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LIGHTLIKE HYPERSURFACES IN INDEFINITE TRANS-SASAKIAN MANIFOLDS

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Abstract

This paper deals with lightlike hypersurfaces of indefinite trans-Sasakian manifolds of type (α, β), tangent to the structure vector field. Characterization Theorems on parallel vector fields, integrable distributions, minimal distributions, Ricci-semi symmetric, geodesibility of lightlike hypersurfaces are obtained. The geometric configuration of lightlike hypersurfaces is established. We prove, under some conditions, that there are no parallel and totally contact umbilical lightlike hypersurfaces of trans-Sasakian space forms, tangent to the structure vector field. We show that there exists a totally umbilical distribution in an Einstein parallel lightlike hypersurface which does not contain the structure vector field. We characterize the normal bundle along any totally contact umbilical leaf of an integrable screen distribution. We finally prove that the geometry of any leaf of an integrable distribution is closely related to the geometry of a normal bundle and its image under $\overline{\phi}$.

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1 Introduction

The Contact and almost contact structures are two of the most interesting examples of differential geometric structures. Their theory is a natural generalization of so-called contact geometry, which has important applications in classical and quantum mechanics. Their study as differential geometric structures dates from works of Chern [6], Gray [13], and Sasaki [29]. Almost contact metric structures are an odd-dimensional analogue of almost Hermitian structures and there exist many important connections between these two classes. This paper is devoted to the geometry of the class of almost contact metric manifolds of Kählerian type known as trans-Sasakian manifolds [28]. They are interesting due to their position in the natural niche between three contact analogues of Kähler manifolds, α-Sasakian, β-Kenmotsu and cosymplectic manifolds. Trans-Sasakian manifolds were introduced by Oubina [28]. In the Gray-Hervella classification of almost Hermitian manifolds [14], there appears a class, $\mathcal{W}_4$, of Hermitian manifolds which are locally conformal Kähler structures (see [7], [18] and many more references therein). Gray and Hervella also introduced two subclasses of trans-Sasakian structures, the $\mathcal{C}_5$ and $\mathcal{C}_6$-structures, which contain, respectively, the β-Kenmotsu and α-Sasakian structures. The class $\mathcal{C}_5 \oplus \mathcal{C}_6$ coincides with the class of trans-Sasakian structures of type (α, β) and their local nature is known on connected differentiable manifolds of dimension greater than or equal to 5 [21]. The literature about the lightlike contact geometry is very limited and some specific discussions on this matter can be found in [5], [12], [23], [24], [25], [26] and [27].

As is well known, the geometry of lightlike submanifolds [10] is different because of the fact that their normal vector bundle intersects with the tangent bundle. Thus, the study becomes more difficult and strikingly different from the study of non-degenerate submanifolds. This means that one cannot use, in the usual way, the classical submanifold theory to define any induced object on a lightlike submanifold. To deal with this anomaly, the lightlike submanifolds were introduced and presented in a book by Duggal and Bejancu [10]. They introduced a non-degenerate screen distribution to construct a nonintersecting lightlike transversal vector bundle of the tangent bundle. Several authors have studied lightlike hypersurfaces of semi-Riemannian manifolds (see [12] and many more references therein).

The growing importance of lightlike geometry is motivated by its extensive use in mathematical physics, in particular in relativity. In fact, semi-Riemannian manifolds $(\overline{M}, \overline{g})$ with $\dim \overline{M} > 4$ are natural generalizations of spacetime of general relativity and lightlike hypersurfaces are models of different types of horizons separating domains of $(\overline{M}, \overline{g})$ with different physical properties. In this case, the relationship between Killing and geodesic notions is well understood. Lightlike hypersurfaces are also studied in the theory of electromagnetism [10].

There are many reasons that motivate the study of the lightlike hypersurfaces of indefinite trans-Sasakian manifolds. In [10], Duggal and Bejancu proved that a lightlike framed hypersurface of a Lorentz $\mathcal{C}$-manifold, with an induced metric connection, is a Killing horizon. Duggal and Sahin in [12] began to work on lightlike submanifolds of indefinite Sasakian manifolds because the contact geometry has a significant use in differential equations, optics and phase spaces of dynamical systems. Furthermore, Duggal [9] shows that a globally hyperbolic spacetime and the de Sitter spacetime can carry a framed structure.

The paper is organized as follows. In section 2, we recall some basic definitions for indefinite trans-Sasakian manifolds of type (α, β) supported by an example and lightlike hypersurfaces of semi-Riemannian manifolds. In section 3, after giving a decomposition of almost contact metrics of lightlike hypersurfaces in indefinite trans-Sasakian manifolds of type (α, β), tangent to the structure vector field, we study its geometric aspects. We prove that the structure vector field is η-conformal Killing on a lightlike hypersurface. Explicit formulae for Ricci tensor in a lightlike hypersurface is obtained. Theorems on parallel vector fields, geodesibility, Ricci-semi symmetric of lightlike hypersurfaces in indefinite trans-Sasakian manifolds of type (α, β) are obtained. We prove that there are no parallel lightlike hypersurfaces of indefinite trans-Sasakian space forms with constant curvature $c \neq \alpha^2 - \beta^2$. By Theorem 3.10, we establish the geometric configuration of lightlike hypersurfaces in trans-Sasakian space forms. We prove the non-existence of totally contact umbilical lightlike hypersurfaces of trans-Sasakian space forms with constant curvature $c \neq -\beta^2 - 3\alpha^2$, tangent to the structure vector field. A characterization of parallel lightlike hypersurfaces is given (Theorem 3.14). We show that there exists a totally umbilical distribution in an Einstein parallel lightlike hypersurface which does not contain the structure vector field (Theorem 3.15). In section 4, we investigate the geometry of integrable distributions. Theorems on integrable distributions, minimal distributions, Killing distributions, geodesibility of lightlike hypersurfaces and of leaves of integrable distributions $S(TM)$, $D_0 \perp \langle\xi\rangle$ and D_0 are stated. We characterize the normal bundle along any totally contact umbilical leaf of an integrable screen distribution (Theorem 4.6). By Theorem 4.10, we characterize the geometry of any leaf of an integrable distribution $D_0 \perp \langle\xi\rangle$. Finally, we discuss the effect of any change of the screen distribution on different results found.

2 Preliminaries

Let $\overline{M}$ be a $(2n+1)$-dimensional manifold endowed with an almost contact structure $(\overline{\phi}, \xi, \eta)$, i.e. $\overline{\phi}$ is a tensor field of type $(1,1)$, ξ is a vector field, and η is a 1-form satisfying

$$\overline{\phi}^2 = -\mathbb{I} + \eta \otimes \xi,\ \ \eta(\xi) = 1,\ \ \eta \circ \overline{\phi} = 0 \text{ and } \overline{\phi}\xi = 0. \tag{2.1}$$

Then $(\overline{\phi}, \xi, \eta, \overline{g})$ is called an almost contact metric structure on $\overline{M}$ if $(\overline{\phi}, \xi, \eta)$ is an almost contact structure on $\overline{M}$ and $\overline{g}$ is a semi-Riemannian metric on $\overline{M}$ such that [2],

$$\eta(\overline{X}) = \overline{g}(\xi, \overline{X}), \quad \overline{g}(\overline{\phi}\,\overline{X}, \overline{\phi}\,\overline{Y}) = \overline{g}(\overline{X}, \overline{Y}) - \eta(\overline{X})\,\eta(\overline{Y}),\ \forall\, \overline{X}, \overline{Y} \in \Gamma(\overline{M}). \tag{2.2}$$

An almost contact metric structure $(\overline{\phi}, \xi, \eta, \overline{g})$ on $\overline{M}$ is called *trans-Sasakian* if $(\overline{M} \times \mathbb{R},\ J,\ \widehat{g})$ belongs to the class $\mathcal{W}_4$, where J is the almost complex structure on $\overline{M} \times \mathbb{R}$ denoted by $J\left(\overline{X},\ f\frac{d}{dt}\right) = \left(\overline{\phi}\,\overline{X} - f\xi,\ \eta(\overline{X})\frac{d}{dt}\right)$ for all vector fields $\overline{X}$ on $\overline{M}$ and f is $\mathcal{C}^\infty$- function on $\overline{M} \times \mathbb{R}$, is integrable, which is equivalent to the condition $N_{\overline{\phi}} + 2d\eta \otimes \xi = 0$, where $N_{\overline{\phi}}$ denotes the Nijenhuis torsion of $\overline{\phi}$ (see [3], for details), and $\widehat{g}$ is the product metric on $\overline{M} \times \mathbb{R}$. This may be expressed by the condition [3]

$$(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y} = \alpha\left\{\overline{g}(\overline{X}, \overline{Y})\xi - \eta(\overline{Y})\overline{X}\right\} + \beta\left\{\overline{g}(\overline{\phi}\,\overline{X}, \overline{Y})\xi - \eta(\overline{Y})\overline{\phi}\,\overline{X}\right\}, \tag{2.3}$$

where $\overline{\nabla}$ is the Levi-Civita connection for the semi-Riemannian metric $\overline{g}$, α and β are smooth functions on $\overline{M}$, and we say that the trans-Sasakian structure is of type $(\alpha,\, \beta)$. From (2.3), it is easy to see that the following equations hold for a trans-Sasakian manifold,

$$
\begin{aligned}
d\eta &= \alpha\Phi, && (2.4)\\
\overline{\nabla}_{\overline{X}}\xi &= -\alpha\overline{\phi}\,\overline{X} + \beta\left\{\overline{X} - \eta(\overline{X})\xi\right\}, && (2.5)
\end{aligned}
$$

where Φ is the fundamental 2-form of $\overline{M}$ defined by $\Phi(\overline{X},\overline{Y}) = \overline{g}(\overline{X},\overline{\phi}\,\overline{Y})$.

If $\alpha = 0$, then $\overline{M}$ is β-Kenmotsu manifold and if $\beta = 0$ then $\overline{M}$ is α-Sasakian manifold [16]. Moreover, if $\alpha = 0$ and $\beta = 1$ then $\overline{M}$ is Kenmotsu manifold [17] and if $\alpha = 1$ and $\beta = 0$ then $\overline{M}$ is Sasakian manifold [22]. Another important kind of trans-Sasakian manifolds is that of cosymplectic manifolds, obtained for $\alpha = \beta = 0$. This is equivalent to $\overline{M}$ being normal with η and Φ closed forms [2] and this implies, using (2.5), that $\overline{\nabla}_{\overline{X}}\xi = 0$. Therefore, ξ is a Killing vector field on cosymplectic manifolds.

In this paper, the functions α and β are non-zero, unless specifically mentioned otherwise.

Let $\overline{R}$ be a curvature tensor of $\overline{\nabla}$. Then, by direct calculations one obtains

$$
\begin{aligned}
\overline{R}(\overline{X},\overline{Y})\xi &= (\alpha^2-\beta^2)\left\{\eta(\overline{Y})\overline{X} - \eta(\overline{X})\overline{Y}\right\} + 2\alpha\beta\left\{\eta(\overline{Y})\overline{\phi}\,\overline{X} - \eta(\overline{X})\overline{\phi}\,\overline{Y}\right\}\\
&+\overline{Y}(\alpha)\overline{\phi}\,\overline{X} - \overline{X}(\alpha)\overline{\phi}\,\overline{Y} + \overline{X}(\beta)\left\{\overline{Y} - \eta(\overline{Y})\xi\right\} - \overline{Y}(\beta)\left\{\overline{X} - \eta(\overline{X})\xi\right\}. \qquad (2.6)
\end{aligned}
$$

Using (2.6) in $\overline{g}(\overline{R}(\xi,\overline{Y})\overline{X},\overline{Z}) = \overline{g}(\overline{R}(\overline{X},\overline{Z})\xi,\overline{Y})$, we obtain

$$
\begin{aligned}
\overline{R}(\xi,\overline{Y})\overline{X} &= (\alpha^2-\beta^2)\left\{\overline{g}(\overline{X},\overline{Y})\xi - \eta(\overline{X})\overline{Y}\right\} + 2\alpha\beta\left\{\overline{g}(\overline{\phi}\,\overline{X},\overline{Y})\xi - \eta(\overline{X})\overline{\phi}\,\overline{Y}\right\}\\
&+\overline{X}(\alpha)\overline{\phi}\,\overline{Y} + \overline{g}(\overline{\phi}\,\overline{X},\overline{Y})\,\mathrm{grad}\alpha + \overline{X}(\beta)\left\{\overline{Y} - \eta(\overline{Y})\xi\right\} - \overline{g}(\overline{\phi}\,\overline{X},\overline{\phi}\,\overline{Y})\,\mathrm{grad}\beta. \qquad (2.7)
\end{aligned}
$$

From (2.8), in view of (2.1), we obtain $\overline{R}(\xi,\overline{Y})\xi = (\alpha^2-\beta^2-\xi(\beta))\{\eta(Y)\xi - Y\} - (2\alpha\beta + \xi(\alpha))\overline{\phi}\,\overline{Y}$, while (2.7) gives $\overline{R}(\xi,\overline{Y})\xi = (\alpha^2-\beta^2-\xi(\beta))\{\eta(Y)\xi - Y\} + (2\alpha\beta + \xi(\alpha))\overline{\phi}\,\overline{Y}$. The above two equations lead, in a trans-Sasakian manifold of type (α,β), to

$$
\begin{aligned}
&\overline{R}(\xi,\overline{X})\xi = \left(\alpha^2-\beta^2-\xi(\beta)\right)\left\{\eta(\overline{X})\xi - \overline{X}\right\} && (2.8)\\
\text{and}\quad &2\alpha\beta + \xi(\alpha) = 0. && (2.9)
\end{aligned}
$$

Note that, by (2.9) a trans-Sasakian manifold of type (α,β) with α a non-zero constant is always α-Sasakian. If $\xi(\alpha) = 0$, the trans-Sasakian structure is of class $\mathcal{C}_5$ or is of class $\mathcal{C}_6$. Therefore, in a trans-Sasakian manifold of type (α,β), the functions α and β are not arbitrary.

