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# Light propagation in statistically homogeneous and isotropic dust universes

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**1. Introduction**

*Observations and clumpiness.* Observations of the universe are inconsistent with homogeneous and isotropic cosmological models with ordinary matter and standard gravity (meaning matter with non-negative pressure and gravity that is described by the four-dimensional Einstein-Hilbert action). Observational results are usually expressed in terms of their interpretation in the context of the homogeneous and isotropic Friedmann-Robertson-Walker (FRW) models, in which there is a one-to-one relationship between the expansion rate and the distance (given the spatial curvature). Analysed this way, observations imply that the expansion has accelerated in the past few billion years [1,2]. A model-independent statement would be that the observed distances at late times are a factor of about 2 higher than expected in FRW models with ordinary matter and gravity. This is usually taken as evidence for exotic matter with negative pressure or a modification of gravity. In particular, the homogeneous and isotropic, spatially flat model  $\Lambda$ CDM model fits observations of the distance scale and the expansion rate well by introducing vacuum energy. (We will refer to all homogeneous and isotropic models which contain only dust and vacuum energy as  $\Lambda$ CDM, whatever the spatial curvature.) However, the universe is known to be far from exact homogeneity and isotropy at late times due to the formation of non-linear structures. Before concluding that new physics is needed, it is necessary to quantify the effect of clumpiness on the observations.

The influence of inhomogeneity and/or anisotropy on the average evolution was first mentioned in [3] and was discussed in detail in [4] under the name “fitting problem”. The effect on the expansion rate is also known as backreaction [5–7]; see [8–10] for reviews. The possibility that backreaction could lead to accelerated expansion and explain the observations without new physics was first considered in [11,12] (and briefly mentioned in [13,14]). Both metric and matter perturbations were taken into account and the observables were correctly identified in [15], where first order perturbation theory was

expanded to second order, and this was extended to a consistent second order calculation in [16]. In [9, 17, 18] it was explained with toy models that the physical reason for average acceleration is that faster expanding regions come to occupy a larger fraction of the volume. Accelerated expansion has also been explicitly demonstrated with the exact Lemaître-Tolman-Bondi (LTB) solution [17, 19, 20]. A semi-realistic model with an evolving distribution of non-linear structures was studied in [21, 22], and it was found that the expansion rate rises (relative to the homogeneous and isotropic case) by the right order of magnitude around the right time, some billions to tens of billions of years, though not rapidly enough to correspond to acceleration. The model involved several approximations, and a more careful treatment would be needed for detailed comparison with observations.

However, most observations, including those of the cosmic microwave background (CMB) [23] and type Ia supernovae [24], do not directly measure the expansion rate, but rather cosmological distances and redshifts, which are defined in terms of light propagation. The few measurements that are sensitive to the expansion rate independent of the distance scale are those of the local Hubble parameter [25, 26], the ages of passively evolving galaxies as a function of redshift [27], the Integrated Sachs-Wolfe (ISW) effect [28, 29] and the growth rate of matter fluctuations [30]. Baryon acoustic oscillations depend on a mixture of the expansion rate and distance [31, 32].

*The distance and the expansion rate.* In addition to changing the expansion of the universe, non-linear structures affect the relationship between the expansion rate and light propagation. In a general spacetime, there is no direct relationship between the expansion rate and the distance scale, and it would in principle be possible to explain the observations without accelerated expansion. For example, in models where we are located near the center of an untypically large spherical void, the distances can be consistent with the observations, but generally there is no acceleration [33, 34]. (See [35] for a review, [9] for more references, and [36–41] for observational constraints related to the inhomogeneity of these models.) Even if the large local void models are not realistic, studying them with the exact LTB solution has established unambiguously that inhomogeneities with density contrast of order unity and sizes smaller than the horizon can have a large impact on the distance scale. This is in contrast to perturbation theory arguments based on the amplitude of metric perturbations in the longitudinal gauge.

The speculative possibility of an unexpectedly large local void aside, the observed universe seems to be statistically homogeneous and isotropic on large scales, with a homogeneity scale of around 100 Mpc [42, 43] (though see [44, 45]). (For discussion of statistically homogeneous and isotropic but locally clumpy dust universes, see [9, 21].) The relevant question is then what is the effect of such a distribution of non-linear structures on light propagation.

It was conjectured in [21] that in a statistically homogeneous and isotropic dust universe, light propagation can be treated in terms of the overall geometry (meaning the

average expansion rate and average spatial curvature) if the structures are realistically small and the observer is not in a special location. Such a conjecture is in agreement with various studies of light propagation (see [21] for an overview and references), and it is also suggested by the fact that different observations are well explained in terms of the evolution of a single scale factor. However, until now there had been no proof of the conjecture, and it was not known whether additional conditions are necessary in addition to statistical homogeneity and isotropy.

In the present work, we clarify the relationship between the expansion rate and the distance scale in statistically homogeneous and isotropic dust universes which may contain non-linear structures. We relate the redshift and the distance to the dust geometry, and confirm that light propagation can be expressed in terms of average geometrical quantities, up to a term related to the null geodesic shear. In fact, the null shear aside, the average expansion rate (and the matter density today) is sufficient to determine the distance, and the spatial curvature enters only via its effect on the expansion rate. This implies that a clumpy model can be consistent with the observed position of the CMB acoustic peaks even when there is significant spatial curvature. The result also shows that the Dyer-Roeder prescription of multiplying the matter density by a smoothness factor does not correctly describe the effect of clumping.

In section 2 we set up the dust geometry, in section 3 we relate the redshift and the distance to the average geometry, and in section 4 we discuss and summarise our results.

## 2. The dust geometry

### 2.1. The local equations

*The gradient decomposition.* We are interested in light propagation in a dust spacetime. The geometry is a solution to the Einstein equation with dust matter,

$$\begin{aligned} R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R &= 8\pi G_N T_{\alpha\beta} \\ &= 8\pi G_N \rho u_\alpha u_\beta, \end{aligned} \tag{1}$$

where  $R_{\alpha\beta}$  is the Ricci tensor,  $R$  is the Ricci scalar,  $G_N$  is Newton's constant,  $T_{\alpha\beta}$  is the energy-momentum tensor,  $\rho$  is the dust energy density and  $u^\alpha$  is the velocity of observers comoving with the dust.

The velocity has unit norm,  $u_\alpha u^\alpha = -1$ . Since the pressure is zero,  $u^\alpha$  is the tangent vector of timelike geodesics,  $u^\beta \nabla_\beta u^\alpha = 0$ . We can define a tensor which projects on the tangent space orthogonal to  $u^\alpha$  by

$$h_{\alpha\beta} \equiv g_{\alpha\beta} + u_\alpha u_\beta, \tag{2}$$

where  $g_{\alpha\beta}$  is the full metric. The three-metric  $h_{\alpha\beta}$  satisfies  $h_{\alpha\beta} u^\beta = 0$ ,  $h_\alpha^\gamma h_{\gamma\beta} = h_{\alpha\beta}$ ,  $h^\alpha_\alpha = 3$ .

It is useful to express the geometry in terms of the decomposition of the covariant derivative of  $u^\alpha$  (for reviews of this covariant approach, see [46–50]):

$$\begin{aligned}\nabla_\beta u_\alpha &= \theta_{\alpha\beta} + \omega_{\alpha\beta} \\ &= \frac{1}{3}h_{\alpha\beta}\theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta} ,\end{aligned}\tag{3}$$

where the symmetric part  $\theta_{\alpha\beta} = \nabla_{(\beta}u_{\alpha)}$  is the expansion tensor and the antisymmetric part  $\omega_{\alpha\beta} = \nabla_{[\beta}u_{\alpha]}$  is the vorticity tensor. The trace of the expansion tensor  $\theta = \nabla_\alpha u^\alpha$  is the volume expansion rate and the trace-free part  $\sigma_{\alpha\beta} = \nabla_{(\alpha}u_{\beta)} - \frac{1}{3}h_{\alpha\beta}\theta$  is the shear tensor. Like  $h_{\alpha\beta}$ , the tensors  $\sigma_{\alpha\beta}$  and  $\omega_{\alpha\beta}$  are spatial in the sense that they are orthogonal to  $u^\alpha$ ,  $\omega_{\alpha\beta}u^\beta = \sigma_{\alpha\beta}u^\beta = 0$ . They are also traceless,  $\omega^\alpha{}_\alpha = \sigma^\alpha{}_\alpha = 0$ .

*The scalar equations.* The Einstein equation (1) can be conveniently written in terms of the decomposition (3) and the electric and magnetic parts of the Weyl tensor. For the full system of equations, see [48] (page 27). We are interested in the overall geometry, in other words in average quantities. Since only scalars can be straightforwardly averaged in a curved spacetime (though see [51–53]), we will consider only the scalar part of the Einstein equation, which reads

$$\dot{\theta} + \frac{1}{3}\theta^2 = -4\pi G_N \rho - 2\sigma^2 + 2\omega^2\tag{4}$$

$$\frac{1}{3}\theta^2 = 8\pi G_N \rho - \frac{1}{2}{}^{(3)}R + \sigma^2 - \omega^2\tag{5}$$

$$\dot{\rho} + \theta\rho = 0 ,\tag{6}$$

where a dot stands for  $\partial_t \equiv u^\alpha \nabla_\alpha$ , the covariant derivative with respect to proper time  $t$  measured by observers comoving with the dust,  $\sigma^2 = \frac{1}{2}\sigma_{\alpha\beta}\sigma^{\alpha\beta} \geq 0$  is the shear scalar,  $\omega^2 = \frac{1}{2}\omega_{\alpha\beta}\omega^{\alpha\beta} \geq 0$  is the vorticity scalar, and  ${}^{(3)}R$  is the Ricci scalar on the tangent space orthogonal to  $u^\alpha$ . The acceleration equation (4) is known as the Raychaudhuri equation, and (5) is the Hamiltonian constraint.

