\sum_{i=1}^m (t_i^2 - \sigma^2)^{j+k-2+\ell} \right] \end{aligned} \tag{1.15}$$

which leads, via the Cauchy-Binet formula, to

$$I_{m,m+\ell}^{\text{BD}} = \frac{1}{\mathcal{N}} \det \left([f_{j+k+\ell-1}]_{j,k=1,\dots,m} \right). \tag{1.16}$$

2 Short-distance and light-cone limits for finite fishnets

2.1 Euclidean short-distance limit ($\sigma \rightarrow \infty$ with φ fixed)

If $\sigma \rightarrow \infty$, then $u \rightarrow 0$ and $v \rightarrow 1$. As $x_{14}^2 x_{23}^2 = x_{13}^2 x_{24}^2$, the limit $u \rightarrow 0$ implies that either $x_3 \rightarrow x_4$ or $x_1 \rightarrow x_2$. By conformal transformation one can achieve that both conditions are satisfied, with

$$|x_{12}|^2, |x_{34}|^2 \sim \sqrt{u} |x_{13}|^2 \quad (u \rightarrow 0, v \rightarrow 1), \tag{2.1}$$

hence $x_1 \sim x_2$ and $x_3 \sim x_4$. This is the Euclidean short-distance, or OPE, limit studied in [16]. For the sake of completeness, I sketch the derivation of the leading log asymptotics given there.

The short-distance limit is achieved by sending $\sigma \rightarrow \infty$ with φ finite. The ladder integrals (1.14) become (after shifting the integration variable $t \rightarrow t - \sigma$)

$$f_k(z, \bar{z}) \xrightarrow{\sigma \gg k} \int_0^\infty (2|\sigma|)^k t^{k-1} e^{-t} dt = (2|\sigma|)^k (k-1)! \tag{2.2}$$

and the determinant formula (1.16) gives [16]

$$I_{m,m+\ell}^{\text{BD}} \rightarrow \frac{(2|\sigma|)^{m(m+\ell)}}{\mathcal{N}} \det_{j,k} [(j+k+\ell-2)!] = \left(\log \frac{1}{u} \right)^{m(m+\ell)} C_{m,m+\ell} \tag{2.3}$$

where

$$C_{m,n} = \frac{G(m+1)G(n+1)}{G(m+n+1)}, \quad G(m) = 1!2! \dots (m-2)!. \tag{2.4}$$

2.2 Double light-cone, or null, limit ($\varphi \rightarrow \infty$ with σ fixed)

Sending $\varphi \rightarrow \infty$ with σ finite implies $\{u, v\} \rightarrow \{0, 0\}$, which means that the Minkowski intervals (but not the Euclidean distances!) x_{12}^2 and x_{14}^2 become simultaneously near light-like,

$$x_{12}^2, x_{34}^2 \sim \sqrt{u} |x_{13}| |x_{24}|, \quad x_{14}^2, x_{23}^2 \sim \sqrt{v} |x_{13}| |x_{24}|. \tag{2.5}$$

The ladder integrals f_k can be approximated by

$$f_k(z, \bar{z}) \xrightarrow{\varphi \gg k} 2 \int_0^\varphi t^{2k-1} dt = \frac{\varphi^{2k}}{k} \tag{2.6}$$

and the determinant formula gives, for $m, \ell \ll \varphi$,

$$I_{m,m+\ell}^{\text{BD}} \rightarrow \frac{\varphi^{2m(m+\ell)}}{\mathcal{N}} \det \left[\frac{1}{i+j-1+\ell} \right]_{i,j=1,\dots,m} \tag{2.7}$$

To compute the determinant, denote $x_i = i, y_j = j + \ell - 1$ and apply Cauchy's identity,¹

$$\begin{aligned} \det \left[\frac{1}{i+j-1+\ell} \right]_{i,j=1,\dots,m} &= \det_{i,j=1,\dots,m} \frac{1}{x_i + y_j} = \frac{\Delta(x)\Delta(y)}{\prod_{i,j=1}^m (x_i + y_j)} \\ &= \frac{\prod_{i=1}^{m-1} i!^2}{\prod_{i,j=1}^m (i+j+\ell-1)} = \frac{G(m+1)^2 G(m+\ell+1)^2}{G(2m+\ell+1)G(\ell+1)} = \mathcal{N} (C_{m,m+\ell})^2. \end{aligned} \tag{2.8}$$

Since $\varphi^2 = \log u \log v$, the fishnet integral takes in the double light-cone limit the following factorised form,

$$I_{m,m+\ell}^{\text{BD}} = C_{m,m+\ell} \left(\log \frac{1}{u} \right)^{m(m+\ell)} \times C_{m,m+\ell} \left(\log \frac{1}{v} \right)^{m(m+\ell)}. \tag{2.9}$$

The two factors are obviously associated with the two pairs of operators which become light-like. A factorised expression very similar to (2.9) was recently obtained in [40] for the leading log singularities of the dimensionally regularised Basso-Dixon fishnet in momentum space.

2.3 Single light-cone limit ($\varphi \rightarrow \infty$ with $\mu \equiv \varphi - \sigma$ fixed)

The limit $\varphi \rightarrow \infty$ with $\mu = \varphi - \sigma$ fixed, or $z \rightarrow 0$ with fixed $\bar{z} = e^\mu$, translates in terms of u and v as

$$u \rightarrow 0, \quad v \rightarrow \frac{1}{1+e^\mu} \quad (\mu \equiv \varphi - \sigma). \tag{2.10}$$

For μ finite, the sides 12 and 34 are close to light-like,

$$x_{12}^2, x_{34}^2 \sim \sqrt{u} |x_{13}| |x_{24}|, \tag{2.11}$$

while the sides 23 and 42 remain in general position.

It is obvious from (2.10) that the light-like limit interpolates continuously between the Euclidean short-distance limit ($\mu \rightarrow -\infty$) and the double light-like limit ($\mu \rightarrow +\infty$). The dependence on z and \bar{z} of the ladder integral (1.14) factorises,

$$\begin{aligned} f_k(z, \bar{z}) &\rightarrow (2|\sigma|)^k \int_0^\infty \frac{1+e^\mu}{e^t+e^\mu} t^{k-1} dt \\ &= -(k-1)! \left(\log \frac{1}{z} \right)^k \left(1 - \frac{1}{\bar{z}} \right) \text{Li}_k(\bar{z}) \end{aligned} \tag{2.12}$$

hence the fishnet integral factorises as well. The dependence on z exhibits the standard leading log singularity, while the dependence on \bar{z} is more involved,

$$I_{m,m+\ell}^{\text{BD}} \xrightarrow{z \rightarrow 0} \left(\log \frac{1}{z} \right)^{m(m+\ell)} F(\bar{z}), \tag{2.13}$$

with $F(\bar{z})$ being an $m \times m$ determinant of polylogs.