Next, we have an specific example of a trans-Sasakian manifold of type (α,β). Let $\overline{M}^3$ be a 3-dimensional manifold defined by $\overline{M}^3 = \left\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_3 \neq 0\right\}$. The vector fields $e_1 = e^{x_3}(\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_3})$, $e_2 = e^{x_3}\frac{\partial}{\partial x_2}$, $e_3 = \frac{\partial}{\partial x_3}$, are linearly independent at each point of $\overline{M}^3$. Let $\overline{g}$ be the semi-Riemannian metric defined by $\overline{g}(e_i,e_j) = 0,\ \forall\, i\neq j,\ i,\ j = 1,2,3$ and $\overline{g}(e_k,e_k) = -1,\ \forall\, k = 1,2$, $\overline{g}(e_3,e_3) = 1$. Let η be the 1-form defined by $\eta(\overline{X}) = \overline{g}(\overline{X},e_3)$, for any $\overline{X}\in\Gamma(T\overline{M})$. Let $\overline{\phi}$ be the $(1,1)$ tensor field defined by $\overline{\phi}e_1 = -e_2,\ \overline{\phi}e_2 = e_1,\ \overline{\phi}e_3 = 0$. Then, using the linearity of $\overline{\phi}$ and $\overline{g}$, we have $\overline{\phi}^2\overline{X} = -\overline{X} + \eta(\overline{X})e_3$, $\overline{g}(\overline{\phi}\,\overline{X}, \overline{\phi}\,\overline{Y}) = \overline{g}(\overline{X}, \overline{Y}) - \eta(\overline{X})\eta(\overline{Y})$. Thus, for $e_3 = \xi$, $(\overline{\phi},\, \xi,\, \eta,\, \overline{g})$ defines an almost contact metric structure on $\overline{M}$. Let $\overline{\nabla}$ be the Levi-Civita connection with respect to the metric $\overline{g}$. Then, we have $[e_i, e_3] = -e_i$, $\forall\, i = 1, 2$. The metric connection $\overline{\nabla}$ of the metric $\overline{g}$ is given by

$$\begin{aligned} 2\overline{g}(\overline{\nabla}_{\overline{X}}\overline{Y}, \overline{Z}) &= \overline{X}(\overline{g}(\overline{Y}, \overline{Z})) + \overline{Y}(\overline{g}(\overline{Z}, \overline{X})) - \overline{Z}(\overline{g}(\overline{X}, \overline{Y})) - \overline{g}(\overline{X}, [\overline{Y}, \overline{Z}]) \\ &\quad -\overline{g}(\overline{Y}, [\overline{X}, \overline{Z}]) + \overline{g}(\overline{Z}, [\overline{X}, \overline{Y}]), \end{aligned}$$

which is known as Koszul's formula. Using this formula, the non-vanishing covariant derivatives are given by $\overline{\nabla}_{e_1}e_1 = -\xi - x_2e^{x_3}e_1$, $\overline{\nabla}_{e_1}e_2 = -x_2e^{x_3}e_2 + \frac{1}{2}e^{2x_3}\xi$, $\overline{\nabla}_{e_1}\xi = -e_1 + \frac{1}{2}e^{2x_3}e_2$, $\overline{\nabla}_{e_2}e_1 = -\frac{1}{2}e^{2x_3}\xi$, $\overline{\nabla}_{e_2}e_2 = -\xi$, $\overline{\nabla}_{e_2}\xi = -\frac{1}{2}e^{2x_3}e_1 - e_2$, $\overline{\nabla}_{\xi}e_1 = \frac{1}{2}e^{2x_3}e_2$, $\overline{\nabla}_{\xi}e_2 = -\frac{1}{2}e^{2x_3}e_1$. From these relations, it is easy to check that $(\overline{\phi}, \xi, \eta, \overline{g})$ is a trans-Sasakian structure of type $(\frac{1}{2}e^{2x_3}, -1)$ in $\overline{M}^3$. Hence, $\overline{M}^3$ is a trans-Sasakian manifold of type $(\frac{1}{2}e^{2x_3}, -1)$. In general, in a 3-dimensional K-contact manifold with structure tensors $(\overline{\phi}, \xi, \eta, \overline{g})$ for a non-constant function f, if we define $\overline{g}' = f\overline{g} + (1 - f)\eta \otimes \eta$; then $(\overline{\phi}, \xi, \eta, \overline{g})$ is a trans-Sasakian structure of type $(1/f, (1/2)\xi(\ln f))$ [21].

A plane section σ in $T_p\overline{M}$ is called a $\overline{\phi}$-section if it is spanned by $\overline{X}$ and $\overline{\phi}\,\overline{X}$, where $\overline{X}$ is a unit tangent vector field orthogonal to ξ. Since $\overline{\phi}\sigma = \sigma$, the $\overline{\phi}$-section σ is a holomorphic $\overline{\phi}$-section and the sectional curvature of a $\overline{\phi}$-section σ is called a $\overline{\phi}$-holomorphic sectional curvature. If a trans-Sasakian manifold of type (α, β), $\overline{M}$, has constant $\overline{\phi}$-holomorphic sectional curvature c, then, by virtue of the Theorem 2.3 in [4], the curvature tensor $\overline{R}$ of $\overline{M}$ satisfies,

$$\begin{aligned} \overline{g}(\overline{R}(\overline{X}, \overline{Y})\overline{Z}, \overline{W}) &= \frac{c + 3(\alpha^2 - \beta^2)}{4}\left\{\overline{g}(\overline{X}, \overline{W})\overline{g}(\overline{Y}, \overline{Z}) - \overline{g}(\overline{X}, \overline{Z})\overline{g}(\overline{Y}, \overline{W})\right\} \\ &\quad + \frac{c - \alpha^2 + \beta^2}{4}\left\{\overline{g}(\overline{X}, \overline{\phi}\,\overline{W})\overline{g}(\overline{Y}, \overline{\phi}\,\overline{Z}) - \overline{g}(\overline{X}, \overline{\phi}\,\overline{Z})\overline{g}(\overline{Y}, \overline{\phi}\,\overline{W})\right. \\ &\quad \left. -2\overline{g}(\overline{X}, \overline{\phi}\,\overline{Y})\overline{g}(\overline{Z}, \overline{\phi}\,\overline{W})\right\}, \quad \forall\, \overline{X},\, \overline{Y},\, \overline{Z},\, \overline{W} \in \Gamma(\ker\eta). \end{aligned} \tag{2.10}$$

A trans-Sasakian manifold $\overline{M}$ of type (α, β) of constant $\overline{\phi}$-holomorphic sectional curvature c will be called *trans-Sasakian space form* and denoted by $\overline{M}(c)$.

Let $(\overline{M}, \overline{g})$ be a $(2n+1)$-dimensional semi-Riemannian manifold with index s, $0 < s < 2n+1$ and let (M, g) be a hypersurface of $\overline{M}$, with $g = \overline{g}_{|M}$. M is a lightlike hypersurface of $\overline{M}$ if the metric g is of constant rank $2n-1$ and the orthogonal complement $TM^\perp$ of tangent space TM, defined as

$$TM^\perp = \bigcup_{p \in M}\left\{Y_p \in T_p\overline{M} : g_p(X_p, Y_p) = 0,\ \forall X_p \in T_pM\right\}, \tag{2.11}$$

is a distribution of rank 1 on M [10]: $TM^\perp \subset TM$ and then coincides with the radical distribution $\mathrm{Rad}\,TM = TM \cap TM^\perp$. A complementary bundle of $TM^\perp$ in TM is a constant rank $2n-1$ non-degenerate distribution over M. It is called a *screen distribution* and denoted by $S(TM)$. The existence of $S(TM)$ is secured provided M be paracompact. However, in general, $S(TM)$ is not canonical (thus it is not unique) and the lightlike geometry depends on its choice but it is canonically isomorphic to the vector bundle $TM/\mathrm{Rad}\,TM$ [20].

A lightlike hypersurface endowed with a specific screen distribution is denoted by the triple $(M, g, S(TM))$. As $TM^\perp$ lies in the tangent bundle, the following result has an important role in studying the geometry of a lightlike hypersurface [10].

Theorem 2.1 [10] *Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\overline{M}, \overline{g})$. Then, there exists a unique vector bundle $N(TM)$ of rank 1 over M such that for any non-zero section E of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique section N of $N(TM)$ on $\mathcal{U}$ satisfying $\overline{g}(N, E) = 1$ and $\overline{g}(N, N) = \overline{g}(N, W) = 0$, $\forall W \in \Gamma(S(TM)|_{\mathcal{U}})$.*

Throughout the paper, all manifolds are supposed to be paracompact and smooth. We denote by $\Gamma(\mathrm{E})$ the smooth sections of the vector bundle E. Also by $\perp$ and $\oplus$ we denote the orthogonal and nonorthogonal direct sum of two vector bundles. By Theorem 2.1 we may write down the following decompositions

$$\begin{aligned} TM &= S(TM) \perp TM^{\perp}, \\ T\overline{M} &= TM \oplus N(TM) = S(TM) \perp (TM^{\perp} \oplus N(TM)). \end{aligned} \tag{2.12}$$

Let $\overline{\nabla}$ be the Levi-Civita connection on $(\overline{M}, \overline{g})$, then, using the second decomposition of (2.12) and considering a normalizing pair $\{E, N\}$ as in Theorem 2.1, we have Gauss and Weingarten formulae in the form, for any $X, Y \in \Gamma(TM|_{\mathcal{U}})$, $V \in \Gamma(N(TM))$,

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \text{ and } \overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{2.13}$$

where $\nabla_X Y$, $A_V X \in \Gamma(TM)$ and $h(X, Y)$, $\nabla^{\perp}_X V \in \Gamma(N(TM))$. ∇ is an induced a symmetric linear connection on M, $\nabla^{\perp}$ is a linear connection on the vector bundle $N(TM)$, h is a $\Gamma(N(TM))$-valued symmetric bilinear form and A_V is the shape operator of M concerning V. Equivalently, consider a normalizing pair $\{E, N\}$ as in Theorem 2.1. Then (2.13) takes the form, for any $X, Y \in \Gamma(TM|_{\mathcal{U}})$,

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)\, N \text{ and } \overline{\nabla}_X N = -A_N X + \tau(X) N, \tag{2.14}$$

where B, A_N, τ and ∇ are called the local second fundamental form, the local shape operator, the transversal differential 1-form and the induced linear torsion free connection, respectively, on $TM_{|\mathcal{U}}$. It is important to mention that B is independent of the choice of screen distribution, in fact, from (2.14), we obtain, $B(X, Y) = \overline{g}(\overline{\nabla}_X Y, E)$ and $\tau(X) = \overline{g}(\nabla^{\perp}_X N, E)$.

Let P be the projection morphism of TM on $S(TM)$ with respect to the orthogonal decomposition of TM. We have, for any $X, Y \in \Gamma(TM|_{\mathcal{U}})$,

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)E \text{ and } \nabla_X E = -A^*_E X - \tau(X)E, \tag{2.15}$$

where $\nabla^*_X PY$ and $A^*_E X$ belong to $\Gamma(S(TM))$. C, A^*_E and ∇^* are called the local second fundamental form, the local shape operator and the induced connection on $S(TM)$. The induced linear connection ∇ is not a metric connection and we have, for any $X, Y \in \Gamma(TM|_{\mathcal{U}})$,

$$(\nabla_X g)(Y, Z) = B(X, Y)\theta(Z) + B(X, Z)\theta(Y), \tag{2.16}$$

where θ is a differential 1-form locally defined on M by $\theta(\cdot) := \overline{g}(N, \cdot)$. Also, we have, $g(A^*_E X, PY) = B(X, PY)$, $g(A_N X, PY) = C(X, PY)$ and $B(X, E) = 0$. Using (2.14), the curvature tensor fields $\overline{R}$

and R of $\overline{M}$ and M, respectively, are related as

$$\overline{R}(X,Y)Z = R(X,Y)Z + B(X,Z)A_N Y - B(Y,Z)A_N X$$
$$+ \ \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z)\}\, N, \qquad (2.17)$$
$$\text{where } \ (\nabla_X B)(Y,Z) = X.B(Y,Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z). \qquad (2.18)$$

3 Lightlike hypersurfaces of indefinite trans-Sasakian manifolds

Let (M,g) be a lightlike hypersurface of an indefinite trans-Sasakian manifold $(\overline{M},\overline{\phi},\,\xi,\eta,\overline{g})$ of type (α,β), *tangent to the structure vector field* ξ *(*$\xi \in TM$*)*. If E is a local section of $TM^{\perp}$, then $\overline{g}(\overline{\phi}E,E) = 0$, and $\overline{\phi}E$ is tangent to M. Thus $\overline{\phi}(TM^{\perp})$ is a distribution on M of rank 1 such that $\overline{\phi}(TM^{\perp}) \cap TM^{\perp} = \{0\}$. This enables us to choose a screen distribution $S(TM)$ such that it contains $\overline{\phi}(TM^{\perp})$ as a vector subbundle. If we consider a local section N of $N(TM)$, since $\overline{g}(\overline{\phi}\,N,E) = -\overline{g}(N,\overline{\phi}\,E) = 0$, we deduce that $\overline{\phi}\,E$ belongs to $S(TM)$. At the same time, since $\overline{g}(\overline{\phi}\,N,N) = 0$, $\overline{\phi}\,N \in \Gamma(S(TM))$. From (2.1), we have $\overline{g}(\overline{\phi}\,N,\overline{\phi}E) = 1$. Therefore, $\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))$ is a non-degenerate vector subbundle of $S(TM)$ of rank 2. If M is tangent to the structure vector field ξ, then, we may choose $S(TM)$ so that ξ belongs to $S(TM)$. Using this, and since $\overline{g}(\overline{\phi}\,\cdot,\xi) = 0$, there exists a non-degenerate distribution D_0 of rank $2n-4$ on M such that

$$S(TM) = \left\{\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))\right\} \perp D_0 \perp <\xi>, \qquad (3.1)$$

where $\langle\xi\rangle = Span\{\xi\}$. The distribution D_0 is invariant under $\overline{\phi}$. Moreover, from (2.12) and (3.1) we obtain the decompositions

$$TM \;=\; \left\{\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))\right\} \perp D_0 \perp <\xi> \perp TM^{\perp}, \qquad (3.2)$$
$$T\overline{M} \;=\; \left\{\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))\right\} \perp D_0 \perp <\xi> \perp (TM^{\perp} \oplus N(TM)). \qquad (3.3)$$

Note that a hypersurface of a 3-dimensional indefinite trans-Sasakian manifold of type (α,β), tangent to the structure vector field ξ is of dimension 1 and its tangent space is reduced to the distribution spanned by ξ which is non-degenerate. This means that the dimension 3 is too low to develop the theory and this agrees with the decomposition (3.3) which requires $n \geq 2$.