## 2.2. The average equations

*Defining the average.* If and only if the vorticity is zero, the tangent spaces orthogonal to  $u^\alpha$  form spatial hypersurfaces, and provide a foliation that fills the spacetime exactly once. These flow-orthogonal hypersurfaces coincide with the hypersurfaces of constant proper time of comoving observers. If the vorticity is non-zero, the hypersurfaces of constant proper time are no longer orthogonal to the fluid flow [46, 47].

We assume in this subsection that the vorticity is zero, and follow the formalism of [6, 7]. The spatial average of a quantity is then its integral over the hypersurface of constant proper time  $t$  orthogonal to  $u^\alpha$ , divided by the volume of the hypersurface,

$$\langle f \rangle(t) \equiv \frac{\int_t \epsilon f}{\int_t \epsilon} ,\tag{7}$$

where  $\epsilon_{\alpha\beta\gamma} \equiv \eta_{\alpha\beta\gamma\delta}u^\delta$  is the volume element on the tangent space orthogonal to  $u^\alpha$ ,  $\eta_{\alpha\beta\gamma\delta}$  being the spacetime volume element.

In particular, the average expansion rate is

$$\begin{aligned} \langle \theta \rangle(t) &= \frac{\int_t \epsilon \theta}{\int_t \epsilon} \\ &= \frac{\partial_t \int_t \epsilon}{\int_t \epsilon} \\ &\equiv 3 \frac{\dot{a}}{a}, \end{aligned} \tag{8}$$

where we have defined the scale factor  $a(t)$  as the volume of the hypersurface of constant proper time to power  $1/3$ ,

$$a(t) \equiv \left( \frac{\int_t \epsilon}{\int_{t_0} \epsilon} \right)^{\frac{1}{3}}, \tag{9}$$

and  $a$  has been normalised to unity at time  $t_0$ , which we take to be today. We will also use the notation  $H \equiv \dot{a}/a$ .

*The Buchert equations.* Let us take the average of the equations (4)–(6). The resulting Buchert equations are [6]

$$3 \frac{\ddot{a}}{a} = -4\pi G_N \langle \rho \rangle + \mathcal{Q} \tag{10}$$

$$3 \frac{\dot{a}^2}{a^2} = 8\pi G_N \langle \rho \rangle - \frac{1}{2} \langle {}^{(3)}R \rangle - \frac{1}{2} \mathcal{Q} \tag{11}$$

$$\partial_t \langle \rho \rangle + 3 \frac{\dot{a}}{a} \langle \rho \rangle = 0, \tag{12}$$

where the backreaction variable  $\mathcal{Q}$  contains the effect of inhomogeneity and anisotropy:

$$\mathcal{Q} \equiv \frac{2}{3} (\langle \theta^2 \rangle - \langle \theta \rangle^2) - 2 \langle \sigma^2 \rangle. \tag{13}$$

The integrability condition between (10) and (11) reads

$$\partial_t \langle {}^{(3)}R \rangle + 2 \frac{\dot{a}}{a} \langle {}^{(3)}R \rangle = -\partial_t \mathcal{Q} - 6 \frac{\dot{a}}{a} \mathcal{Q}. \tag{14}$$

The Buchert equations (10)–(12) describe the evolution of the volume  $a^3$  of a spatial domain, or equivalently its average expansion rate  $\langle \theta \rangle$ . They differ from the FRW equations by the presence of the backreaction variable  $\mathcal{Q}$ , and the related fact that the average spatial curvature  $\langle {}^{(3)}R \rangle$  can evolve in a non-trivial manner, as indicated by the integrability condition (14), whereas in FRW universes it is always proportional to  $a^{-2}$  [54] (page 720), [55]. If the backreaction variable  $\mathcal{Q}$  is large enough, the expansion will accelerate, as indicated by (10), and the spatial curvature will be correspondingly large, as indicated by (11) and (14) [9, 56].

### 3. The light propagation

#### 3.1. The redshift

*Geometrical optics.* When the wavelength of light is much shorter than both the local curvature radius and the typical scale over which the amplitude and the

wavelength change appreciably, light propagation can be treated in the geometrical optics approximation [54] (page 570), [57] (page 93). In geometrical optics, light travels along null geodesics, and the light rays have no effect on the geometry. The tangent vector of the null geodesics is given by the gradient of the wave phase  $S$ , and it is identified with the photon momentum,  $k_\alpha = \partial_\alpha S$ . The null geodesic tangent vector satisfies  $k_\alpha k^\alpha = 0$  and  $k^\alpha \nabla_\alpha k^\beta = 0$ .

We will consider the propagation of a bundle of nearby null geodesics. We are interested in the redshift and the surface area of the bundle, the latter of which gives the angular diameter distance. We will not consider caustics, which are not expected to be important for typical light rays in cosmology (though see [58]).

*The general expression for the redshift.* The spacetime geometry is determined dynamically by the Einstein equation (1) and traced by the dust geodesics with tangent vector  $u^\alpha$ . Light propagation involves a derived geometrical structure, given by the solution of the null geodesic equations in the fixed spacetime geometry, traced by the photon geodesics with tangent vector  $k^\alpha$ . The tangent vectors  $u^\alpha$  and  $k^\alpha$  are parallel propagated with respect to the dust and photon geodesics, respectively, but not with respect to each other. It follows that the photon momentum changes along the dust geodesics. The redshift  $z$  of a source is defined as the observed photon wavelength divided by the wavelength at the source, minus one. The wavelength is inversely proportional to the energy  $E$ , so

$$1 + z = \frac{E_s}{E_o} , \quad (15)$$

where s refers to the source and o to the observer. The energy is the projection of the momentum onto the observer's velocity, given by the tangent vector of the dust geodesic,

$$E = -u_\alpha k^\alpha . \quad (16)$$

It is convenient to decompose  $k^\alpha$  into an amplitude and the direction, and split the direction into components orthogonal and parallel to the dust geodesics,

$$k^\alpha = E(u^\alpha + e^\alpha) , \quad (17)$$

where  $u_\alpha e^\alpha = 0$ ,  $e_\alpha e^\alpha = 1$ . The vector  $e^\alpha$  is spatial in the sense that it lies in the three-space which has the metric  $h_{\alpha\beta}$ ,  $h^\alpha_\beta e^\beta = e^\alpha$ .

To find out how the energy evolves along the null geodesic, we take the derivative with respect to the affine parameter  $\lambda$ ,

$$\begin{aligned} \partial_\lambda E &\equiv k^\alpha \nabla_\alpha E \\ &= -k^\alpha k^\beta \nabla_\alpha u_\beta \\ &= -k^\alpha k^\beta (\theta_{\beta\alpha} + \omega_{\beta\alpha}) \\ &= -E^2 e^\alpha e^\beta \theta_{\alpha\beta} \\ &= -E^2 \left( \frac{1}{3} \theta + \sigma_{\alpha\beta} e^\alpha e^\beta \right) , \end{aligned} \quad (18)$$



where we have applied the decompositions (3) and (17) and taken into account  $k^\alpha \nabla_\alpha k^\beta = 0$ . Note that the vorticity tensor drops out due to its asymmetry. We can integrate (18) to obtain  $E \propto \exp\left(-\int d\lambda E \theta_{\alpha\beta} e^\alpha e^\beta\right)$  (the factor  $E$  is retained in the integrand for later convenience). Using (15) we then have for the redshift

$$\begin{aligned} 1+z &= \exp\left(\int_{\lambda_s}^{\lambda_o} d\lambda E \theta_{\alpha\beta} e^\alpha e^\beta\right) \\ &= \exp\left(\int_{\lambda_s}^{\lambda_o} d\lambda E \left[\frac{1}{3}\theta + \sigma_{\alpha\beta} e^\alpha e^\beta\right]\right), \end{aligned} \quad (19)$$

where the integral is from the source to the observer along a specific geodesic. The above relation gives the redshift in a general dust spacetime in terms of the dust geometry and the spatial direction  $e^\alpha$  of the null geodesics. (Vorticity enters indirectly via the geodesic equation which determines  $e^\alpha$ .)

Looking at the redshift from the viewpoint of observers on dust geodesics,  $u^\alpha$  is constant, and the momentum  $k^\alpha$  decreases. However, following the null geodesics instead makes the relation to the change in the dust geometry more transparent. Along a null geodesic the momentum  $k^\alpha$  is constant, and the product  $-u_\alpha k^\alpha$  changes because  $u_\alpha$  is turning along the photon path, which is precisely what  $\nabla_\beta u_\alpha$  quantifies. (Following the dust geodesics by taking the derivative  $\partial_t = u^\alpha \nabla_\alpha$  instead of  $\partial_\lambda$  would lead to an expression for the redshift in terms of the turning of the null geodesics.)

Let us assume that the vorticity vanishes. Then the hypersurfaces of constant proper time are orthogonal to  $u^\alpha$ , and  $t(\lambda)$  is monotonic. We can therefore parametrise points along the geodesic with the value of the proper time  $t$  on the hypersurface that the null geodesic is crossing<sup>‡</sup>. We can then invert  $\partial_\lambda = E(u^\alpha + e^\alpha)\partial_\alpha$  to obtain  $\int d\lambda = \int_\lambda dt E^{-1}$ , so we have

$$1+z = \exp\left(\int_{t,\lambda}^{t_o} dt \left[\frac{1}{3}\theta(t, \mathbf{x}(t)) + \sigma_{\alpha\beta}(t, \mathbf{x}(t))e^\alpha(t, \mathbf{x}(t))e^\beta(t, \mathbf{x}(t))\right]\right), \quad (20)$$

where the subscript  $\lambda$  indicates that the integral is from the source to the observer along a specific geodesic, which crosses the hypersurface of proper time  $t$  at spatial position  $\mathbf{x}(t)$ .

*Statistical homogeneity and isotropy.* In a FRW universe, the shear is zero, and only the expansion rate remains in the integral (20). Since the universe is homogeneous and isotropic, it does not matter in which direction the geodesic goes, and the result  $1+z = a(t)^{-1}$  immediately follows from the definition of the scale factor (9).

