

Lightlike Hypersurfaces of Indefinite Cosymplectic Manifolds

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Abstract

We shall study lightlike hypersurfaces of indefinite cosymplectic manifolds and prove nonexistence of totally umbilical lightlike hypersurfaces of indefinite cosymplectic space forms $\bar{M}(c)$ with $c \neq 0$.

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1. Introduction

Let M be a hypersurface of an m -dimensional semi-Riemannian manifold (\bar{M}, \bar{g}) of constant index $0 < \nu < m$. In case of *degenerate* (or *lightlike*) hypersurfaces, the normal bundle has a nontrivial intersection with the tangent bundle, i.e., the normal bundle is a subbundle of rank one of the tangent bundle over M . So, we fail to use the theory of nonnull geometry. To use the theory of nondegenerate submanifolds we need an alternative vector bundle (the so-called *transversal vector bundle*) which plays the role of the normal bundle in case of nondegenerate hypersurfaces. But the situation is very different from that of nondegenerate submanifolds (see section 4).

In [6], Duggal and Bejancu studied real hypersurfaces of indefinite Kaehler manifolds, and in particular proved that there exist no totally umbilical lightlike real hypersurfaces of indefinite complex space forms with constant holomorphic sectional curvature $c \neq 0$. The odd-dimensional counterpart of indefinite Kaehler manifolds are indefinite cosymplectic manifolds. It is natural that we expect analogous results as in indefinite Kaehler manifolds. Also, lightlike hypersurfaces of indefinite Sasakian manifolds are studied in [10]. In

addition, there exist some papers dealing with lightlike(null) hypersurfaces, eg, [1,3,4,6,7,8,9,12,13].

2. Indefinite Cosymplectic Manifolds

An odd dimensional semi-Riemannian manifold (\bar{M}, \bar{g}) is called an *almost contact metric manifold* if there is a (1,1) tensor field $\bar{\phi}$, a vector field ξ , called *characteristic vector field* and a 1-form η such that for any vector fields X, Y on \bar{M}

$$\bar{g}(\bar{\phi}X, \bar{\phi}Y) = \bar{g}(X, Y) - \epsilon\eta(X)\eta(Y), \quad (2.1)$$

$$\bar{g}(\xi, \xi) = \epsilon, \quad \epsilon = 1 \quad \text{or} \quad \epsilon = -1, \quad (2.2)$$

$$\bar{\phi}^2(X) = -X + \eta(X)\xi, \quad \bar{g}(X, \xi) = \epsilon\eta(X). \quad (2.3)$$

Then $(\bar{\phi}, \xi, \eta, \bar{g})$ is called an *almost contact metric structure* of \bar{M} (cf. [2,5,9]). It follows that

$$\bar{\phi}(\xi) = 0, \quad \eta \circ \bar{\phi} = 0, \quad \eta(\xi) = \epsilon. \quad (2.4)$$

Also, the almost contact metric structure $(\bar{\phi}, \xi, \eta, \bar{g})$ is normal if $N_{\bar{\phi}} + d\eta \otimes \xi = 0$, where $N_{\bar{\phi}}$ is the Nijenhuis tensor field, which is defined by $N_{\bar{\phi}}(X, Y) = \bar{\phi}^2[X, Y] + [\bar{\phi}X, \bar{\phi}Y] - \bar{\phi}[\bar{\phi}X, Y] - \bar{\phi}[X, \bar{\phi}Y]$. Define a 2-form Φ on \bar{M} by $\Phi(X, Y) = \bar{g}(X, \bar{\phi}Y)$. A normal almost contact metric structure $(\bar{\phi}, \xi, \eta, \bar{g})$ on \bar{M} such that Φ is closed and $d\eta = 0$ is called a *cosymplectic structure*. It is characterized by $\bar{\nabla}_X \bar{\phi} = 0$ and $\bar{\nabla}_X \eta = 0$ for any vector field X on \bar{M} , where $\bar{\nabla}$ is the Levi-Civita connection of \bar{g} (cf.[2,11]). A semi-Riemannian manifold \bar{M} with a cosymplectic structure $(\bar{\phi}, \xi, \eta, \bar{g})$ is called an *indefinite cosymplectic manifold*.

Lemma 2.1. *For an indefinite cosymplectic manifold, we have*

$$\bar{\nabla}_X \xi = 0, \quad \forall X \in \Gamma(T\bar{M}), \quad (2.5)$$

where ξ is the characteristic vector field.