Example 3.1 We consider the 7-dimensional manifold $\overline{M}^7 = \left\{x \in \mathbb{R}^7 : x_7 \neq 0\right\}$, where $x = (x_1, x_2, ..., x_7)$ are the standard coordinates in $\mathbb{R}^7$. Let us consider the vector fields $e_1, e_2, ..., e_7$, linearly independent at each point of $\overline{M}^7$, as a combination of frames $\left\{\frac{\partial}{\partial x_i}\right\}$. Let $\overline{g}$ be the semi-Riemannian metric defined by $\overline{g}(e_i,e_j) = 0$, $\forall\, i \neq j$, $i,j = 1,2,...,7$ and $\overline{g}(e_k,e_k) = 1$, $\forall\, k = 1,2,3,4,7$, $\overline{g}(e_m,e_m) = -1$, $\forall\, m = 5,6$. Let η be the 1-form defined by $\eta(\cdot) = \overline{g}(\cdot,e_7)$. Let $\overline{\phi}$ be the $(1,1)$ tensor field defined by $\overline{\phi}e_1 = -e_2$, $\overline{\phi}e_2 = e_1$, $\overline{\phi}e_3 = -e_4$, $\overline{\phi}e_4 = e_3$, $\overline{\phi}e_5 = -e_6$, $\overline{\phi}e_6 = e_5$, $\overline{\phi}e_7 = 0$. Using the linearity of $\overline{\phi}$ and $\overline{g}$, we have $\overline{\phi}^2\overline{X} = -\overline{X} + \eta(\overline{X})e_7$, $\overline{g}(\overline{\phi}\,\overline{X},\overline{\phi}\,\overline{Y}) = \overline{g}(\overline{X},\overline{Y}) - \eta(\overline{X})\eta(\overline{Y})$. Thus, for $e_7 = \xi$, $(\overline{\phi},\,\xi,\,\eta,\,\overline{g})$ defines an almost contact metric structure on $\overline{M}$. Let $\overline{\nabla}$ be the Levi-Civita connection with respect to the metric $\overline{g}$ and let us choose the vector fields $e_1, e_2, ..., e_7$ to be

$$e_i = e^{x_7} \sum_{j=1}^{7} f_{ij}(x_1, ..., x_6) \frac{\partial}{\partial x_j}, \ \ \det(f_{ij}) \neq 0 \text{ and } e_7 = \xi,$$

where functions f_{ij} are defined such that the action of $\overline{\nabla}$, on the basis $\{e_1, e_2, ..., e_7\}$, is given by

$$\begin{aligned}
&\overline{\nabla}_{e_1} e_1 = \xi, \ \overline{\nabla}_{e_1} e_2 = -\frac{1}{2} e^{2x_7} \xi, \ \overline{\nabla}_{e_2} e_1 = -x_2 e^{x_7} e_2 + \frac{1}{2} e^{2x_7} \xi, \ \overline{\nabla}_{e_2} e_2 = x_2 e^{x_7} e_1 + \xi, \\
&\overline{\nabla}_{e_3} e_4 = x_3 e^{x_7} e_3 - \frac{1}{2} e^{2x_7} \xi, \ \overline{\nabla}_{e_3} e_3 = -x_3 e^{x_7} e_4 + \xi, \ \overline{\nabla}_{e_4} e_3 = \frac{1}{2} e^{2x_7} \xi, \overline{\nabla}_{e_4} e_4 = \xi, \\
&\overline{\nabla}_{e_5} e_5 = -\xi, \ \overline{\nabla}_{e_5} e_6 = \frac{1}{2} e^{2x_7} \xi, \ \overline{\nabla}_{e_6} e_5 = x_6 e^{x_7} e_6 - \frac{1}{2} e^{2x_7} \xi, \ \overline{\nabla}_{e_6} e_6 = -x_6 e^{x_7} e_5 - \xi, \\
&\overline{\nabla}_{e_i} e_j = 0, \ \ \forall\, i \neq j, \ i, j = 1, 2, 3, ..., 6, \ \text{such that} \ \ \overline{g}(\overline{\phi} e_i, e_j) = 0.
\end{aligned}$$

The non-vanishing brackets are given by, for $i = 1, 2, 3, ..., 6$, $[e_i, e_7] = -e_i$ and $[e_1, e_2] = x_2 e^{x_7} e_2 - e^{2x_7} \xi$, $[e_3, e_4] = x_3 e^{x_7} e_3 - e^{2x_7} \xi$, $[e_5, e_6] = -x_6 e^{x_7} e_6 + e^{2x_7} \xi$. The m^{th}-component of the Lie brackets $[e_i, e_j]$ is given by, for $i, j = 1, 2, ..., 6$,

$$[e_i, e_j]_m = e^{2x_7} \sum_{k=1}^{6} (f_{ik} \frac{\partial}{\partial x_k}(f_{jm}) - f_{jk} \frac{\partial}{\partial x_k}(f_{im})) + e^{2x_7}(f_{i7} f_{jm} - f_{j7} f_{im}).$$

By Koszul's formula, we have $\overline{\nabla}_{e_1} e_7 = -e_1 + \frac{1}{2} e^{2x_7} e_2$, $\overline{\nabla}_{e_2} e_7 = -\frac{1}{2} e^{2x_7} e_1 - e_2$, $\overline{\nabla}_{e_3} e_7 = -e_3 + \frac{1}{2} e^{2x_7} e_4$, $\overline{\nabla}_{e_4} e_7 = -\frac{1}{2} e^{2x_7} e_3 - e_4$, $\overline{\nabla}_{e_5} e_7 = -e_5 + \frac{1}{2} e^{2x_7} e_6$, $\overline{\nabla}_{e_6} e_7 = -\frac{1}{2} e^{2x_7} e_5 - e_6$. From these relations, it is easy to see that $(\overline{\phi}, \xi, \eta, \overline{g})$ is a trans-Sasakian structure of type $(\frac{1}{2} e^{2x_7}, -1)$ in $\overline{M}^7$. Therefore, $(\overline{M}^7, \overline{\phi}, \xi, \eta, \overline{g})$ is a trans-Sasakian manifold of type $(\frac{1}{2} e^{2x_7}, -1)$. Let M be a hypersurface of $(\overline{M}^7, \overline{\phi}, \xi, \eta, \overline{g})$ defined by $M = \{x \in \overline{M}^7 : x_5 = x_4, \ f_{4i} = f_{5j} = 0, f_{44} \neq 0 \text{ and } f_{55} \neq 0\}$. Thus, the tangent space TM is spanned by $\{U_i\}$, where $U_1 = e_1$, $U_2 = e_2$, $U_3 = e_3$, $U_4 = e_4 - e_5$, $U_5 = e_6$, $U_6 = \xi$ and the 1-dimensional distribution $TM^\perp$ of rank 1 is spanned by E, where $E = e_4 - e_5$. It follows that $TM^\perp \subset TM$. Then M is a 6-dimensional lightlike hypersurface of $\overline{M}^7$. Also, the transversal bundle $N(TM)$ is spanned by $N = \frac{1}{2}(e_4 + e_5)$. Using the almost contact structure of $\overline{M}^7$ and the decomposition (3.1), D_0 is spanned by $\{F, \overline{\phi} F\}$, where $F = U_1$, $\overline{\phi} F = -U_2$ and the distributions $\langle \xi \rangle$, $\overline{\phi}(TM^\perp)$ and $\overline{\phi}(N(TM))$ are spanned, respectively, by ξ, $\overline{\phi} E = U_3 + U_5$ and $\overline{\phi} N = \frac{1}{2}(U_3 - U_5)$. Hence, M is a lightlike hypersurface of $\overline{M}^7$.

Now, we consider the distributions on M, $D := TM^\perp \perp \overline{\phi}(TM^\perp) \perp D_0$, $D' := \overline{\phi}(N(TM))$. Then, D is invariant under $\overline{\phi}$ and

$$TM = D \oplus D' \perp \langle \xi \rangle. \tag{3.4}$$

Let us consider the local lightlike vector fields $U := -\overline{\phi} N$, $\ \ V := -\overline{\phi} E$. Then, from (3.4), any $X \in \Gamma(TM)$ is written as $\ X = RX + QX + \eta(X)\xi$, $\ \ QX = u(X) U$, where R and Q are the projection morphisms of TM into D and D', respectively, and u is a differential 1-form locally defined on M by

$$u(\cdot) := g(V, \cdot). \tag{3.5}$$

Applying $\overline{\phi}$ to X and (2.1), one obtains $\overline{\phi}\,X = \phi\,X + u(X)\,N$, where ϕ is a tensor field of type $(1,1)$ defined on M by $\phi\,X := \overline{\phi}\,RX$. Also, we obtain, for any $X \in \Gamma(TM)$,

$$\begin{aligned} B(X,\xi) &= -\alpha u(X), && (3.6)\\ B(X,U) &= C(X,V), && (3.7)\\ C(X,\xi) &= -\alpha v(X) + \beta\theta(X), && (3.8)\\ \phi^2\,X &= -X + \eta(X)\xi + u(X)U, && (3.9)\\ \text{and} \quad \nabla_X\xi &= -\alpha\phi X + \beta\left\{X - \eta(X)\xi\right\}. && (3.10)\end{aligned}$$

By using (2.1) we derive $g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) - u(Y)v(X) - u(X)v(Y)$, where v is a differential 1-form locally defined on M by $v(\cdot) = g(U,\cdot)$. We note that

$$g(\phi X, Y) + g(X, \phi Y) = -u(X)\theta(Y) - u(Y)\theta(X). \qquad (3.11)$$

For the sake of future use, we have the following identities: for any X, $Y \in \Gamma(TM)$,

$$\begin{aligned} (\nabla_X u)Y &= -B(X,\phi Y) - \tau(X)u(Y) - \beta u(X)\eta(Y), && (3.12)\\ (\nabla_X v)Y &= -C(X,\phi Y) + \tau(X)v(Y) - \alpha\eta(Y)\theta(X) - \beta\eta(Y)v(X), && (3.13)\\ (\nabla_X\phi)Y &= \alpha\left\{g(X,Y)\xi - \eta(Y)X\right\} + \beta\left\{\overline{g}(\overline{\phi}X,Y)\xi - \eta(Y)\phi X\right\} && \\ &\quad -B(X,Y)U + u(Y)A_N X, && (3.14)\\ R(X,Y)\xi &= (\alpha^2 - \beta^2)\left\{\eta(Y)X - \eta(X)Y\right\} + 2\alpha\beta\left\{\eta(Y)\phi X - \eta(X)\phi Y\right\} && \\ &\quad +(Y.\alpha)\phi X - (X.\alpha)\phi Y + (X.\beta)\left\{Y - \eta(Y)\xi\right\} - (Y.\beta)\left\{X - \eta(X)\xi\right\} && \\ &\quad +\alpha\left\{u(X)A_N Y - u(Y)A_N X\right\}, && (3.15)\\ R(\xi,X)\xi &= (\alpha^2 - \beta^2 - \xi.\beta)\left\{\eta(X)\xi - X\right\} - \alpha\,u(X)A_N\xi. && (3.16)\end{aligned}$$

The Lie derivative of g with respect to the vector field V is given by, for any X, $Y \in \Gamma(TM)$,

$$(L_V g)(X,Y) = X(u(Y)) + Y(u(X)) + u([X,Y]) - 2u(\nabla_X Y). \qquad (3.17)$$

The relation (3.17) can be written in terms of B and τ using the relation (3.12) and we have

$$\begin{aligned} (L_V g)(X,Y) &= -\beta\left\{\eta(X)u(Y) + \eta(Y)u(X)\right\} - \left\{B(\phi X,Y) + B(X,\phi Y)\right\} && \\ &\quad - \left\{\tau(X)u(Y) + \tau(Y)u(X)\right\} && (3.18)\end{aligned}$$

As the geometry of a lightlike hypersurface depends on the chosen screen distribution, it is important to investigate the relationship between geometric objects induced by two screen distributions. The Lie derivative L_V (3.17) is not independent of the choice of a screen distribution $S(TM)$ and this is proven as follows. Suppose a screen $S(TM)$ changes to another screen $S(TM)'$. The following are the transformation equations due to this change (see [10], page 87)

$$K_i' = \sum_{j=1}^{2n-1} K_i^j (K_j - \epsilon_j c_j E),$$

$$
\begin{aligned}
N' &= N - \frac{1}{2}\{\sum_{i=1}^{2n-1} \epsilon_i (c_i)^2\}E + K, \\
\tau'(X) &= \tau(X) + B(X, K), \\
\nabla'_X Y &= \nabla_X Y + B(X, Y)\{\frac{1}{2}(\sum_{i=1}^{2n-1} \epsilon_i (c_i)^2)E - K\},
\end{aligned} \tag{3.19}
$$

where $K = \sum_{i=1}^{2n-1} c_i K_i$, $\{K_i\}$ and $\{K'_i\}$ are the local orthonormal bases of $S(TM)$ and $S(TM)'$ with respective transversal sections N and N' for the same null section E. Here c_i and K_i^j are smooth functions on $\mathcal{U}$ and $\{\epsilon_1, ..., \epsilon_{2n-1}\}$ is the signature of the basis $\{K_1, ..., K_{2n-1}\}$. The Lie derivatives L_V and L'_V of the screen distributions $S(TM)$ and $S(TM)'$, respectively, are related through the relation [22]:

$$
(L'_V g)(X, Y) = (L_V g)(X, Y) - u(X)B(Y, K) - u(Y)B(X, K).
$$

The Lie derivative L_V is unique, that is, L_V is independent of $S(TM)$, if and only if, the second fundamental form h of M vanishes identically on M.

A section $X \in \Gamma(TM)$ is said to be an η-conformal Killing vector field if

$$
L_X g = \Omega(g - \eta \otimes \eta), \tag{3.20}
$$

where Ω is a smooth function on $\mathcal{U} \subset M$.

Lemma 3.2 *Let M be a lightlike hypersurface of an indefinite trans-Sasakian manifold $\overline{M}$ of type (α, β) with $\xi \in TM$. Then ξ is an η-conformal Killing vector field on M, that is, $L_\xi g = \Omega(g - \eta \otimes \eta)$, with $\Omega = 2\beta$.*

Proof: The proof follows by direct calculation. □

Let M be a lightlike hypersurface of an indefinite trans-Sasakian manifold $\overline{M}$ of type (α, β) with $\xi \in TM$. Let us consider the $\{E, N\}$ on $\mathcal{U} \subset M$ (Theorem 2.1) and using (2.17), we obtain,

$$
(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = \tau(Y)B(X, Z) - \tau(X)B(Y, Z) + \overline{g}(\overline{R}(X, Y)Z, E). \tag{3.21}
$$

Lemma 3.3 *Let M be a lightlike hypersurface of an indefinite trans-Sasakian manifold $\overline{M}$ of type (α, β) with $\xi \in TM$. Then the Lie derivative of the second fundamental form B with respect to ξ is given by, for any X, $Y \in \Gamma(TM)$,*

$$
\begin{aligned}
(L_\xi B)(X, Y) &= Y(\alpha)u(X) - X(\alpha)u(Y) + 4\alpha\beta\eta(Y)u(X) + (\beta - \tau(\xi))B(X, Y) \\
&\quad + E(\alpha)\overline{g}(\overline{\phi}Y, X) - E(\beta)g(\overline{\phi}X, \overline{\phi}Y).
\end{aligned} \tag{3.22}
$$

Proof: Using (2.18) and (3.10), we obtain, for any X, $Y \in \Gamma(TM)$,

$$
\begin{aligned}
&(\nabla_\xi B)(X, Y) = (L_\xi B)(X, Y) + \alpha\{B(\phi X, Y) + B(X, \phi Y)\} - 2\beta B(X, Y) \\
&-\alpha\beta\{\eta(X)u(Y) + \eta(Y)u(X)\}.
\end{aligned} \tag{3.23}
$$

Likewise, using (2.18), (3.6), (3.10) and (3.12), we have

$$(\nabla_X B)(\xi, Y) = -(X.\alpha)u(Y) + \alpha\{B(\phi X, Y) + B(X, \phi Y)\} - \beta B(X, Y)$$
$$+\alpha\tau(X)u(Y) + \alpha\beta\{u(X)\eta(Y) - \eta(X)u(Y)\}. \tag{3.24}$$

Subtracting (3.23) and (3.24), we obtain

$$(\nabla_\xi B)(X, Y) - (\nabla_X B)(\xi, Y) = (L_\xi B)(X, Y) - \beta B(X, Y)$$
$$-2\alpha\beta\eta(Y)u(X) + X(\alpha)u(Y) - \alpha\tau(X)u(Y). \tag{3.25}$$

From (3.21), and after calculations, the left-hand side of (3.25) becomes

$$(\nabla_\xi B)(X, Y) - (\nabla_X B)(\xi, Y) = \tau(X)B(\xi, Y) - \tau(\xi)B(X, Y) + \overline{g}(\overline{R}(Y, E)\xi, X)$$
$$= -\alpha\tau(X)u(Y) - \tau(\xi)B(X, Y) + E(\alpha)\overline{g}(\overline{\phi}Y, X) + Y(\alpha)u(X) + 2\alpha\beta\eta(Y)u(X)$$
$$-E(\beta)\{g(X, Y) - \eta(X)\eta(Y)\}. \tag{3.26}$$

The expressions (3.25) and (3.26) imply (3.22). □

A submanifold M is said to be parallel if its second fundamental form $h = B \otimes N$ satisfies $\nabla h = 0$, that is, for any $X, Y, Z \in \Gamma(TM)$,

$$(\nabla_X h)(Y, Z) = 0. \tag{3.27}$$

That is, $(\nabla_X B)(Y, Z) = -\tau(X)B(Y, Z)$. This means that, in general, the parallelism of h does not imply the parallelism of B and vice versa.