In a general dust universe, the shear can be important, and the result depends on the direction of the null geodesic. In [57] (page 136), it was assumed that the average of  $\theta_{\alpha\beta} e^\alpha e^\beta$  along the ray reduces to the FRW expansion rate, and that this may be considered as part of the incomplete definition of a model being “on average FRW”. We

<sup>‡</sup> When rotation is present, it is not obvious that a light ray could not pass from a higher value of  $t$  to a smaller value. Note that light propagating from an observer with a larger proper time to one with a smaller proper time does not necessarily violate causality.

will try to make this reasoning somewhat more explicit, and take into account that the average expansion rate does not necessarily reduce to the FRW one.

In a general dust spacetime, there is no direct relationship between the redshift (20) and the scale factor (9). The redshift is given by the integral of the local expansion rate and projected shear along a null geodesic, while the scale factor is given by the time integral of the spatially averaged expansion rate. However, in a statistically homogeneous and isotropic universe, the two quantities are closely related. If structures are identical in all directions up to statistical fluctuations, and if the coherence scale of the distribution (related to the homogeneity scale) is much smaller than the distance over which we consider light propagation, then the redshift should be the same in all directions, and it should not depend on the specific geodesic we are looking at. We can view this in the following manner.

The dust shear  $\sigma_{\alpha\beta}$  is related to the structures through which the light travels. If the structures have no preferred orientation, the shear is uncorrelated with the direction of the null geodesic, and the integral of the shear projected on the geodesic should vanish over long distances. In other words, the integral of  $\theta_{\alpha\beta}e^\alpha e^\beta$  should be the same for all  $e^\alpha$ , which implies that only the trace contributes. Furthermore, the integral of the local expansion rate along the null geodesic can be related to the integral of the average expansion rate over time. Neglecting the shear, we split the expansion rate at each point along the geodesic into the average value on the hypersurface of constant proper time at that point (given by (8)) and the variation,

$$1 + z = \exp \left( \int_{t,\lambda}^{t_0} dt \left[ \frac{1}{3} \langle \theta \rangle (t) + \frac{1}{3} \Delta \theta (t, \mathbf{x}(t)) \right] \right), \quad (21)$$

where  $\Delta \theta \equiv \theta(t, \mathbf{x}(t)) - \langle \theta \rangle (t)$ . (In what follows, we will use the notation  $f = \langle f \rangle + \Delta f$  for any scalar quantity  $f$ ; there is no assumption that  $\Delta f$  is small, and hence no loss of generality.) If the distribution of structures evolves slowly compared to the time it takes for light to cross the homogeneity scale (i.e. for the null geodesic to integrate over a statistically homogeneous and isotropic sample), then the contribution of the variation  $\Delta \theta$  should be small compared to the contribution of the average  $\langle \theta \rangle$ . Consider two geodesics passing from the hypersurface with proper time  $t$  to one with proper time  $t + \Delta t$ . If the geodesics cross the hypersurfaces at different points, they will go through different structures on the way and will in general gain different amounts of redshift. However, if the structures evolve slowly compared to the passage time  $\Delta t$ , the distribution of structures is essentially static between  $t$  and  $t + \Delta t$ . If the distance  $\Delta t$  is at least as large as the homogeneity scale, the redshift gained is the same for both geodesics, up to statistical fluctuations.

In the real universe, the size of structures and the homogeneity scale are indeed small compared to the Hubble time, which is the timescale of the evolution of the distribution of structures. For typical supersymmetric weakly interacting dark matter candidates, the size of the first structures, which form around  $z \sim 40\text{--}60$ , is of the order  $10^{-8}H^{-1}$  [59]. The size of structures relative to the Hubble length grows as structure formation proceeds, and today the largest typical structures have sizes of around 10 Mpc

$\approx 10^{-3}H^{-1}$ . The homogeneity scale today is of the order of 100 Mpc  $\approx 10^{-2}H^{-1}$  [42,43] (though see [44,45]).

If the contribution of  $\Delta\theta$  in (21) can be neglected, the redshift does not depend on the specific geodesic, and we have

$$1 + z \approx 1 + \langle z \rangle = a(t)^{-1} , \quad (22)$$

where  $a(t)$  is the scale factor defined in (9). The average redshift on the hypersurface of proper time  $\langle z \rangle$  can by statistical homogeneity and isotropy also be understood as the average along a specific geodesic taken over a distance longer than the homogeneity scale (but much shorter than the Hubble scale). In a statistically homogeneous and isotropic universe, the redshift is independent of direction, and it is related to the volume expansion rate in the same way as in the exactly homogeneous and isotropic FRW models. The change in the wavelength of a typical photon is, over long distances, only determined by the overall expansion of the volume. (This conclusion disagrees with [60,61], where the redshift was considered using ad hoc treatments for the spatial curvature; see also [55].)

While the relation (22) between the expansion rate and the redshift is simple, the result is not entirely obvious. For example, it is vital that the shear and the expansion rate appear linearly in the redshift integral (20). In comparison, the shear and the expansion rate contribute quadratically to the dust equations of motion (4) and (5), so the shear does not drop out when considering large regions, and the variance of the expansion rate plays an important role.

The parametrisation of the null geodesics in terms of the proper time is crucial. The evolution of structures is governed by proper time, so the hypersurface of proper time is also the hypersurface of statistical homogeneity and isotropy. It is the statistical homogeneity and isotropy which makes it possible to neglect the shear and  $\Delta\theta$ , and relate the scale factor (9) to observables. This expresses in more detail the relation between the redshift, the scale factor and statistical homogeneity and isotropy discussed in [9,21]. For making this argument sharper, it would be worthwhile to have a more precise notion of statistical homogeneity and isotropy in a general dust spacetime.

This connection with the observable redshift establishes the observational relevance of the expansion rate averaged over the hypersurface of constant proper time. Studying the average expansion rate has been criticised [62] on the grounds that it depends on the hypersurface of averaging [63,64], and also because the average is taken on a spacelike hypersurface, while observations are made along and inside the past light cone. Though there is a preferred foliation for dust, given by the hypersurface of constant proper time of comoving observers [9,21,65], ultimately the usefulness of the expansion rate averaged on that hypersurface is determined by relating it to observed quantities, which we have now done for the redshift.

The expansion rate and the shear can have large variations along the null geodesic, of the order of the average values. A crucial reason for why it is sufficient to consider only the averages is that the coherence length of the variations is much smaller than

the scale over which the averages change significantly. (The large amplitude of local variations in the expansion rate is clear from (4)–(6) and the fact that there are large differences in the local density. This can be seen explicitly in the spherical collapse model and its underdense equivalent [66, 67].)

*Deviations from the mean.* In a universe which is only statistically and not exactly homogeneous and isotropic, the contribution of the shear to the redshift integral (20) is not zero, only suppressed compared to the contribution of the expansion rate. The fact that the null geodesics are affected by the dust shear also leads to a correlation between the shear and the spatial direction of the null geodesic  $e^\alpha$ . However, from observations it is known that the change in  $e^\alpha$  for typical light rays is small, at the percent level [68]. Similarly, because there are statistical fluctuations and because the distribution is not exactly static, the integral of the spatial variation of the expansion rate will not completely vanish even over scales much longer than the homogeneity scale, though its contribution to (21) will be small compared to the contribution of the average expansion rate. In perturbed FRW models, the contribution of the shear and  $\Delta\theta$  reduces, in addition to the local dipole term, to the ISW effect and the Rees-Sciama effect. (For covariant treatment of CMB perturbation theory, see [69]. The covariant formalism was also recently applied to the CMB in [70].)

Another source of variation is the fact that the cancellations discussed above only happen when integrating over long distances. For distances smaller than the homogeneity scale, directional variation in the redshift due to differences in the local expansion rate and shear should be expected. Typical variation of the expansion rate in different directions within 70 Mpc around our location was found to be 20% in [71] (see also [72]). For the nearby supernovae, these variations are well known under the name peculiar velocities. (In the comoving approach we follow, peculiar velocities are zero by definition. For a covariant way of defining peculiar velocities, see [73].) Because the homogeneity scale is small compared to the distance over which most cosmological observations are made, the effect of these deviations is expected to be small.

A rough estimate of the contribution of the local variation in the geometry to the redshift (20) would be  $\int dt \Delta\theta \approx L \langle \Delta\theta \rangle$ , where  $L$  is the size of the region where there is significant uncanceled variation, and  $\langle \Delta\theta \rangle$  is the typical magnitude of the variation. For  $L = 70$  Mpc and  $|\langle \Delta\theta \rangle| = 0.2H_0$ , where  $H_0$  is the Hubble parameter today, we get  $10^{-3}$ . This may be an overestimate, because in the linear regime perturbations in the expansion rate and the shear mostly cancel, apart from the dipole (for which the contribution of local structures is indeed observed to be  $10^{-3}$ ). There does not appear to be a reason for such a cancellation in the non-linear regime, but the importance of the deviation from the linear results for the local universe remains to be determined§.

§ In the average expansion rate, the deviation from FRW behaviour does not become significant until fourth order, because of a similar cancellation between the local expansion rate and the shear at second order. This can be understood in terms of the Newtonian limit [9, 74]. It is not clear whether the cancellation in the redshift can be understood in a similar manner.

In any case, the correction is negligible for cosmological probes other than the CMB. For the CMB, the effect might give an important contribution at large angles (where the variation in the expansion rate is large), and could possibly explain some of the observed anomalies [75, 76] (see [9] for discussion and more references), particularly since some of them are correlated with the dipole.

Observationally, we know from the CMB that the redshift of the last scattering surface is the same in all directions, i.e. for different geodesics, to the level  $10^{-3}$ , or  $10^{-5}$  if we exclude the dipole. Assuming that our location is typical in space, this is the variation over the hypersurface of constant proper time. The variation is likely to be at least as small at earlier times, because structure formation is less advanced. In addition, deviation of the CMB spectrum from the blackbody shape is observed to be small [77]. The sum of two blackbody spectra at different temperatures is not a blackbody, so finite angular resolution inevitably changes the spectral shape when different directions have different CMB temperatures [78]. Scattering of the CMB leads to a similar effect [79] (this was used to constrain local void models in [36]). The limit on the  $y$ -distortion relevant for both cases is  $|y| < 1.5 \times 10^{-5}$  [77]. So the cancellations discussed above are well realised in the universe.