Proof. Differentiating $\bar{\phi}(\xi) = 0$, we get $\bar{\phi}(\bar{\nabla}_X \xi) = 0$. Transvecting this with $\bar{\phi}$ and using (2.4) complete the proof. \square

Let \bar{M} be an indefinite cosymplectic manifold. Let

$$D_p := \{ \bar{X} \in T_p(\bar{M}) ; \eta(\bar{X}) = 0 \}.$$

For a non-null vector \bar{X} in D_p , \bar{X} and $\bar{\phi}\bar{X}$ span a non-degenerate 2-plane, and hence we can consider a sectional curvature

$$K(\bar{X}) := \frac{\bar{g}(\bar{R}(\bar{X}, \bar{\phi}\bar{X})\bar{\phi}\bar{X}, \bar{X})}{\bar{g}(\bar{X}, \bar{X})\bar{g}(\bar{\phi}\bar{X}, \bar{\phi}\bar{X}) - \bar{g}(\bar{X}, \bar{\phi}\bar{X})^2},$$

where \bar{R} denotes the curvature tensor of \bar{M} . If $K(\bar{X})$ is constant for all non-null vector \bar{X} in D_p , \bar{M} is said to be of constant $\bar{\phi}$ -sectional curvature at the point p . $K(\bar{X})$ is a function of $p \in M$, say $k(p)$. Moreover, if $k(p)$ is a constant on \bar{M} , \bar{M} is said to be of constant $\bar{\phi}$ -sectional curvature $k(p) = c$. In this case, it is known [2,9] that the curvature tensor \bar{R} has the form.

$$\begin{aligned} \bar{R}(\bar{X}, \bar{Y})\bar{Z} = \frac{\epsilon}{4} \{ & \bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{X}, \bar{Z})\bar{Y} + \eta(\bar{X})\eta(\bar{Z})\bar{Y} - \eta(\bar{Y})\eta(\bar{Z})\bar{X} \\ & + \bar{g}(\bar{X}, \bar{Z})\eta(\bar{Y})\xi - \bar{g}(\bar{Y}, \bar{Z})\eta(\bar{X})\xi \\ & + \bar{g}(\bar{\phi}\bar{Y}, \bar{Z})\bar{\phi}\bar{X} + \bar{g}(\bar{\phi}\bar{Z}, \bar{X})\bar{\phi}\bar{Y} \\ & - 2\bar{g}(\bar{\phi}\bar{X}, \bar{Y})\bar{\phi}\bar{Z} \} \end{aligned} \tag{2.6}$$

for any tangent vectors \bar{X}, \bar{Y} and \bar{Z} , where we have assumed $\epsilon = 1$ (by a suitable conformal transformation if necessary). An indefinite cosymplectic manifold \bar{M} is said to be an *indefinite cosymplectic space form* if \bar{M} is of constant $\bar{\phi}$ -sectional curvature c , and will be denoted by $\bar{M}(c)$.

3. Decompositions of the tangent bundle $T\bar{M}$

Let $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$ be a $(2n + 1)$ -dimensional almost contact metric manifold, where \bar{g} is a semi-Riemannian metric of index ν , $0 < \nu < 2n + 1$.

Let M be a hypersurface of \bar{M} . We consider for any $p \in M$

$$T_pM^\perp = \{E \in T_p\bar{M}; \bar{g}_p(E, W) = 0, \quad \forall W \in T_pM\},$$

$$Rad T_pM = T_pM \cap T_pM^\perp,$$

where T_pM is a hyperplane of a semi-Euclidean space $(T_p\bar{M}, \bar{g}_p)$. M is said to be a *lightlike*(or *degenerate*) *hypersurface* of \bar{M} (or the immersion is *lightlike*(*degenerate*)) if $Rad T_pM \neq \{0\}$ at any point $p \in M$. In this case the induced metric g on M from the semi-Riemannian metric \bar{g} on \bar{M} is degenerate. Let (M, g) be a lightlike hypersurface of (\bar{M}, \bar{g}) . Then there exists a non-zero $E_p \in T_pM$ such that $g_p(E_p, X_p) = 0$ for any $X_p \in T_pM$. It follows that $E_p \in T_pM^\perp$. Note that $Rad T_pM = T_pM^\perp$, since $\dim T_pM^\perp = 1$. For any a local section $E \in \Gamma(TM^\perp)$ we get $\bar{g}(\bar{\phi}E, E) = 0$, and so $\bar{\phi}E$ is tangent to M . Hence we get a distribution $\bar{\phi}(TM^\perp)$ on M of rank 1. Now we choose a complementary distribution $S(TM)$ (called a *screen distribution*) to TM^\perp in