Lemma 3.4 *There exist no lightlike hypersurfaces of indefinite trans-Sasakian space forms $(\overline{M}(c), c \neq \alpha^2 - \beta^2)$ with $\xi \in TM$ and parallel second fundamental form.*

Proof: Suppose $c \neq \alpha^2 - \beta^2$ and h is parallel. Then, putting $X = U$, $Y = E$ and $Z = U$ into (3.21) and using (2.10), one obtains $0 = \overline{g}(\overline{R}(U, E)U, E) = -\frac{3}{4}(c - \alpha^2 + \beta^2)$ and we have $c = \alpha^2 - \beta^2$, which is a contradiction. □

Theorem 3.5 *Let M be a parallel lightlike hypersurface of an indefinite trans-Sasakian manifold $\overline{M}$ of type (α, β) with $\xi \in TM$. Then, for any X, $Y \in \Gamma(TM)$,*

$$(L_V g)(X, Y) = -\frac{\beta}{\alpha}B(X, Y) - \{\tau(X)u(Y) + \tau(Y)u(X)\}. \tag{3.28}$$

Moreover, $\overline{\phi}(TM^\perp)$ is a Killing distribution if and only if the local second fundamental form B is proportional to $\tau \otimes u + u \otimes \tau$.

Proof: Using (2.18), (3.6), (3.10), (3.12) and (3.22),

$$-\tau(\xi)B(X, Y) = (\nabla_\xi B)(X, Y) = (Y.\alpha)u(X) - (X.\alpha)u(Y) - (\beta + \tau(\xi))B(X, Y)$$
$$+(E.\alpha)\overline{g}(\overline{\phi}Y, X) - (E.\beta)\overline{g}(\overline{\phi}X, \overline{\phi}Y) + 2\alpha\beta\{\eta(Y)u(X) - \eta(X)u(Y)\}$$
$$-\alpha(L_V g)(X, Y) - \alpha\{\tau(X)u(Y) + \tau(Y)u(X)\}. \tag{3.29}$$

Likewise, using (3.18), we have

$$\begin{aligned} \alpha\tau(Y)u(X) &= (\nabla_Y B)(\xi, X) = -Y(\alpha)u(X) - 2\alpha\beta\eta(Y)u(X) - \alpha(L_V g)(X,Y) \\ &\quad -\beta B(X,Y) - \alpha\tau(X)u(Y), \qquad (3.30) \\ \text{and } \alpha\tau(X)u(Y) &= (\nabla_X B)(Y,\xi) = -X(\alpha)u(Y) - 2\alpha\beta\eta(X)u(Y) - \alpha(L_V g)(X,Y) \\ &\quad -\beta B(X,Y) - \alpha\tau(Y)u(X). \qquad (3.31) \end{aligned}$$

Subtracting (3.30) and (3.31), we have

$$Y(\alpha)u(X) - X(\alpha)u(Y) + 2\alpha\beta\left\{\eta(Y)u(X) - \eta(X)u(Y)\right\} = 0. \qquad (3.32)$$

Substituting (3.32) into (3.29), we obtain

$$\begin{aligned} -\tau(\xi)B(X,Y) &= -(\beta + \tau(\xi))B(X,Y) + E(\alpha)\overline{g}(\overline{\phi}Y, X) - E(\beta)g(\overline{\phi}X, \overline{\phi}Y) \\ &\quad -\alpha(L_V g)(X,Y) - \alpha\left\{\tau(X)u(Y) + \tau(Y)u(X)\right\}. \qquad (3.33) \end{aligned}$$

If $\alpha \neq 0$, the expression (3.33) leads to

$$\begin{aligned} (L_V g)(X,Y) &= -\frac{\beta}{\alpha}B(X,Y) + E(\ln|\alpha|)\overline{g}(\overline{\phi}Y, X) - \frac{1}{\alpha}E(\beta)g(\overline{\phi}X, \overline{\phi}Y) \\ &\quad - \left\{\tau(X)u(Y) + \tau(Y)u(X)\right\}. \qquad (3.34) \end{aligned}$$

Since $-E(\beta) = \overline{g}(\overline{R}(\xi, V)U, E) = \overline{g}((\nabla_\xi h)(V,U), E) = 0$ and $E(\alpha) = \overline{g}(\overline{R}(U,E)\xi, E) = \overline{g}((\nabla_U h)(E,\xi), E) = 0$. Putting these relations into (3.34), we have the relation (3.28). The last assertion is obvious. □

Let M be a lightlike hypersurface of an indefinite trans-Sasakian space form $\overline{M}(c)$ with $\xi \in TM$. By definition $Ric(X,Y) = \mathrm{trace}(Z \longrightarrow R(X,Y)Z)$, we have, for any $X, Y \in \Gamma(TM)$,

$$\begin{aligned} Ric(X,Y) &= \sum_{i=1}^{2n-4} \varepsilon_i \overline{g}(R(F_i, X)Y, F_i) + \overline{g}(R(\xi, X)Y, \xi) + \overline{g}(R(E,X)Y, N) \\ &+ \overline{g}(R(\overline{\phi}E, X)Y, \overline{\phi}N) + \overline{g}(R(\overline{\phi}N, X)Y, \overline{\phi}E), \qquad (3.35) \end{aligned}$$

where $\{F_i\}_{1\leq i \leq 2n-4}$ is an orthogonal basis of D_0 and $\varepsilon_i = g(F_i, F_i) \neq 0$, since D_0 is non-degenerate. Using (2.6), (2.10), (2.17) and (3.35), one obtains,

$$\begin{aligned} Ric(\xi,\xi) &= (2n-1)\left\{\alpha^2 - \beta^2 - \xi(\beta)\right\} - B(A_N\xi, \xi), \qquad (3.36) \\ Ric(X,\xi) &= (2n-1)(\alpha^2 - \beta^2)\eta(X) - 2(n-1)X(\beta) - \overline{\phi}X(\alpha) - \xi(\beta)\eta(X) \\ &\quad -B(A_N X, \xi) - \alpha u(X) trA_N, \ \forall\, X \in \Gamma(TM), \qquad (3.37) \\ Ric(\xi,X) &= (2n-1)(\alpha^2 - \beta^2)\eta(X) - 2(n-1)X(\beta) - \overline{\phi}X(\alpha) - \xi(\beta)\eta(X) \\ &\quad -B(A_N\xi, X) - \alpha u(X) trA_N, \ \forall\, X \in \Gamma(TM), \qquad (3.38) \\ Ric(Y,Z) &= \{\frac{(2n+1)c}{4} + \frac{3n-5}{2}(\alpha^2 - \beta^2) + \xi(\beta)\}g(Y,Z) - B(A_N Y, Z) \\ &\quad +B(Y,Z)trA_N, \ \forall\, Y,\, Z \in \Gamma(\ker\eta). \qquad (3.39) \end{aligned}$$

Proposition 3.6 *Let M be a lightlike hypersurface of an indefinite trans-Sasakian space form $\overline{M}(c)$ with $\xi \in TM$. Then the Ricci tensor Ric is given by, for any X, $Y \in \Gamma(TM)$,*

$$\begin{aligned} Ric(X,Y) &= \{\frac{(2n+1)c}{4} + \frac{3n-5}{2}(\alpha^2-\beta^2) + \xi(\beta)\}g(X,Y) \\ &-\{\frac{(2n+1)c}{4} - \frac{n+3}{2}(\alpha^2-\beta^2) - 2(n-2)\xi(\beta)\}\eta(X)\eta(Y) \\ &-2(n-1)\{\eta(Y)X(\beta) + \eta(X)Y(\beta)\} - \eta(Y)\overline{\phi}X(\alpha) - \eta(X)\overline{\phi}Y(\alpha) \\ &-B(A_N X, Y) + B(X,Y)trA_N, \end{aligned} \tag{3.40}$$

where trace tr is written with respect to g restricted to $S(TM)$.

Proof: The proof follows by direct calculation using (3.36), (3.37), (3.38) and (3.39). □

From (3.40), we have

$$Ric(X,Y) - Ric(Y,X) = B(A_N X, Y) - B(A_N Y, X). \tag{3.41}$$

This means that the Ricci tensor of a lightlike hypersurface M of an indefinite trans-Sasakian space form $\overline{M}(c)$ is not symmetric in general. So, only some privileged conditions on B may enable the Ricci tensor to be symmetric. It is easy to check that the Ricci tensor (3.40) is symmetric if and only if the shape operator A_N is symmetric with respect to B [15]. Also, the Ricci tensor (3.40) of any totally geodesic lightlike hypersurface is symmetric.

Next, we study Ricci-semi Symmetric lightlike hypersurfaces of indefinite trans-Sasakian space forms, tangent to the structure vector field ξ.

A lightlike hypersurface M of an indefinite trans-Sasakian space form $\overline{M}(c)$ with $\xi \in TM$ is said to be Ricci semi-symmetric if the following condition is satisfied [8],

$$(R(X_1, X).Ric)(X_2, Y) = 0, \;\; \forall X_1, X, X_2, Y \in \Gamma(TM). \tag{3.42}$$

The latter condition is equivalent to

$$-Ric(R(X_1,X)X_2, Y) - Ric(X_2, R(X_1,X)Y) = 0. \tag{3.43}$$

By straightforward calculation, we have, for any $X, Y \in \Gamma(TM)$,

$$\begin{aligned} R(E,X)Y &= \theta(R(E,X)Y)E + \eta(R(E,X)Y)\xi + \sum_{i=1}^{2n-4} \varepsilon_i g(R(E,X)Y, F_i)F_i \\ &+v(R(E,X)Y)V + u(R(E,X)Y)U, \end{aligned} \tag{3.44}$$

where

$$\begin{aligned} &\theta(R(E,X)Y) = \frac{c+3(\alpha^2-\beta^2)}{4}\{g(X,Y) - \eta(X)\eta(Y)\} + \frac{c-\alpha^2+\beta^2}{4}\{2u(X)v(Y) \\ &+u(Y)v(X)\} - E(\alpha)\eta(Y)v(X) + E(\beta)\eta(Y)\theta(X) - X(\beta)\eta(Y) - Y(\beta)\eta(X) \\ &-N(\alpha)\eta(X)u(Y) + \eta(X)\eta(Y)\left\{\alpha^2 - \beta^2 + \xi(\beta)\right\}, \end{aligned}$$

$$g(R(E,X)Y,F_i)=-E(\alpha)\eta(Y)\overline{g}(\overline{\phi}X,F_i)+E(\beta)\eta(Y)g(X,F_i)-\eta(X)u(Y)F_i(\alpha),$$
$$\eta(R(E,X)Y)=u(Y)X(\alpha)+E(\alpha)\overline{g}(\overline{\phi}X,Y)-E(\beta)\{g(X,Y)-\eta(X)\eta(Y)\}$$
$$+2\alpha\beta\eta(X)u(Y)+\beta B(X,Y),$$
$$u(R(E,X)Y)=E(\beta)\eta(Y)u(X)-V(\alpha)\eta(X)u(Y),$$
$$v(R(E,X)Y)=-\frac{c-\alpha^2+\beta^2}{4}\{2u(X)\theta(Y)+u(Y)\theta(X)\}-\eta(Y)X(\alpha)+\eta(X)Y(\alpha)$$
$$+E(\alpha)\eta(Y)\theta(X)+E(\beta)\eta(Y)v(X)-U(\alpha)\eta(X)u(Y)+B(X,Y)C(E,U).$$

Taking $X_1=X_2=E$, we have

$$R(E,X)E=-E(\beta)\eta(X)E+E(\alpha)u(X)\xi-\{\frac{1}{2}(c-\alpha^2+\beta^2)u(X)-E(\alpha)\eta(X)\}V.$$

Using this relation, we deduce

$$\begin{aligned}
&Ric(R(E,X)E,Y)=\{\frac{(2n+1)c}{4}+\frac{3n-5}{2}(\alpha^2-\beta^2)+\xi(\beta)\}E(\alpha)u(X)\eta(Y)\\
&-\{\frac{(2n+1)c}{4}+\frac{3n-5}{2}(\alpha^2-\beta^2)+\xi(\beta)\}\{\frac{1}{2}(c-\alpha^2+\beta^2)u(X)-E(\alpha)\eta(X)\}u(Y)\\
&-\{\frac{(2n+1)c}{4}-\frac{n+3}{2}(\alpha^2-\beta^2)-2(n-2)\xi(\beta)\}E(\alpha)u(X)\eta(Y)\\
&+2(n-1)(E(\beta))^2\eta(X)\eta(Y)-2(n-1)E(\alpha)\xi(\beta)\eta(Y)u(X)\\
&+2(n-1)\{\frac{1}{2}(c-\alpha^2+\beta^2)u(X)-E(\alpha)\eta(X)\}V(\beta)\eta(Y)-2(n-1)E(\alpha)u(X)Y(\beta)\\
&-E(\beta)V(\alpha)\eta(X)\eta(Y)+\{\frac{1}{2}(c-\alpha^2+\beta^2)u(X)-E(\alpha)\eta(X)\}E(\alpha)\eta(Y)\\
&-E(\alpha)u(X)\overline{\phi}Y(\alpha)+E(\beta)\eta(X)B(A_NE,Y)-E(\alpha)u(X)B(A_N\xi,Y)\\
&+\{\frac{1}{2}(c-\alpha^2+\beta^2)u(X)-E(\alpha)\eta(X)\}B(A_NV,Y)+E(\alpha)u(X)B(\xi,Y)trA_N,\\
&-\{\frac{1}{2}(c-\alpha^2+\beta^2)u(X)-E(\alpha)\eta(X)\}B(V,Y)trA_N.
\end{aligned}\tag{3.45}$$

Also, using $B(A_NE,\xi)=0$, we have

$$\begin{aligned}
&Ric(E,R(E,X)Y)=-2(n-1)E(\beta)X(\alpha)u(Y)-2(n-1)E(\beta)E(\alpha)\overline{g}(\overline{\phi}X,Y)\\
&+2(n-1)(E(\beta))^2\{g(X,Y)-\eta(X)\eta(Y)\}-4(n-1)\alpha\beta E(\beta)\eta(X)u(Y)\\
&-2(n-1)E(\beta)B(X,Y)+V(\alpha)X(\alpha)u(Y)+V(\alpha)E(\alpha)\overline{g}(\overline{\phi}X,Y)\\
&-V(\alpha)E(\beta)\{g(X,Y)-\eta(X)\eta(Y)\}+2\alpha\beta V(\alpha)\eta(X)u(Y)+\beta V(\alpha)B(X,Y)\\
&-\sum_{i=1}^{2n-4}\varepsilon_i g(R(E,X)Y,F_i)B(A_NE,F_i)-v(R(E,X)Y)B(A_NE,V)\\
&-u(R(E,X)Y)B(A_NE,U).
\end{aligned}\tag{3.46}$$