*Inapplicability of the ‘almost EGS theorem’.* There is an ‘almost Ehlers-Geren-Sachs theorem’, which states that if the observed CMB anisotropy is everywhere small, then the universe is close to FRW [80] (see also [81]). This is contrary to our conclusion above. The crucial assumption in the proof of the theorem which is not satisfied in the real universe is that the magnitude of the spatial and time derivatives of the CMB anisotropy  $\Delta T / \langle T \rangle$  is at most  $|\Delta T| / \langle T \rangle$  times the expansion rate<sup>||</sup>. Here  $T \propto E$  is the CMB temperature, and the average on the hypersurface of constant proper time  $\langle T \rangle$  can via statistical homogeneity and isotropy be also understood as the average over the sky measured at one position.

As we have seen, it is true that  $|\Delta E| \ll \langle E \rangle$  at every point, but this does not imply that the derivatives of  $\Delta E$  would be smaller than the derivatives of  $E$ . Since the local variation in the expansion rate is of the order of the average expansion rate,  $|\Delta \theta| \sim \langle \theta \rangle$ , it follows from (18) that the derivative  $(u^\alpha + e^\alpha) \partial_\alpha (\Delta E)$  is of the order  $|\theta| \langle E \rangle$ , rather than  $|\theta| |\Delta E|$ . Stated the other way round, large derivatives of the CMB temperature anisotropies do not imply a large variation in the CMB temperature between nearby points. The temperature difference is the integral of the derivative over some length, and as long as the spatial derivative remains large only over length scales small compared to the inverse derivative, the difference is small. The small correlation length of spatial variations is at the heart of the argument which leads from the general expression for the redshift (20) to the simple relation (22).

<sup>||</sup> It has been earlier pointed out [21] that the theorem is not valid in the real universe, because it also indicates that the gradient of the local matter density is at most the density divided by the Hubble length, times the CMB anisotropy of (neglecting the dipole)  $10^{-5}$ . However, the reason why the theorem fails was not identified.

### 3.2. The distance

*The two-metric.* Cosmological distances are defined in terms of observations of light. Two commonly used distance measures are the angular diameter distance  $D_A$ , which measures apparent size, and the luminosity distance  $D_L$ , which measures apparent brightness. These distance measures are in a general spacetime related to each other by the reciprocity relation  $D_L = (1+z)^2 D_A$ , so there is only one independent distance [47], [57] (page 111), [82]. Other distances can be defined by multiplying with different powers of  $1+z$ ; for discussion of different distance measures in the context of FRW models, see [2, 83].

It is convenient to analyse the null photon geodesics in terms of the decomposition of the covariant derivative of the tangent vector, like we did with the timelike dust geodesics. In the case of timelike geodesics, the tensor  $h_{\alpha\beta}$  defined in (2) provides a natural three-metric orthogonal to the flow. For null geodesics, the situation is more involved, because it is not possible to construct a metric orthogonal to  $k^\alpha$  using only  $g_{\alpha\beta}$  and  $k^\alpha$ : a new vector field is needed. There is no unique choice, and different vectors are used in the literature [57] (page 106), [84, 85]. For example, we can use the timelike vector  $u^\alpha$ , which is already defined. It is straightforward to identify the observed area of a source as the projection onto the two-space orthogonal to both  $k^\alpha$  and the observer's velocity  $u^\alpha$ . This corresponds to the two-metric

$$\begin{aligned}\tilde{h}_{\alpha\beta} &\equiv g_{\alpha\beta} - E^{-2}k_\alpha k_\beta + E^{-1}u_\alpha k_\beta + E^{-1}k_\alpha u_\beta \\ &= g_{\alpha\beta} + u_\alpha u_\beta - e_\alpha e_\beta ,\end{aligned}\tag{23}$$

where we have applied the decomposition (17) on the second line. The expression in terms of  $e^\alpha$  is particularly transparent: the two-metric  $\tilde{h}_{\alpha\beta}$  spans the subspace of the three-space orthogonal to  $u^\alpha$  that is also orthogonal to the spatial direction of the null geodesic  $e^\alpha$ . The two-metric (23) satisfies  $\tilde{h}_{\alpha\beta}u^\beta = \tilde{h}_{\alpha\beta}e^\beta = 0$ ,  $\tilde{h}_{\alpha\beta}k^\beta = 0$ ,  $\tilde{h}_\alpha^\gamma \tilde{h}_{\gamma\beta} = \tilde{h}_{\alpha\beta}$ ,  $\tilde{h}^\alpha_\alpha = 2$ . While conceptually clear, the choice (23) is not the most convenient for practical calculations, because  $u^\alpha$  is not parallel propagated along the null geodesic. However, the area is independent of the choice of two-metric [84]. We will mostly keep the two-metric completely general and use only the condition  $\tilde{h}_{\alpha\beta}k^\beta = 0$ .

*The angular diameter distance.* Given any two-metric orthogonal to  $k^\alpha$ , we can decompose the covariant derivative of  $k^\alpha$  as follows,

$$\begin{aligned}\nabla_\beta k_\alpha &= \tilde{\theta}_{\alpha\beta} \\ &= \frac{1}{2}\tilde{h}_{\alpha\beta}\tilde{\theta} + \tilde{\sigma}_{\alpha\beta} + k_{(\alpha}P_{\beta)} ,\end{aligned}\tag{24}$$

where the trace  $\tilde{\theta} = \tilde{h}^\alpha_\beta \nabla_\alpha k^\beta = \nabla_\alpha k^\alpha$  is the area expansion rate,  $\tilde{\sigma}_{\alpha\beta} = \tilde{h}_\alpha^\gamma \tilde{h}_\beta^\delta \nabla_\gamma k_\delta - \frac{1}{2}\tilde{h}_{\alpha\beta}\tilde{\theta}$  is the shear and  $P_\alpha$  is a vector which depends on the choice of  $\tilde{h}_{\alpha\beta}$  and plays no role in what follows. We have  $\tilde{\sigma}_{\alpha\beta}k^\beta = 0$ ,  $P_\alpha k^\alpha = 0$ . The vorticity is automatically zero (unlike for  $u^\alpha$ ), because  $k^\alpha$  is the gradient of a scalar. Also in contrast to the decomposition of  $u^\alpha$ , the shear is not the symmetric trace-free part of the full  $\nabla_\beta k_\alpha$ ,

but rather the symmetric trace-free part of  $\nabla_\beta k_\alpha$  projected onto the two-space with the metric  $\tilde{h}_{\alpha\beta}$ . Thus the shear, unlike the area expansion rate, depends on the choice of two-metric.

Denoting the local scale factor which describes the linear size of the null geodesic bundle two-space by  $s(t, \mathbf{x})$ , the area expansion rate is  $\tilde{\theta} = 2\partial_\lambda s/s$ . The angular diameter distance is proportional to the linear size,  $D_A \propto s$  (see for example [85]), so

$$D_A \propto \exp\left(\frac{1}{2} \int d\lambda \tilde{\theta}\right). \quad (25)$$

As noted, the distance is independent of the choice of the two-metric  $\tilde{h}_{\alpha\beta}$  (in particular, it does not depend on the observer's velocity  $u^\alpha$ ).

*Evolution of the angular diameter distance.* To determine how the angular diameter distance changes along the null geodesic, we need the evolution equation of  $\tilde{\theta}$ . As in the case of the redshift, we take a derivative with respect to  $\lambda$ ,

$$\begin{aligned} \partial_\lambda \tilde{\theta} &= k^\alpha \nabla_\alpha \nabla_\beta k^\beta \\ &= k^\alpha R_{\alpha\beta}{}^{\beta\gamma} k_\gamma + k^\alpha \nabla_\beta \nabla_\alpha k^\beta \\ &= -R_{\alpha\beta} k^\alpha k^\beta - \nabla_\beta k^\alpha \nabla_\alpha k^\beta \\ &= -8\pi G_N \rho E^2 - 2\tilde{\sigma}^2 - \frac{1}{2}\tilde{\theta}^2 \\ &\equiv -2\mu^2 - \frac{1}{2}\tilde{\theta}^2, \end{aligned} \quad (26)$$

where  $R_{\alpha\beta\gamma\delta}$  is the Riemann tensor, and we have used the condition  $k^\alpha \nabla_\alpha k^\beta = 0$  and the decomposition (24). On the next to last line, we have used the Einstein equation (1). We have defined  $\tilde{\sigma}^2 \equiv \frac{1}{2}\tilde{\sigma}_{\alpha\beta}\tilde{\sigma}^{\alpha\beta} \geq 0$  and  $\mu^2 \equiv 4\pi G_N \rho E^2 + \tilde{\sigma}^2 \geq 0$ . Note that  $\tilde{\sigma}^2$ , unlike  $\tilde{\sigma}_{\alpha\beta}$ , is independent of the choice of  $\tilde{h}_{\alpha\beta}$ . The Raychaudhuri equation (26) for the null geodesics is analogous to the Raychaudhuri equation (4) for the timelike geodesics.

Given the relation (25) and the equation (26), we obtain the equation satisfied by the angular diameter distance:

$$\begin{aligned} \partial_\lambda^2 D_A &= -(4\pi G_N \rho E^2 + \tilde{\sigma}^2) D_A \\ &= -\mu^2 D_A. \end{aligned} \quad (27)$$

The right-hand side is non-positive (and the initial condition for  $\partial_\lambda D_A$  is negative), so the distance is monotonic along the null geodesic. We also see that the dust energy density and the photon shear can only make distances smaller (i.e. objects appear larger and brighter) compared to the non-sheared vacuum case. In particular, neglecting the null shear would give an upper bound on the distance.