¹I thank Philippe Di Francesco for suggesting that.

3 Double scaling limit

Now let us consider the most general double scaling limit achieved by combining thermodynamical limit $m \rightarrow \infty$ with the scaling (1.8) of the cross ratios u and v ,

$$\ell, m, \sigma, \varphi \rightarrow \infty \quad \text{with} \quad \hat{\sigma} \equiv \frac{\sigma}{2\pi m}, \quad \hat{\varphi} \equiv \frac{\varphi}{2\pi m}, \quad \hat{\ell} \equiv \frac{\ell}{2\pi m} \quad \text{fixed.} \quad (3.1)$$

The third parameter $\hat{\ell}$ controls the aspect ratio, $n/m = (1 + \hat{\ell})/(1 - \hat{\ell})$. The so defined scaling parameters $\hat{\varphi}$ and $\hat{\sigma}$ are invariant under exchanging $m \leftrightarrow n$ while $\hat{\ell}$ changes sign. I will stick most of the time to the original non-normalised variables ℓ, σ, φ , assuming the scaling (3.1), i.e. $\ell, \sigma, \varphi \sim m$ with m sufficiently large. This will make more obvious the comparison of the results obtained for different scales.

The goal is to compute the leading contribution to the free energy $\mathcal{F}_m = \log \mathcal{Z}_m$, or more strictly the scaling function which defines free energy per vertex

$$\text{Lim}_{m \rightarrow \infty} \frac{\mathcal{F}_m(\ell, \sigma, \varphi)}{m(m + \ell)} = \hat{\mathcal{F}}(\hat{\sigma}, \hat{\varphi}, \hat{\ell}). \quad (3.2)$$

The multiple integral (1.11) describes a statistical ensemble of m identical particles characterised by a repulsive two-body Coulomb interaction and a confining potential (1.12),

$$V(t) = V_0(t) - \ell \log(t^2 - \sigma^2), \quad V_0(t) = \log \frac{\cosh t + \cosh \varphi}{\cosh \sigma + \cosh \varphi}. \quad (3.3)$$

In the thermodynamical limit, the fluctuations are suppressed and the partition function is determined by the configuration minimising the energy. The positions $t_1 > \dots > t_m \geq |\sigma|$ of the m particles at equilibrium are determined by the saddle-point equations

$$\partial \mathcal{S} / \partial t_i = 0 \quad (i = 1, \dots, m) \quad (3.4)$$

where \mathcal{S} is the total energy

$$\mathcal{S} \equiv \sum_{j=1}^m V(t_j) - \sum_{k \neq j}^m \log(t_k^2 - t_j^2) - \sum_{j=1}^m \log(2t_j). \quad (3.5)$$

The saddle-point equations read explicitly

$$V'(t_j) = \sum_{k \neq j}^m \frac{2}{t_j - t_k} + \sum_{k=1}^m \frac{2}{t_j + t_k} = 0 \quad (i = 1, \dots, m). \quad (3.6)$$

With the potential (1.11), the roots t_j of the saddle-point equations are real and scale as $t_j \sim m$. In the thermodynamical limit the sum can be approximated by an integral with a continuous density $\rho(t)$. At macroscopic scale the density is supported by a compact interval $[a, b]$ with $|\sigma| \leq b < a < |\varphi|$. It is useful to extend the density to the whole real axis by the symmetry $\rho(t) = \rho(-t)$ and consider the saddle-point solution as a symmetric distribution of $2m$ particles with support $[-a, -b] \cup [b, a]$. Then the saddle-point equations take the form (at macroscopic scale)

$$V'(t) = 2 \int_{\mathbb{R}} \frac{dt' \rho(t')}{t - t'} \quad (b < |t| < a). \quad (3.7)$$

Obviously $a, b \sim m$ while ρ remains finite when $m \rightarrow \infty$.

The standard technique to solve the saddle-point equations is by reformulating the integral equation (3.7) as a Riemann-Hilbert like problem. For that introduce the resolvent

$$G(t) = \sum_{k=1}^m \frac{1}{t - t_k} = \int_b^a \frac{dt' \rho(t')}{t - t'} \quad (3.8)$$

and the function (giving the force acting on a probe particle at the point $t \in \mathbb{C}$)

$$H(t) = -\frac{1}{2}V'(t) + G(t) - G(-t). \quad (3.9)$$

The meromorphic function $H(t)$ has large- t asymptotics

$$H(t) = -\frac{1}{2}V'(t) + \frac{2m}{t} + O(t^{-3}), \quad (3.10)$$

and, apart from the singularities inherited from the external potential, two cuts $[-a, -b]$ and $[b, a]$ on the real axis. The integral equation (3.7) can be formulated as the boundary condition

$$\frac{H(t - i0) - H(t + i0)}{2\pi i} \times (H(t + i0) + H(t - i0)) = 0, \quad t \in \mathbb{R}. \quad (3.11)$$

As a consequence, the square $H^2(t)$ is analytic in the vicinity of the real axis which, together with the asymptotics (3.10) at $t \rightarrow \infty$ determines the function $H(t)$ uniquely. By a standard argument² (see e.g. section 2.1 of [41]), the function $H(t)$ can be written as a linear integral

$$H(t) = \int_b^a \frac{dt_1}{2\pi} \frac{2t V'(t_1)}{t^2 - t_1^2} \frac{y(t)}{y(t_1)} = -2 \int_b^a \frac{dt_1}{2\pi} \frac{y(t)}{y(t_1)} \frac{tV'(t) - t_1 V'(t_1)}{t^2 - t_1^2}, \quad (3.12)$$

where $y(t)$ is the positive root of the equation

$$a^2 y^2 = (a^2 - t^2)(t^2 - b^2). \quad (3.13)$$

The endpoints a and b of the eigenvalue distribution are determined by the asymptotics at infinity (3.10) which imposes the constraints

$$\int_b^a \frac{dt}{y(t)} V'(t) = 0, \quad \int_b^a \frac{dt}{y(t)} t^2 V'(t) = 2\pi m a. \quad (3.14)$$

For the computation of the free energy it is useful to introduce the *effective potential* $\phi(t)$ of a probe particle at the point $t \in \mathbb{C}$,

$$\phi(t) = \phi(-t) = V(t) + 2 \int_t^\infty G(t) dt = 2 \int_t^\infty H(t) dt. \quad (3.15)$$

Obviously the effective potential must be constant on the two cuts,

$$\phi(t) = \phi_0, \quad b \leq |t| \leq a. \quad (3.16)$$

²For the rescaled variable $\hat{t} = t/2\pi m$ the width of the analyticity strip vanishes as $1/m$ because of the arrays of poles at $\Re \hat{t} = \pm \varphi$ of $V'(\hat{t})$, but these poles are in general at macroscopic distance from the branch points and the standard argument still works.