TM such that it contains both $\bar{\phi}E$ and ξ . Since the screen distribution $S(TM)$ is non-degenerate, there exists a complementary orthogonal vector subbundle $S(TM)^\perp$ to $S(TM)$ in $T\bar{M}$ over M . Hence we have the orthogonal decomposition

$$T\bar{M} = S(TM) \perp S(TM)^\perp. \quad (3.1)$$

Note that TM^\perp is a lightlike vector subbundle of the non-degenerate vector bundle $S(TM)^\perp$ with two-dimensional fibers. Then, for any local section $E \in \Gamma(TM^\perp)$ there exists a unique lightlike local section $N \in \Gamma(S(TM)^\perp)$ (cf. [7]) such that

$$\bar{g}(N, E) = 1. \quad (3.2)$$

Hence, N is not tangent to M and $\{E, N\}$ are a local field of frames of $S(TM)^\perp$. Moreover we have a 1-dimensional vector subbundle $tr(TM)$ of $T\bar{M}$ over M , which is locally spanned by N . Then we set

$$S(TM)^\perp = TM^\perp \oplus tr(TM), \quad (3.3)$$

where the decomposition is not orthogonal. Thus we have the following decomposition of $T\bar{M}$

$$T\bar{M} = S(TM) \perp (TM^\perp \oplus tr(TM)) = TM \oplus tr(TM). \quad (3.4)$$

Then N is orthogonal to $\bar{\phi}E$ and we have

$$\bar{g}(\bar{\phi}N, E) = -\bar{g}(N, \bar{\phi}E) = 0, \quad \bar{g}(\bar{\phi}N, N) = 0, \quad (3.5)$$

which means that $\bar{\phi}N$ is also tangent to M and belongs to $S(TM)$. From (2.1) we have

$$\bar{g}(\bar{\phi}E, \bar{\phi}N) = 1. \quad (3.6)$$

Therefore, $\bar{\phi}(TM^\perp) \oplus \bar{\phi}(tr(TM))$ (a direct sum but not orthogonal) is a non-degenerate vector subbundle of $S(TM)$ of rank 2. Then there exists a non-degenerate distribution D_0 on M such that

$$S(TM) = \{\bar{\phi}(TM^\perp) \oplus \bar{\phi}(tr(TM))\} \perp D_0, \quad (3.7)$$

where $\xi \in \Gamma(D_0)$ and D_0 is an invariant distribution with respect to $\bar{\phi}$, i.e., $\bar{\phi}(D_0) = D_0$. Hence, from (3.1), (3.3), (3.4) and (3.7) we obtain the decompositions

$$TM = \{\bar{\phi}(TM^\perp) \oplus \bar{\phi}(tr(TM))\} \perp D_0 \perp TM^\perp, \quad (3.8)$$

$$T\bar{M} = \{\bar{\phi}(TM^\perp) \oplus \bar{\phi}(tr(TM))\} \perp D_0 \perp \{TM^\perp \oplus tr(TM)\}. \tag{3.9}$$

4. A Brief Review of Lightlike Immersions

In the present section, we recall some results from the general theory of lightlike hypersurfaces(cf. [7]). Let (M, g) be a lightlike hypersurface of an indefinite cosymplectic manifold $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$. Then, according to the decomposition (3.4), we write for any $X, Y \in \Gamma(TM)$

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{4.1}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \tag{4.2}$$

where $\nabla_X Y$ and $A_N X$ belong to $\Gamma(TM)$, while $B(X, Y)$ and $\tau(X)$ are smooth functions on M . We call B, A_N and τ the second fundamental form, the shape operator, and the transversal 1-form, respectively, for the lightlike immersion of M in \bar{M} . As in the non-degenerate case we call (4.1) and (4.2) the formulae of Gauss and Weingarten of the lightlike hypersurface M , respectively. It is easy to see that B is a symmetric tensor field of type $(0,2)$, τ is a differential 1-form, A_N is a tensor field of type $(1,1)$ and ∇ is a torsion-free linear connection on M . Moreover the second fundamental form B is independent of the choice of screen distribution, in fact, from (4.1) and (3.2) we obtain $B(X, Y) = \bar{g}(\bar{\nabla}_X Y, E)$ for any $X, Y \in \Gamma(TM)$. The tensor fields B and A_N are not related by means of g , and therefore, in general, A_N is not symmetric with respect to g . The 1-form τ , in general, does not disappear as it does in the non-degenerate case. Furthermore, the induced linear connection ∇ is not a metric connection. More precisely, we obtain from (4.1) and the fact that $\bar{\nabla}$ is a metric connection.

