Lightlike hypersurfaces of indefinite Sasakian manifolds with parallel symmetric bilinear forms

Fortuné Massamba

Abstract. In this paper, we investigate lightlike hypersurfaces of indefinite Sasakian manifold which are tangent to the structure vector field ξ. Some necessary and sufficient conditions have been given for lightlike hypersurface to be totally geodesic, $D \perp \langle\xi\rangle$-totally geodesic and D'-totally geodesic. We prove that, under some conditions, the geometry of lightlike hypersurface M of indefinite Sasakian manifold has a close relation with the geometry of structure vector field ξ, the distributions $TM^{\perp}$ and $\overline{\phi}(TM^{\perp})$.

M.S.C. 2000: 53C15, 53C25, 53C50.
Key words: lightlike hypersurfaces, indefinite Sasakian, screen distribution, Killing distribution, parallel distribution.

1 Introduction

The general theory of lightlike (degenerate) submanifolds of semi-Riemannian (or Riemannian) manifolds is one of the interesting topics of differential geometry. Many authors have studied lightlike hypersurfaces (or submanifolds) of semi-Riemannian manifold [3], [5] and others. The growing importance of geometry in mathematical physics and very limited information available on lightlike hypersurfaces of indefinite Sasakian manifold are the motivation for the study of this topic. The aim of this note is to study those lightlike hypersurfaces of indefinite Sasakian manifold which are tangent to the structure vector field. In Section 2 and section 3, we give the basic information needed for the rest of the note and the decomposition of almost contact metric, respectively. In Section 4, we prove, under some conditions, that the geometry of M is closely related with the geometry of structure vector field ξ, the distributions $TM^{\perp}$ and $\overline{\phi}(TM^{\perp})$. Finally, we end this section by a characterization of the $D \perp \langle\xi\rangle$-totally geodesic lightlike hypersurfaces of indefinite Sasakian manifold, a classification and also discuss the effect of the change of the screen distribution on different results.

Differential Geometry - Dynamical Systems, Vol.10, 2008, pp. 226-234.

2 Preliminaries

A $(2n+1)$-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ is said to be an *indefinite Sasakian manifold* if it admits an almost contact structure $(\overline{\phi}, \xi, \eta)$, i.e. $\overline{\phi}$ is a tensor field of type $(1,1)$ of rank $2n$, ξ is a vector field, and η is a 1-form, satisfying

$$\begin{aligned} &\overline{\phi}^2 = -\mathbb{I} + \eta \otimes \xi,\ \eta(\xi) = 1,\ \eta \circ \overline{\phi} = 0,\ \overline{\phi}\xi = 0,\ \eta(\overline{X}) = \overline{g}(\xi, \overline{X}), \\ &\overline{g}(\overline{\phi}\,\overline{X}, \overline{\phi}\,\overline{Y}) = \overline{g}(\overline{X}, \overline{Y}) - \eta(\overline{X})\,\eta(\overline{Y}), (\overline{\nabla}_{\overline{X}}\eta)\overline{Y} = \overline{g}(\overline{\phi}\,\overline{X}, \overline{Y}), \\ &(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y} = \overline{g}(\overline{X}, \overline{Y})\xi - \eta(\overline{Y})\overline{X},\ \overline{\nabla}_{\overline{X}}\xi = -\overline{\phi}(\overline{X}),\ \forall\, \overline{X}, \overline{Y} \in \Gamma(T\overline{M})), \end{aligned} \tag{2.1}$$

where $\overline{\nabla}$ is the Levi-Civita connection for a semi-Riemannian metric $\overline{g}$.

A plane section σ in $T_p\overline{M}$ is called a $\overline{\phi}$-section if it is spanned by $\overline{X}$ and $\overline{\phi}\,\overline{X}$, where $\overline{X}$ is a unit tangent vector field orthogonal to ξ. The sectional curvature of a $\overline{\phi}$-section σ is called a $\overline{\phi}$-sectional curvature. A Sasakian manifold $\overline{M}$ with constant $\overline{\phi}$-sectional curvature c is said to be a *Sasakian space form* and is denoted by $\overline{M}(c)$. The curvature tensor $\overline{R}$ of a Sasakian space form $\overline{M}(c)$ is given by [7]