The constant $\phi_0 = \phi(a)$ is an important collective variable. It gives the increase of the critical action upon bringing a new particle from infinity and therefore is conjugate to the number of particles m ,

$$\partial_m \mathcal{S}_c = \phi_0. \tag{3.17}$$

The constant ϕ_0 can be computed once the expression for the derivative $\partial_m H(t)$ is known. By (3.10), the derivative $\partial_m H(t)$ behaves at infinity as $2/t$ and thus depends on the external potential only through the positions of the branch points. It defines a normalised Abelian differential of first kind on the elliptic curve with equation (3.13) with singular point at infinity and has a standard form

$$d\omega(t) = \partial_m H(t) dt = \frac{2tdt}{\sqrt{t^2 - a^2}\sqrt{t^2 - b^2}}, \quad \omega(t) = 2 \log \frac{\sqrt{t^2 - a^2} + \sqrt{t^2 - b^2}}{2}. \tag{3.18}$$

Now let us compute how saddle-point energy \mathcal{S}_c changes with the number of particles. Starting with (3.5), the saddle-point energy can be written as an integral with the density,

$$\mathcal{S}_c = \int_b^a dt \rho(t) \left(V(t) + \frac{1}{2}[\phi(t) - V(t)] \right) = \frac{1}{2} \int_b^a dt \rho(t) V(t) + \frac{1}{2} m \phi_0, \tag{3.19}$$

where the equilibrium condition (3.16) have been used. Combining the derivative of (3.19) in m and (3.17), $\phi_0 = \phi(a)$ can be expressed in terms of the Abelian differential (3.18) as

$$\begin{aligned} \phi_0 &= m \partial_m \phi(a) + \int_b^a dt \partial_m \rho(t) V(t) \\ &= -2m \omega(a) + \frac{1}{\pi a} \int_b^a \frac{2tdt}{y(t)} V(t). \end{aligned} \tag{3.20}$$

From here one obtains for the first derivative of the free energy

$$\begin{aligned} \partial_m \mathcal{F} &= -\partial_m \log \mathcal{N} - \phi_0 \\ &= -\partial_m \log \mathcal{N} + 2m \log \frac{a^2 - b^2}{4} - \frac{1}{\pi} \int_b^a dt \frac{2tV(t)}{\sqrt{a^2 - t^2}\sqrt{t^2 - b^2}}. \end{aligned} \tag{3.21}$$

The integral in the last term is relatively easy, unlike the integral in (3.19). Differentiating once again, one obtains for the second derivative of the free energy

$$\partial_m^2 \mathcal{F} = -\partial_m^2 \log \mathcal{N} - \partial_m \phi(a) = 2 \log \frac{a^2 - b^2}{4(2m + \ell)^2}. \tag{3.22}$$

For the free energy itself,

$$\mathcal{F} = -\log \mathcal{N} - \frac{1}{2} \int_b^a dt \rho(t) V(t) - \frac{1}{2} \phi_0, \tag{3.23}$$

explicit formulas can be obtained only at particular points in the parameter space characterised by single scaling limits.

4 Solution of the saddle-point equations in the double scaling limit

At macroscopic scale $t \sim m$, the piece V_0 in (3.3) as a function of the complex variable t is approximated by

$$V_0(t) = \max(|t|, |\varphi|) - \max(|\sigma|, |\varphi|), \quad V'(t) = \text{sgn}(t) \theta(|t| - |\varphi|) \quad (t \in \mathbb{R}) \quad (4.1)$$

where $\theta(t)$ is the Heaviside function. The extension to the complex plane is done by simply replacing $t \rightarrow \Re t$ in (4.1). Since at macroscopic scale the derivative of the external potential develops a discontinuity $t = \pm\varphi$, the solution for the saddle-point equations have different analytical properties depending on whether the point φ belongs to the support of the equilibrium density. The parameter space splits into two domains depicted in figure 1,

- I. $b \leq |\varphi| < a$
- II. $|\varphi| \leq b < a$.

In the domain I the density is a smooth function while in the domain II it develops a logarithmic cusp.

• **Regime I ($|\varphi| < b$).** If $|\varphi| < b$, the saddle-point equations do not depend on the parameter φ . In [16] it was noticed that in this regime the integral equation (3.7) coincide with the finite-gap equation [32–34] for the Frolov-Tseytlin folded string [35] rotating in $\text{AdS}_3 \times S^1$ with quantum numbers $\{S, J\} = \{2m, \ell\}$ and $\sigma \sim$ string tension. The saddle-point equations (3.6) can be obtained starting with the Bethe equations

$$\left(\frac{t_j - \frac{\sigma^2}{t_j} + i\pi}{t_j - \frac{\sigma^2}{t_j} - i\pi} \right)^\ell \prod_{k \neq j}^{2m} \frac{t_j - t_k + 2\pi i}{t_j - t_k - 2\pi i} = 1, \quad (j = 1, \dots, 2m) \quad (4.2)$$

by assuming that all roots are large and imposing the constraint that the configuration of the roots is even, $\{t_j\} = \{-t_j\}$. The solution is completely characterised by the choice of the mode numbers which define the branch of the logarithms when the Bethe equations are written in logarithmic form. With the symmetry of the roots taken into account, the equations for the positive roots read

$$\frac{2\ell t_j}{t_j^2 - \sigma^2} + \sum_{k \neq j}^m \frac{2}{t_j - t_k} + \sum_{k=1}^m \frac{2}{t_j + t_k} = n_j \quad (j = 1, \dots, m). \quad (4.3)$$

The folded string solution corresponds to choosing $n_1 = \dots = n_m = 1$ and taking the limit of large charges m and ℓ .

As the free energy is sensitive to the constant mode of the piece V_0 of the potential, eq. (4.1), the regime I splits into two subdomains separated by the lines $|\sigma| = |\varphi|$,

- Regime Ia ($|\sigma| \leq |\varphi| \leq b < a$)
- Regime Ib ($|\varphi| \leq |\sigma| \leq b < a$).