$$(\nabla_X g)(Y, Z) = B(X, Y)\theta(Z) + B(X, Z)\theta(Y) \tag{4.3}$$

for any $X, Y \in \Gamma(TM)$, where θ is a differential 1-form locally defined on M by

$$\theta(X) := \bar{g}(X, N), \quad \forall X \in \Gamma(TM). \tag{4.4}$$

Next, we denote by P the projection morphism of TM on $S(TM)$ with respect to the orthogonal decomposition $TM = S(TM) \perp TM^\perp$. Taking into account of this decomposition, we can put

$$\nabla_X PY = \nabla_X^* PY + C(X, PY)E, \tag{4.5}$$

$$\nabla_X E = -A_E^* X + n(X)E \tag{4.6}$$

for any $X, Y \in \Gamma(TM), E \in \Gamma(TM^\perp)$, where $\nabla_X^* PY$ and $A_E^* X$ belong to $\Gamma(S(TM))$, while $C(X, PY)$ and $n(X)$ are smooth functions on M . It follows that C is $C^\infty(M)$ -bilinear on $\Gamma(TM) \times \Gamma(S(TM))$, but in general, it is not symmetric on $\Gamma(S(TM)) \times \Gamma(S(TM))$, A_E^* is a tensor field of type (1,1) on M and ∇^* is a linear connection on the screen distribution $S(TM)$, respectively. Then, using (3.2), (4.1), (4.2) and (4.6) we obtain

$$n(X) = \bar{g}(\nabla_X E, N) = \bar{g}(\bar{\nabla}_X E, N) = -\bar{g}(E, \bar{\nabla}_X N) = -\tau(X).$$

Hence (4.6) becomes

$$\nabla_X E = -A_E^* X - \tau(X)E. \tag{4.7}$$

It follows from (4.1),(4.2),(4.5) and (4.6) that

$$g(A_N X, PY) = C(X, PY), \quad \bar{g}(A_N X, N) = 0, \tag{4.8}$$

$$g(A_E^* X, PY) = B(X, PY), \quad \bar{g}(A_E^* X, N) = 0 \tag{4.9}$$

for any $X, Y \in \Gamma(TM)$. From $\bar{g}(\bar{\nabla}_X E, E) = 0$ we get

$$B(X, E) = 0, \quad \forall X \in \Gamma(TM) \tag{4.10}$$

Finally, we are concerned with the structure equations of the immersion of a lightlike hypersurface into an indefinite cosymplectic manifold. Denote by \bar{R} and R the the curvature tensor of $\bar{\nabla}$ and ∇ , respectively. By definition of curvature tensor, we obtain from (4.1) and (4.2).

$$\begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z + B(X, Z)A_N Y - B(Y, Z)A_N X \\ &\quad + \{(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \\ &\quad + B(Y, Z)\tau(X) - B(X, Z)\tau(Y)\}N \end{aligned} \tag{4.11}$$

for any $X, Y, Z \in \Gamma(TM)$, where we have put

$$(\nabla_X B)(Y, Z) = XB(Y, Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z). \tag{4.12}$$

Then we have the Gauss-Codazzi equations of the lightlike hypersurface $(M, g, S(TM))$.

$$\begin{aligned} \bar{g}(\bar{R}(X, Y)Z, PW) &= g(R(X, Y)Z, PW) + B(X, Z)C(Y, PW) \\ &\quad - B(Y, Z)C(X, PW). \end{aligned} \tag{4.13}$$

$$\begin{aligned} \bar{g}(\bar{R}(X, Y)Z, E) &= (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \\ &\quad + B(Y, Z)\tau(X) - B(X, Z)\tau(Y). \end{aligned} \tag{4.14}$$

$$\bar{g}(\bar{R}(X, Y)Z, N) = \bar{g}(R(X, Y)Z, N) \tag{4.15}$$

for any $X, Y, Z, W \in \Gamma(TM)$, where we have used (3.2), (4.8), (4.10) and (4.11).

