Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2007, Article ID 85145, 8 pages
doi:10.1155/2007/85145

Research Article

Limit Cycle for the Brusselator by He's Variational Method

Juan Zhang

Received 10 May 2006; Accepted 5 February 2007

Recommended by Katica R. (Stevanovic) Hedrih

He's variational method for finding limit cycles is applied to the Brusselator. The technique developed in this paper is similar to Kantorovitch's method in variational theory. The present theory can be applied not only to weakly nonlinear equations, but also to strongly ones, and the obtained results are valid for the whole solution domain.

1. Introduction

The Brusselator originates from a chemical reaction which consists of four steps:

$$\overline{A} \longrightarrow X, \quad \overline{B} + X \longrightarrow \overline{D} + Y, \quad 2X + Y \longrightarrow 3X, \quad X \longrightarrow \overline{E}, \tag{1.1}$$

where $\overline{A}$, $\overline{B}$, $\overline{D}$, $\overline{E}$, X, and Y are all species. The differential equations given in dimensionless form for these species are

$$\dot{X} = A - (1+B)X + X^2Y, \tag{1.a}$$

$$\dot{Y} = BX - X^2Y, \tag{2.a}$$

where all rate constants are assumed to be equal to 1, and the reactants $\overline{A}$ and $\overline{B}$ are assumed to be in large excess so that their concentrations do not change with time. The parameters A and B are the controllable parameters.

For this analysis, the dynamics of the Brusselator reaction can be described by a system of two ODEs. In dimensionless forms, they are

$$\dot{x} = A - (1+B)x + x^2 y, \tag{1.b}$$

$$\dot{y} = Bx - x^2 y, \tag{2.b}$$

where $x, y \in \mathbb{R}$, and $A, B \in \mathbb{R}$ are constants with $A, B > 0$, x and y stand for the dimensionless concentrations of reference reactants.

System (1.b)-(2.b) has been extensively studied in a mathematical view [1–3], but rarely in an engineering approach. In engineering, we need a design formulation embodying the essential relationships needed by engineers who have to design practical systems.

System (1.b)-(2.b) has no possible small parameters, so the traditional perturbation methods [4] cannot be directly applied. Recently, some new perturbation methods and nonperturbative methods are proposed, for example, nonperturbative method [5], δ-method [6, 7], artificial small parameter method [8], homotopy perturbation method [9–14], variational iteration methods [15–18], perturbation-incremental method [21, 22], various modified Lindstedt-Poincare methods [23–25], a review of the recently developed analytical methods are given by He [19, 20].

The determination of amplitude and period of limit cycles is a crucial question in nonlinear problems [26–35]. Ji-Huan He suggested an energy approach to limit cycles [26, 27], it is a simple but powerful method. The method is similar to Kantorovitch's method in variational theory, so the method was called as He's variational method by D'Acunto [28, 29]. In this paper, we apply He's variational method to the Brusselator, revealing that the method is very effective and convenient.

2. An illustrative example

Generally speaking, limit cycles can be determined approximately in the form [4, 19, 20, 26, 27]

$$x = b + a(t)\cos\omega t + \sum_{n=1}^{m} \left(C_n \cos n\omega t + D_n \sin n\omega t\right), \tag{2.1}$$

where b, C_n, and D_n are constants.

In order to best illustrate the theory, we consider Duffing equation as an illustrative example,

$$\dot{x} = y, \tag{2.2}$$

$$\dot{y} = -x - \varepsilon x^3. \tag{2.3}$$

Suppose that $x = a\cos\omega t$, where a is a constant. From (2.2), we have $y = -a\omega\sin\omega t$. Substituting the results into (2.3), we get the following residual:

$$R(t) = \dot{y} + x + \varepsilon x^3 = -a\omega^2\cos\omega t + a\cos\omega t + \varepsilon a^3\cos^3\omega t. \tag{2.4}$$