$$\begin{aligned} \overline{R}(\overline{X}, \overline{Y})\overline{Z} &= \frac{c+3}{4}\left(\overline{g}(\overline{Y}, \overline{Z})\overline{X} - \overline{g}(\overline{X}, \overline{Z})\overline{Y}\right) + \frac{c-1}{4}\Big(\eta(\overline{X})\eta(\overline{Z})\overline{Y} - \eta(\overline{Y})\eta(\overline{Z})\overline{X} \\ &+ \overline{g}(\overline{X}, \overline{Z})\eta(\overline{Y})\xi - \overline{g}(\overline{Y}, \overline{Z})\eta(\overline{X})\xi + \overline{g}(\overline{\phi}\,\overline{Y}, \overline{Z})\overline{\phi}\,\overline{X} - \overline{g}(\overline{\phi}\,\overline{X}, \overline{Z})\overline{\phi}\,\overline{Y} \\ &- 2\overline{g}(\overline{\phi}\,\overline{X}, \overline{Y})\overline{\phi}\,\overline{Z}\Big),\ \overline{X},\ \overline{Y},\ \overline{Z} \in \Gamma(T\overline{M}). \end{aligned} \tag{2.2}$$

Let $(\overline{M}, \overline{g})$ be a $(2n+1)$-dimensional semi-Riemannian manifold with index s, $0 < s < 2n+1$ and let (M, g) be a hypersurface of $\overline{M}$, with $g = \overline{g}_{|M}$. M is a lightlike hypersurface of $\overline{M}$ if g is of constant rank $2n-1$ and the normal bundle $TM^{\perp}$ is a distribution of rank 1 on M [3]. A complementary bundle of $TM^{\perp}$ in TM is a rank $2n-1$ non-degenerate distribution over M. It is called a *screen distribution* and is often denoted by $S(TM)$. A lightlike hypersurface endowed with a specific screen distribution is denoted by the triple $(M, g, S(TM))$. As $TM^{\perp}$ lies in the tangent bundle, the following result has an important role in studying the geometry of a lightlike hypersurface.

Theorem 2.1. [3] *Let $(M, g, S(TM))$ be a lightlike hypersurface of $\overline{M}$. Then, there exists a unique vector bundle $N(TM)$ of rank 1 over M such that for any non-zero section E of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exist a unique section N of $N(TM)$ on $\mathcal{U}$ satisfying $g(N, E) = 1$ and $\overline{g}(N, N) = \overline{g}(N, W) = 0$, $\forall\, W \in \Gamma(S(TM)|_{\mathcal{U}})$.*

Throughout the paper, all manifolds are supposed to be paracompact and smooth. We denote Γ(E) the smooth sections of the vector bundle E. Also by $\perp$ and $\oplus$ we denote the orthogonal and nonorthogonal direct sum of two vector bundles. By Theorem 2.1 we may write down the following decomposition [3]

$$TM = S(TM) \perp TM^{\perp}, \quad T\overline{M} = S(TM) \perp (TM^{\perp} \oplus N(TM)). \tag{2.3}$$

Let $\overline{\nabla}$ be the Levi-Civita connection on $(\overline{M}, \overline{g})$, then by using the second decomposition of (2.3), we have Gauss and Weingarten formulae in the form

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM), \tag{2.4}$$

$$\text{and} \quad \overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \quad \forall X, Y \in \Gamma(TM),\ V \in \Gamma(N(TM)), \tag{2.5}$$

where $\nabla_X Y$, $A_V X \in \Gamma(TM)$ and $h(X,Y)$, $\nabla^{\perp}_X V \in \Gamma(N(TM))$. ∇ is a symmetric linear connection on M called an induced linear connection, $\nabla^{\perp}$ is a linear connection on the vector bundle $N(TM)$. h is a $\Gamma(N(TM))$-valued symmetric bilinear form and A_V is the shape operator of M concerning V.

Equivalently, consider a normalizing pair $\{E, N\}$ as in Theorem 2.1. Then (2.4) and (2.5) take the form

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y)\,N \text{ and } \overline{\nabla}_X N = -A_N X + \tau(X)N. \tag{2.6}$$

It is important to mention that the second fundamental form B is independent of the choice of screen distribution, in fact, from (2.6), we obtain $B(X,Y) = \overline{g}(\overline{\nabla}_X Y, E)$ and $\tau(X) = \overline{g}(\nabla^{\perp}_X N, E)$.
Let P be the projection morphism of TM on $S(TM)$ with respect to the orthogonal decomposition of TM. We have