Using (3.45) and (3.46), one obtains

$$Ric(R(E,X)E,E)=-2(n-1)E(\alpha)E(\beta)u(X)+E(\alpha)V(\alpha)u(X),\tag{3.47}$$

and

$$\begin{aligned}
&Ric(E,R(E,X)E)=-2(n-1)E(\alpha)E(\beta)u(X)+E(\alpha)V(\alpha)u(X)\\
&-\{\frac{1}{2}(c-\alpha^2+\beta^2)u(X)-E(\alpha)\eta(X)\}B(A_NE,V).
\end{aligned}\tag{3.48}$$

If M is Ricci semi-symmetric, then, putting the pieces (3.47) and (3.48) together into (3.43) with $X_1 = X_2 = E$, we have

$$
\begin{aligned}
&2E(\alpha)V(\alpha)u(X) - \{\frac{1}{2}(c - \alpha^2 + \beta^2)u(X) - E(\alpha)\eta(X)\}B(A_N E, V) \\
&-4(n-1)E(\alpha)E(\beta)u(X) = 0. \qquad (3.49)
\end{aligned}
$$

Using (3.45), (3.46), (3.47), (3.48) and (3.49), it is easy to show that if $Ric(E, V) \neq 0$

$$E(\alpha) = 0, \;\; V(\alpha) = 0, \;\; U(\alpha) = 0, \;\; F_i(\alpha) = 0 \;\text{ and }\; E(\beta) = 0. \qquad (3.50)$$

Theorem 3.7 *Let M be a Ricci semi-symmetric lightlike hypersurface of an indefinite trans-Sasakian space form $\overline{M}(c)$ with $\xi \in TM$ and $Ric(E, V) \neq 0$. Then $c = \alpha^2 - \beta^2$. Moreover, if $c = \alpha^2 - \beta^2$, then either M is totally geodesic or $C(E, U) = 0$.*

Proof: Suppose that $Ric(E, V) \neq 0$. Using the relation (3.50), the relation (3.49) becomes $(c - \alpha^2 + \beta^2)u(X)B(A_N E, V) = 0$ which implies, for $X = U$, that $c = \alpha^2 - \beta^2$. If $c = \alpha^2 - \beta^2$, then, using (3.50), (3.45) and (3.46) become respectively, $Ric(R(E, X)E, Y) = 0$ and $Ric(E, R(E, X)Y) = -\eta(Y)X(\alpha) + \eta(X)Y(\alpha) + B(X, Y)C(E, U)$. Since M is Ricci semi-symmetric, we have $B(X, Y)C(E, U) = \eta(Y)X(\alpha) - \eta(X)Y(\alpha)$. As the left-hand side of this relation is anti-symmetric, and since B is symmetric, one obtains $B(X, Y)C(E, U) = 0$ which completes the proof. $\square$

Corollary 3.8 *There exist no Ricci semi-symmetric lightlike hypersurfaces M of indefinite trans-Sasakian space forms $(\overline{M}(c),\ c \neq \alpha^2 - \beta^2)$ with $\xi \in TM$ and $Ric(E, V) \neq 0$.*

A lightlike submanifold (M, g) of $(\overline{M}, \overline{g})$ [10] is totally umbilical in $\overline{M}$ if the local second fundamental form B of M satisfies, for any $X, Y \in \Gamma(TM)$,

$$B(X, Y) = \zeta g(X, Y), \qquad (3.51)$$

where ζ is a smooth function on M. Since $\overline{\nabla}_X \xi = \nabla_X \xi + B(X, \xi)N$, we have $B(\xi, \xi) = 0$. If M is a totally umbilical lightlike hypersurface of an indefinite trans-Sasakian manifold $\overline{M}$, then h satisfies (3.51) and we have $0 = B(\xi, \xi) = \zeta$. Hence, M is totally geodesic.

Proposition 3.9 *Let (M, g) be a totally umbilical lightlike hypersurface of an indefinite trans-Sasakian manifold $(\overline{M}, \overline{g})$ of type (α, β) with $\xi \in TM$. Then M is totally geodesic.*

It follows from the Proposition 3.9 that a trans-Sasakian manifold $\overline{M}$ does not admit any non-totally geodesic, totally umbilical lightlike hypersurface. From this point of view, Bejancu [1] considered the concept of totally contact umbilical semi-invariant submanifolds. The notion of totally contact umbilical submanifolds was first defined by Kon [19]. We now follow the Bejancu [1] definition of totally contact umbilical submanifolds and state it for totally contact umbilical lightlike hypersurfaces.

A submanifold M is said to be a totally contact umbilical lightlike hypersurface of a semi-Riemannian manifold $\overline{M}$ if the second fundamental form h of M satisfies [24],

$$h(X, Y) = \{g(X, Y) - \eta(X)\eta(Y)\}\, H + \eta(X)h(Y, \xi) + \eta(Y)h(X, \xi), \qquad (3.52)$$

for any X, $Y \in \Gamma(TM)$, where $H = \lambda N$ is a normal vector field on M, λ is a smooth function on $\mathcal{U} \subset M$. From the totally contact umbilical condition (3.52), we have

$$B(X,Y) = \lambda \{g(X,Y) - \eta(X)\eta(Y)\} - \alpha \{\eta(X)u(Y) + \eta(Y)u(X)\}. \tag{3.53}$$

If $\lambda = 0$, then the lightlike hypersurface M is said to be totally contact geodesic.

Let M be a lightlike hypersurface of an indefinite trans-Sasakian space form $\overline{M}(c)$ with $\xi \in TM$. Suppose that M is parallel. Then, h satisfies $(\nabla_X h)(Y,Z) = 0$, for any X, Y, $Z \in \Gamma(TM)$. Using (3.21), we have $0 = \overline{g}((\nabla_X h)(Y,Z), E) = \overline{g}(\overline{R}(X,Y)Z, E)$ which implies, by taking $Z = \xi$ and using (2.6),

$$2\alpha\beta \{\eta(Y)u(X) - \eta(X)u(Y)\} + Y(\alpha)u(X) - X(\alpha)u(Y) = 0. \tag{3.54}$$

This implies $X(\alpha) = 0$, for any $X \in \Gamma(D)$ by replacing $Y = U$. If M is totally contact umbilical, then, $\forall\, X \in \Gamma(TM)$, $(\nabla_X B)(\xi, U) = -X(\alpha) - \alpha\lambda\theta(X) - \beta\lambda v(X) + \alpha\tau(X)$. Since M is parallel, then $(\nabla_X B)(\xi, U) = \alpha\tau(X) = -X(\alpha) - \alpha\lambda\theta(X) - \beta\lambda v(X) + \alpha\tau(X)$, that is $-X(\alpha) - \alpha\lambda\theta(X) - \beta\lambda v(X) = 0$ which implies $\xi(\alpha) = 0$ and $U(\alpha) = 0$. Using (3.54), we get $X(\alpha) = 0$. Thus, $\lambda\left(\alpha\theta(X) + \beta v(X)\right) = 0$, then $\lambda = 0$, that is M is totally contact geodesic. Since $\xi(\alpha) = 0$, by (2.9), we have $\alpha = 0$ or $\beta = 0$. This means that M is a submanifold of an indefinite cosympletic manifold $\overline{M}$ which is not our interest in this paper.

In the sequel, we need the following identities, for $X, Y \in \Gamma(TM)$,

$$(\nabla_X \eta)Y = -\alpha\overline{g}(\overline{\phi}X, Y) + \beta \{g(X,Y) - \eta(X)\eta(Y)\}, \tag{3.55}$$

and the covariant derivative of (3.53) is given by

$$\begin{aligned}(\nabla_X B)(Y,Z) &= X(\lambda)\{g(Y,Z) - \eta(Y)\eta(Z)\} + \lambda\{B(X,Y)\theta(Z) + B(X,Z)\theta(Y)\}\\ &+\alpha\lambda\left\{\eta(Z)\overline{g}(\overline{\phi}X,Y) + \eta(Y)\overline{g}(\overline{\phi}X,Z)\right\} - \beta\lambda\{g(X,Y)\eta(Z) + g(X,Z)\eta(Y)\}\\ &+2\alpha\lambda\eta(X)\eta(Y)\eta(Z) - (X.\alpha)\{\eta(Y)u(Z) + \eta(Z)u(Y)\} + \alpha^2\left\{u(Z)\overline{g}(\overline{\phi}X,Y)\right.\\ &\left.+u(Y)\overline{g}(\overline{\phi}X,Z)\right\} - \alpha\beta\{u(Z)g(X,Y) + u(Y)g(X,Z)\} + \alpha\beta\eta(Y)\{\eta(X)u(Z)\\ &+\eta(Z)u(X)\} + \alpha\{\eta(Y)B(X,\phi Z) + \eta(Z)B(X,\phi Y)\} + \alpha\tau(X)\{\eta(Y)u(Z)\\ &+\eta(Z)u(Y)\} + \alpha\beta\eta(Z)\{u(X)\eta(Y) + u(X)\eta(Y)\}.\end{aligned} \tag{3.56}$$

Theorem 3.10 *Let M be a totally contact umbilical lightlike hypersurface of an indefinite trans-Sasakian space form $\overline{M}(c)$ with $\xi \in TM$. Then, $c = -\beta^2 - 3\alpha^2$ and λ satisfies the partial differential equations*

$$E(\lambda) + \lambda\tau(E) - \lambda^2 = 0, \tag{3.57}$$

$$\xi(\lambda) + \lambda(\beta + \tau(\xi)) + V(\alpha) + E(\beta) = 0, \tag{3.58}$$

$$\textit{and} \quad PX(\lambda) + \lambda\tau(PX) + \alpha\beta\, u(PX) = 0, \quad PX \neq \xi, \ \ \forall\, X \in \Gamma(TM). \tag{3.59}$$

Proof: Let M be a totally contact umbilical lightlike hypersurface. From (2.17), we have, for any X, $Y \in \Gamma(TM)$,

$$(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = \overline{g}(\overline{R}(X,Y)Z,E) + \tau(Y)B(X,Z) - \tau(X)B(Y,Z). \qquad (3.60)$$

Using (3.56), equation (3.60) becomes

$$\begin{aligned}
&X(\lambda)\{g(Y,Z) - \eta(Y)\eta(Z)\} - Y(\lambda)\{g(X,Z) - \eta(X)\eta(Z)\} + \lambda\{B(X,Z)\theta(Y)\\
&-B(Y,Z)\theta(X)\} + \alpha\lambda\left\{\eta(Z)\overline{g}(\overline{\phi}X,Y) + \eta(Y)\overline{g}(\overline{\phi}X,Z) - \eta(Z)\overline{g}(\overline{\phi}Y,X)\right.\\
&\left.-\eta(X)\overline{g}(\overline{\phi}Y,Z)\right\} - \beta\lambda\{\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\} - X(\alpha)\{\eta(Y)u(Z)\\
&+\eta(Z)u(Y)\} + Y(\alpha)\{\eta(X)u(Z) + \eta(Z)u(X)\} + \alpha^2\left\{u(Z)\overline{g}(\overline{\phi}X,Y)\right.\\
&\left.+u(Y)\overline{g}(\overline{\phi}X,Z) - u(Z)\overline{g}(\overline{\phi}Y,X) - u(X)\overline{g}(\overline{\phi}Y,Z)\right\} - \alpha\beta\{u(Y)g(X,Z)\\
&-u(X)g(Y,Z)\} + \alpha\beta\eta(Z)\{\eta(Y)u(X) - \eta(X)u(Y)\} + \alpha\{\eta(Y)B(X,\phi Z)\\
&+\eta(Z)B(X,\phi Y) - \eta(X)B(Y,\phi Z) - \eta(Z)B(Y,\phi X)\} + \alpha\tau(X)\{\eta(Y)u(Z)\\
&+\eta(Z)u(Y)\} - \alpha\tau(Y)\{\eta(X)u(Z) + \eta(Z)u(X)\} = \overline{g}(\overline{R}(X,Y)Z,E)\\
&+\tau(Y)B(X,Z) - \tau(X)B(Y,Z). \qquad (3.61)
\end{aligned}$$

Putting $X = E$ in (3.61) and using $B(X,V) = \lambda u(X)$, one obtains

$$\begin{aligned}
&E(\lambda)\{g(Y,Z) - \eta(Y)\eta(Z)\} - \lambda B(Y,Z) - \alpha\lambda\{2\eta(Z)u(Y) + \eta(Y)u(Z)\}\\
&-E(\alpha)\{\eta(Y)u(Z) + \eta(Z)u(Y)\} - 3\alpha^2 u(Y)u(Z) + \alpha\lambda\eta(Z)u(Y)\\
&+\alpha\tau(E)\{\eta(Y)u(Z) + \eta(Z)u(Y)\} = \overline{g}(\overline{R}(E,Y)Z,E) - \tau(E)B(Y,Z). \qquad (3.62)
\end{aligned}$$

Taking $Y = Z = U$ in (3.62) and using (2.10), we have $-3\alpha^2 = \overline{g}(\overline{R}(E,U)U,E) = \frac{3}{4}\left(c - \alpha^2 + \beta^2\right)$, which implies $c = -\beta^2 - 3\alpha^2$. On the other hand, by taking $Y = V$ and $Z = U$ in (3.62), using the fact that $B(V,U) = \lambda$ and (2.10), we have $E(\lambda) + \lambda\tau(E) - \lambda^2 = 0$. Putting $X = \xi$ into (3.61) and using (3.10), we have $\xi(\lambda) + \lambda\beta + V(\alpha) = \overline{g}(\overline{R}(\xi,V)U,E) - \alpha\tau(\xi)$, since $\overline{g}(\overline{R}(\xi,V)U,E) = -E(\beta)$, one obtains $\xi(\lambda) + \lambda(\beta + \tau(\xi)) + V(\alpha) + E(\beta) = 0$.