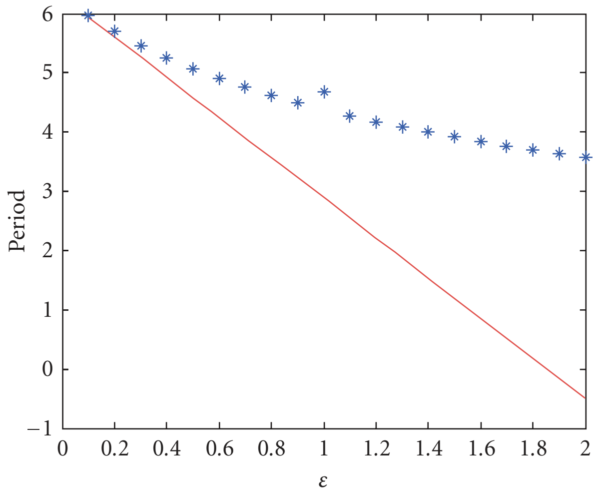


Figure 2.1. Comparison of perturbation period (T_{pert}) of Duffing equation (continuous line) with the exact one (T_{ex}) (discontinuous line).

In general, the residual might not be vanishingly small at all points. The best approximation for the solution is to minimize the residuum R, and the simplest method of obtaining the solution is the weighted residual method [26, 27], which requires that

$$\int_0^T R\cos\omega t dt = 0, \tag{2.5}$$

where T is the period.

From (2.5), we readily obtain the following result:

$$\omega = \sqrt{1 + \frac{3}{4}\varepsilon a^2}. \tag{2.6}$$

We, therefore, obtain the following approximate period:

$$T = \frac{2\pi}{\sqrt{1 + 0.75\varepsilon a^2}}. \tag{2.7}$$

In addition, from [4], we know that the perturbation solution is

$$T_{\text{pert}} = 2\pi\left(1 - \frac{3}{8}\varepsilon a^2\right), \quad \varepsilon \ll 1, \tag{2.8}$$

and the exact solution is

$$T_{\text{ex}} = \frac{4}{\sqrt{1+\varepsilon a^2}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k\sin^2 x}}, \quad k = \frac{\varepsilon a^2}{2(1+\varepsilon a^2)}. \tag{2.9}$$

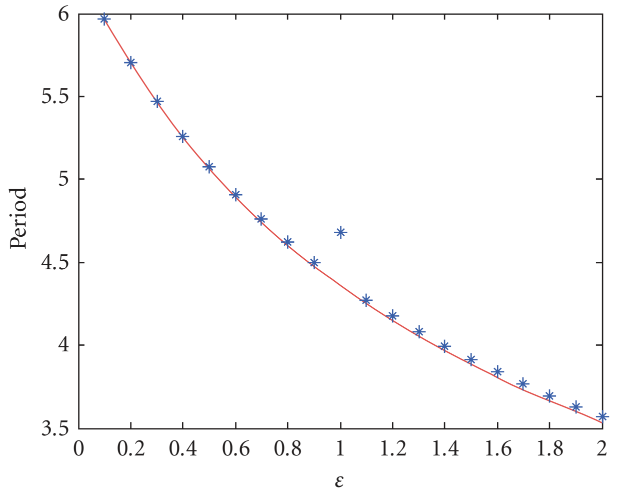


Figure 2.2. Comparison of our result (2.7) of Duffing equation with the exact one. Our result: continuous line; exact solution: discontinuous line.

From Figures 2.1 and 2.2, it is obvious that perturbation solution becomes invalid for large values of ε, however, our result is valid for the whole solution domain, that is, $0 < \varepsilon < \infty$. In case $\varepsilon \to \infty$, we have

$$\lim_{\varepsilon\to\infty} \frac{T_{\text{ex}}}{T} = \frac{2\sqrt{0.75}}{\pi} \int_0^{\pi/2} \frac{dx}{\sqrt{1-0.5\sin^2 x}} = \frac{2\sqrt{0.75}}{\pi} \times 1.68575 = 0.929. \tag{2.10}$$

The 7.6% accuracy is remarkably good in view of the simplest trial function, $x = a\cos\omega t$, when $\varepsilon \to \infty$. The accuracy can be dramatically improved if we choose the trial function in the form $x = a\cos\omega t + b\cos 3\omega t$.

In order to improve the accuracy, we can begin with $x_0 = a\cos\omega t$, then from (2.3) we can obtain y_0; substituting y_0 into (2.2), the function x can be updated as x_1. The procedure can be continued before we use the weighted residual method to identify the frequency. The technique developed in this paper is similar to Kantorovitch's method in variational theory [4].