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)E \text{ and } \nabla_X E = -A^*_E X - \tau(X)E, \tag{2.7}$$

where $\nabla^*_X PY$ and $A^*_E X$ belong to $\Gamma(S(TM))$. C, A^*_E and ∇^* are called the local second fundamental form, the local shape operator and the induced connection on $S(TM)$. The induced linear connection ∇ is not a metric connection and we have

$$(\nabla_X g)(Y,Z) = B(X,Y)\theta(Z) + B(X,Z)\theta(Y), \ \forall\, X,\ Y \in \Gamma(TM|_{\mathcal{U}}), \tag{2.8}$$

where θ is a differential 1-form locally defined on M by $\theta(\cdot) := \overline{g}(N, \cdot)$. Also, we have the following identities, $g(A^*_E X, PY) = B(X, PY)$, $g(A^*_E X, N) = 0$, $B(X,E) = 0$. Finally, using (2.6), $\overline{R}$ and R are the curvature tensor of $\overline{M}$ and M are related as

$$\begin{aligned} \overline{R}(X,Y)Z &= R(X,Y)Z + B(X,Z)A_N Y - B(Y,Z)A_N X \\ &+ \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z)\}\,N, \end{aligned} \tag{2.9}$$

$$(\nabla_X B)(Y,Z) = X.B(Y,Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z). \tag{2.10}$$

3 Lightlike hypersurfaces of indefinite Sasakian manifolds

Let $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$ be an indefinite Sasakian manifold and (M, g) be its lightlike hypersurface, *tangent to the structure vector field* ξ *(*$\xi \in TM$*)*[1]. If E is a local section of $TM^{\perp}$, then $\overline{g}(\overline{\phi}E, E) = 0$, and $\overline{\phi}E$ is tangent to M. Thus $\overline{\phi}(TM^{\perp})$ is a distribution on M of rank 1 such that $\overline{\phi}(TM^{\perp}) \cap TM^{\perp} = \{0\}$. This enables us to choose a screen distribution $S(TM)$ such that it contains $\overline{\phi}(TM^{\perp})$ as vector subbundle. We consider local section N of $N(TM)$. Since $\overline{g}(\overline{\phi}\,N, E) = -\overline{g}(N, \overline{\phi}\,E) = 0$, we deduce that $\overline{\phi}\,N$ is also tangent to M and belongs to $S(TM)$. On the other hand, since $\overline{g}(\overline{\phi}\,N, N) = 0$, we see that the components of $\overline{\phi}\,N$ with respect to E vanishes. Thus $\overline{\phi}\,N \in \Gamma(S(TM))$. From (2.1), we have $\overline{g}(\overline{\phi}\,N, \overline{\phi}E) = 1$. Therefore, $\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))$ (direct sum but not orthogonal) is a nondegenerate vector subbundle of $S(TM)$ of rank 2.

[1] Many geometers use to consider ξ tangent to the manifold because in the theory of CR submanifolds the condition M normal to ξ leads to M anti-invariant submanifold (see [8]; Proposition 1.1, p. 43) and the condition ξ oblique gives very complicated embedding equations

It is known [2] that if M is tangent to the structure vector field ξ, then, ξ *belongs to* $S(TM)$. Using this, and since $\overline{g}(\overline{\phi}E, \xi) = \overline{g}(\overline{\phi}N, \xi) = 0$, there exists a nondegenerate distribution D_0 of rank $2n-4$ on M such that

$$S(TM) = \left\{\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))\right\} \perp D_0 \perp \langle\xi\rangle, \tag{3.1}$$

where $\langle\xi\rangle = Span\{\xi\}$. The distribution D_0 is an invariant under $\overline{\phi}$, that is, $\overline{\phi}(D_0) = D_0$.

Moreover, from (2.3) and (3.1) we obtain the decomposition

$$TM = \left\{\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))\right\} \perp D_0 \perp \langle\xi\rangle \perp TM^{\perp}. \tag{3.2}$$

Now, we consider the distributions on M, $D := TM^{\perp} \perp \overline{\phi}(TM^{\perp}) \perp D_0$, $D' := \overline{\phi}(N(TM))$. Then D is invariant under $\overline{\phi}$ and