Finally, substituting $X = PX$, $Y = PY$ and $Z = PZ$, PX, PY, $PZ \in \Gamma(S(TM) - \langle\xi\rangle)$, into (3.61) with $c = -\beta^2 - 3\alpha^2$ and since $S(TM)$ is non-degenerate, we obtain

$$\{PX(\lambda) + \lambda\tau(PX) + \alpha\beta\, u(PX)\}\, PY = \{PY(\lambda) + \lambda\tau(PY) + \alpha\beta\, u(PY)\}\, PX. \qquad (3.63)$$

Now suppose that there exists a vector field X_0 on some neighborhood of M such that $PX_0(\lambda) + \lambda\tau(PX_0) + \alpha\beta\, u(PX_0) \neq 0$ at some point p in the neighborhood. Then, from (3.63) it follows that all vectors of the fibre $(S(TM) - \langle\xi\rangle)_p := \left(\overline{\phi}(TM^\perp) \oplus \overline{\phi}(N(TM)) \perp D_0\right)_p \subset S(TM)_p$ are collinear with $(PX_0)_p$. This contradicts $\dim(S(TM) - \langle\xi\rangle)_p > 1$, since $(S(TM) - \langle\xi\rangle)_p$ is a non-degenerate distribution of rank $2n-2$, $n \geq 2$. This implies (3.59). □

Corollary 3.11 *There exist no totally contact umbilical lightlike hypersurfaces M of indefinite trans-Sasakian space forms ($\overline{M}(c)$, $c \neq -\beta^2 - 3\alpha^2$) with $\xi \in TM$.*

Note that the expressions (3.57), (3.58) and (3.59) generalize the ones found in [24] and [25] in case of totally contact umbilical lighlike hypersurfaces of 1-Sasakian and 1-Kenmostu manifolds, respectively. From (3.57), (3.58) and (3.59), we have $\nabla^{\perp}_{E}H = \overline{g}(H, E)^2 N$, $\nabla^{\perp}_{\xi}H = -\beta\overline{g}(H, E)N - (V(\alpha) + E(\beta))N$ and $\nabla^{\perp}_{PX}H = -\alpha\beta u(PX)$, $PX \neq \xi$, $\forall X \in \Gamma(TM)$. This means that H is not parallel on M and consequently, we have

Lemma 3.12 *Let M be a totally contact umbilical lightlike hypersurface of an indefinite trans-Sasakian space form $\overline{M}(c)$ with $\xi \in TM$. Then, M cannot be an extrinsic sphere.*

It is known, in [22], that M is $D \perp \langle\xi\rangle$-totally geodesic if and only if

$$A^*_E X = u(A_N X)V. \tag{3.64}$$

If M is a parallel totally contact umbilical lightlike hypersurface, by (3.18) and (3.28), we have

$$\beta B(X, Y) = \alpha\beta\left\{\eta(X)u(Y) + \eta(Y)u(X)\right\} + \alpha\left\{B(\phi X, Y) + B(X, \phi Y)\right\}. \tag{3.65}$$

Lemma 3.13 *Let $(M, g, S(TM))$ be a parallel lightlike hypersurface of an indefinite trans-Sasakian manifold $\overline{M}$ of type (α, β) with $\xi \in TM$. Then, for any $X \in \Gamma(TM)$, $B(X, V) = 0$.*

Proof: Taking $Y = V$ in (3.65), we have, for any $X \in \Gamma(TM)$, $\beta B(X, V) = \alpha\{B(\phi X, V) + B(X, \phi V)\} = \alpha B(\phi X, V)$, which implies, for $X = \phi X$, $\beta B(\phi X, V) = \alpha B(\phi^2 X, V) = -\alpha B(X, V) + \alpha u(X)B(U, V)$. These lead to $B(X, V) = \frac{\alpha^2}{\alpha^2+\beta^2}u(X)B(U, V) = 0$, since, by the using the first one, $B(U, V) = 0$ and this completes the proof. □

Now, say that the screen distribution $S(TM)$ is totally umbilical if on any coordinate neighborhood $\mathcal{U} \subset M$, there exists a smooth function φ such that

$$C(X, PY) = \varphi g(X, PY),\ \forall X,\ Y \in \Gamma(TM_{|\mathcal{U}}). \tag{3.66}$$

If $S(TM)$ is totally umbilical, then, C is symmetric on $\Gamma(S(TM)_{|\mathcal{U}})$ and by Theorem 2.3 in [10], $S(TM)$ is integrable. We also have $A_N X = \varphi PX$ and $C(E, PX) = 0$. Using (3.8), we have $\varphi = \eta(A_N\xi) = 0$, so $S(TM)$ is totally geodesic.

Theorem 3.14 *Let $(M, g, S(TM))$ be a parallel lightlike hypersurface of an indefinite trans-Sasakian manifold $\overline{M}$ of type (α, β) with $\xi \in TM$ such that $S(TM)$ is totally umbilical. Then, the following assertions are equivalent:*

(i) *M is $D \perp \langle\xi\rangle$-totally geodesic,*

(ii) *$A^*_E X = 0$, $\forall X \in \Gamma(D \perp \langle\xi\rangle)$,*

(iii) *$\overline{\phi}(TM^{\perp})$ is a $D \perp \langle\xi\rangle$-parallel distribution on M,*

(iv) *$\overline{\phi}(TM^{\perp})$ is a $D \perp \langle\xi\rangle$-Killing distribution on M.*

Proof: Suppose that $S(TM)$ is totally umbilical. Then, $\forall\, X,\, Y \in \Gamma(S(TM))$, $C(X,Y) = 0$. In particular, for any $X \in \Gamma(\overline{\phi}(TM^{\perp}) \perp D_0 \perp \langle\xi\rangle)$, $C(X,V) = u(A_N X) = 0$. Since $C(E,V) = 0$, for any $X_0 \in \Gamma(D \perp \langle\xi\rangle)$, $u(A_N X_0) = 0$ and the equivalence of (i) and (ii) follows from (3.64). The equivalence of (i) and (iv) follows from (3.28), since M is parallel. Now, we want to show the equivalence of (ii) and (iii). First of all, we have

$$\overline{\nabla}_{X_0} V = -(\overline{\nabla}_{X_0}\overline{\phi})E - \overline{\phi}(\nabla_{X_0} E) = -\beta u(X_0)\xi + \overline{\phi}(A^*_E X_0) - \tau(X_0)V. \tag{3.67}$$

Writing the left-hand side of (3.67) as $\overline{\nabla}_{X_0} V = \nabla_{X_0} V + u(A^*_E X_0)N$, we deduce

$$\nabla_{X_0} V = \overline{\phi}(A^*_E X_0) - \beta u(X_0)\xi - u(A^*_E X_0)N - \tau(X_0)V. \tag{3.68}$$

Suppose $A^*_E X_0 = 0$, $\forall\, X_0 \in \Gamma(D \perp \langle\xi\rangle_{|\mathcal{U}})$. Then, the relation (3.68) becomes, $\nabla_{X_0} V = -\tau(X_0)V$. Since $\overline{\phi}(TM^{\perp})$ is a distribution on M of rank 1 and spanned by V, then, for any $Y_0 \in \Gamma(\overline{\phi}(TM^{\perp}))$, $\nabla_{X_0} Y_0 = (X_0(v(Y_0)) - v(Y_0)\tau(X_0))V \in \Gamma(\overline{\phi}(TM^{\perp}))$, since $Y_0 = v(Y_0)V$. We have, $\overline{\phi}(TM^{\perp})$ is $D \perp \langle\xi\rangle$-parallel. Conversely, suppose $\overline{\phi}(TM^{\perp})$ is $D \perp \langle\xi\rangle$-parallel. Then, for any $X_0 \in \Gamma(D \perp \langle\xi\rangle)$ and $Y_0 = v(Y_0)V \in \Gamma(\overline{\phi}(TM^{\perp})_{|\mathcal{U}})$, $\nabla_{X_0} Y_0 \in \Gamma(\overline{\phi}(TM^{\perp})_{|\mathcal{U}})$. In particular, we have $\nabla_{X_0} V \in \Gamma(\overline{\phi}(TM^{\perp})_{|\mathcal{U}})$. Since $\overline{\phi}(TM^{\perp})$ is spanned by V, there exists a smooth function $\varepsilon \neq 0$ on M such that $\nabla_{X_0} V = \varepsilon V$. Using (3.68), we have $\varepsilon = \overline{g}(\overline{\phi}(A^*_E X_0), U) - \tau(X_0) = -\overline{g}(A^*_E X_0, N) - \tau(X_0) = -\tau(X_0)$. Since $\nabla_{X_0} V = -\tau(X_0)\, V$ and $u|_{D\perp\langle\xi\rangle} = 0$, (3.68) leads to $\overline{\phi}(A^*_E X_0) = u(A^*_E X_0)N$. which implies, by applying $\overline{\phi}$ and Lemma 3.13, that $A^*_E X_0 = B(X_0, V)U = 0$. This completes the proof. $\square$

Theorem 3.14 can be extended by using Theorem 2.2 [10] in order to get more information about the geometry of M. The totally umbilical distribution $S(TM) - \langle\xi\rangle$ is not totally geodesic.

Theorem 3.15 *Let $(M, g, S(TM))$ be a parallel lightlike hypersurface of an indefinite trans-Sasakian space form $\overline{M}(c)$ with $\xi \in TM$. If M is Einstein on the distribution $D \oplus D'$ and A_N has no components in $\langle\xi\rangle$, then the distribution $S(TM) - \langle\xi\rangle$ is totally umbilical.*

Proof: Let M be a parallel lightlike hypersurface of $\overline{M}(c)$ with $\xi \in TM$. Then, $c = \alpha^2 - \beta^2$. If M is Einstein on $D \oplus D'$, the Ricci tensor Ric satisfies, for any $X, Y \in \Gamma(D \oplus D')$, $Ric(X,Y) = kg(X,Y)$, where $k = \{\frac{8n-9}{4}(\alpha^2 - \beta^2) + \xi(\beta)\} - B(A_N V, U) + B(V,U)trA_N$. Since Ric is symmetric, using (3.41), we have $B(A_N X, Y) = B(X, A_N Y)$. By Lemma 3.13, $B(A_N V, U) = B(V, A_N U) = 0$ and $B(V,U) = 0$ and $k = \{\frac{8n-9}{4}(\alpha^2 - \beta^2) + \xi(\beta)\}$. Using (3.40), $-B(A_N X, Y) + B(X,Y)trA_N = 0$, which leads to $B(X,Y)trA_N - B(A_N X, Y) = 0$, that is, $g((trA_N)X - A_N X, Y) = 0$ and we have $(trA_N)X - A_N X = \theta(X)(trA_N)E - \eta(A_N X)\xi$. Since $g(A_N X, \xi) = 0$, $(trA_N)X - A_N X = \theta(X)(trA_N)E$, that is, $A_N X = (trA_N)(X - \theta(X)E) = (trA_N)PX$ and $S(TM) - \langle\xi\rangle$ is totally umbilical. $\square$

4 Screen integrable lightlike hypersurfaces of indefinite trans-Sasakian manifolds

Let M be a lightlike hypersurface of an indefinite trans-Sasakian space form $\overline{M}(c)$ with $\xi \in TM$. From the differential geometry of lightlike hypersurfaces, we recall the following desirable property for lightlike geometry. It is known that lightlike submanifolds whose screen distribution is integrable have interesting properties. For any $X, Y \in \Gamma(TM)$,

$$u([X,Y]) = B(X,\phi Y) - B(\phi X, Y) + \beta\{\eta(Y)u(X) - \eta(X)u(Y)\}. \tag{4.1}$$

Since $u(X) = 0$, $\forall X \in \Gamma(D \perp \langle\xi\rangle)$, then, the relation (4.1) becomes, X, $Y \in \Gamma(D \perp \langle\xi\rangle)$, $u([X,Y]) = B(X,\phi Y) - B(\phi X, Y)$. So, it is easy to check that the distribution $D \perp \langle\xi\rangle$ is integrable if and only if $B(X,\phi Y) = B(\phi X, Y)$.

From (3.12) and (4.1), the differential of the 1-form u is given by, for any $X, Y \in \Gamma(TM)$,

$$2du(X,Y) = (\nabla_X u)Y - (\nabla_Y u)X = -u([X,Y]) + \tau(Y)u(X) - \tau(X)u(Y), \tag{4.2}$$

and this relation leads to the following result.

Theorem 4.1 *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite trans-Sasakian manifold $\overline{M}$ of type (α,β) with $\xi \in TM$. Then, the distribution $D \perp \langle\xi\rangle$ is integrable if and only if the 1-form u (3.5) is closed on $D \perp \langle\xi\rangle$, that is, $du(X,Y) = 0$, $\forall\, X$, $Y \in \Gamma(D \perp \langle\xi\rangle)$.*

Proposition 4.2 *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite trans-Sasakian manifold $\overline{M}$ of type (α,β) with $\xi \in TM$ such that the distribution $D \perp \langle\xi\rangle$ is integrable. Then, M is $D \perp \langle\xi\rangle$-totally geodesic if and only if $\overline{\phi}(TM^{\perp})$ is a $D \perp \langle\xi\rangle$-Killing distribution.*

Proof: Since $D \perp \langle\xi\rangle$ is integrable, using (3.18), one obtains, for any $X, Y \in \Gamma(D \perp \langle\xi\rangle)$, $(L_V g)(X,Y) = -\alpha\{B(X,\phi Y) + B(\phi X, Y)\} = -2\alpha B(X,\phi Y)$. Using (3.6) and the fact that $\overline{\phi}(D \perp \langle\xi\rangle) = D$, we complete the proof. □

Proposition 4.3 *Let $(M, g, S(TM))$ be a totally contact umbilical lightlike hypersurface of an indefinite trans-Sasakian manifold $\overline{M}$ of type (α,β) with $\xi \in TM$. Then, the distribution $D \perp \langle\xi\rangle$ is integrable if and only if M is totally contact geodesic.*

Proof: Using (3.11), (4.1) and the fact that M is totally contact umbilical, we have, for any X, $Y \in \Gamma(D \perp \langle\xi\rangle)$, $u([X,Y]) = B(X,\phi Y) - B(\phi X, Y) = \lambda\{g(X,\phi Y) - g(\phi X, Y)\} = 2\lambda g(X,\phi Y)$. If $D \perp \langle\xi\rangle$ is integrable, then, $2\lambda g(X,\phi Y) = 0$ which implies, for $X = \overline{\phi}F_i$ and $Y = F_i$, that $\lambda = 0$, since $g(F_i, F_i) \neq 0$. The converse is obvious. □

Let us assume that the screen distribution $S(TM)$ of M is integrable and let M' be a leaf of $S(TM)$. Then, using (2.14) and (2.15), we obtain, for any $X, Y \in \Gamma(TM')$,

$$\overline{\nabla}_X Y = \nabla^*_X Y + C(X,Y)E + B(X,Y)N = \nabla'_X Y + h'(X,Y), \tag{4.3}$$

where ∇' and h' are the Levi-Civita connection and the second fundamental form of M' in $\overline{M}$. Thus, for any $X, Y \in \Gamma(TM')$

$$h'(X,Y) = C(X,Y)E + B(X,Y)N. \tag{4.4}$$

In the sequel, we need the following identities, for any $X \in \Gamma(TM')$,

$$\begin{aligned}
\nabla'_X \xi &= -\alpha\phi X + \beta\{PX - \eta(X)\xi\} + \alpha v(X)E, && (4.5)\\
\nabla'_X U &= -\beta v(X)\xi - v(A_N X)E - v(A^*_E X)N + \overline{\phi}(A_N X) + \tau(X)U, && (4.6)\\
\nabla'_X V &= -\beta u(X)\xi - u(A_N X)E - u(A^*_E X)N + \overline{\phi}(A^*_E X) - \tau(X)V. && (4.7)
\end{aligned}$$

It is well known that the second fundamental form and the shape operators of a non-degenerate hypersurface (in general, submanifold) are related by means of the metric tensor field. Contrary to this, we see from Section 2 that in the case of lightlike hypersurfaces, the second fundamental forms on M and their screen distribution $S(TM)$ are related to their respective shape operators A_N and A^*_E. As the shape operator is an information tool for studying the geometry of submanifolds, their study turns out to be very important. For instance, in [11] a class of lightlike hypersurfaces was considered, with shape operators the same as the ones of their screen distribution up to a conformal non zero smooth factor in $\mathcal{F}(M)$. That work gave a way to generate, under some geometric conditions, an integrable canonical screen (see [11] for more details).