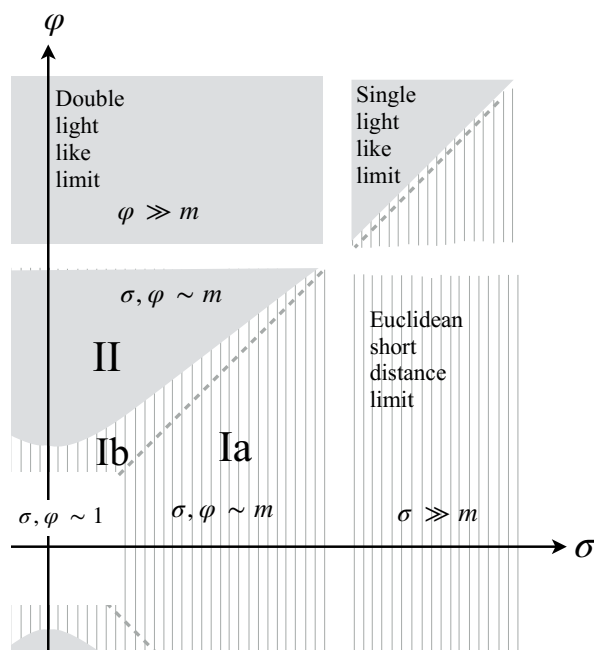


Figure 1. The scaling regimes of a large rectangular fishnet. The double scaling domain $\sigma, \varphi \sim m$ is split into regimes I and II. In regime I (the hatched area $|\varphi| \leq b$) the solution for the spectral density does not depend on φ . In the domain $|\varphi| < |\sigma|$ (regime Ia) this is so also for the free energy while for $|\sigma| < \varphi \leq b$ (regime Ib) the free energy contains a term $m\varphi$. The solution in regime I relates analytically the bulk thermodynamical limit c $\sigma, \varphi \sim 1$ and the Euclidean short distance limit $\sigma \gg m$. In regime II (the grey domain $|\varphi| \geq b$) the solution depends on both σ and φ . The double light-cone limits is attained in regime II by taking $\varphi \gg m$ with σ finite. The single light-cone limit is reached by taking $\varphi \gg m$ with $\mu = \varphi - \sigma$ fixed.

• **Regime II**($b < |\varphi| < a$). If $|\varphi| > b$, the non-analyticity of the external potential at $t = \varphi$ affects the saddle-point equations and the spectral density develops a cusp at $t = \varphi$. To get more intuition about the origin of the cusp, let us turn to the interpretation of the saddle-point equations as Bethe equations for the generalised $sl(2)$ spin chain, eqs. (4.2)–(4.3). Regime II is again characterised by an even distribution of the Bethe roots, but with different choice for the mode numbers, namely $n_j = 1$ if $t_j > |\varphi|$ and $n_j = 0$ if $t_j < |\varphi|$. This choice for the mode numbers is formally allowed but does not describe a finite gap solution in the limit of large charges. The two groups of eigenvalues characterised by mode numbers 1 and 0 do not condense to separated cuts but instead collide at the point $t = \varphi$ producing a logarithmic cusp of the density.

Although the solutions of the saddle-point equations in regime II and in regime I are not related analytically, they match continuously on the curve with equation $b = |\varphi|$ which separates the two regimes. Since the solution in the regime I does not depend on φ , it can be computed for $\varphi = b$. In this sense the regime I is contained in the regime II as a boundary value.

4.1 Solution in regime I

The solution in regime I has been found in [32–34] and adjusted for the fishnet in [16]. The equations for the branch points, eq. (3.14), are expressed in terms of the complete elliptic integrals of first and second kind $\mathbb{E} = E(k^2)$ and $\mathbb{K} = K(k^2)$ (the notations for the elliptic integrals are collected in appendix A), namely

$$\begin{aligned} \sqrt{(a^2 - \sigma^2)(b^2 - \sigma^2)} \mathbb{K} &= \pi \ell a, \\ a^2 \mathbb{E} - \sigma^2 \mathbb{K} &= \pi(2m + \ell)a; \\ k^2 &= 1 - k'^2, \quad k' = b/a. \end{aligned} \tag{4.4}$$

The spectral density is expressed in terms of the complete elliptic integral of third kind,

$$\rho(t) = \frac{1}{\pi} \frac{\ell t}{t^2 - \sigma^2} \sqrt{\frac{(a^2 - t^2)(t^2 - b^2)}{(a^2 - \sigma^2)(b^2 - \sigma^2)}} + \frac{1}{\pi^2} \frac{t}{a} \sqrt{\frac{t^2 - b^2}{a^2 - t^2}} \Pi \left(\frac{a^2 - b^2}{a^2 - t^2} \middle| 1 - \frac{b^2}{a^2} \right). \tag{4.5}$$

For the derivative of the free energy one finds from (3.21)

$$\begin{aligned} \partial_m \mathcal{F} &= (2m + \ell) \log \frac{(a^2 - b^2)}{4(2m + \ell)^2} + 2\ell \operatorname{arctanh} \frac{\sqrt{b^2 - \sigma^2}}{\sqrt{a^2 - \sigma^2}} \\ &\quad - \frac{2\ell \sigma^2}{\sqrt{(a^2 - \sigma^2)(b^2 - \sigma^2)}} + \max(|\varphi|, |\sigma|), \end{aligned} \tag{4.6}$$

where the asymptotics

$$\partial_m \log \mathcal{N} \approx 2(2m + \ell) \log \frac{2m + \ell}{e} \tag{4.7}$$

is taken into account. There is no reasons to believe that the integration of (4.6) can be done explicitly for general σ . The special cases $\sigma = 0$ and $\sigma \rightarrow \infty$ have been solved completely in [16], to be reviewed below.

• **Bulk thermodynamical limit ($\sigma = 0$).** This case correspond to the origin in figure 1. The equations for the branch points are

$$\frac{\mathbb{E}}{\pi} = \frac{2m + \ell}{a}, \quad \frac{\mathbb{K}}{\pi} = \frac{\ell}{b} \quad (\sigma = \varphi = 0) \tag{4.8}$$

and the resolvent is essentially Heuman’s Lambda function $\Lambda_0(\psi, k^2)$,

$$\begin{aligned} H(t) &= \frac{1}{2} \Lambda_0 \left(\arcsin \frac{a}{t}, k^2 \right) - \frac{1}{2} \operatorname{sgn}(\Re t), \\ \rho(t) &= -\frac{1}{2\pi} \Im \Lambda_0 \left(\arcsin \frac{a}{t}, k^2 \right). \end{aligned} \tag{4.9}$$

The derivative of the free energy,

$$\partial_m \mathcal{F} = (2m + \ell) \log \frac{(a^2 - b^2)}{4(2m + \ell)^2} + 2\ell \operatorname{arctanh}(b/a), \tag{4.10}$$

integrates to [16]

$$\begin{aligned} \mathcal{F} &= m^2 \log \frac{a-b}{2} + (m+\ell)^2 \log \frac{a+b}{2} - \frac{\ell^2}{2} \log \frac{ab}{\ell} - \frac{(2m+\ell)^2}{2} \log(2m+\ell) \\ &= m^2 \log \frac{1-k'}{2} + n^2 \log \frac{1+k'}{2} + 2mn \log \pi \\ &\quad + \frac{1}{2}(m-n)^2 \log \mathbb{K} - \frac{1}{2}(m+n)^2 \log \mathbb{E}. \end{aligned} \tag{4.11}$$

The constant of integration was determined by the requirement that the free energy must vanish when $m = 0$.