$$TM = D \oplus D' \perp \langle\xi\rangle. \tag{3.3}$$

Now we consider the local lightlike vector fields $U := -\overline{\phi}\,N$, $V := -\overline{\phi}\,E$. Then, from (3.3), any $X \in \Gamma(TM)$ is written as $X = RX + QX + \eta(X)\xi$, $QX = u(X)\,U$, where R and Q are the projection morphisms of TM into D and D', respectively, and u is a differential 1-form locally defined on M by $u(\cdot) := g(V, \cdot)$. Applying $\overline{\phi}$ to X and (2.1) (note that $\overline{\phi}^2\,N = -N$), we obtain $\overline{\phi}\,X = \phi\,X + u(X)\,N$, where ϕ is a tensor field of type $(1,1)$ defined on M by $\phi\,X := \overline{\phi}\,RX$ and we also have $\phi^2\,X = -\,X + \eta(X)\xi + u(X)\,U$, $\forall\,X \in \Gamma(TM)$. Now applying ϕ to $\phi^2\,X$ and since $\phi U = 0$, we obtain $\phi^3 + \phi = 0$, which shows that ϕ is an f-structure [7] of constant rank. By using (2.1) we derive $g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) - u(Y)v(X) - u(X)v(Y)$, where v is a 1-form locally defined on M by $v(\cdot) = g(U, \cdot)$. From direct calculations, we have the following useful identities

$$\begin{aligned}
\nabla_X\xi &= -\phi X, && (3.4)\\
B(X,\xi) &= -u(X), && (3.5)\\
(\nabla_X u)Y &= -B(X,\phi Y) - u(Y)\tau(X), && (3.6)\\
(\nabla_X\phi)Y &= g(X,Y)\xi - \eta(Y)X - B(X,Y)U + u(Y)A_N X. && (3.7)
\end{aligned}$$

Proposition 3.1. *Let M be a lightlike hypersurface of an indefinite Sasakian manifold $\overline{M}$ with $\xi \in TM$. The Lie derivative with respect to the vector fields V is given by, for any X, $Y \in \Gamma(TM)$,*

$$(L_V g)(X,Y) = X.u(Y) + Y.u(X) + u([X,Y]) - 2u(\nabla_X Y). \tag{3.8}$$

Proof. The proof follows by direct calculation. □

4 Lightlike real hypersurfaces with parallel symmetric bilinear forms

Let $\overline{M}(c)$ be an indefinite Sasakian space form and M be a real lightlike hypersurface of $\overline{M}(c)$. Let us consider the pair $\{E, N\}$ on $\mathcal{U} \subset M$ (see Theorem 2.1) and by using the (2.9), we obtain the local expression

$$\overline{g}(\overline{R}(X,Y)Z,E) = (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z), \tag{4.1}$$

for any X, Y, $Z \in \Gamma(TM_{|\mathcal{U}})$. From (4.1) and by using (2.2), we have

$$(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = \tau(Y)B(X,Z) - \tau(X)B(Y,Z) + \frac{c-1}{4}\left\{\overline{g}(\overline{\phi}Y,Z)u(X) - \overline{g}(\overline{\phi}X,Z)u(Y) - 2\overline{g}(\overline{\phi}X,Y)u(Z)\right\} \tag{4.2}$$

Definition 4.1. (a) A distribution Ξ on M is a Killing distribution (respectively $D \perp \langle\xi\rangle$-Killing distribution) if $(L_X g)(Y,Z) = 0$, for any $X \in \Gamma(\Xi)$ and Y, $Z \in \Gamma(TM)$ (respectively Y, $Z \in \Gamma(D \perp \langle\xi\rangle)$).
(b) A distribution Ξ on M is parallel (respectively $D \perp \langle\xi\rangle$-parallel) if $\nabla_X Y \in \Gamma(\Xi)$, for any $X \in \Gamma(TM)$ (respectively $X \in \Gamma(D \perp \langle\xi\rangle)$) and $Y \in \Gamma(\Xi)$.

Theorem 4.2. *Let M be a lightlike hypersurface of an indefinite Sasakian space form $\overline{M}(c)$ of constant curvature c. Then, the Lie derivative of the second fundamental form B with respect to ξ is given by*

$$L_\xi B(X,Y) = -\tau(\xi)B(X,Y),\ \forall\, X,\, Y \in \Gamma(TM). \tag{4.3}$$

Proof. By replacing Z with ξ into (2.10) and using (3.4), we obtain

$$(\nabla_\xi B)(X,Y) = (L_\xi B)(X,Y) + B(\phi X,Y) + B(X,\phi Y). \tag{4.4}$$

Likewise, by replacing Z with X and X with ξ into (2.10) and also using (3.4) and (3.5), we have

$$(\nabla_X B)(\xi,Y) = -X.u(Y) + B(\phi X,Y) + u(\nabla_X Y). \tag{4.5}$$

Subtracting (4.4) and (4.5), and using (3.6) we obtain

$$(\nabla_\xi B)(X,Y) - (\nabla_X B)(\xi,Y) = (L_\xi B)(X,Y) - u(Y)\tau(X). \tag{4.6}$$

From (4.2), the left hand side of (4.6) becomes

$$(\nabla_\xi B)(X,Y) - (\nabla_X B)(\xi,Y) = -u(Y)\tau(X) - \tau(\xi)B(X,Y) \tag{4.7}$$

The expressions (4.6) and (4.7) implies $(L_\xi B)(X,Y) = -\tau(\xi)B(X,Y)$. □

A lightlike hypersurface M is totally geodesic (respectively $D \perp \langle\xi\rangle$ or D'-totally geodesic) if the local second fundamental form B satisfies $B(X,Y) = 0$, for any X, $Y \in \Gamma(TM)$ (respectively X, $Y \in \Gamma(D \perp \langle\xi\rangle)$ or $\Gamma(D')$).