3. The Brusellator

To simplify the procedure, from (1.2) and (1.3) we can obtain the following equation:

$$\dot{y} = -\dot{x} + A - x. \tag{3.1}$$

System (1.b)-(2.b) is equivalent to (1.b) and (3.1), or (2.b) and (3.1). Now we begin with

$$x = a_0\cos\omega t + a_1, \tag{3.2}$$

where a_0, a_1, and ω are unknown constants. Substituting (3.2) into (3.1) results in

$$\dot{y} = a_0\omega\sin\omega t + A - a_0\cos\omega t - a_1. \tag{3.3}$$

No secular terms in y requires that

$$a_1 = A. \tag{3.4}$$

Solving (3.3), we have

$$y = -a_0 \cos \omega t - \frac{a_0}{\omega} \sin \omega t + b, \tag{3.5}$$

where b is a constant to be further determined.

In view of (3.2) and (3.5), we obtain the following residuum:

$$\begin{aligned}
R = -\dot{y} + Bx - x^2 y &= -a_0 \omega \sin \omega t + a_0 \cos \omega t + B(a_0 \cos \omega t + A) \\
&\quad + (a_0 \cos \omega t + A)^2 \left(a_0 \cos \omega t + \frac{a_0}{\omega} \sin \omega t - b\right) \\
&= -a_0 \omega \sin \omega t + a_0 \cos \omega t + Ba_0 \cos \omega t + AB \\
&\quad + (a_0^2 \cos^2 \omega t + 2Aa_0 \cos \omega t + A^2)\left(a_0 \cos \omega t + \frac{a_0}{\omega} \sin \omega t - b\right) \\
&= -a_0 \omega \sin \omega t + a_0 \cos \omega t + Ba_0 \cos \omega t + AB \\
&\quad + (a_0^3 \cos^3 \omega t + 2Aa_0^2 \cos^2 \omega t + A^2 a_0 \cos \omega t) \\
&\quad + a_0^2 \frac{a_0}{\omega} \sin \omega t \cos^2 \omega t + 2Aa_0 \frac{a_0}{\omega} \sin \omega t \cos \omega t + A^2 \frac{a_0}{\omega} \sin \omega t \\
&\quad - ba_0^2 \cos^2 \omega t - 2Aa_0 b \cos \omega t - A^2 b.
\end{aligned} \tag{3.6}$$

In order to identify the constants a_0, b, and ω, we set

$$\begin{gathered}
\int_0^T R dt = 0, \\
\int_0^T R \cos \omega t dt = 0, \\
\int_0^T R \sin \omega t dt = 0,
\end{gathered} \tag{3.7}$$

where T is the period.

From (3.7), we have

$$\begin{gathered}
AB + Aa_0^2 - \frac{1}{2} ba_0^2 - A^2 b = 0, \\
a_0 + Ba_0 + \frac{3}{4} a_0^3 + A^2 a_0 - 2Aa_0 b = 0, \\
-a_0 \omega + \frac{a_0^3}{4\omega} + A^2 \frac{a_0}{\omega} = 0.
\end{gathered} \tag{3.8}$$

Solving (3.8), simultaneously, we have

$$b = \frac{(7/2)A^2 + B + 1 \pm \sqrt{-(15/4)A^4 + 3A^2(B-3) + (B+1)^2}}{4A},$$
$$a_0^2 = \frac{AB - A - A^3}{b - (5/4)A} = \frac{B - 1 - A^2}{(b/A) - (5/4)}, \tag{3.9}$$
$$\omega = \sqrt{\frac{AB - A - A^3}{4b - 5A} + A^2}.$$

Note that b and a_0 are real numbers, so there are

$$\Delta = -\frac{15}{4}A^4 + 3A^2(B-3) + (B+1)^2 \geq 0,$$
$$\frac{B - 1 - A^2}{(b/A) - (5/4)} \geq 0. \tag{3.10}$$

By a simple analysis, we can obtain the following results.