5. Integrability

Let $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$ be an indefinite cosymplectic manifold and (M, g) be its lightlike hypersurface. We consider the distribution on M

$$D := TM^\perp \perp \bar{\phi}(TM^\perp) \perp D_0, \quad D' := \bar{\phi}(tr(TM)).$$

Then D is invariant under $\bar{\phi}$ and

$$TM = D \oplus D' \tag{5.1}$$

Now we consider the local lightlike vector fields

$$U := -\bar{\phi}N, \quad V := -\bar{\phi}E. \tag{5.2}$$

Then, from (5.1), any $X \in \Gamma(TM)$, is written as

$$X = SX + QX, \quad QX = u(X)U, \tag{5.3}$$

where S and Q are the projection morphisms of TM into D and D' , respectively, and u is a 1-form locally defined on M by

$$u(X) := g(X, V). \tag{5.4}$$

Applying $\bar{\phi}$ to (5.3) and using (2.1) (note that $\bar{\phi}^2N = -N$), we obtain

$$\bar{\phi}X = \phi X + u(X)N, \tag{5.5}$$

where ϕ is a tensor field of type (1,1) defined on M by

$$\phi X := \bar{\phi}SX, \quad X \in \Gamma(TM). \tag{5.6}$$

Again, applying $\bar{\phi}$ to (5.5) and using (2.1), we also have

$$\phi^2X = -X + \eta(X)\xi + u(X)U, \quad X \in \Gamma(TM). \tag{5.7}$$

Differentiating (5.5) and comparing both sides with respect to the decomposition (3.4), we obtain for any $X, Y \in \Gamma(TM)$

$$(\nabla_X \phi)Y = u(Y)A_N X - B(X, Y)U, \tag{5.8}$$

$$(\nabla_X u)Y = -B(X, \phi Y) - u(Y)\tau(X), \tag{5.9}$$

where we have used (4.1), (4.2) and (5.4).

Theorem 5.1. *A lightlike hypersurface M of an indefinite cosymplectic manifold \bar{M} is totally geodesic, i.e., $B = 0$ if and only if*

$$(\nabla_X \phi)Y = 0, \quad \forall X \in \Gamma(TM), Y \in \Gamma(D), \quad (5.10)$$

$$A_N X = -\phi(\nabla_X U), \quad \forall X \in \Gamma(TM) \quad (5.11)$$

Proof. Note that $u(Y) = 0, \forall Y \in \Gamma(D)$. Then (5.8) is reduced to the equation

$$(\nabla_X \phi)Y = -B(X, Y)U, \quad (5.12)$$

where $Y \in \Gamma(D)$. On the other hand, replacing Y by U in (5.8), we also obtain

$$A_N X = -\phi(\nabla_X U) + B(X, U)U \quad (5.13)$$

with the aid of $u(U) = 1$. Therefore if we assume that M is totally geodesic, then (5.10) and (5.11) follow from (5.12) and (5.13), respectively. The converse is clear. Thus we complete the proof. \square

Proposition 5.2. *Let M be a lightlike hypersurface of an indefinite cosymplectic manifold \bar{M} . Then we have for any $X \in \Gamma(TM)$*

(i) *if the vector field U is parallel, then*

$$A_N X = \eta(A_N X)\xi + u(A_N X)U, \quad \tau(X) = 0, \quad (5.14)$$

(ii) *if the vector V is parallel, then*

$$A_E^* X = \eta(A_E^* X)\xi + u(A_E^* X)U, \quad \tau(X) = 0. \quad (5.15)$$

Proof. (i) Applying ϕ to (5.13) and using (5.7), we have

$$\phi(A_N X) = \nabla_X U - \eta(\nabla_X U)\xi - u(\nabla_X U), \quad X \in \Gamma(TM).$$

If U is parallel, i.e., $\nabla_X U = 0$, then this equation reduces to $\phi(A_N X) = 0$, from which and (5.5), we have $\bar{\phi}(A_N X) = u(A_N X)N$. Applying $\bar{\phi}$ to this equation and using (2.3), we get the first equation in (5.14). Replacing Y in (5.9) by U and noting that $u(U) = 1$, we have $(\nabla_X u)(U) = -\tau(X)$, and so $\tau(X) = 0$, since $(\nabla_X u)(U) = -u(\nabla_X U) = 0$.

(ii) Suppose that the vector field V is parallel. Replacing Y by E in (5.8) and remembering $E \in \Gamma(TM^\perp)$ and (4.10), we have $(\nabla_X \phi)E = 0$. Hence

$$\begin{aligned} 0 &= (\nabla_X \phi)E = \nabla_X(\phi(E)) - \phi(\nabla_X E) \\ &= \nabla_X(\bar{\phi}(E)) - \nabla_X(u(E)N) - \phi(\nabla_X E) \\ &= -\nabla_X V - \phi(-A_E^* X - \tau(X)E) = \phi(A_E^* X + \tau(X)E). \end{aligned}$$