From the Theorem 4.2, we have the following corollary.

Corollary 4.3. *Let M be a lightlike hypersurface of an indefinite Sasakian space form $\overline{M}(c)$ of constant curvature c, with $\xi \in TM$. Then ξ is a Killing vector field with respect to the second fundamental form B if and only if $\tau(\xi) = 0$ or M is totally geodesic.*

The second fundamental form h of M is said to be parallel if $(\nabla_Z h)(X,Y) = 0$, $\forall\, X,\, Y,\, Z \in \Gamma(TM)$. That is, $(\nabla_Z B)(X,Y) = -\tau(Z)B(X,Y)$. This means that, in general, the parallelism of h does not imply the parallelism of B and vice versa. We note that $(\nabla_Z h)(X,E) = (\nabla_Z B)(X,E)N$.

Theorem 4.4. *Let M be a lightlike hypersurface of an indefinite Sasakian space form $\overline{M}(c)$ of constant curvature c. If the local second fundamental form B of M is parallel, then,*

$$(L_V g)(X,Y) = \tau(\xi)B(X,Y),\ \forall\, X,\, Y \in \Gamma(TM). \tag{4.8}$$

Moreover, if $\tau(\xi) \neq 0$, then M is totally geodesic if and only if $\overline{\phi}(TM^{\perp})$ is a Killing distribution on M.

Proof. Using (4.3), (4.4), (3.4), (3.5) and (3.6), after some computations we have, for any $X,\, Y \in \Gamma(TM)$,

$$\begin{aligned} 0 &= (\nabla_\xi B)(X,Y) = L_\xi B(X,Y) + B(\phi X,Y) + B(X,\phi Y) \\ &= -\tau(\xi)B(X,Y) - (L_V g)(X,Y) - u(X)\tau(Y) - u(Y)\tau(X). \end{aligned} \tag{4.9}$$

On the other hand, using (2.10), we obtain

$$\begin{aligned} 0 = (\nabla_Y B)(\xi,X) &= -Y.u(X) + B(X,\phi Y) + u(\nabla_Y X) \\ &= -(L_V g)(X,Y) - u(Y)\tau(X), \end{aligned} \tag{4.10}$$

$$\begin{aligned} \text{and}\quad 0 = (\nabla_X B)(Y,\xi) &= X.B(\xi,Y) - B(\nabla_X Y,\xi) - B(Y,\nabla_X \xi) \\ &= -(L_V g)(X,Y) - u(X)\tau(Y). \end{aligned} \tag{4.11}$$

So substituting (4.10) and (4.11) in (4.9), we obtain (4.8). If $\tau(\xi) \neq 0$, the equivalence follows. □

Lemma 4.5. *There exist no lightlike hypersuface of indefinite Sasakian space forms $\overline{M}(c)(c \neq 1)$ with parallel second fundamental form.*

Proof. Suppose $c \neq 1$ and second fundamental form is parallel. Then, if we take $Y = E$ and $Z = U$ in (4.2), we obtain $\frac{c-1}{4}u(X) = 0$. Taking $X = U$, we have $c = 1$, which is a contradiction. □

Lemma 4.6. *Let M be a lightlike hypersurface of an indefinite Sasakian space form $\overline{M}(c)$ of constant curvature c such that its local second fundamental form B is parallel. If $\tau(E) \neq 0$, then $c = 1$ if and only if M is D'-totally geodesic.*

Proof. Suppose B is parallel. Then, taking $Y = E$ in (4.2), we obtain $3\frac{c-1}{4}u(X)u(Z) = \tau(E)B(X,Z)$. Taking $X = Z = U$, we have $3\frac{c-1}{4} = \tau(E)B(U,U)$ and if $\tau(E) \neq 0$, the equivalence follows. □