(1) When $B > 1 + A^2$, the constant b can be finally determined as

$$b = \frac{(7/2)A^2 + B + 1 + \sqrt{-(15/4)A^4 + 3A^2(B-3) + (B+1)^2}}{4A}. \tag{3.11}$$

(2) When $B \leq 1 + A^2$ and $A^2 > 4$, the constant b can be finally determined as

$$b = \frac{(7/2)A^2 + B + 1 \pm \sqrt{-(15/4)A^4 + 3A^2(B-3) + (B+1)^2}}{4A}. \tag{3.12}$$

The approximate period can be written in the form

$$T = \frac{2\pi}{\sqrt{((AB - A - A^3)/(4b - 5A)) + A^2}}, \tag{3.13}$$

where b is defined by (3.11) or (3.12).

4. Conclusion

To summarize, we can conclude from the results thus obtained that the method developed here is extremely simple in its principle, quite easy to use, and gives a very good accuracy in the whole solution domain, even with the simplest trial functions. Theoretically, any accuracy can be arrived at by suitable choice of trial functions or by iterations before weighted residual method is applied.

References

[1] Y. X. Qin and X. W. Zhen, "Qualitative investigation of the differential equation of the Brusselator in biochemistry," *Kexue Tongbao*, vol. 25, no. 4, pp. 273–276, 1980.

[2] Z. Jing, X. Zeng, and K. Y. Chan, "Harmonic and subharmonic bifurcation in the Brusselator model with periodic force," *Acta Mathematicae Applicatae Sinica*, vol. 13, no. 3, pp. 289–301, 1997.

[3] P. Yu and A. B. Gumel, "Bifurcation and stability analyses for a coupled Brusselator model," *Journal of Sound and Vibration*, vol. 244, no. 5, pp. 795–820, 2001.

[4] J.-H. He, *Non-Perturbative Methods for Strongly Nonlinear Problems*, Dissertation.de-Verlag im Internet GmbH, Berlin, Germany, 2006.

[5] B. Delamotte, "Nonperturbative (but approximate) method for solving differential equations and finding limit cycles," *Physical Review Letters*, vol. 70, no. 22, pp. 3361–3364, 1993.

[6] I. Andrianov and J. Awrejcewicz, "Construction of periodic solutions to partial differential equations with non-linear boundary conditions," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 1, no. 4, pp. 327–332, 2000.

[7] C. M. Bender, K. A. Milton, S. S. Pinsky, and L. M. Simmons Jr., "A new perturbative approach to nonlinear problems," *Journal of Mathematical Physics*, vol. 30, no. 7, pp. 1447–1455, 1989.

[8] I. V. Andrianov, J. Awrejcewicz, and A. Ivankov, "Artificial small parameter method—solving mixed boundary value problems," *Mathematical Problems in Engineering*, vol. 2005, no. 3, pp. 325–340, 2005.

[9] J.-H. He, "Homotopy perturbation technique," *Computer Methods in Applied Mechanics and Engineering*, vol. 178, no. 3-4, pp. 257–262, 1999.

[10] J.-H. He, "A coupling method of a homotopy technique and a perturbation technique for non-linear problems," *International Journal of Non-Linear Mechanics*, vol. 35, no. 1, pp. 37–43, 2000.

[11] A. M. Siddiqui, R. Mahmood, and Q. K. Ghori, "Thin film flow of a third grade fluid on a moving belt by He's homotopy perturbation method," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 7, no. 1, pp. 7–14, 2006.

[12] J.-H. He, "Homotopy perturbation method for solving boundary value problems," *Physics Letters. A*, vol. 350, no. 1-2, pp. 87–88, 2006.

[13] J.-H. He, "Application of homotopy perturbation method to nonlinear wave equations," *Chaos, Solitons & Fractals*, vol. 26, no. 3, pp. 695–700, 2005.

[14] J.-H. He, "Homotopy perturbation method for bifurcation of nonlinear problems," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 6, no. 2, pp. 207–208, 2005.

[15] J.-H. He, "Variational iteration method: a kind of nonlinear analytical technique: some examples," *International Journal of Nonlinear Mechanics*, vol. 34, no. 4, pp. 699–708, 1999.

[16] J.-H. He and X.-H. Wu, "Construction of solitary solution and compacton-like solution by variational iteration method," *Chaos, Solitons & Fractals*, vol. 29, no. 1, pp. 108–113, 2006.