We need the evolution equation for  $\tilde{\sigma}_{\alpha\beta}$ , or at least for  $\tilde{\sigma}^2$ . The equation for  $\tilde{\sigma}_{\alpha\beta}$  will be different for different choices of  $\tilde{h}_{\alpha\beta}$ , since the components  $\tilde{\sigma}_{\alpha\beta}$  depend on  $\tilde{h}_{\alpha\beta}$ . However, the equation for  $\tilde{\sigma}^2$  can be written in the simple form

$$\partial_\lambda \tilde{\sigma}^2 = \tilde{\sigma}^{\alpha\beta} \partial_\lambda \tilde{\sigma}_{\alpha\beta}$$

$$\begin{aligned}
&= \tilde{\sigma}^{\alpha\beta} k^\gamma \nabla_\gamma \left( \nabla_\alpha k_\beta - \frac{1}{2} \tilde{h}_{\alpha\beta} \tilde{\theta} - k_{(\alpha} P_{\beta)} \right) \\
&= \tilde{\sigma}^{\alpha\beta} k^\gamma \nabla_\gamma \nabla_\alpha k_\beta \\
&= \tilde{\sigma}^{\alpha\beta} k^\gamma R_{\gamma\alpha\beta\delta} k^\delta + \tilde{\sigma}^{\alpha\beta} k^\gamma \nabla_\alpha \nabla_\gamma k_\beta \\
&= -\tilde{\sigma}^{\alpha\beta} k^\gamma k^\delta C_{\alpha\gamma\beta\delta} - 2\tilde{\theta} \tilde{\sigma}^2, \tag{28}
\end{aligned}$$

where  $C_{\alpha\beta\gamma\delta}$  is the Weyl tensor. We have used the properties  $k^\alpha \nabla_\alpha k^\beta = 0$ ,  $\tilde{\sigma}_{\alpha\beta} k^\beta = 0$  and the relations  $\tilde{\sigma}_\alpha{}^\gamma \tilde{\sigma}_{\gamma\beta} = \tilde{h}_{\alpha\beta} \tilde{\sigma}^2$  and  $\tilde{\sigma}^{\alpha\beta} \partial_\lambda \tilde{h}_{\alpha\beta} = 0$ . Equation (28) is not closed; as in the case of the dust geodesics, the shear equation of motion cannot be reduced to scalar form.

For practical use, (27) should be written in terms of the observed redshift rather than  $\lambda$ , and the contribution of the null shear term (28) should be evaluated. We will first go through this in the FRW case and then consider the clumpy situation.

*Exact homogeneity and isotropy.* The symmetry of a FRW universe implies that (in the appropriate coordinate system) the diagonal components of  $\tilde{\sigma}^\alpha_\beta$  should be equal, and the off-diagonal components should vanish, since  $\tilde{\sigma}^\alpha_\beta$  is a spatial tensor. This implies that  $\tilde{\sigma}^\alpha_\beta$  vanishes, since it is traceless. However, the shape of the source can break the symmetry, generating non-zero shear. Since the Weyl tensor vanishes in FRW models, (28) gives  $\tilde{\sigma}^2 \propto s^{-4} \propto D_A^{-4}$ . So  $\tilde{\sigma}^2$  can only decrease along the geodesic, and if it is zero initially, it will remain zero. We can thus neglect the null shear.

Because of the symmetry, the distance is independent of spatial position, so  $\partial_\lambda D_A = E \partial_t D_A = -E(1+z)H \partial_z D_A$ . Since  $E \propto 1+z$  and  $\rho \propto a^{-3} \propto (1+z)^3$ , (27) reduces to

$$\begin{aligned}
H \partial_z [(1+z)^2 H \partial_z D_A] &= -4\pi G_N \rho D_A \\
&= -4\pi G_N \rho_0 (1+z)^3 D_A. \tag{29}
\end{aligned}$$

Given the initial conditions  $D_A(0) = 0$ ,  $\partial_z D_A(0) = H_0^{-1}$ , the distance  $D_A$  is completely determined by the evolution history  $H(z)$  and the present value of the matter density  $\rho_0$ . The spatial curvature enters only via its effect on  $H$ . For general matter content,  $\rho$  on the right-hand side of (29) would be replaced by  $\rho + p$ , where  $p$  is the pressure.

Instead of using the Einstein equation to substitute the energy-momentum tensor for  $R_{\alpha\beta}$  in (26), we can express  $R_{\alpha\beta}$  directly in terms of  $H$  and the spatial curvature  ${}^{(3)}R = 6K(1+z)^2$ . Essentially, we are swapping  $\rho + p$  for the spatial curvature. This makes it possible to integrate (29) in a closed form, regardless of the matter content or the theory of gravity: the result only depends on the metric having the FRW form. Substituting  $4\pi G_N(\rho + p) = -\dot{H} + K(1+z)^2$  on the left-hand side of (29) and making the change of variable  $v = \int_0^z dz'/H(z')$ , we obtain

$$\partial_v^2 [(1+z) D_A] = -K(1+z) D_A, \tag{30}$$

which integrates into the well-known expression

$$D_A = (1+z)^{-1} \frac{1}{\sqrt{-K}} \sinh \left( \sqrt{-K} \int_0^z \frac{dz'}{H(z')} \right). \tag{31}$$



The fact that the spatial curvature evolves like  $(1+z)^2$  in all FRW universes is the reason why  $D_A$  can be written in this universal form, which depends only on  $H$  and  $K$ . However, we may equally view  $D_A$  as being determined by  $H$  and  $\rho + p$ . While these quantities are related to the spatial curvature in a simple manner in the homogeneous and isotropic case, the situation is different in a clumpy space.

In [86], the universal FRW relation (31) was formulated as a consistency condition between  $D_A$  and  $H$ . (A similar consistency condition, but specific to the  $\Lambda$ CDM model, was presented in [87].) If the relation (31) between  $D_A$  and  $H$  is violated, the metric cannot be the FRW one. This is a null test: if the condition is not violated, we cannot conclude that the metric has the FRW form, especially as recovering  $H$  from the observations requires assumptions about the geometry, with the FRW metric usually adopted to begin with.

*Statistical homogeneity and isotropy.* In a general dust spacetime, the first obstruction to finding  $D_A(z)$  is that no such function exists. While the angular diameter distance is a monotonic function of  $\lambda$ , the redshift is not. As (18) shows, the redshift can both increase and decrease along the null geodesic. Physically this is clear: in a region which is collapsing (or strongly negatively sheared in the direction along the null geodesic), the light gains a blueshift, i.e. a negative redshift. There is a unique redshift at each point along the null geodesic, but more than one point may share the same redshift. Therefore, while the function  $D_A(\lambda)$  exists (and is monotonic), the expression  $D_A(z)$  is not single-valued, and the same redshift may correspond to several distances. This is true even for a single geodesic, variation in different directions on the sky aside (see [88] for an example).

In a statistically homogeneous and isotropic universe which expands on average, the redshift accumulated over a section of the null geodesic passing through a homogeneity-scale sized region is always positive. So while  $D_A(z)$  does not exist, there is a function  $D_A(\langle z \rangle)$ . As before, by statistical homogeneity and isotropy,  $\langle z \rangle$  can be interpreted as either the spatial average or the average over a section of the null geodesic longer than the homogeneity scale, but much shorter than the Hubble scale.

In the case of the redshift, it was possible to directly integrate equation (18) in terms of the variable  $\lambda$ , so we could straightforwardly discuss the smoothing in terms of average quantities related to the dust geometry. For the area expansion rate  $\tilde{\theta}$ , or equivalently the angular diameter distance  $D_A$ , we cannot write down the solution explicitly<sup>¶</sup>. Nevertheless, we can still follow a similar line of reasoning than with the redshift, but consider smoothing at the level of the equation instead of its solution. The objective is to see whether it follows from statistical homogeneity and isotropy that the mean of the area expansion rate  $\tilde{\theta}$  dominates over the variation,  $|\Delta\tilde{\theta}| \ll |\langle\tilde{\theta}\rangle|$  (and if not, what additional assumptions are needed), and find the equation relating  $\langle\tilde{\theta}\rangle$  to the

<sup>¶</sup> While the equation (26) for  $\tilde{\theta}$  looks simple, it is a sub-case of the Riccati equation which does not have a general solution. Switching to  $D_A$  gives (27), the one-dimensional Schrödinger equation, for which there is no general solution either.

average dust geometry.

The source term  $\mu^2$  in the equation (26) for  $\tilde{\theta}$  has large local variations, just like the right-hand side of the redshift equation (18). In fact, the variation is stronger than in the redshift case, since the matter density can change by orders of magnitude,  $|\Delta\rho| \gg \langle\rho\rangle$ . However, analogously to the redshift case,  $\tilde{\theta}$  can depend on  $\mu^2$  only via the integral  $\int d\lambda\mu^2$  (and its further integrals). (This is transparent with the substitution  $\tilde{\theta} = f(\lambda) - 2 \int d\lambda\mu^2$  into (26).) We can write this quantity as

$$\begin{aligned} \int d\lambda\mu^2 &= \int d\lambda(4\pi G_N \rho E^2 + \tilde{\sigma}^2) \\ &= \int_{\lambda} dt \frac{1}{E} (4\pi G_N \rho E^2 + \tilde{\sigma}^2) \\ &\approx \int_{\lambda} dt \frac{1}{\langle E \rangle} [4\pi G_N (\langle \rho \rangle + \Delta\rho) \langle E \rangle^2 + \langle \tilde{\sigma}^2 \rangle + \Delta\tilde{\sigma}^2] , \end{aligned} \quad (32)$$

where we have again assumed that vorticity vanishes so that we can parametrise points along the geodesic uniquely with  $t$ , and we have taken into account  $|\Delta E| \ll \langle E \rangle$ .

For the matter density, we can use the same reasoning as with the expansion rate in (21) to argue that the contribution of  $\Delta\rho$  to (32) is subdominant to the contribution of  $\langle\rho\rangle$  over sufficiently long distances. (Via (6), the matter density is related to the expansion rate by  $\rho(t, \mathbf{x}) = \rho(t_0, \mathbf{x}) e^{-\int_{t_0}^t dt\theta}$ .) In fact, the argument is now stronger, because fluctuations in the density necessarily cancel due to conservation of mass. In [89,90], the line average of the density in a Swiss cheese model was found to be smaller than the volume average. However, the structures in the model are not distributed in a statistically homogeneous and isotropic manner (and they are also unrealistically large). Randomising the structures leads to a suppression of the deviation from the FRW result for the distance [90–92], and we expect this to hold also for the density.