Lemma 4.7. *Let M be a lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M},\overline{g})$, with $\xi \in TM$. If the second fundamental form h of M is parallel, then, $\overline{\phi}(TM^{\perp})$ is a $D \perp \langle\xi\rangle$-Killing distribution and*

$$(L_E B)(X,Y) = -\tau(E)B(X,Y),\ \forall\, X,\, Y \in \Gamma(TM). \tag{4.12}$$

Proof. Taking $Z = \xi$ in $(\nabla_Z h)(X,Y) = 0$, we have $(L_V g)(X,Y) = -u(X)\tau(Y) - u(Y)\tau(X)$ and for any X, $Y \in \Gamma(D \perp \langle\xi\rangle)$, $(L_V g)(X,Y) = 0$. Also by taking $Z = E$ in $(\nabla_Z h)(X,Y) = 0$, we have $(L_E B)(X,Y) = -\tau(E)B(X,Y) + 2B(A^*_E X, Y)$. On the other hand $0 = \overline{g}((\nabla_X h)(Y,E),E) = B(A^*_E X, Y)$. So, we have $(L_E B)(X,Y) = -\tau(E)B(X,Y)$. □

Theorem 4.8. *Let M be a lightlike hypersurface of an indefinite Sasakian manifold $\overline{M}$ with $\xi \in TM$. Then, M is $D \perp \langle\xi\rangle$-totally geodesic if and only if, for any $X \in \Gamma(D \perp \langle\xi\rangle)$, $A^*_E X = u(A_N X)V$.*

Proof. By direct calculation, the shape operator A^*_E is given by $A^*_E X = \sum_{i=1}^{2n-4} \frac{B(X,F_i)}{g(F_i,F_i)} F_i + B(X,\xi)\xi + B(X,U)V + B(X,V)U$, $\forall\ X \in \Gamma(TM)$ and the equivalence follows from the latter. □

It is well known that if the lightlike hypersurface (M, g) is totally geodesic, the induced connection ∇ on M is torsion-free and g-metric. Also, the shape operator A^*_E vanishes identically on M (see Theorem 2.2. [3] p. 88). This vanishing property failed when the lightlike hypersurface M, with $\xi \in TM$, is $D \perp \langle\xi\rangle$-totally geodesic. That is, only some privileged conditions on the screen distribution of M may enable to get the $D \perp \langle\xi\rangle$-version of the Theorem 2.2 in ([3], p. 88).

Now, say that the screen distribution $S(TM)$ is totally umbilical if on any coordinates neighborhood $\mathcal{U} \subset M$, there exists a smooth function ρ such that $C(X, PY) = \rho g(X, PY)$, $\forall X$, $Y \in \Gamma(TM_{|\mathcal{U}})$. If we assume that the screen distribution $S(TM)$ of the lightlike hypersurface M with $\xi \in TM$ is totally umbilical, then it follows that C is symmetric on $\Gamma(S(TM)_{|\mathcal{U}})$ and hence according to Theorem 2.3 in [3], the distribution $S(TM)$ is integrable. Also, we have $A_N X = \rho PX$ and $C(E, PX) = 0$. Since $\overline{\phi}\xi = 0$ and by using $\eta(A_N X) = -v(X)$, we have $\eta(A_N \xi) = \rho\overline{g}(\xi,\xi) = -v(\xi) = 0$ which implies that $\rho = 0$, so the screen distribution $S(TM)$ is totally geodesic. Therefore, we have the following result.

Theorem 4.9. *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$, with $\xi \in TM$, such that $S(TM)$ is totally umbilical. If the second fundamental form h is parallel, then, M is $D \perp \langle\xi\rangle$-totally geodesic if and only if the distribution $\overline{\phi}(TM^\perp)$ is $D \perp \langle\xi\rangle$-parallel.*