[17] Z. M. Odibat and S. Momani, "Application of variational iteration method to nonlinear differential equations of fractional order," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 7, no. 1, pp. 27–34, 2006.

[18] N. Bildik and A. Konuralp, "The use of variational iteration method, differential transform method and adomian decomposition method for solving different types of nonlinear partial differential equations," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 7, no. 1, pp. 65–70, 2006.

[19] J.-H. He, "Some asymptotic methods for strongly nonlinear equations," *International Journal of Modern Physics B*, vol. 20, no. 10, pp. 1141–1199, 2006.

[20] J.-H. He, "Addendum: new interpretation of homotopy perturbation method," *International Journal of Modern Physics B*, vol. 20, no. 18, pp. 2561–2568, 2006.

[21] H. S. Y. Chan, K. W. Chung, and Z. Xu, "A perturbation-incremental method for strongly non-linear oscillators," *International Journal of Non-Linear Mechanics*, vol. 31, no. 1, pp. 59–72, 1996.

[22] H. S. Y. Chan, K. W. Chung, and Z. Xu, "Calculation of limit cycles," in *Proceedings of the 3rd International Conference on Nonlinear Mechanics (ICNM '98)*, pp. 597–601, Shanghai, China, August 1998.

[23] J.-H. He, "Modified Lindstedt-Poincaré methods for some strongly non-linear oscillations—part I: expansion of a constant," *International Journal of Non-Linear Mechanics*, vol. 37, no. 2, pp. 309–314, 2002.

[24] J.-H. He, "Modified Lindstedt-Poincaré methods for some strongly non-linear oscillations—part II: a new transformation," *International Journal of Non-Linear Mechanics*, vol. 37, no. 2, pp. 315–320, 2002.

[25] J.-H. He, "Modified Lindstedt-Poincaré methods for some strongly nonlinear oscillations—part III: double series expansion," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 2, no. 4, pp. 317–320, 2001.

[26] J.-H. He, "Determination of limit cycles for strongly nonlinear oscillators," *Physical Review Letters*, vol. 90, no. 17, Article ID 174301, 3 pages, 2003.

[27] J.-H. He, "Erratum: determination of limit cycles for strongly nonlinear oscillators," *Physical Review Letters*, vol. 91, no. 19, Article ID 199902, 1 page, 2003.

[28] M. D'Acunto, "Determination of limit cycles for a modified van der Pol oscillator," *Mechanics Research Communications*, vol. 33, no. 1, pp. 93–98, 2006.

[29] M. D'Acunto, "Self-excited systems: analytical determination of limit cycles," *Chaos, Solitons & Fractals*, vol. 30, no. 3, pp. 719–724, 2006.

[30] J.-H. He, "Periodic solutions and bifurcations of delay-differential equations," *Physics Letters. A*, vol. 347, no. 4–6, pp. 228–230, 2005.

[31] J.-H. He, "Limit cycle and bifurcation of nonlinear problems," *Chaos, Solitons & Fractals*, vol. 26, no. 3, pp. 827–833, 2005.