There is one additional complication compared to the redshift case, namely correlation between  $\Delta\rho$  and  $\Delta E$ . We have neglected  $\Delta E$  because it is small compared to  $\langle E \rangle$ , and  $\Delta\rho$  because its mean vanishes. However,  $\Delta E$  and  $\Delta\rho$  are correlated (the density is anti-correlated with the expansion rate, which is correlated with the redshift), so the contribution of the term  $8\pi G_N \Delta\rho \Delta E \langle E \rangle$  in (32) does not vanish over long distances, and  $|\Delta\rho|/\langle\rho\rangle$  can be locally orders of magnitude higher than unity. However, highly overdense regions can take up only a small fraction of the volume, because mass is conserved. In particular, since  $|\Delta\rho|/\langle\rho\rangle$  in underdense regions cannot exceed unity, the typical value in overdense regions is also at most unity, and the overall mean amplitude of  $|\Delta\rho|/\langle\rho\rangle$  cannot compensate for the suppression factor  $|\Delta E|/\langle E \rangle \sim 10^{-5}$ .

For the null geodesic shear, there is no such simple argument. Integrating (28), we have

$$\tilde{\sigma}^2(\lambda) = \tilde{\sigma}^2(\lambda_0) e^{-\int_{\lambda_0}^{\lambda} d\lambda' 2\tilde{\theta}} - e^{-\int_{\lambda_0}^{\lambda} d\lambda' 2\tilde{\theta}} \int_{\lambda_0}^{\lambda} d\lambda' e^{\int_{\lambda_0}^{\lambda'} d\lambda'' 2\tilde{\theta}} k^{\gamma} k^{\delta} \tilde{\sigma}^{\alpha\beta} C_{\alpha\gamma\beta\delta} . \quad (33)$$

The first term vanishes if the initial shear is zero, and anyway decreases along the null geodesic, as in the FRW case. The second term is less straightforward. Using the

decomposition (17), the Weyl term in the integrand can be written as

$$\begin{aligned} k^\gamma k^\delta C_{\alpha\gamma\beta\delta} \tilde{\sigma}^{\alpha\beta} &= E^2 (u^\gamma u^\delta + e^\gamma e^\delta + 2u^{(\gamma} e^{\delta)}) C_{\alpha\gamma\beta\delta} \tilde{\sigma}^{\alpha\beta} \\ &= 2E^2 (E^{\alpha\beta} + \tilde{\epsilon}^{\mu(\alpha} H^{\beta)}) \tilde{\sigma}_{\alpha\beta} , \end{aligned} \quad (34)$$

where we have decomposed the Weyl tensor in terms of its electric and magnetic components, defined as  $E_{\alpha\beta} \equiv C_{\alpha\gamma\beta\delta} u^\gamma u^\delta$ ,  $H_{\alpha\beta} \equiv \frac{1}{2} \epsilon_\alpha^{\gamma\delta} C_{\gamma\delta\beta\mu} u^\mu$ , and we have denoted the volume element on the two-space orthogonal to both  $u^\alpha$  and  $e^\alpha$  as  $\tilde{\epsilon}_{\alpha\beta} \equiv \epsilon_{\alpha\beta\gamma} e^\gamma$ . The tensors  $E_{\alpha\beta}$  and  $H_{\alpha\beta}$  are traceless, symmetric, and orthogonal to  $u^\alpha$ . Writing the null shear in terms of  $e^\alpha$  and quantities related to the dust geometry using (3) and (17) and adopting the two-metric (23), we have

$$\begin{aligned} k^\gamma k_\delta C_{\alpha\gamma\beta\delta} \tilde{\sigma}^{\alpha\beta} &= 2E^3 (E^{\alpha\beta} + \tilde{\epsilon}^{\mu(\alpha} H^{\beta)}) \left( \tilde{h}_\alpha^\gamma \tilde{h}_\beta^\delta - \frac{1}{2} \tilde{h}_{\alpha\beta} \tilde{h}^{\gamma\delta} \right) (\sigma_{\gamma\delta} + \nabla_{(\gamma} e_{\delta)}) \\ &= 2E^3 E^{\alpha\beta} \left( \tilde{h}_\alpha^\gamma \tilde{h}_\beta^\delta + \frac{1}{2} e_\alpha e_\beta \tilde{h}^{\gamma\delta} \right) (\sigma_{\gamma\delta} + \nabla_{(\gamma} e_{\delta)}) \\ &\quad + 2E^3 H^{\alpha\beta} \tilde{h}_\alpha^\gamma \tilde{\epsilon}_\beta^\delta (\sigma_{\gamma\delta} + \nabla_{(\gamma} e_{\delta)}) . \end{aligned} \quad (35)$$

We could argue that if there is no preferred direction on the two-space orthogonal to  $u^\alpha$  and  $e^\alpha$ , the diagonal components of  $H_{\alpha\beta}$ , as well as  $\sigma_{\alpha\beta} + \nabla_{(\alpha} e_{\beta)}$ , in the directions parallel to that space should contribute equally when integrated over long distances. Then the contribution of the magnetic Weyl term on the second line of (35) to the integral (33) would vanish. A similar argument for  $E_{\alpha\beta}$  would leave a product of the off-diagonal terms of  $E_{\alpha\beta}$  and  $\sigma_{\alpha\beta} + \nabla_{(\alpha} e_{\beta)}$ , and it is not clear why its integral would vanish. We could try to formulate an argument along the lines that the contribution of the term involving  $\nabla_{(\alpha} e_{\beta)}$  should vanish in a statistically homogeneous and isotropic space, since the result should not depend on  $e^\alpha$  (though this is not obvious), but this would still leave the dust shear. There is no clear symmetry reason for  $E_{\alpha\beta}$  and the dust shear  $\sigma_{\alpha\beta}$ , or in terms of (34), the null shear  $\tilde{\sigma}_{\alpha\beta}$ , to be uncorrelated, especially as the Weyl tensor acts as a source for the null shear. We cannot get rid of all terms not directly related to the average dust geometry with symmetry arguments leading to a lack of correlation over long distances, unlike in the case of the redshift integral (20).

However, if the amplitude of the Weyl tensor is highly suppressed compared to the Ricci tensor (specifically, to  $8\pi G_N \langle \rho \rangle$ ), the null shear can be neglected in (32), regardless of the correlations of the Weyl tensor. In general, the components of the Weyl tensor are locally not smaller than the matter density. (In particular, in vacuum regions the Ricci tensor is zero, and the curvature is manifested entirely via the Weyl tensor; see [93] for an example in a Swiss cheese model.) However, it is possible that in a statistically homogeneous and isotropic space the contribution of the Weyl tensor to scalar observables is small compared to the contribution of the Ricci tensor, when integrated over scales larger than the homogeneity scale.

From observations, the null shear is known to be small in the real universe [68]. The smallness of the shear is theoretically supported by studies of various models which find only small effects on the distance when the expansion rate is the FRW one (see [21]

for an overview and references). Nevertheless, since we do not have a general theoretical argument for the smallness of the Weyl contribution, we will retain  $\tilde{\sigma}^2$  in the equations. Even though the Weyl tensor (and its relative contribution to  $\mu^2$ ) can vary strongly between different regions, we assume that since the Weyl tensor affects  $\tilde{\theta}$  only via a double integral, the contribution of the variation  $\Delta\tilde{\sigma}^2$  is small compared to the mean value  $\langle\tilde{\sigma}^2\rangle$ , when integrated over long distances.

In addition to  $\int d\lambda\mu^2$ , the area expansion rate  $\tilde{\theta}$  can explicitly depend on the affine parameter  $\lambda$ . So we should also divide  $\lambda$  into the mean and the variation. We have  $\int d\lambda = \int_\lambda \frac{dt}{E} \approx \int \frac{dt}{\langle E \rangle} - \int_\lambda dt \frac{\Delta E}{\langle E \rangle^2}$ , again assuming that vorticity vanishes so that we can parametrise the null geodesic with  $t$ , and taking into account  $|\Delta E| \ll \langle E \rangle$ . This gives  $\langle \lambda \rangle$  and  $\Delta\lambda$ , and shows that the variation of  $\lambda$  on hypersurfaces of constant proper time is small,  $|\Delta\lambda| \ll \langle \lambda \rangle$ . In the first term, we have dropped the subscript  $\lambda$  to indicate that it is independent of the specific geodesic. We can write<sup>+</sup> (with some abuse of notation)

$$\tilde{\theta} \left( \lambda, \int d\lambda\mu^2 \right) \approx \tilde{\theta} \left( \langle \lambda \rangle + \Delta\lambda, \int \frac{dt}{\langle E \rangle} \langle \mu^2 \rangle + \int_\lambda dt \frac{\Delta\mu^2}{\langle E \rangle} \right), \quad (36)$$

where we have taken into account  $|\Delta E| \ll \langle E \rangle$ . The correction terms in both arguments of  $\tilde{\theta}$  are small compared to the mean, so if we expand  $\tilde{\theta}$  in a Taylor series around the average values of the arguments, the next order terms are suppressed and we have the result  $|\Delta\tilde{\theta}| \ll |\langle\tilde{\theta}\rangle|$ . (It is crucial that  $\tilde{\theta}$  depends on  $\mu^2$  only via an integral: otherwise the result would not hold, since  $|\Delta\mu^2| \gg \langle\mu^2\rangle$ .)