Proof. Since the screen distribution $S(TM)$ is totally umbilical, $S(TM)$ is totally geodesic, that is, for any $X, Y \in \Gamma(S(TM))$, $C(X,Y) = 0$. In particular, for any $X \in \Gamma(\overline{\phi}(TM^\perp) \perp D_0 \perp \langle\xi\rangle)$, $C(X,V) = u(A_N X) = 0$. Since $C(E,V) = 0$, for any $X_0 \in \Gamma(D \perp \langle\xi\rangle)$, $u(A_N X_0) = 0$. From the Theorem 4.8, M is $D \perp \langle\xi\rangle$-totally geodesic if and only if, for any $X_0 \in \Gamma(D \perp \langle\xi\rangle_{|\mathcal{U}})$, $A^*_E X_0 = 0$. So $A^*_{\overline{\phi}E} X_0 = \overline{\phi}(A^*_E X_0) - u(A^*_E X_0)N = 0$. Also, if $A^*_{\overline{\phi}E} X_0 = 0$ and using (3.5), it is easy to check that $A^*_E X_0 = -u(X_0)\xi + u(A^*_E X_0)U = u(A^*_E X_0)U = -B(A^*_E X_0, \xi)U = -(\nabla_{X_0} B)(\xi, E)U = -\overline{g}((\nabla_{X_0} h)(\xi, E), E)U = 0$, since $X_0 \in \Gamma(D \perp \langle\xi\rangle)$ and h is parallel. Now, we want to show that $A^*_{\overline{\phi}E} X_0 = 0$ if and only if $\nabla_{X_0} Y_0 \in \Gamma(\overline{\phi}(TM^\perp))$, for any $Y_0 \in \Gamma(\overline{\phi}(TM^\perp)$. Suppose, for any $X_0 \in \Gamma(D \perp \langle\xi\rangle_{|\mathcal{U}})$, $A^*_{\overline{\phi}E} X_0 = 0$. Since the normal bundle $\overline{\phi}(TM^\perp)$ is a distribution on M of rank 1 and spanned by $\overline{\phi}E$, then, for any, $\nabla_{X_0} Y_0 = (X_0.v(Y_0) - v(Y_0)\tau(X_0))\overline{\phi}E \in \Gamma(\overline{\phi}(TM^\perp))$, since $Y_0 = v(Y_0)\overline{\phi}E$. So, the distribution $\overline{\phi}(TM^\perp)$ is $D \perp \langle\xi\rangle$-parallel. Conversely, suppose the distribution $\overline{\phi}(TM^\perp)$ is $D \perp \langle\xi\rangle$-parallel. Then, for any $X_0 \in \Gamma(D \perp \langle\xi\rangle)$

and $Y_0 = v(Y_0)\overline{\phi}E \in \Gamma(\overline{\phi}(TM^{\perp})_{|\mathcal{U}})$, $\nabla_{X_0}Y_0 \in \Gamma(\overline{\phi}(TM^{\perp})_{|\mathcal{U}})$. Since $\overline{\phi}(TM^{\perp})$ is spanned by $\overline{\phi}E$, there exist a smooth functions on M $\lambda \neq 0$ such that $\nabla_{X_0}Y_0 = \lambda\,\overline{\phi}E$. We have $\lambda = \overline{g}(X_0.v(Y_0)\overline{\phi}E + v(Y_0)\nabla_{X_0}\overline{\phi}E, \overline{\phi}N) = X_0.v(Y_0) - v(Y_0)\tau(X_0)$, since $\overline{\phi}N \perp (TM^{\perp} \perp D_0)$ and $A^*_{\overline{\phi}E}X_0 \in \Gamma(TM^{\perp} \perp D_0)$. On the other hand, $\nabla_{X_0}Y_0 = (X_0.v(Y_0) - v(Y_0)\tau(X_0))\overline{\phi}E - v(Y_0)A^*_{\overline{\phi}E}X_0$. So, we have $\nabla_{X_0}Y_0 = (X_0.v(Y_0) - v(Y_0)\tau(X_0))\overline{\phi}E - v(Y_0)A^*_{\overline{\phi}E}X_0 = (X_0.v(Y_0) - v(Y_0)\tau(X_0))\overline{\phi}E$, that is, $v(Y_0)A^*_{\overline{\phi}E}X_0 = 0$. Taking $Y_0 = V$, we have $A^*_{\overline{\phi}E}X_0 = 0$. This completes the proof. □

From the Theorems 4.2, 4.8, 4.4 and 4.9, we have the following characterization of $D \perp \langle\xi\rangle$-totally geodesic of lightlike hypersurfaces M of an indefinite Sasakian manifold $\overline{M}$.

Theorem 4.10. *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$, with $\xi \in TM$, such the $S(TM)$ is totally umbilical. If the local second fundamental form B is parallel and $\tau(\xi) \neq 0$, then the following assertions are equivalent:*
(i) M is $D \perp \langle\xi\rangle$-totally geodesic.
(ii) $A^*_E X = 0$, $\forall\, X \in \Gamma(D \perp \langle\xi\rangle)$.
(iii) $A^*_V X = 0$, $\forall\, X \in \Gamma(D \perp \langle\xi\rangle)$.
(iv) The connection $\widehat{\nabla} = \nabla|_{D\perp\langle\xi\rangle}$ induced by $\overline{\nabla}$ on M is torsion-free and metric.
(v) $\overline{\phi}(TM^{\perp})$ is a $D \perp \langle\xi\rangle$-parallel distribution with respect to ∇.
(vi) $\overline{\phi}(TM^{\perp})$ is a $D \perp \langle\xi\rangle$-Killing distribution on M.