Next, we study these operators and give their implications in a lightlike hypersurface of indefinite trans-Sasakian manifold $(\overline{M}, \overline{g})$ of type (α, β) with $\xi \in TM$. Let $\widehat{W}$ be an element of $TM^\perp \oplus N(TM)$ which is a non-degenerate distribution of rank 2. Then there exist non zero smooth functions μ and ν such that

$$\widehat{W} = \mu\, E + \nu\, N. \qquad (4.8)$$

It is easy to check that $\mu = \overline{g}(\widehat{W}, N)$ and $\nu = \overline{g}(\widehat{W}, E)$. The Lie derivative $L_{\widehat{W}}\overline{g}$ is given by,

$$\begin{aligned}
&(L_{\widehat{W}}\overline{g})(X,Y) = -2\mu B(X,Y) - \nu\{C(X,Y) + C(Y,X)\}\\
&+\nu\{\tau(X)\theta(Y) + \tau(Y)\theta(X)\} + X(\nu)\theta(Y) + Y(\nu)\theta(X), \ \forall\, X,\, Y \in \Gamma(TM). \qquad (4.9)
\end{aligned}$$

Let $\mathcal{A}_{\widehat{W}}$ be a tensor field of type $(1,1)$ locally defined, in terms of A^*_E and A_N, by

$$\mathcal{A}_{\widehat{W}} X = \mu\, A^*_E X + \nu\, A_N X, \ \ \forall X \in \Gamma(TM). \qquad (4.10)$$

The action of $\overline{\nabla}$ on the normal bundle $TM^\perp \oplus N(TM)$ is defined as, for any $X, Y \in \Gamma(TM)$,

$$\overline{g}(\mathcal{A}_{\widehat{W}} X, PY) = \overline{g}(\mu A^*_E X + \nu A_N X, PY) = -\overline{g}(\overline{\nabla}_X \widehat{W}, PY), \ \forall\, X,\, Y \in \Gamma(TM). \qquad (4.11)$$

From (4.11), we deduce that

$$\overline{\nabla}_X \widehat{W} = -\mathcal{A}_{\widehat{W}} X + \nabla^{*\perp}_X \widehat{W}, \qquad (4.12)$$

where $\nabla^{*\perp}_X \widehat{W} = \{X(\mu) - \mu\tau(X)\}E + \{X(\nu) + \nu\tau(X)\}N$.

Let $(M, g, S(TM))$ be a screen integrable lightlike hypersurface of an indefinite trans-Sasakian manifold $(\overline{M}, \overline{g})$ of type (α, β) with $\xi \in TM$. Then, (4.9) becomes, for any $X,\, Y \in \Gamma(TM)$,

$$\begin{aligned}
(L_{\widehat{W}}\overline{g})(X,Y) &= -2\mu B(X,Y) - 2\nu\, C(X,Y) + \nu\{\tau(X)\theta(Y) + \tau(Y)\theta(X)\}\\
&\quad + X(\nu)\theta(Y) + Y(\nu)\theta(X). \qquad (4.13)
\end{aligned}$$

Let M' be a leaf of $S(TM)$. Then, on M', the relation (4.13) becomes,

$$(L_{\widehat{W}}\overline{g})(X,Y) = -2\mu B(X,Y) - 2\nu\, C(X,Y), \;\; \forall\, X,\, Y \in \Gamma(TM'). \tag{4.14}$$

From (4.6) and (4.7), we have the following combination, for any $X \in \Gamma(TM')$,

$$\begin{aligned}&\mu\nabla'_X V + \nu\nabla'_X U = -\beta\{\mu\, u(X) + \nu\, v(X)\}\xi - v(\mathcal{A}_{\widehat{W}}X)E - u(\mathcal{A}_{\widehat{W}}X)N \\ &+\overline{\phi}(\mathcal{A}_{\widehat{W}}X) - \mu\,\tau(X)V + \nu\,\tau(X)U.\end{aligned} \tag{4.15}$$

Theorem 4.4 *Let $(M, g, S(TM))$ be a screen integrable lightlike hypersurface of an indefinite trans-Sasakian manifold $(\overline{M}, \overline{g})$ of type (α, β) with $\xi \in TM$ and let M' be a leaf of $S(TM)$. Then, the vector fields V and U are parallel with respect to the Levi-Civita connection ∇' on M' if and only if, for any $X \in \Gamma(TM')$,*

$$\mathcal{A}_{\widehat{W}}X = u(\mathcal{A}_{\widehat{W}}X)U + v(\mathcal{A}_{\widehat{W}}X)V,$$

and τ and $\mu\, u + \nu\, v$ vanish on M'.

Proof: Suppose V and U are parallel with respect to the Levi-Civita connection ∇' on M'. Then, for any $X \in \Gamma(TM')$, $\nabla'_X V = 0$ and $\nabla'_X U = 0$. Using (4.6) and (4.7), we have,

$$\begin{aligned}0 \;\; = \;\; &\mu\nabla'_X V + \nu\nabla'_X U = -\beta\{\mu\, u(X) + \nu\, v(X)\}\xi - v(\mathcal{A}_{\widehat{W}}X)E - u(\mathcal{A}_{\widehat{W}}X)N \\ &+\overline{\phi}(\mathcal{A}_{\widehat{W}}X) - \mu\,\tau(X)V + \nu\,\tau(X)U,\end{aligned} \tag{4.16}$$

which leads, by using (3.7), to

$$\begin{aligned}\overline{\phi}(\mathcal{A}_{\widehat{W}}X) \;\; = \;\; &\beta\{\mu\, u(X) + \nu\, v(X)\}\xi + v(\mathcal{A}_{\widehat{W}}X)E + u(\mathcal{A}_{\widehat{W}}X)N \\ &+\mu\,\tau(X)V - \nu\,\tau(X)U.\end{aligned} \tag{4.17}$$

Since $\overline{\phi}(\mathcal{A}_{\widehat{W}}X) = \phi(\mathcal{A}_{\widehat{W}}X) + u(\mathcal{A}_{\widehat{W}}X)N$, we obtain

$$\phi(\mathcal{A}_{\widehat{W}}X) = \beta\{\mu\, u(X) + \nu\, v(X)\}\xi + v(\mathcal{A}_{\widehat{W}}X)E + \mu\,\tau(X)V - \nu\,\tau(X)U. \tag{4.18}$$

Apply ϕ to (4.18) and using (3.9) and the fact that $\phi U = 0$, we obtain

$$\begin{aligned}\mathcal{A}_{\widehat{W}}X \;\; &= \;\; \eta(\mathcal{A}_{\widehat{W}}X)\xi + u(\mathcal{A}_{\widehat{W}}X)U + v(\mathcal{A}_{\widehat{W}}X)V - \mu\,\tau(X)E \\ &= \;\; -\alpha\{\mu\, u(X) + \nu\, v(X)\}\xi + u(\mathcal{A}_{\widehat{W}}X)U + v(\mathcal{A}_{\widehat{W}}X)V - \mu\,\tau(X)E.\end{aligned} \tag{4.19}$$

Putting (4.19) into (4.15) and using (3.7), one obtains, $-\beta\{\mu\, u(X) + \nu\, v(X)\}\xi + \nu\,\tau(X)U = 0$, which is equivalent to $\mu\, u(X) + \nu\, v(X) = 0$ and $\tau(X) = 0$ and (4.19) is reduced to

$$\mathcal{A}_{\widehat{W}}X = u(\mathcal{A}_{\widehat{W}}X)U + v(\mathcal{A}_{\widehat{W}}X)V. \tag{4.20}$$

The converse is obvious, using (4.15). $\square$

Corollary 4.5 *Let $(M, g, S(TM))$ be a screen integrable lightlike hypersurface of an indefinite trans-Sasakian manifold $(\overline{M}, \overline{g})$ of type (α, β) with $\xi \in TM$ and let M' be a leaf of $S(TM)$ such that U and V are parallel with respect to the Levi-Civita connection ∇' on M'. Then, the type number $t'(x)$ of M' (with $x \in M'$) satisfies $t'(x) \leq 2$.*

Proof: The proof follows from Theorem 4.4. □

Next, we deal with the geometry of the normal bundle $TM^{\perp} \oplus N(TM)$ and we show there exists a close relationship between its geometry and the geometries of $\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))$ and of any leaf of an integrable distribution. Denote by H' the mean curvature vector of M', a leaf of an integrable screen distribution $S(TM)$. As $N(TM) \oplus TM^{\perp}$ is the normal bundle of M', there exist smooth functions δ and ρ such that $H' = \delta E + \rho N$. If M' is totally contact umbilical immersed in $\overline{M}$, we have, for any X, $Y \in \Gamma(TM')$,

$$h'(X, Y) = (g(X, Y) - \eta(X)\eta(Y))\, H' + \eta(X)h'(Y, \xi) + \eta(Y)h'(X, \xi), \tag{4.21}$$

which implies, using (4.4), $B = \rho\{g - \eta \otimes \eta\} - \alpha\{\eta \otimes u + u \otimes \eta\}$ and $C = \delta\{g - \eta \otimes \eta\} - \alpha\{\eta \otimes v + v \otimes \eta\}$ along M', and by putting these relations together into (4.22), one obtains,

$$\begin{aligned}(L_{\widehat{W}}\overline{g})(X, Y) &= -2(\mu\rho + \nu\delta)\{g(X, Y) - \eta(X)\eta(Y)\} + 2\alpha\eta(X)\{\mu\, u(Y) + \nu\, v(Y)\} \\ &\quad + 2\alpha\eta(Y)\{\mu\, u(X) + \nu\, v(X)\},\ \forall\, X,\, Y \in \Gamma(TM').\end{aligned} \tag{4.22}$$

Theorem 4.6 *Let $(M, g, S(TM))$ be a screen integrable lightlike hypersurface of an indefinite trans-Sasakian manifold $\overline{M}$ of type (α, β) with $\xi \in TM$. Suppose any leaf M' of $S(TM)$ is totally contact umbilical immersed in $\overline{M}$ as a non-degenerate submanifold. Then,*

(i) *$TM^{\perp} \oplus N(TM)$ is an η-conformal Killing distribution on M' if and only if $\mu\, u + \nu\, v$ vanishes identically on M'.*

(ii) *Moreover, if the vector fields V and U are parallel with respect to the Levi-Civita connection ∇' on M', $TM^{\perp} \oplus N(TM)$ is an η-conformal Killing distribution on M'.*

Proof: The proof of (i) and (ii) follows from (4.22) and Theorem 4.4. □

Note that if a leaf M' of an integrable screen distribution $S(TM)$ is η-totally umbilical immersed in $\overline{M}$, then, the section $\widehat{W}$ is an η-conformal Killing vector field on M', that is, $TM^{\perp} \oplus N(TM)$ is an η-conformal Killing distribution on M'.

Let W be an element of $\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))$, a non-degenerate vector subbundle of $S(TM)$ of rank 2. Then, there exist non-zero functions a and b such that

$$W = aV + bU. \tag{4.23}$$

We have $a = v(W)$ and $b = u(W)$. Let ω be a differential 1-form locally defined by $\omega(\cdot) = g(W, \cdot)$. Using (3.12) and (3.13), the covariant derivative of ω and the Lie derivative of g with respect to the vector field W are given, respectively, by

$$\begin{aligned}(\nabla_X\omega)Y &= -v(W)B(X,\phi Y)-u(W)C(X,\phi Y)-\beta\omega(X)\eta(Y)\\ &\quad -\alpha u(W)\eta(Y)\theta(X), \quad (4.24)\\ (L_W g)(X,Y) &= X(\omega(Y))+Y(\omega(X))+\omega([X,Y])-2\omega(\nabla_X Y), \quad (4.25)\end{aligned}$$

for any X, $Y\in\Gamma(TM)$. Referring to (3.2), for any $X\in\Gamma(TM)$, $Y\in\Gamma(D_0\perp\langle\xi\rangle)$, we have

$$\nabla_X Y=\widetilde{\nabla}_X Y+\widetilde{h}(X,Y), \quad (4.26)$$

where $\widetilde{\nabla}$ is a linear connection on the bundle $D_0\perp\langle\xi\rangle$ and $\widetilde{h}:\Gamma(TM)\times\Gamma(D_0\perp\langle\xi\rangle)\longrightarrow\Gamma(\overline{\phi}(TM^\perp)\oplus\overline{\phi}(N(TM))\perp TM^\perp)$ is $\mathcal{F}(M)$-bilinear. Let $\mathcal{U}\subset M$ be a coordinate neighbourhood as fixed in Theorem 2.1. Then, using (3.2), (4.26) can be rewritten (locally) as,

$$\nabla_X Y=\widetilde{\nabla}_X Y+C(X,Y)E+C(X,\phi Y)V+B(X,\phi Y)U, \quad (4.27)$$

for any X, $Y\in\Gamma((D_0\perp\langle\xi\rangle)|_{\mathcal{U}})$ and the local expression of $\widetilde{h}$ is

$$\widetilde{h}(X,Y)=C(X,Y)E+C(X,\phi Y)V+B(X,\phi Y)U. \quad (4.28)$$

Using this relation, we obtain the following result.

Theorem 4.7 *Let $(M,\,g,\,S(TM))$ be a lightlike hypersurface of an indefinite trans-Sasakian manifold $\overline{M}$ of type (α,β) with $\xi\in TM$. Then, the distribution $D_0\perp\langle\xi\rangle$ is integrable if and only if, for any X, $Y\in\Gamma(D_0\perp\langle\xi\rangle)$, $C(\phi X,Y)=C(X,\phi Y)$, $B(\phi X,Y)=B(X,\phi Y)$ and $C(X,Y)=C(Y,X)$.*

Proof: The proof follows from a direct calculation, using (4.27). □

Note that Theorem 4.7 remains valid when the distribution $D_0\perp\langle\xi\rangle$ is replaced by D_0. Also, looking at (4.28) and using the above theorem, we deduce that $\widetilde{h}$ is symmetric on $D_0\perp\langle\xi\rangle$ if and only if $D_0\perp\langle\xi\rangle$ is integrable. Moreover, the integrability of $D_0\perp\langle\xi\rangle$ implies that $\widetilde{\nabla}$ is a linear symmetric connection on the integral manifolds.

By direct calculation, the differential of the 1-form ω is given by

$$2d\omega(X,Y)=(\nabla_X\omega)Y-(\nabla_Y\omega)X=-\omega([X,Y]),\ \forall X,\,Y\in\Gamma(TM). \quad (4.29)$$

This means that the integrability condition on $D_0\perp\langle\xi\rangle$ is related to the 1-form ω, that is, the distribution $D_0\perp\langle\xi\rangle$ is integrable if and only if ω is closed along $D_0\perp\langle\xi\rangle$, that is, $d\omega(X,Y)=0$, $\forall\,X,\,Y\in\Gamma(D_0\perp\langle\xi\rangle)$.