We cannot simply substitute  $\tilde{\theta} = \langle\tilde{\theta}\rangle + \Delta\tilde{\theta}$  into (26) and drop all terms involving  $\Delta\tilde{\theta}$ , because, as in the redshift case, we do not in general have  $|\partial_\lambda(\Delta\tilde{\theta})| \ll |\partial_\lambda\langle\tilde{\theta}\rangle|$ . In fact,  $|\partial_\lambda(\Delta\tilde{\theta})| \sim |\Delta\mu^2| \gg \langle\mu^2\rangle \sim |\partial_\lambda\langle\tilde{\theta}\rangle|$ . However, we can integrate (26) once to get an equation where  $\int d\lambda\tilde{\theta}^2$  appears instead of  $\partial_\lambda\tilde{\theta}$ . All terms involving  $\Delta\tilde{\theta}$  are then subdominant, and can be dropped. The resulting equation depends only on the time  $t$ , not on the spatial coordinates. Taking a time derivative, we obtain

$$\begin{aligned} \langle E \rangle \partial_t \langle \tilde{\theta} \rangle + \frac{1}{2} \langle \tilde{\theta} \rangle^2 &= -2 \langle \mu^2 \rangle \\ &= -8\pi G_N \langle \rho E^2 \rangle - 2 \langle \tilde{\sigma}^2 \rangle \\ &\approx -8\pi G_N \langle \rho \rangle \langle E \rangle^2 - 2 \langle \tilde{\sigma}^2 \rangle, \end{aligned} \quad (37)$$

where we have taken into account  $|\Delta E| \ll \langle E \rangle$ . This is the equation that we would have gotten by naively replacing all quantities by their averages in (26), putting  $\partial_\lambda \rightarrow \langle E \rangle \partial_t$ , and neglecting variance and the non-commutation of taking the derivative and averaging.

We now have all the ingredients for the equation for the angular diameter distance in terms of the average redshift. From the relation (25) we get, using similar reasoning as above,  $\langle D_A \rangle \propto \exp\left(\frac{1}{2} \int \frac{dt}{\langle E \rangle} \langle \tilde{\theta} \rangle\right)$  and  $|\Delta D_A| \ll \langle D_A \rangle$ . Inverting, we have  $\langle \tilde{\theta} \rangle = 2 \langle E \rangle \partial_t \langle D_A \rangle / \langle D_A \rangle$ . Recall that  $E \propto (1+z) \approx 1 + \langle z \rangle = a^{-1}$ , and  $H = \dot{a}/a$ . From (12) we have  $\langle \rho \rangle \propto a^{-3} \propto (1 + \langle z \rangle)^3$ . Putting the pieces together, we obtain

$$\begin{aligned} H \partial_{\langle z \rangle} \left[ (1 + \langle z \rangle)^2 H \partial_{\langle z \rangle} \langle D_A \rangle \right] &= - \left[ 4\pi G_N \langle \rho \rangle + \langle E \rangle^{-2} \langle \tilde{\sigma}^2 \rangle \right] \langle D_A \rangle \\ &= - \left[ 4\pi G_N \langle \rho_0 \rangle (1 + \langle z \rangle)^3 + \langle E \rangle^{-2} \langle \tilde{\sigma}^2 \rangle \right] \langle D_A \rangle, \end{aligned} \quad (38)$$

<sup>+</sup> We omit possible dependence on further integrals of  $\mu^2$ .

where

$$\begin{aligned} \langle E \rangle^{-2} \langle \tilde{\sigma}^2 \rangle &= 2(1 + \langle z \rangle)^{-2} \langle D_A \rangle^{-4} \int_0^{\langle z \rangle} \frac{dz'}{H} (1 + z') \langle D_A \rangle^4 \times \\ &\times \left\langle \left[ E^{\alpha\beta} \left( \tilde{h}_\alpha^\gamma \tilde{h}_\beta^\delta + \frac{1}{2} e_\alpha e_\beta \tilde{h}^{\gamma\delta} \right) + H^{\alpha\beta} \tilde{h}_\alpha^\gamma \tilde{\epsilon}_\beta^\delta \right] (\sigma_{\gamma\delta} + \nabla_{(\gamma} e_{\delta)}) \right\rangle, \end{aligned} \quad (39)$$

where we have assumed that the initial shear is small and can be neglected, and we have taken into account  $|\Delta D_A| \ll \langle D_A \rangle$ .

The equation (38) is our main result for the distance. It shows that the average angular diameter distance can be written in terms of the average dust geometry, plus the null shear. Aside from the null shear term, (38) has the same form as the corresponding FRW equation (29) in the case when  $\rho + p \propto (1 + z)^3$ , i.e. when the matter consists of dust and vacuum energy. When the null shear is negligible, differences between the distances of a clumpy model and the  $\Lambda$ CDM FRW model are completely encoded in the expansion rate and the redshift. A clumpy model with the same  $H(\langle z \rangle)$  (and present-day matter density) as the  $\Lambda$ CDM model has the same average distance-redshift relation, even though the spatial curvature will in general evolve quite differently, as discussed in section 2.2. (This conclusion disagrees with [60, 61, 94], where the distance was considered using ad hoc treatments for the spatial curvature; see also [55].) This is in contrast to FRW models with matter other than dust plus vacuum energy, where the distance deviates from the  $\Lambda$ CDM case because of a different  $\rho + p$  in addition to a different  $H$ . Of course, this depends on writing the equation for the angular diameter distance in terms of  $H$  and  $\rho + p$  instead of  $H$  and the spatial curvature. In the FRW case, it is possible to eliminate  $\rho + p$  in favour of the spatial curvature. Let us see why this does not work in a clumpy universe.

Analogously to the FRW case, we can use (10) and (11) to substitute  $4\pi G_N \langle \rho \rangle = -\dot{H} + \langle {}^{(3)}R \rangle / 6 + \mathcal{Q} / 2$  on the left-hand side of (38) and make the change of variable  $v = \int_0^{\langle z \rangle} dz' / H(z')$  to obtain

$$\partial_v^2 D = - \left( \frac{1}{6} \langle {}^{(3)}R \rangle + \frac{1}{2} \mathcal{Q} + \langle E \rangle^{-2} \langle \tilde{\sigma}^2 \rangle \right) (1 + \langle z \rangle)^{-2} D, \quad (40)$$

where we have denoted  $D \equiv (1 + \langle z \rangle) \langle D_A \rangle$ . We could express  $\mathcal{Q}$  in terms of  $\langle {}^{(3)}R \rangle$  using the integrability condition (14). However, there is a simple integral (31) in terms of  $v$  only when the right-hand side of (40) is independent of  $\langle z \rangle$ , which requires the expression inside the parentheses to be proportional to  $(1 + \langle z \rangle)^2$ , such as when  $\mathcal{Q} = 0$  and  $\langle \tilde{\sigma}^2 \rangle = 0$ . (When  $\mathcal{Q} = 0$ , it follows from (14) that  $\langle {}^{(3)}R \rangle \propto a^{-2}$ .)

## 4. Discussion

### 4.1. Observational and modelling issues

*Spatial curvature and the position of the CMB peaks.* The position of the CMB acoustic peaks is often considered to be a measurement of spatial curvature. The reason is that the peak location in multipole space corresponds to the apparent size of the last

scattering sound horizon, which provides a measure of the angular diameter distance to the last scattering surface at  $z \approx 1100$  [95] (page 99). The observed position of the peaks is consistent with spatial flatness in a  $\Lambda$ CDM universe [23].

However, the peak position does not imply spatial flatness, even in a FRW universe. It is clear from the way  $H$  and  $K$  enter the distance (31) that for any value of  $K$ , it is possible to adjust  $H$  to compensate for the spatial curvature so as to keep the distance fixed. For example, the peak position is consistent with a FRW universe with large positive spatial curvature [96]. Such a model is not viable due to other constraints, such as the value of the Hubble parameter today. By replacing the vacuum energy with exotic matter with a time-dependent equation of state, it is possible to do a similar adjustment and allow spatial curvature without changing the expansion history as radically [97, 98].

In the FRW case, it is true that given the expansion history  $H(z)$ , the peak position provides a measurement of the spatial curvature. The root of this argument is the relation (31), which expresses the distance in terms of the expansion rate and the spatial curvature. In a clumpy universe the distance is instead completely fixed by the expansion rate and the matter density (as well as the null shear), according to (38)\*. As (40) shows, the expression (31) is inapplicable due to the non-trivial evolution of the spatial curvature as well as the fact that clumping contributes to the expansion rate via  $\mathcal{Q}$ . In terms of the density parameters defined by dividing the expansion law (11) by  $3H^2$ , the density of matter and the density of curvature do not sum to unity, because of the contribution of  $\mathcal{Q}$  [6, 99]. In a FRW model, any additional contribution that changes  $H$  enters the distance also via  $\rho + p$  (which is related to the spatial curvature), but that is not the case here. In the FRW case with arbitrary matter, the spatial curvature is always proportional to  $(1+z)^2$ , and the evolution of  $\rho + p$  is not fixed, while in a clumpy dust universe,  $\rho + p$  is always proportional to  $(1+z)^3$ , but the evolution of the spatial curvature is complicated.

It is somewhat trivial that the spatial curvature enters the distance only via its effect on the expansion rate, since the spatial curvature can be written in terms of  $H$ ,  $\dot{H}$  and  $\langle \rho \rangle \propto (1 + \langle z \rangle)^3$  using the Buchert equations (10) and (11). However, it is not obvious that the equation for the average distance (38) depends only on  $H$  and the matter density, or that the dependence on  $H$  is the same as in the FRW case, so that we recover the  $\Lambda$ CDM equation (because  $\rho + p$  is the same in both cases).

In summary, the CMB peak position can be consistent with large spatial curvature in a clumpy model. All that is required is that the expansion history and the matter density today is sufficiently close to that of the spatially flat  $\Lambda$ CDM model. (Since the Hubble rate enters via an integral, significant variation in  $H(\langle z \rangle)$  is still allowed [100].)

*The effective equations of state.* Observations of distances are typically analysed in terms of an effective equation of state in a FRW model. (Regarding the dependence of

\* Note that we are comparing a clumpy universe with only dust to a FRW universe with arbitrary matter content. If we allowed other matter in the clumpy situation, the distance would also depend on the non-trivial  $\langle \rho \rangle + \langle p \rangle$ . In this case, the Buchert equations would also be more complicated [7].

the results on the adopted parametrisation for the equation of state, see [100].) The equation of state determines the evolution history  $H(z)$ , which then gives the distance  $D_A(z)$  via (31). If we want to express the evolution of a clumpy dust model this way, there are two different effective equations of state, because the relation between  $H$  and  $\langle D_A \rangle$  is different than in FRW models.