• **Euclidean short-distance limit $\sigma \gg m$.** When $\sigma \gg m$, the positions of the branch points scale as $a - \sigma \sim m, \beta - \sigma \sim m$ so that $k^2 \sim 1/\sigma$ and the elliptic curve (3.13) degenerates into a gaussian one. Setting

$$a = \sigma + S + 2R, \quad b = \sigma + S - 2R, \tag{4.12}$$

where $S, R \sim m$, and expanding (4.4) at large σ , one obtains in the leading order

$$R = \sqrt{m(m+\ell)}, \quad S = 2m + \ell \quad k^2 = 1 - b^2/a^2 \approx 8R/\sigma. \tag{4.13}$$

One obtains for the leading large σ asymptotics the free energy

$$\begin{aligned} \mathcal{F} &= m(m+\ell) \log(2\sigma) + \frac{3}{2}m(m+\ell) + \frac{1}{2}m^2 \log(m) \\ &\quad + \frac{1}{2}(m+\ell)^2 \log(m+\ell) - \frac{1}{2}(2m+\ell)^2 \log(2m+\ell) \quad (\sigma \gg m, \ell \gg 1). \end{aligned} \tag{4.14}$$

The expression (4.14) matches the large m asymptotics of (2.3). It is not difficult to work out the $1/\sigma$ corrections to the leading log asymptotics, see [16].

4.2 Solution in regime II: square fishnet with $\sigma = 0$ and $\varphi \sim m$

Before giving the general solution, it is instructive to consider a special case which in spite of its simplicity exhibits the main new features in the regime II. This is the case $\ell = \sigma = 0$ which corresponds to a square fishnet with a kinematical constraint $x_{12}^2 x_{34}^2 = x_{14}^2 x_{23}^2$.

The solution depends on the remaining parameter φ and connects analytically the bulk thermodynamical limit ($\varphi = 0$) and the double light-like limit ($\varphi \rightarrow \infty$) of the square fishnet. Because the derivative of the external potential in this case is piecewise constant,

$$V'(t) = \theta(|\tau| - |\varphi|) \operatorname{sgn}(t), \tag{4.15}$$

all the integrals evaluate to elementary functions. The equations (3.14) for the two branch points give $b = 0$ and $a = \sqrt{\varphi^2 + 4\pi^2 m^2}$, and the function (3.12) reads

$$H(t) = \frac{i}{2\pi} \operatorname{sgn}(\Re t) \log \left(\frac{\sqrt{a^2 - \varphi^2} + i\sqrt{t^2 - a^2}}{\sqrt{a^2 - \varphi^2} - i\sqrt{t^2 - a^2}} \right) + \frac{1}{2} \operatorname{sgn}(\Re t) \theta(|\varphi| - |\Re t|). \tag{4.16}$$

The saddle-point density

$$\rho(t) = \frac{1}{2\pi^2} \log \left| \frac{\sqrt{\varphi^2 + 4\pi^2 m^2 - t^2} + 2\pi m}{\sqrt{\varphi^2 + 4\pi^2 m^2 - t^2} - 2\pi m} \right|, \quad 0 = b \leq |t| \leq a = \sqrt{\varphi^2 + 4\pi^2 m^2}, \tag{4.17}$$

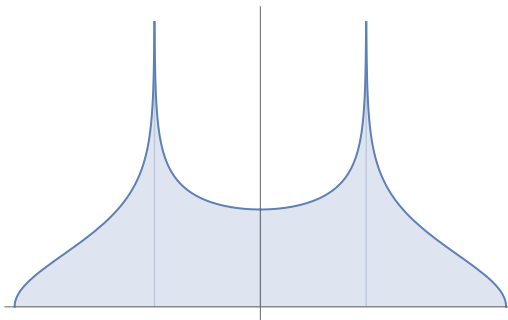


Figure 2. Profile of the spectral density for a large square fishnet with $\sigma = 0$. The density is finite at $t = 0$ and develops a cusp at $t = \varphi$.



Figure 3. When $\varphi \rightarrow 0$, the cusp moves to the origin and the density becomes singular at $t = 0$ (left). When $\varphi \rightarrow \infty$, the (right).

continued by symmetry to negative t has a profile shown in figure 2. It exhibits a logarithmic cusp localised at $t = \varphi$ which is a consequence of the non-analyticity of the external potential at this point at scale $t \sim m$. Near the cusp the density behaves as

$$\rho(t)_{\text{sing}} \approx \frac{1}{2\pi^2} \log |t - \varphi| + \text{smooth function}, \quad 1 \ll |t - \varphi| \ll m. \quad (4.18)$$

In this simple case the expression for the derivative of the free energy

$$\partial_m \mathcal{F} = 2m \log \left(\frac{\varphi^2 + 4\pi^2 m^2}{16m^2} \right) + \frac{2\varphi}{\pi} \text{arccot} \left(\frac{\varphi}{2\pi m} \right) - \log(2\pi m) + O(1) \quad (4.19)$$

can be integrated explicitly, with the integration constant fixed by the condition that the free energy vanishes at $m = 0$,

$$\mathcal{F} = m^2 \log \left(\frac{\varphi^2 + 4\pi^2 m^2}{16m^2} \right) - \frac{\varphi^2}{4\pi^2} \log \left(\frac{\varphi^2 + 4\pi^2 m^2}{\varphi^2} \right) + \frac{2\varphi m}{\pi} \text{arccot} \left(\frac{\varphi}{2\pi m} \right). \quad (4.20)$$

One can check that the $\varphi \gg m$ asymptotics of (4.20) coincides with the large m asymptotics of the solution in the double light-like limit (2.9) with $u = v = e^{-|\varphi|}$,

$$\mathcal{F} = m^2 \log \frac{\varphi^2}{16m^2} + 3m^2 + \frac{2\pi^2 m^4}{3\varphi^2} + O(\varphi^{-4}). \quad (4.21)$$

The solution (4.20) interpolates smoothly between the bulk thermodynamical limit ($\varphi \rightarrow 0$) and the double light-like limit ($\varphi \rightarrow \infty$) of the large square fishnet. The first term in the small φ expansion

$$\mathcal{F} = m^2 \log \frac{\pi^2}{4} + m\varphi + \left(\log \frac{\varphi^2}{4\pi^2 m^2} - 3 \right) \frac{\varphi^2}{4\pi^2} + O(\varphi^4) \quad (4.22)$$

matches the value $\log(\pi^2/4)$ for the free energy density, computed in [16], as it should. The typical shape of the spectral density in the two limits is shown in figure 3. Curiously, the expansion coefficients for large φ and for small φ are almost identical after an appropriate rescaling. A similar phenomenon has been observed for the weak/strong coupling expansions of the dressing phase in $\mathcal{N} = 4$ SYM [42]. Here there is a simple technical explanation this phenomenon. It is easy to check that the free-energy density as a function of the scaling variable $\hat{\varphi}$ normalised as in (1.8),

$$\hat{\mathcal{F}}(\hat{\varphi}) = \frac{\mathcal{F}}{m^2}, \quad \hat{\varphi} = \frac{\varphi}{2\pi m}, \tag{4.23}$$

transform under inversion $\hat{\varphi} \rightarrow 1/\hat{\varphi}$ in a very simple way, namely

$$\frac{\hat{\mathcal{F}}(\hat{\varphi}) - \log \frac{\pi^2}{4}}{\hat{\varphi}} + \frac{\hat{\mathcal{F}}(\hat{\varphi}^{-1}) - \log \frac{\pi^2}{4}}{\hat{\varphi}^{-1}} = 2\pi. \tag{4.24}$$