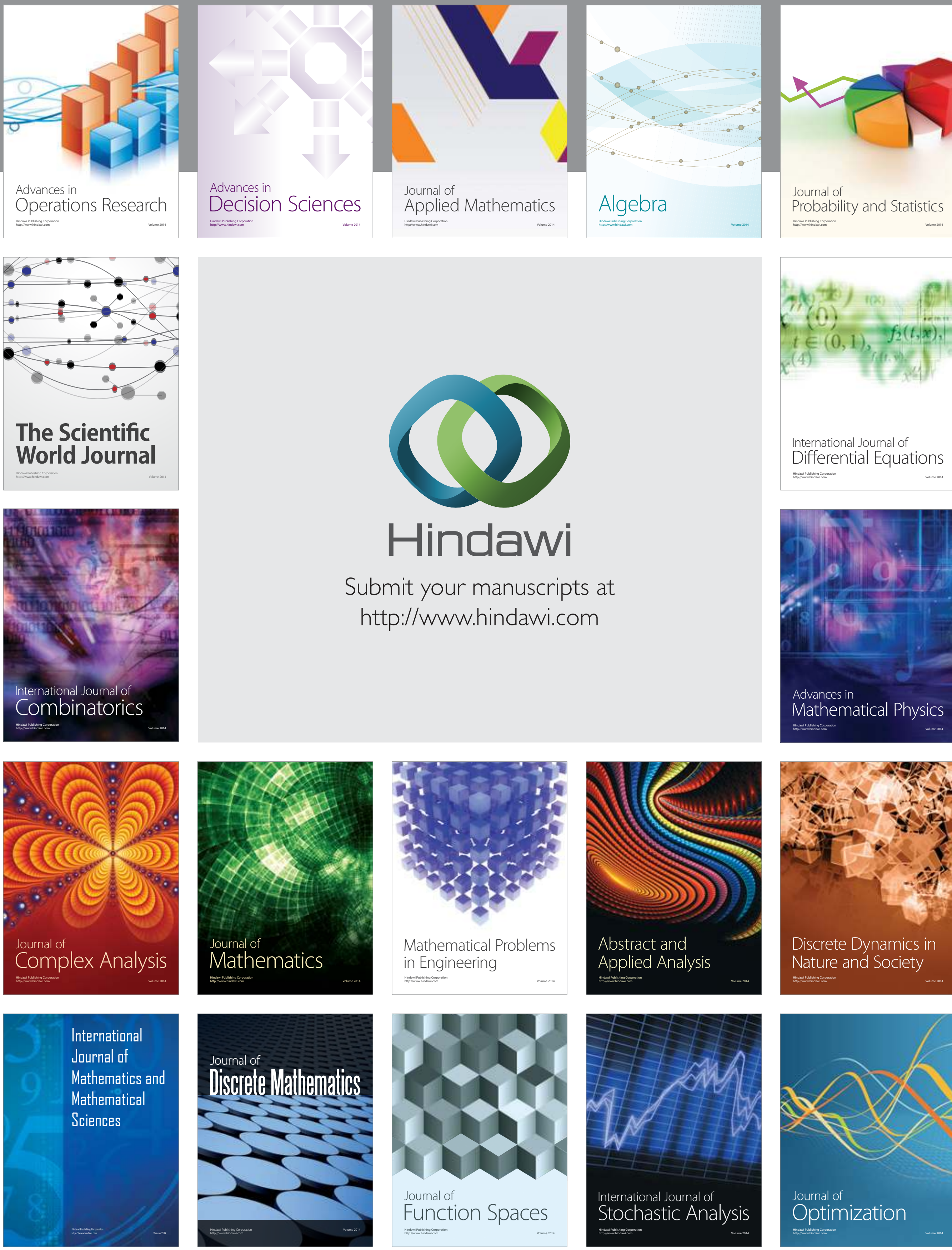
[32] M. S. Padin, F. I. Robbio, J. L. Moiola, and G. Chen, "On limit cycle approximations in the van der Pol oscillator," *Chaos, Solitons & Fractals*, vol. 23, no. 1, pp. 207–220, 2005.

[33] T. Ozis and A. Yildirim, "Determination of limit cycles by a modified straightforward expansion for nonlinear oscillators," *Chaos, Solitons & Fractals*, vol. 32, no. 2, pp. 445–448, 2007.

[34] M. Marhl and M. Perc, "Determining the flexibility of regular and chaotic attractors," *Chaos, Solitons & Fractals*, vol. 28, no. 3, pp. 822–833, 2006.

[35] I. Bashkirtseva and L. Ryashko, "Sensitivity and chaos control for the forced nonlinear oscillations," *Chaos, Solitons & Fractals*, vol. 26, no. 5, pp. 1437–1451, 2005.

Juan Zhang: Department of Applied Mathematics, College of Sciences, Donghua University, 1882 Yan'an Xilu Road, Shanghai 200051, China
Email address: zhangjuan@dhu.edu.cn



Advances in Operations Research
Advances in Decision Sciences
Journal of Applied Mathematics
Algebra
Journal of Probability and Statistics
The Scientific World Journal
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of Differential Equations
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
Mathematical Problems in Engineering
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
International Journal of Stochastic Analysis
Journal of Optimization