Applying ϕ to this equation and using (5.7)

$$-A_E^* X + \eta(A_E^* X)\xi + u(A_E^* X)U = \tau(X)E.$$

But the left and right hand side lie in the screen distribution $\Gamma(S(TM))$ and $\Gamma(TM^\perp)$, respectively. Hence (5.15) follows from (3.4). \square

Now, according to the decomposition (3.8), we consider a local field of frames on M ,

$$\{\bar{\phi}E, \bar{\phi}N, \xi, E_a, \bar{\phi}E_a, E; a = 1, 2, \dots, n - 2\}$$

such that $\{\xi, E_a, \bar{\phi}E_a; a = 1, 2, \dots, n - 2\}$ is an orthonormal field of frames of D_0 . Put

$$D_0 = D'_0 \perp [\xi].$$

Then, using (2.1)~(2.4),(4.1),(4.2) and (4.7)~(4.10), we obtain

Proposition 5.3. *Let M be a lightlike hypersurface of an indefinite cosymplectic manifold \bar{M} .*

(i) $TM^\perp \perp \bar{\phi}(TM^\perp)$ is integrable if and only if

$$B(X, Y) = 0, \quad \forall X \in \Gamma(\bar{\phi}(TM^\perp)), Y \in \Gamma(\bar{\phi}(TM^\perp) \perp D'_0). \quad (5.16)$$

(ii) $TM^\perp \perp \bar{\phi}(TM^\perp) \perp [\xi]$ is integrable if and only if (5.16) holds.

(iii) $\bar{\phi}(TM^\perp) \oplus \bar{\phi}(tr(TM))$ is integrable if and only if

$$C(U, V) = C(V, U), \quad (5.17)$$

$$B(U, \bar{\phi}X) = C(V, \bar{\phi}X), \quad \forall X \in \Gamma(D'_0). \quad (5.18)$$

(iv) D_0 is interable if and only if

$$C(\xi, X) = 0, \quad \forall X \in \Gamma(D'_0), \quad (5.19)$$

$$C(X, Y) = C(Y, X), \quad \forall X, Y \in \Gamma(D'_0), \quad (5.20)$$

$$C(X, \bar{\phi}Y) = C(Y, \bar{\phi}X), \quad \forall X, Y \in \Gamma(D'_0) \quad (5.21)$$

and

$$B(X, \bar{\phi}Y) = B(Y, \bar{\phi}X), \quad \forall X, Y \in \Gamma(D'_0). \quad (5.22)$$

(v) $TM^\perp \perp D_0$ is integrable if and only if (5.19),(5.21) and (5.22) hold and

$$B(X, V) = 0, \quad \forall X \in \Gamma(D'_0), \quad (5.23)$$

$$C(E, \bar{\phi}X) + B(X, U) = 0, \quad \forall X \in \Gamma(D'_0). \quad (5.24)$$

(vi) $\bar{\phi}(TM^\perp) \perp D_0$ is integrable if and only if (5.19), (5.20) and (5.22) hold and

$$C(X, V) = C(V, X), \quad \forall X \in \Gamma(D'_0), \quad (5.25)$$

$$B(V, \bar{\phi}X) = 0, \quad \forall X \in \Gamma(D'_0). \quad (5.26)$$

(vii) $\bar{\phi}(\text{tr}(TM)) \perp D_0$ is integrable if and only if (5.19), (5.20) and (5.21) hold and

$$C(\xi, U) = 0, \quad (5.27)$$

$$C(X, U) = C(U, X), \quad \forall X \in \Gamma(D'_0), \quad (5.28)$$

$$C(U, \bar{\phi}X) = 0 \quad \forall X \in \Gamma(D'_0). \quad (5.29)$$

(viii) D is integrable if and only if (5.22), (5.23) and (5.26) hold and

$$B(V, V) = 0. \quad (5.30)$$

(ix) $\bar{\phi}(\text{tr}(TM)) \perp TM^\perp \perp D_0$ is integrable if and only if (5.19), (5.21), (5.24) and (5.29) hold and

$$B(U, U) = 0. \quad (5.31)$$

Remark. Proposition 5.3 indicates that the integrability of distributions involved is expressed as both second fundamental forms of M and $S(TM)$. In the proof of Proposition 5.3 we have used the following identities.

$$B(X, \xi) = 0, \quad C(X, \xi) = 0, \quad \forall X \in \Gamma(TM), \quad (5.32)$$

where the first identity follows from (2.5), (3.2), (4.1), and the second one follows from (2.5), (3.2), (4.1), (4.5), respectively.