4.3 Solution in the regime II: the general case

Now let us consider the most general case with all the three parameters ℓ, σ, φ scaling linearly with m . With $V'(t)$ given by (4.1), the meromorphic function (3.12) and the equations for the branch points are expressed in terms of incomplete elliptic integrals (see appendix A)

$$H(t) = \frac{t\ell}{t^2 - \sigma^2} \frac{\sqrt{a^2 - t^2}\sqrt{b^2 - t^2}}{\sqrt{a^2 - \sigma^2}\sqrt{b^2 - \sigma^2}} + \frac{t}{\pi a} \frac{\sqrt{t^2 - b^2}}{\sqrt{t^2 - a^2}} \Pi\left(\frac{a^2 - b^2}{a^2 - t^2}; \psi | k^2\right), \tag{4.25}$$

$$F(\psi | k^2) \sqrt{(a^2 - \sigma^2)(b^2 - \sigma^2)} = \pi \ell a, \tag{4.26}$$

$$a^2 E(\psi | k^2) - \sigma^2 F(\psi | k^2) = \pi(2m + \ell)a, \tag{4.27}$$

$$k^2 = 1 - \frac{b^2}{a^2}, \quad \psi = \arcsin \frac{\sqrt{a^2 - \varphi^2}}{\sqrt{a^2 - b^2}}. \tag{4.28}$$

The semiclassical spectral density

$$\rho(t) = \frac{1}{\pi} \frac{\ell t}{t^2 - \sigma^2} \sqrt{\frac{(a^2 - t^2)(t^2 - b^2)}{(a^2 - \sigma^2)(b^2 - \sigma^2)}} + \frac{1}{\pi^2} \frac{t}{a} \sqrt{\frac{t^2 - b^2}{a^2 - t^2}} \Pi\left(\frac{a^2 - b^2}{a^2 - t^2}; \psi | k^2\right), \tag{4.29}$$

has a cusp at $t = \varphi$ (figure 4). The regime I is attained when the cusp moves to the left edge and disappears (figure 5, left). In the light-cone limit $\varphi \gg m$, the cusp moves to the right edge and changes the square-root singularity to a logarithmic one (figure 5, right).

The expression for the derivative of the free energy,

$$\begin{aligned} \partial_m \mathcal{F} &= (2m + \ell) \log \frac{(a^2 - b^2)}{4(2m + \ell)^2} + \frac{2\varphi}{\pi} \arctan \frac{\sqrt{a^2 - \varphi^2}}{\sqrt{\varphi^2 - b^2}} \\ &+ 2\ell \operatorname{arctanh} \frac{\sqrt{b^2 - \sigma^2}}{\sqrt{a^2 - \sigma^2}} - \frac{2\ell \sigma^2}{\sqrt{(a^2 - \sigma^2)(b^2 - \sigma^2)}}, \end{aligned} \tag{4.30}$$

is obtained by using the relations (4.26) and (4.27) to express the elliptic integrals in eq. (3.21) in terms of a and b .

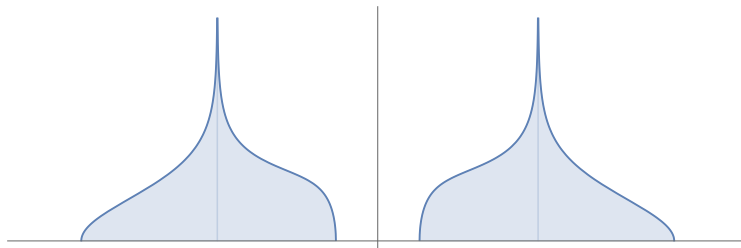


Figure 4. Profile of the spectral density $\rho(t)$ in regime II.



Figure 5. Profile of the spectral density when $b = |\varphi|$ (left) and for $a \rightarrow |\varphi|$ (right).

Eq. (4.30) can be used to generate series expansions of the free energy in different limits of regime II, as the bulk thermodynamical limit and the double light-cone limit, but this task is beyond the scope of this paper. Below I will only check that the limit $\sigma/m \gg 1$ of (4.30) along the line $\sigma = 0$ indeed reproduces the large m asymptotics of the expression (2.9) for the double light-like limit.

• **Double light-cone limit $\sigma = 0, \varphi \gg m$.** If $\varphi \gg m$, then the right branch point is pushed far as well, $a \gg m$. The left branch point can be anywhere depending on the value of ℓ . The two conditions (4.26)–(4.27) are compatible with $\psi \ll 1$. Retaining only the leading linear order in the expansion of the elliptic integrals in ψ , they read

$$\pi \frac{\ell}{a} = \sqrt{1 - k^2} \psi, \quad \pi \frac{2m + \ell}{a} = \psi, \quad \frac{\varphi}{a} = \sqrt{1 - k^2 \psi^2}, \quad (4.31)$$

with solution to the leading order at $\psi \ll 1$

$$\begin{aligned} \psi &\rightarrow \frac{\pi(2m + \ell)}{\varphi}, & k &\rightarrow \frac{2\sqrt{m(m + \ell)}}{2m + \ell}, \\ a &= \varphi + \frac{2\pi^2 m(m + \ell)}{\varphi}, & b &= \frac{\varphi \ell}{2m + \ell}. \end{aligned} \quad (4.32)$$

At $m \rightarrow 0$, $a = \varphi$ is the position of the minimum of the external potential. With the condition $\mathcal{F}_{m \rightarrow 0} = 0$, the derivative of the free energy can be integrated to

$$\begin{aligned} \mathcal{F} &= 2m(m + \ell) \log \varphi + 3m(m + \ell) \\ &\quad + m^2 \log(m) + (m + \ell)^2 \log(m + \ell) - (2m + \ell)^2 \log(2m + \ell) \\ &= 2mn \log(\varphi) + 3mn + m^2 \log(m) + n^2 \log(n) - (m + n)^2 \log(m + n). \end{aligned} \quad (4.33)$$

This expression matches the large- m asymptotics of (2.9).

5 Discussion

This short note addresses the question how the thermodynamical limit of Basso-Dixon integral for the $m \times n$ rectangular fishnets is affected by the boundary conditions.