For the expansion rate, we can define an effective equation of state  $w_H(z)$  such that a FRW model with this equation of state would have the same expansion history as the clumpy model. Formally this is done by writing the spatial curvature and the backreaction variable  $\mathcal{Q}$  as a single component in the Buchert equations (10)–(12) [7, 101]. For comparison to distance observations, we should introduce another equation of state  $w_D(z)$ , defined so that the resulting  $H$  reproduces the real distance function  $\langle D_A \rangle$  when plugged into the FRW distance formula (31). In both cases, we have to make a choice for the spatial curvature of the FRW fitting model. The simplest choice, which does not involve any loss of generality, is to take the FRW model to be spatially flat. In general,  $w_H \neq w_D$ , so the effect of clumpiness cannot be treated just as an effective source in the FRW equations. This limits the usefulness of an effective description of backreaction in terms of a scalar field [101] (also,  $w_H$  can violate the null energy condition, unlike the equation of state of a scalar field [21]).

Until calculations of the impact of structure formation are accurate to more than an order of magnitude [21, 22], it is not known how large the expected difference between  $w_H$  and  $w_D$  is. However, even without a theoretical prediction, it is possible to test the null hypothesis that the equations of state inferred separately from observations of the expansion rate and distance do not show any difference. This is the essence of the FRW consistency check proposed in [86].

At the moment, while distances are measured relatively well, there are few observations probing the expansion rate as a function of redshift independent of the distance scale. The ages of passively evolving galaxies provide an interesting way to measure the expansion history, but at the moment the constraints are rather weak [27]‡. Another measure is provided by baryon acoustic oscillations, which are sensitive to a combination of expansion rate and distance [32]. (This was used to constrain local void models in [41].) There does appear to be some discrepancy between the observations of the luminosity distance of type Ia supernovae and measurements of baryon acoustic oscillations, but only at the  $2\sigma$  level [102], so it is not statistically significant. When analysing the data in the context of FRW models, this discrepancy would be interpreted as a violation of the reciprocity relation  $D_L = (1+z)^2 D_A$  instead of the FRW relation (31) between  $H$  and  $D_A$ .

The expansion rate also has a role in the ISW effect, i.e. deviations of the redshift from the mean value  $\langle z \rangle$  in different directions, as well as in the growth of density perturbations [30]. The ISW signal is slightly higher than the spatially flat  $\Lambda$ CDM

‡ In a space which is not statistically homogeneous and isotropic, this data does not measure purely the expansion rate, since the shear also contributes to the redshift (20). This was used to constrain local void models in [39].

prediction, but the difference is not statistically significant (it is evaluated as  $2\sigma$  in [28] and  $1\sigma$  in [29]). Neither the ISW signal nor the growth factor can be used at present to put accurate constraints on  $H$  as a function of redshift.

*Average observables.* The equation (38) determines the average angular diameter distance as a function of the average redshift, given the average expansion rate. The averages are taken on the hypersurface of constant proper time. However, we observe the redshift and the distance only in one fixed location. Nevertheless, for practical purposes, the averages  $\langle D_A \rangle$  and  $\langle z \rangle$  do correspond to directly observable quantities.

In order to model what could be observed in principle, we would have to know the details of the structures along each line of sight to calculate the relation between  $D_A$  and  $z$  for each direction. As noted earlier, the distance-redshift relation cannot in general be expressed as a function  $D_A(z)$ , because several values of  $D_A$  can correspond to the same redshift. In practice differences between redshifts corresponding to the same distance are likely to be smaller than the observational resolution, except for nearby sources (or the CMB, for which the redshift is very accurately measured), because the regions which are collapsing (or have strong negative shear in the direction of the null geodesic) are small. Using the redshift integral (20), a naive estimate of the blueshift due to a region one Mpc across which is collapsing with a rate of the present-day Hubble parameter is  $10^{-3}$ . As long as the variations in  $D_A$  and  $z$  are smaller than the observational errors, we can safely say that the averages correspond to the observed quantities. Observationally, the variation of the CMB peak position with direction is known to be small [103]. Note that standard CMB analysis also predicts only an ensemble average, and that in practice cosmological observations are often analysed using an average over the full sky, which by statistical homogeneity and isotropy corresponds to the average over the spatial hypersurface.

*The Dyer-Roeder approximation.* An approach where the effect of clumping on light propagation is modeled assuming that the light rays encounter only a fraction of the mass in the universe was introduced by Zel'dovich [104] and is known as the Dyer-Roeder approximation [105]. In this prescription, the FRW equation (29) for the distance is modified by multiplying the matter density by a constant  $\alpha$ , which varies between 0 and 1, corresponding to a universe where the lines of sight are completely empty or completely filled with matter, respectively. The smoothness parameter  $\alpha$  was generalised to a function of redshift in [106] to account for the evolution of structures, and change of the expansion rate due to clumping was added to the equation in [107].

According to our result (38), clumping is irrelevant for the contribution of the matter density, which is always proportional to  $(1 + \langle z \rangle)^3$ , with no extra prefactor. The reason is that mass is conserved, so if the line of sight goes through an underdense region somewhere, it must correspondingly go through overdensities elsewhere when considering distances of the homogeneity scale or larger, as discussed in connection with equations (21) and (32). The Dyer-Roeder parameter  $\alpha$  is always unity, and clumping



enters instead by changing the expansion rate. In addition, there is the null shear term; if interpreted as an effective, redshift-dependent  $\alpha$ , it would correspond to  $\alpha > 1$ , contrary to the Dyer-Roeder case. In summary, the Dyer-Roeder prescription does not correctly describe the effect of clumping.

One possible caveat is that we have not taken into account the possibility that the observer or the sources could be in a special location, or that observations could be made along special lines of sight. For example, we would expect supernovae to be preferentially located in very overdense and thus highly untypical regions. The possibility that observations might be biased towards empty lines of sight has been brought up in [107, 108].

The effect of the location of the observer is likely to be small, because the deviation from the mean is significant only over regions which are small compared to the overall distance travelled by the light, and the amplitude of the deviation is typically not correspondingly large (except in very special locations, such as near a black hole). This is another way of saying that the homogeneity scale is small. For the same reason, the location of the sources is not expected to have a large impact, though there could be a secular effect, as the degree to which the source locations are untypical could evolve with redshift. In any case, cosmological observations rely on various different sources, not only supernovae. Since different observations roughly agree, these kind of selection effects must be subdominant. This is also an argument against large effects due to special lines of sight used in observations. In particular, the CMB covers the full sky, so it is not subject to this kind of bias (apart from some uncertainty in the direction of the Galactic plane). Furthermore, since most cosmological observations (apart from nearby objects) are made over scales much larger than the homogeneity scale, the variation in the density along lines of sight should be small, and empty lines of sight should be very rare. Note that a clear line of sight is not necessarily empty, because most of the dust is dark matter, which is transparent. (See [102] and references therein for more on cosmic transparency.)

#### 4.2. Conclusion

*Summary.* It was conjectured in [21] that light propagation in a statistically homogeneous and isotropic dust universe which may contain non-linear structures can be expressed in terms of average geometrical quantities (namely the expansion rate and the spatial curvature), assuming that the observer is not in a special location, and that structures have realistically small sizes. As reviewed in [21], the literature on light propagation is mostly in agreement with this statement, but there had been no proof thus far.

We have now derived the equation for the angular diameter distance, assuming that the dust universe is statistically homogeneous and isotropic as well as rotationless, and that structures are small and their distribution evolves slowly compared to the time it takes for light to cross the homogeneity scale. It follows from these assumptions that

the distance can be expressed in terms of the average dust geometry, apart from a term related to the null geodesic shear. Of the average geometry, the only term that is required is the average expansion rate, along with the present value of the matter density. In particular, the spatial curvature enters only via its effect on the expansion rate, unlike in Friedmann-Robertson-Walker (FRW) models with arbitrary matter content. Therefore, significant spatial curvature is not necessarily inconsistent with the position of the cosmic microwave background acoustic peaks. If the expansion history and present-day matter density of a clumpy model is close to that of the spatially flat  $\Lambda$ CDM FRW model, the peak position will also be near the  $\Lambda$ CDM case, and thus consistent with observations. This is important, because accelerated expansion due to structure formation involves large negative spatial curvature [9, 21, 56]. The result also shows that the clumping is not correctly described by the Dyer-Roeder approximation, which changes the evolution of the matter density (which is in fact fixed) and misses the change of the expansion rate.

To complete the proof that the redshift and the distance can be expressed in terms of the average geometry alone, it would be necessary to show that the contribution of the null shear can be neglected. Observationally, the shear is known to be small [68], but this should be shown to follow from statistical homogeneity and isotropy (or, if this is not the case, the required additional assumptions should be identified).

Since the distance-redshift relation is determined by the average expansion rate, we should calculate the average expansion rate in a realistic model to evaluate the effect of structure formation on cosmological observations. Using the Buchert equations [6], this backreaction can be determined from purely statistical quantities (the variance of the expansion rate and the average dust shear). Evaluating these quantities in a realistic model is a challenging task, especially as the backreaction is a general relativistic effect related to spatial curvature and has no counterpart in Newtonian gravity [5, 6, 10, 21]††. A first step was taken in [21, 22], where the average expansion rate was calculated in a semi-realistic model with an evolving distribution of structures. To compare with observations of distance and the expansion rate in detail, a more rigorous treatment is needed, with well-quantified errors. After that, the calculation of fluctuations around the average should follow, in order to evaluate the Integrated Sachs-Wolfe effect and the growth of density perturbations.

Before detailed analytical results on the effect of structure formation are worked out, it is possible to make general tests. The relation between the expansion rate and the distance scale in clumpy models is different than in FRW universes. Therefore, observations which probe the expansion rate and the distance separately can be used to test the null hypothesis that the two are related by the FRW consistency condition, as proposed in [86]. Comparing observations of type Ia supernovae, baryon acoustic oscillations and the ages of passively evolving galaxies seems promising in this respect.

†† Regarding the differences between Newtonian gravity and the weak field limit of general relativity, see [47, 109–111].

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