Corollary 5.4. *Let M be a totally geodesic, lightlike hypersurface of an indefinite cosymplectic manifold \bar{M} . Then the distributions $TM^\perp \perp \bar{\phi}(TM^\perp)$, $TM^\perp \perp \bar{\phi}(TM^\perp) \perp [\xi]$ and D on M are integrable.*

Corollary 5.5. *Let M be a totally geodesic, lightlike hypersurface of an indefinite cosymplectic manifold \bar{M} . Then*

(i) *On each leaf L of the distribution D on M the followings are valid :*

$$\phi^2(X) = -X + \eta(X)\xi,$$

$$(\nabla_X \phi)Y = 0, \quad \forall X, Y \in \Gamma(D).$$

(ii) *The distribution D is parallel with respect to the induced connection ∇ .*

Proof. (i) follows from (5.7) with $u = 0$ and (5.10). For the assertion (ii) the D' -components of $\nabla_X Y$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D_0)$ vanish, i.e.,

$$g(\nabla_X E, \bar{\phi}E) = g(\nabla_X \bar{\phi}E, \bar{\phi}E) = g(\nabla_X Y, \bar{\phi}E) = 0$$

Hence D is parallel with respect to ∇ . □

6. Totally Umbilical Lightlike Hypersurfaces.

A submanifold M is called a *totally umbilical lightlike hypersurface* of a semi-Riemannian manifold \bar{M} if the local second fundamental form B of M satisfies

$$B(PX, PY) = \rho g(PX, PY), \quad \forall X, Y \in \Gamma(TM), \tag{6.1}$$

where ρ is a smooth function on M .

Theorem 6.1. *Let M be a totally umbilical lightlike hypersurface of an indefinite cosymplectic space form $\bar{M}(c)$. Then $c = 0$ (i.e., $\bar{M}(c)$ is of constant curvature 0) and ρ satisfies the partial differential equations*

$$E\rho + \rho\tau(E) - \rho^2 = 0, \tag{6.2}$$

and

$$PX(\rho) + \rho\tau(PX) = 0, \quad \forall X \in \Gamma(TM). \tag{6.3}$$

Proof. From (2.6) we get

$$\begin{aligned} \bar{g}(\bar{R}(X, Y)Z, E) &= \frac{c}{4} \{ \bar{g}(\bar{\phi}Y, Z)\bar{g}(\bar{\phi}X, E) + \bar{g}(\bar{\phi}Z, X)\bar{g}(\bar{\phi}Y, E) \\ &\quad - 2\bar{g}(\bar{\phi}X, Y)\bar{g}(\bar{\phi}Z, E) \} \end{aligned} \tag{6.4}$$

for any $X, Y, Z \in \Gamma(TM)$. Substituting (4.14) into the left hand side of (6.4) and using (5.4) yield

$$\begin{aligned} &\frac{c}{4} \{ \bar{g}(\bar{\phi}Y, Z)u(X) + \bar{g}(\bar{\phi}Z, X)u(Y) - 2\bar{g}(\bar{\phi}X, Y)u(Z) \} \\ &= (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) - \tau(Y)B(X, Z). \end{aligned} \tag{6.5}$$

Replacing X, Y and Z in (6.5) by PX, E and PZ respectively, we deduce

$$-\frac{3}{4} c u(PX)u(PZ) = \{-E(\rho) - \rho\tau(E) + \rho^2\}g(PX, PZ), \tag{6.6}$$

where we have used (4.3), (4.4), (4.10), (4.12) and (6.1).

Taking $PX = PZ = U$ in (6.6) yields $c = 0$. Next, substituting $X = E, Y = PY, Z = PY$ into (6.5) with $c = 0$ gives

$$\{E(\rho) - \rho\tau(E) + \rho^2\}g(PY, PY) = 0,$$

which means (6.2), since we can take Y such that $g(PY, PY) \neq 0$ (locally).

Finally, substituting $X = PY, Y = PY, Z = PZ$ into (6.5) with $c = 0$ and remembering that $S(TM)$ is nondegenerate, we get

$$\{PX(\rho) + \rho\tau(PX)\}PY = \{PY(\rho) + \rho\tau(PY)\}PX. \tag{6.7}$$

Now suppose that there exists a vector field X_0 on some neighborhood of M such that $PX_0(\rho) + \rho\tau(PX_0) \neq 0$ at some point p in the neighborhood. Then from (6.7) it follows that all vectors of the fibre $S(TM)_p$ are collinear with $\dim S(TM)_p > 1$. This implies (6.3). □

From Theorem 6.1 we obtain

Corollary 6.2. *There exist no totally umbilical lightlike hypersurfaces of an indefinite cosymplectic space form $\bar{M}(c)$ with $c \neq 0$.*

Next, we say that the screen distribution $S(TM)$ is *totally umbilical* if

$$C(X, PY) = \lambda g(X, PY), \quad \forall X, Y \in \Gamma(TM) \tag{6.8}$$

holds, where λ is a smooth function on M .