The saddle point for the Basso-Dixon integral described by a density function is found in the double scaling limit (3.1) where the spacetime parameters σ and φ scale large with the fishnet lengths m and n . This is the most general scaling regime which contains the Euclidean OPE, the light-cone and the bulk thermodynamical limits as particular cases.

In the double scaling limit there are two regimes, labeled here by I and II and characterised by different analytic solutions. In regime II, a closed analytic expression, eq. (4.30), is derived for the logarithmic derivative of the fishnet in m with fixed $\ell = n - m, \sigma$ and φ . The latter determines how the fishnet changes upon adding a new row and a new column to the rectangle grid. With this observation, eq. (4.30) can be written as

$$\begin{aligned} \log \left[\frac{I_{m+1, n+1}^{\text{BD}}}{I_{m, n}^{\text{BD}}} \right] &= (m+n) \log \frac{a^2 - b^2}{4(m+n)^2} + \frac{2\varphi}{\pi} \arctan \frac{\sqrt{a^2 - \varphi^2}}{\sqrt{\varphi^2 - b^2}} \\ &\quad + 2|n - m| \left(\operatorname{arctanh} \frac{\sqrt{b^2 - \sigma^2}}{\sqrt{a^2 - \sigma^2}} - \frac{\sigma^2}{\sqrt{(a^2 - \sigma^2)(b^2 - \sigma^2)}} \right), \\ F(\psi|k^2) \sqrt{(a^2 - \sigma^2)(b^2 - \sigma^2)} &= \pi a |n - m|, \\ a^2 E(\psi|k^2) - \sigma^2 F(\psi|k^2) &= \pi a (m + n), \\ k^2 = 1 - \frac{b^2}{a^2}, \quad \psi = \arcsin \frac{\sqrt{a^2 - \varphi^2}}{\sqrt{a^2 - b^2}} &\quad (m, n \gg 1; \varphi \geq b \leq \sigma \sim m). \end{aligned} \tag{5.1}$$

Although m and n enter in a non-symmetric way in the original integral, the result is symmetric under $m \leftrightarrow n$. A direct integration is possible only in some cases as e.g. for $\sigma = \ell = 0$ where the solution is given in a closed form in section 4.2. The solution for the regime I is obtained by taking $\psi = \pi/2$ in (5.1).

The saddle-point equations for the large m limit of the Basso-Dixon integral take the form of the Bethe equations for a generalised $sl(2)$ spin chain, with the role of the BMN coupling constant played by the parameter σ . The second parameter φ appears only after the Bethe equations are written in a logarithmic form.

In the regime I which has been investigated in [16], the equation for the density of the Bethe roots is a finite gap equation for a symmetric configuration where all positive Bethe roots have mode number 1. The solution describes the Frolov-Tseytlin folded string rotating in $AdS_3 \times S^1$. In regime II, the solution of the Bethe equations exhibits a completely new feature. The positive roots split into two groups with mode numbers 1 and 0. The roots larger than φ have mode number 1 while the root smaller than φ have mode number 0. Such a choice of the mode numbers is mathematically possible, but does not lead to a finite-gap solution because the two groups of Bethe roots do not repel but attract. As a consequence, the eigenvalue density develops a logarithmic cusp at the point φ . Such a solution does not seem to correspond to any string motion in AdS.

The dual AdS description of the open fishnets thus remains an open problem. Looking ahead, one possible direction is to try to adjust the non-linear sigma model [7] or the string-

bit [8, 9] formulations of the cylindrical fishnet. Another direction would be to connect to the holographic description of the octagon [43] for which fascinating exact results have been obtained recently [44–49], using the fact that the fishnet is obtained from the octagon by truncation. A more relevant quantity in this respect would be the grand partition function for rectangular fishnet obtained as discrete Laplace transform with respect to the fishnet lengths m and n .

Another new in my knowledge result concerns the double light-cone limit with the four points approaching the cusps of a null square, where an exact expression of the rectangular fishnet is found for any m and n . This expression, eq. (2.9), factorises into two pieces associated with the two cross ratios which become singular. The numerical factor in each piece is the same as the one obtained in [16] for the Euclidean short-distance limit. The origin of this factorisation is yet to be understood. In the case of the Euclidean OPE limit, a combinatorial interpretation of the leading log asymptotics (2.3) was recently given in [50]. The rectangular fishnet was interpreted as an amplitude for hopping magnons named ‘stampede’.³ The similarity of (2.3) and (2.9) suggests that a description in terms of hopping magnons is possible also in the double light-cone limit.

The ‘stampede’ interpretation could unveil interesting physics. For example, the conjecture made in [16] about the possibility of arctic-curve phenomenon like in the six-vertex model with domain-wall boundary conditions [53], can be given more concrete shape here. Namely it is very likely that the typical ‘stampede’ exhibits in the thermodynamical limit fluctuating and frozen phases separated by an ‘arctic curve’.

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A Elliptic integrals

$\mathbb{K} = \text{EllipticF}[\psi, k^2]$ and $\mathbb{E} = \text{EllipticE}[\psi, k^2]$ are respectively the complete elliptic integrals of first and second kind,

$$\mathbb{K} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}, \quad \mathbb{E} = \int_0^{\pi/2} d\theta \sqrt{1 - k^2 \sin^2 \theta}, \quad (\text{A.1})$$

$\Pi(\alpha^2|k^2) = \text{EllipticPi}[\alpha^2, k^2]$ is the complete elliptic integral of third kind,

$$\Pi(\alpha^2|k^2) = \int_0^{\pi/2} \frac{d\theta}{(1 - \alpha^2 \sin^2 \theta)\sqrt{1 - k^2 \sin^2 \theta}}. \quad (\text{A.2})$$

$\Lambda_0(\psi, k^2)$ denotes Heuman’s Lambda function,

$$\Lambda_0(\psi, k^2) \equiv \frac{2}{\pi} [\mathbb{E} F(\psi, k') + \mathbb{K} E(\psi, k') - \mathbb{K} F(\psi, k')]. \quad (\text{A.3})$$

³Curiously, a similar combinatorics occurs in the evaluation of the Jordan block spectrum of the ‘hyperelectic spin chain’ [51, 52].