If $D_0\perp\langle\xi\rangle$ is integrable, by Theorem 4.7 and (3.9), we have, for any X, $Y\in\Gamma(D_0\perp\langle\xi\rangle)$,

$$B(\phi X,\phi Y)=-B(X,Y)\ \text{ and }\ C(\phi X,\phi Y)=-C(X,Y). \quad (4.30)$$

Let us define the unsymmetrized second fundamental form of $D_0\perp\langle\xi\rangle$, $A^{D_0\perp\langle\xi\rangle}$ by

$$A_X^{D_0\perp\langle\xi\rangle}Y=p_1(\nabla_{X_0}Y_0),\ \forall\,X,\ Y\in\Gamma(TM), \quad (4.31)$$

where $p_1 : TM \longrightarrow \overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))$ is the canonical projection on $\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))$ and X_0, Y_0 are the projections of X and Y onto $D_0 \perp \langle\xi\rangle$. Then, using (4.26), since $p_1(\widetilde{\nabla}_{X_0} Y_0) = 0$, we obtain $A_X^{D_0\perp\langle\xi\rangle} Y = \widetilde{h}(X_0, Y_0)$, and the symmetric second fundamental form $B^{D_0\perp\langle\xi\rangle}$ is given by

$$B^{D_0\perp\langle\xi\rangle}(X,Y) = \frac{1}{2}\left\{\widetilde{h}(X_0,Y_0) + \widetilde{h}(Y_0,X_0)\right\}, \ \forall X,\, Y \in \Gamma(TM). \tag{4.32}$$

Furthermore, the mean curvature vector of the distribution $D_0 \perp \langle\xi\rangle$, $D_0 \perp \langle\xi\rangle$, being integrable or not integrable, is defined as

$$\lambda^{D_0\perp\langle\xi\rangle} = \frac{1}{rank(D_0 \perp \langle\xi\rangle)} tr(B^{D_0\perp\langle\xi\rangle}), \tag{4.33}$$

and $D_0 \perp \langle\xi\rangle$ is called minimal (respectively, totally geodesic) if $\lambda^{D_0\perp\langle\xi\rangle}$ (respectively, $B^{D_0\perp\langle\xi\rangle}$) vanishes.

Proposition 4.8 *Let $(M,\, g,\, S(TM))$ be a lightlike hypersurface of an indefinite trans-Sasakian manifold $\overline{M}$ of type (α, β) with $\xi \in TM$. Suppose the distribution $D_0 \perp \langle\xi\rangle$ is integrable. Then, $D_0 \perp \langle\xi\rangle$ is minimal with respect to the induced symmetric connection ∇ on M and all its integral manifolds are minimal submanifolds of M with respect to ∇.*

Proof: Suppose that $D_0 \perp \langle\xi\rangle$ is integrable. Then, $\widetilde{h}$ is symmetric and the mean curvature vector of $D_0 \perp \langle\xi\rangle$ is

$$\lambda^{D_0\perp\langle\xi\rangle} = \frac{1}{rank(D_0 \perp \langle\xi\rangle)} tr(B^{D_0\perp\langle\xi\rangle}) = \frac{1}{2n-3} tr(\widetilde{h}).$$

We consider an adapted frame in $D_0 \perp \langle\xi\rangle$, $\{X_i, \phi X_i, \xi\}$ with $i = 1, 2, ..., n-2$, using (4.30) and $\widetilde{h}(\xi,\xi) = 0$ we have,

$$\begin{aligned} tr(\widetilde{h}) &= \sum_{i=1}^{n-2} \varepsilon_i \left(\widetilde{h}(X_i, X_i) + \widetilde{h}(\phi X_i, \phi X_i)\right) + \widetilde{h}(\xi,\xi) \\ &= \sum_{i=1}^{n-2} \varepsilon_i \left(C(X_i,X_i)E + C(X_i,\phi X_i)V + B(X_i,\phi X_i)U + C(\phi X_i,\phi X_i)E\right. \\ &\quad \left. + C(\phi X_i, \phi^2 X_i)V + B(\phi X_i, \phi^2 X_i)U\right) = 0. \end{aligned}$$

This completes the proof. □

Suppose that the distribution $D_0 \perp \langle\xi\rangle$ is integrable and let $\widetilde{M}'$ be a leaf of $D_0 \perp \langle\xi\rangle$. Then, using the first equation of (2.14) and (4.26), we have, for any $X, Y \in \Gamma(T\widetilde{M}')$,

$$\overline{\nabla}_X Y = \widetilde{\nabla}_X Y + \widetilde{h}(X,Y) + B(X,Y)N = \widetilde{\nabla}'_X Y + \widetilde{h}'(X,Y), \tag{4.34}$$

where $\widetilde{\nabla}'$ and $\widetilde{h}'$ are the Levi-Civita connection and second fundamental form of $\widetilde{M}'$ in $\overline{M}$. Thus, for any $X, Y \in \Gamma(T\widetilde{M}')$,

$$\widetilde{h}'(X,Y) = \widetilde{h}(X,Y) + B(X,Y)N. \tag{4.35}$$

Here, the second fundamental form $\widetilde{h}'$ coincides with $B^{D_0\perp\langle\xi\rangle}$ defined in (4.32). $\widetilde{h}'$ can also be seen as a restriction of h' defined in (4.3) by $D_0 \perp \langle\xi\rangle$.

Proposition 4.9 *Let $(M,\, g,\, S(TM))$ be a lightlike hypersurface of an indefinite trans-Sasakian manifold $\overline{M}$ of type (α, β) with $\xi \in TM$. Suppose the distribution $D_0 \perp \langle\xi\rangle$ is integrable and let $\widetilde{M}'$ be a leaf of $D_0 \perp \langle\xi\rangle$. Then, the trace of the second fundamental form $\widetilde{h}'$ of $\widetilde{M}'$ vanishes, that is, $tr(\widetilde{h}') = 0$.*

Proof: The proof follows from Proposition 4.8 and (4.35). □

The result of this proposition means that all integral manifolds of $D_0 \perp \langle\xi\rangle$ are minimal submanifolds of $\overline{M}$ and $D_0 \perp \langle\xi\rangle$ is minimal. The result also is similar to the one found in [5] in case of lightlike hypersurfaces in $\mathcal{S}$-manifolds.

The non-zero functions μ, ν, a and b of (4.8) and (4.23), respectively, are related as follows. Since $-\overline{\phi}\widehat{W} = \mu V + \nu U \in \Gamma(\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM)))$, then there exists a non-zero smooth function γ on M such that $-\overline{\phi}\widehat{W} = \mu V + \nu U = \gamma W$ which implies that $\mu = \gamma\, a$ and $\nu = \gamma\, b$. The Lie derivatives L_W (4.25) is rewritten as,

$$\begin{aligned}(L_W g)(X,Y) &= -\frac{\mu}{\gamma}\{B(X,\phi Y) + B(\phi X, Y)\} - \frac{\nu}{\gamma}\{C(X,\phi Y) + C(Y,\phi X)\}\\ &-\beta\{\omega(X)\eta(Y) + \omega(Y)\eta(X)\} - \frac{\alpha\nu}{\gamma}\{\eta(X)\theta(Y) + \eta(Y)\theta(X)\},\end{aligned} \tag{4.36}$$

for any $X,\, Y \in \Gamma(TM)$. If the distribution $D_0 \perp \langle\xi\rangle$ is integrable, using Theorem 4.7, the Lie derivatives $L_{\widehat{W}}$ (4.13) and L_W (4.36) are related with, for any $X,\, Y \in \Gamma(D_0 \perp \langle\xi\rangle)$,

$$\begin{aligned}(L_{\widehat{W}}\overline{g})(X,Y) &= -2\mu B(X,Y) - 2\nu C(X,Y),\\ \text{and}\ \ (L_W g)(X,Y) &= -\frac{2\mu}{\gamma}B(X,\phi Y) - \frac{2\nu}{\gamma}C(X,\phi Y).\end{aligned} \tag{4.37}$$

Theorem 4.10 *Let $(M,\, g,\, S(TM))$ be a lightlike hypersurface of an indefinite trans-Sasakian manifold $\overline{M}$ of type (α, β) with $\xi \in TM$. Suppose the distribution $D_0 \perp \langle\xi\rangle$ is integrable. Let $\widetilde{M}'$ be a leaf of $D_0 \perp \langle\xi\rangle$. Then, the following assertions are equivalent.*

(i) *$\widetilde{M}'$ is totally geodesic in M,*

(ii) *$\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))$ is a Killing distribution on $\widetilde{M}'$,*

(iii) *$TM^{\perp} \oplus N(TM)$ is a Killing distribution on $\widetilde{M}'$.*

Proof: The proof follows by direct calculation, using (4.35) and (4.37). □

Note that if a leaf $\widetilde{M}'$ of an integrable distribution $D_0 \perp \langle\xi\rangle$ is totally contact geodesic, then, by relation (4.22) and (4.36) and using (3.11), we have, for any $X,\, Y \in \Gamma(T\widetilde{M}')$,

$$(L_{\widehat{W}}\overline{g})(X,Y) = -2(\mu\rho + \nu\delta)\{g(X,Y) - \eta(X)\eta(Y)\}\ \text{ and }\ (L_W g)(X,Y) = 0,$$

which imply that the section $\widehat{W}$ is an η-conformal Killing vector field and W is a Killing vector field on $\widetilde{M}'$. This means that the concept of η-conformal Killing on a totally contact geodesic leaf $\widetilde{M}'$ of an integrable distribution $D_0 \perp \langle\xi\rangle$ is not invariant under $\overline{\phi}$.

It is well known that the interrelation between the second fundamental forms of the lightlike M and its screen distribution and their respective shape operators indicates that the lightlike geometry depends on the choice of a screen distribution. Therefore, it is important to investigate the relationship between some geometric objects induced, studied above, with the change of the screen distributions. We know that the local second fundamental form B of M on $\mathcal{U}$ is independent of the vector bundles $(S(TM),\ S(TM^{\perp}))$ and $N(TM)$. This means that all results of this paper, which depend only on B, are stable with respect to any change of those vector bundles. We discuss now the effect of the change of screen distribution on the results which also depend on other geometric objects. Denote by κ the dual 1-form of K (defined in (3.19)), the characteristic vector field of the screen change, with respect to the induced metric g of M, that is, $\kappa(\cdot) = g(X, \cdot)$. Let P and P' be projections of TM on $S(TM)$ and $S(TM)'$, respectively, with respect to the orthogonal decomposition of TM. Any vector field X on M can be written as $X = PX + \theta(X)E = P'X + \theta'(X)E$, where $\theta'(X) = \theta(X) + \kappa(X)$. Then, we have, $P'X = PX - \kappa(X)E$ and $C'(X, P'Y) = C'(X, PY)$. The relationship between the local second fundamental forms C and C' of the screen distributions $S(TM)$ and $S(TM)'$, respectively, is given using (3.19) by

$$C'(X, PY) = C(X, PY) - \frac{1}{2}\kappa(\nabla_X PY + B(X, Y)K),\ \forall\, X,\, Y \in \Gamma(TM). \tag{4.38}$$

All equations above, depending only on the local second fundamental form C (making equations non unique), are independent of $S(TM)$ if and only if $\omega(\nabla_X PY + B(X, Y)K) = 0$. Also, equations (4.25) and (4.9) are not unique as they depend on C, θ and τ which depend on the choice of a screen vector bundle. The Lie derivatives $L_{(\cdot)}$ and $L'_{(\cdot)}$ of the screen distributions $S(TM)$ and $S(TM)'$, respectively, are related through the relations:

$$\begin{aligned}
&(L'_{W'}g)(X, Y) = (L_W g)(X, Y) + (\frac{\mu}{\gamma} - \frac{\mu'}{\gamma'})\{B(X, \phi Y) + B(\phi X, Y)\} \\
&+(\frac{\nu}{\gamma} - \frac{\nu'}{\gamma'})\{C(X, \phi Y) + C(\phi X, Y)\} + \frac{\nu'}{2\gamma'}\kappa(\nabla_{\{X} P\phi Y\} + (B(X, \phi Y) + B(Y, \phi X)K) \\
&+\beta\eta(X_{(\omega-\omega')(Y)}) + \frac{\alpha\nu}{\gamma}\theta(X_{\eta(Y)}) - \frac{\alpha\nu'}{\gamma'}(\theta + \kappa)(X_{\eta(Y)}), \\
&(L'_{\widehat{W}'}\overline{g})(X, Y) = (L_{\widehat{W}}\overline{g})(X, Y) + 2(\mu - \mu')B(X, Y) + (\nu' - \nu)\theta(X_{\tau(Y)}) \\
&+\theta(X)Y(\nu' - \nu) + \theta(Y)X(\nu' - \nu) + \nu'\kappa(X_{\tau(Y)}) + (\nu - \nu')\{C(X, Y) + C(Y, X)\} \\
&+\frac{1}{2}\nu'\kappa(\nabla_{\{X} PY\} + 2B(X, Y)K) + \nu'(\theta + \kappa)_X(Y) + X(\nu')\kappa(Y) + Y(\nu')\kappa(X),
\end{aligned}$$

where $f_X(Y) = f(X)B(Y, K) + f(Y)B(X, K)$, $\nabla_{\{X} PY\} = \nabla_X PY + \nabla_Y PX$ and $X_{f(Y)} = Xf(Y) + Yf(X)$, f denoting a 1-form.

The covariant derivative of h depends on ∇, N and τ which depend on the choice of the screen vector bundle. The covariant derivatives ∇ of $h = B \otimes N$ and ∇' of $h' = B \otimes N'$ in the screen distributions $S(TM)$ and $S(TM)'$, respectively, are related as follows, for any $X,\ Y,\ Z \in \Gamma(TM)$, $\overline{g}((\nabla'_X h')(Y, Z), E) = \overline{g}((\nabla_X h)(Y, Z), E) + \mathcal{L}_{(X,Y)}Z$, where $\mathcal{L}_{(X,Y)}Z = B(X, Y)B(Z, K) + B(X, Z)B(Y, K) + B(Y, Z)B(X, K)$. It is easy to see that the parallelism of h is independent of the screen distribution $S(TM)$, i.e., $\nabla' h' \equiv \nabla h$, if and only if $B = 0$.

Now we discuss the stability of the result of Proposition 4.8 with respect to the change of screen distribution. From the last equation of (3.19), we have, for any $X,\ Y \in \Gamma(TM)$, $\nabla'_X Y = \nabla_X Y +$

$B(X,Y)(N-N')$. From this relation and (4.5), we have, for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D_0 \perp \langle\xi\rangle)$, $\nabla'_X Y = \widetilde{\nabla}_X Y + \widetilde{h}(X,Y) + B(X,Y)(N-N')$, and by denoting by k' the normal part of $\nabla'_X Y$ with respect to $D_0 \perp \langle\xi\rangle$, we obtain

$$k'(X,Y) = \widetilde{h}(X,Y) + B(X,Y)(N-N').$$

Let $\{X_i, \phi X_i, \xi\}_{1\leq i\leq n-2}$ be an adapted frame of $D_0 \perp \langle\xi\rangle$. The trace of k', $tr(k')$, is given by $tr(k') = tr(\widetilde{h}) + \{\sum_{i=1}^{n-2} \varepsilon_i(B(X_i,X_i) + B(\phi X_i, \phi X_i)) + B(\xi,\xi)\}(N-N')$. Under the same hypotheses of Proposition 4.8 and (4.30), we get $tr(k') = 0$.

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