Proposition 6.3. *Let $(M, g, S(TM))$ be a lightlike hypersurface of $\bar{M}(c)$ such that $S(TM)$ is totally umbilical. Then $S(TM)$ is totally geodesic, i.e. $C(X, PY) = 0$ for any $X, Y \in \Gamma(TM)$.*

Proof. First, the direct calculation of the right hand side in (4.15) shows that

$$\begin{aligned} \bar{g}(\bar{R}(X, Y)PZ, N) = & (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \\ & + \tau(Y)C(X, PZ) - \tau(X)C(Y, PZ) \end{aligned} \tag{6.9}$$

with the aid of (4.5) and (4.7), where we have put

$$(\nabla_X C)(Y, PZ) := X(C(Y, PZ)) - C(\nabla_X Y, PZ) - C(Y, \nabla_X^* PZ).$$

Taking $X = E, Y = PZ = U$ in (6.9) and using (6.8) we have

$$\begin{aligned}
 \bar{g}(\bar{R}(E, U)U, N) &= (\nabla_E \lambda g)(U, U) - (\nabla_U \lambda g)(E, U) \\
 &= \lambda(\nabla_E g)(U, U) - \lambda(\nabla_U g)(E, U) = \lambda g(\nabla_U E, U) \\
 &= -\lambda \bar{g}(\bar{\nabla}_U E, \bar{\phi}N) = -\lambda \bar{g}(\bar{\phi}E, \bar{\nabla}_U N) \\
 &= \lambda \bar{g}(A_N U, \bar{\phi}E) = \lambda C(U, \bar{\phi}E) \\
 &= \lambda^2 \bar{g}(U, \bar{\phi}E) = -\lambda^2.
 \end{aligned}$$

On the other hand, it is clear from (2.6) that $\bar{g}(\bar{R}(E, U)U, N) = 0$, which means that $S(TM)$ is totally geodesic. □

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REFERENCES

- [1] A.Bejancu and K.L.Dugall, Degenerate hypersurfaces of semi-Riemannian manifolds, Bull. Inst. Politechnic. Iasi 37(1991), 13-22.
- [2] D.E.Blair, Riemannian geometry of contact and symplectic manifolds, Birkh  user, 2002.
- [3] W.B.Bonnor, Null hypersurfaces in Minkowski space-times, Tensor(N.S.) 24(1972), 329-345.
- [4] Fr.Cagnac, Geometrie des hypersurfaces isotropes, C.R. Acad. Sci. Paris 201(1965), 3045-3048.
- [5] K.L.Dugall, Spacetime manifolds and contact structures, Internat. J. Math. & Math. Sci., 13(1990), 545-554.
- [6] K.L.Dugall and A.Bejancu, Light-like CR hypersurfaces of indefinite Kaehler manifolds, Acta Appl. Math. 31(1993), 171-190.
- [7] K.L.Dugall and A.Bejancu, Lightlike submanifolds of semi-Riemannian manifolds and applications, Kluwer Academic Publishers, Dordrecht. 1996.
- [8] J.B.Kamerer, Theoreme de Peeling et hypersurfaces isotropes, Rend. Circ. Mat. Palermo(2) 16(1967), 129-202.

[9] T.H.Kang, Lightlike real hypersurfaces of indefinite quaternionic Kaehler manifolds, *Indian J. pure and appl. Math.* 33(12) (2002), 1755-1766.

[10] T.H.Kang, S.D.Jung, B.H.Kim, H.K.Pak and J.S.Pak, Lightlike hypersurfaces of indefinite Sasakian manifolds, *Indian J. pure and appl. Math.* 34(9) (2003), 1369-1380.

[11] G.D.Ludden, Submanifolds of cosymplectic manifolds, *J. Diff. Geom.* 4(1970), 237-244.

[12] R.Rosca, Sur les hypersurfaces isotropes de default incluses dans une variate Lorentzienne, *C.R. Acad. Sci. Paris* 272(1979), 393-396.

[13] R.Rosca, On null hypersurfaces of a Lorentzian manifold, *Tensor(N.S.)* 23(1972), 66-74.

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