$F(\psi|k^2) = \text{EllipticF}[\psi, k^2]$ and $E(\psi|k^2) = \text{EllipticE}[\psi, k^2]$ are respectively the incomplete elliptic integrals of first and second kind,

$$F(\psi|k^2) = \int_0^\psi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}, \quad E(\psi|k^2) = \int_0^\psi d\theta \sqrt{1 - k^2 \sin^2 \theta}. \quad (\text{A.4})$$

$\Pi(\alpha^2; \psi|k^2) = \text{EllipticPi}[\alpha^2, \psi, k^2]$ is the incomplete elliptic integral of third kind;

$$\Pi(\alpha^2; \psi|k^2) = \int_0^\psi \frac{d\theta}{(1 - \alpha^2 \sin^2 \theta) \sqrt{1 - k^2 \sin^2 \theta}}. \quad (\text{A.5})$$

B Fermionic representation of the $(m + \ell) \times m$ rectangular fishnet and derivation of the dual integral

Consider the Fock space of a complex fermion

$$\psi(u) = \sum_{n \geq 0} \psi_n u^{-n-1}, \quad \psi^*(u) = \sum_{n \geq 0} \psi_n^* u^n, \quad [\psi_m, \psi_n^*]_+ = \delta_{m,n} \quad (\text{B.1})$$

with vacuum states defined by $\langle 0 | \psi_n^* = 0$ and $\psi_n | \ell \rangle = 0$. The vacuum states of charge ℓ are constructed as $\langle \ell | = \langle 0 | \psi_0 \psi_1 \dots \psi_\ell$ and $|\ell \rangle = \psi_\ell^* \dots \psi_0^* | 0 \rangle$. The statement is that the Basso-Dixon integral (1.9) equals the matrix element

$$I_{m,n}^{\text{BD}} = \langle \ell | e^{\mathbf{H}} | \ell + 2m \rangle \quad (\text{B.2})$$

of the evolution operator is constructed from the fermion bilinear

$$\begin{aligned} \mathbf{H} &= \frac{1}{\sqrt{uv}} \sum_{a \geq 1} \frac{\sin a\varphi}{\sin \varphi} \int \frac{du}{2\pi} e^{2i\sigma u} \psi(u + ia/2) \psi(u - ia/2) \\ &= \int_{\mathbb{R}+i0} \frac{du}{2\pi i} \psi(u + i0) e^{2i\sigma u} \frac{\cosh \sigma + \cosh \varphi}{\cos \partial_u + \cosh \varphi} \psi(u - i0). \end{aligned} \quad (\text{B.3})$$

The fermionic representation (B.2), which is similar to that of the octagon [39], can be proved by expanding the exponent and using the expression for the two-point function

$$\langle \ell | \psi(u) \psi(v) | \ell + 2 \rangle = u^{-\ell-1} v^{-\ell-2} - v^{-\ell-1} u^{-\ell-2} = \frac{u - v}{(uv)^{2+\ell}}. \quad (\text{B.4})$$

The second line in (B.3) is a formal expression for the sum in the first line. It can be given precise meaning by honestly performing Fourier transformation and using the identity

$$(uv)^{-1/2} \sum_{a \geq 1} e^{-at} \frac{\sinh a\varphi}{\sinh \varphi} = \frac{\cosh \sigma + \cosh \varphi}{\cosh t + \cosh \varphi} \equiv e^{-V_0(t)}. \quad (\text{B.5})$$

The Fourier transformed fermion is given by two different expressions depending on whether the argument is below or above the real axis:

$$\psi(u \pm i0) = \int_{-\infty}^{\infty} dt e^{-iut} \tilde{\psi}_\pm(t), \quad \tilde{\psi}_\pm(t) = \int_{-\infty}^{\infty} \frac{du}{2\pi} e^{iut} \psi(u \pm i0), \quad (\text{B.6})$$

or in terms of the mode expansion (B.1),

$$\psi(u \pm i0) = \sum_{n \geq 0} \psi_n (u \pm i0)^{-n-1}, \quad \tilde{\psi}_{\pm}(t) = \mp i \theta(\mp t) \sum_{n \geq 0} \psi_n \frac{(it)^n}{n!}. \quad (\text{B.7})$$

The two-point function of the Fourier-transformed fermion $\tilde{\psi}_{\pm}$ is

$$\begin{aligned} \langle \ell | \tilde{\psi}_+(t_1) \tilde{\psi}_-(t_2) | \ell + 2 \rangle &= -\frac{i^{2\ell+1}}{\ell!(\ell+1)!} t_1^{\ell} t_2^{\ell} (t_2 - t_1) \theta(-t_1) \theta(t_2) \\ \langle \ell | \tilde{\psi}_-(t_1) \tilde{\psi}_-(t_2) | \ell + 2 \rangle &= -\frac{i^{2\ell+1}}{\ell!(\ell+1)!} t_1^{\ell} t_2^{\ell} (t_2 - t_1) \theta(t_1) \theta(t_2), \quad \text{etc.} \end{aligned} \quad (\text{B.8})$$

The shifts in $\pm ia/2$ acts in the Fourier space diagonally while the factor $e^{i\sigma u}$ transforms into a shift $t \rightarrow t + \sigma$,

$$\begin{aligned} e^{i\sigma u} \psi\left(u \pm \frac{i}{2}a\right) &= \int_{-\infty}^{\infty} dt \theta(\mp t \mp \sigma) e^{\pm at/2} e^{-iut} \tilde{\psi}_{\pm}(t + \sigma) \\ &= \mp \int_{\mp \sigma}^{\infty} dt e^{-at/2} e^{\pm iut} \tilde{\psi}_{\pm}(\sigma \mp t), \end{aligned} \quad (\text{B.9})$$

where we have used that $\tilde{\psi}_{\pm}(t) = \theta(\mp t) \tilde{\psi}_{\pm}(t)$. Therefore

$$\int \frac{du}{2\pi} e^{2i\sigma u} \psi\left(u + \frac{i}{2}a\right) \psi\left(u - \frac{i}{2}a\right) = - \int_{|\sigma|}^{\infty} dt \tilde{\psi}_+(\sigma - t) \tilde{\psi}_-(\sigma + t) e^{-at}. \quad (\text{B.10})$$

The weighted sum in $a \geq 1$ can be performed in the exponent, producing

$$\tilde{\mathbf{H}} = - \int_{|\sigma|}^{\infty} dt e^{-V_0(t)} \tilde{\psi}_+(\sigma - t) \tilde{\psi}_-(\sigma + t), \quad (\text{B.11})$$

with $V_0(t)$ defined in (B.5). Evaluating the expectation value by performing all Wick contractions, one obtains the dual representation of the rectangular fishnet in the Fourier space for the rapidities, eq. (1.11),

$$\begin{aligned} I_{m,n}^{\text{BD}} &= \langle \ell | e^{\tilde{\mathbf{H}}} | \ell + 2m \rangle \\ &= \prod_{j=1}^m \int_{|\sigma|}^{\infty} dt_j \frac{e^{-V_0(t_j)} (\sigma^2 - t_j^2)^{\ell}}{(\ell + 2j - 2)! (\ell + 2j - 1)!} \prod_{j,k=1}^m (t_j + t_k) \prod_{j < k} (t_j - t_k)^2. \end{aligned} \quad (\text{B.12})$$

The determinant representation (1.16) follows from the expression of the expectation value as a $2m \times 2m$ pfaffian, which can be turned into a determinant.

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