The Theorem 4.10 can be extended by using Theorem 2.2 in [3] (p.88) in order to know more about the geometry of lightlike hypersurface M. Finally, we have the following classification for lightlike hypersurface of an indefinite Sasakian manifold such that V (respectively U) and h are parallel.

Theorem 4.11. *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$ such that V (respectively U) is parallel. If the second fundamental form h is parallel, then,*
(i) ξ and E are Killing vector fields with respect to B.
(ii) $\overline{\phi}(TM^{\perp})$ is Killing distribution on M.

Proof. V is parallel, $\nabla_X V = 0$, then $\tau(X) = 0$ and $(\nabla_Z h)(X, Y) = (\nabla_Z B)(X, Y)N$. If h is parallel, from (4.3), (4.8) and (4.12), the proof is complete. □

Concluding remarks. It is known that the second fundamental form of a lightlike hypersurface M on $\mathcal{U}$ is independent of the choice of the screen distribution [3]. Thus, the results of Theorems 4.2, 4.4, 4.9, 4.10 and 4.11 are stable with respect to any change of the screen distribution. The relationship between the second fundamental forms C and C' of the screen distribution $S(TM)$ and $S(TM)'$, respectively, is given by $C'(X, PY) = C(X, PY) - \frac{1}{2}\omega(\nabla_X PY + B(X, Y)W)$, $\forall\, X,\ Y \in \Gamma(TM)$. where $W = \sum_{i=1}^{2n-1} c_i W_i$ is the characteristic vector field of the screen change and ω is the dual 1-form of W with respect to the induced metric g of M, that is, $\omega(\cdot) = g(W, \cdot)$. The Theorem 4.8 is independent of the screen distribution $S(TM)$ if and if $\omega(\nabla_X V + B(X, V)W) = 0$, $\forall\, X \in \Gamma(D \perp \langle\xi\rangle)$. On the other hand, the effect of the change of the screen distribution on the Lie derivative (3.8) is given by the following results.

Proposition 4.12. *The Lie derivatives L_V and L'_V of the screen distributions $S(TM)$ and $S(TM)'$, respectively, are related as: $(L'_V g)(X,Y) = (L_V g)(X,Y) - H(X,Y)$, where H is a bilinear form defined by $H(X,Y) = u(X)B(Y,W) + u(Y)B(X,W)$.*

Proof. The proof follows from $\tau'(X) = \tau(X) + B(X,W)$. □

From this Proposition, we have the following result

Theorem 4.13. *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. Then, the Lie derivative L_V is unique, that is, L_V is independent of $S(TM)$, if and only if, the second fundamental form h (or equivalently B) of M vanishes identically on M.*

Proof. $(L_V g)(X,Y) = (\nabla_X u)Y + (\nabla_Y u)X$ and using Theorem 2.2 in [3], we complete the proof. □

Acknowledgements.
I would like to thank The Abdus Salam International Centre for Theoretical Physics for the support during this work.

References

[1] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Mathematics 203, Birkhauser Boston, Inc., Boston, MA, 2002.

[2] C. Calin, *Contribution to geometry of CR-submanifold*, Ph.D. Thesis, University of Iasi, Iasi, Romania, 1998.

[3] K. L. Duggal and A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Its Applications*, Kluwer, Dordrecht, 1996.

[4] T. H. Kang, S.D. Jung, B. H. Kim, H. K. Pak and J. S. Pak, *Lightlike hypersurfaces of indefinite Sasakian manifolds*, Indian J. Pure Appl. Math. 34(9) (2003), 1369-1380.

[5] D. N. Kupeli, *Singular Semi-Riemannian Geometry*, Kluwer, Dortrecht, 1996.

[6] T. Takahashi, *Sasakian manifolds with pseudo-Riemannian metric*, Tôhoku Math. J., 21 (1969), 271-290.

[7] K. Yano and M. Kon, *Structures on Manifolds*, World Scientific, Singapore, 1984.

[8] K. Yano and M. Kon, *CR-submanifolds of Kählerian and Sasakian manifolds*, Progr. Math. 30 Birkhäuser, Boston, Basel, Stuttgart, 1983.

Author's address:

Fortuné Massamba
University of Botswana, Department of Mathematics,
Private Bag 0022, Gaborone, Botswana.
E-mail: massfort@yahoo.fr, massambaf@mopipi.ub.bw