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Limit cycles bifurcating from the periodic orbits of a discontinuous piecewise linear differential center with two zones

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We study a class of discontinuous piecewise linear differential systems with two zones separated by the straight line $x = 0$. In $x > 0$ we have a linear saddle with its equilibrium point living in $x > 0$, and in $x < 0$ we have a linear differential center. Let p be the equilibrium point of this linear center, when p lives in $x < 0$, we say that its is real, and when p lives in $x > 0$ we say that it is virtual. We assume that this discontinuous piecewise linear differential systems formed by the center and the saddle has a center q surrounded by periodic orbits ending in a homoclinic orbit of the saddle, independent if p is real, virtual or p is in $x = 0$. Note that $q = p$ if p is real or p is in $x = 0$. We perturb these three classes of systems, according to the position of p, inside the class of all discontinuous piecewise linear differential systems with two zones separated by $x = 0$. Let N be the maximum number of limit cycles which can bifurcate from the periodic solutions of the center q with these perturbations. Our main results show that $N = 2$ when p is on $x = 0$, and $N \geq 2$ when p is a real or virtual center.

Keywords: discontinuous differential system, limit cycle, piecewise linear differential system

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1. Introduction and statement of the main results

For a given differential system a *limit cycle* is a periodic orbit isolated in the set of all its periodic orbits of the system. One of the main problems of the qualitative theory of the differential systems in the plane is the study of their limit cycles. A *center* is a point having a neighborhood, except itself, filled by periodic solutions. A classical way to produce and study limit cycles is perturbing the periodic solutions of a center. This problem for the continuous differential systems in the plane has been studied intensively, see, for instance, the hundred of references in the book [Christopher *et al*, 2007].

The main objective of this paper is to study the limit cycles that can bifurcate from a center of a discontinuous piecewise linear differential systems with two zones separated by the straight line $x = 0$ when the center is perturbed inside the class of all discontinuous piecewise linear differential systems with two zones separated by $x = 0$.

The study of the piecewise linear differential systems goes back to Andronov and coworkers [Andronov *et al*, 1966], and in the present days still continues receiving a big attention by researchers. Thus, in these last years there has been a big interest from the mathematical community in understanding their dynamical richness. On the other hand such systems are widely used to model many real phenomena and different modern devices, see for instance the books [di Bernardo *et al*, 2008] and [Simpson, 2010], the paper [Makarenkov & Lamb, 2012], and the references quoted in there.

Of course, the case of *continuous piecewise linear systems*, when they have only two pieces separated by a straight line is the simplest possible configuration of piecewise linear systems. We note that even in this simple case, only after a huge analysis it was possible to establish the existence of at most one limit cycle for such systems, see [Freire *et al*, 1988], and for a recent shorter proof see [Llibre *et al*, 2012]. There are two reasons for that misleading simplicity of piecewise linear systems. First, even being easily the computations of the solutions in any linear region, the time that each orbit requires to pass from one linear region to the other is not easy to compute, and consequently the matching of the corresponding solutions is a difficult problem. Second, the number of parameters to consider in order to be sure that one controls all possible configurations is generally not small, so the obtention of efficient canonical forms with fewer parameters is important. See also [Liang *et al*, 2013] and the references quoted there.

In this paper we deal with *discontinuous piecewise linear systems* having the form $\dot{X} = F(X) + \text{sign}(x)G(X)$, where $X = (x, y) \in \mathbb{R}^2$ and F and G are linear vector fields. Systems of this kind have been studied recently in [Giannakopoulos & Pliete, 2001; Han & Zhang, 2010; Shui *et al*, 2010; Freire *et al*, 2012; Huan & Yang, 2012; Llibre & Ponce, 2012; Llibre *et al*, 2012; Artés *et al*, 2013; Braga & Mello, 2013; Buzzi *et al*, 2013; Huan & Yang, 2013a,b; Freire *et al*, 2014; Llibre *et al*, 2014; Novaes, 2014; Xiong & Han, 2014], among other papers. In [Han & Zhang, 2010] some results about the existence of two limit cycles surrounding a unique equilibrium point appeared, and the authors conjectured that the maximum number of limit cycles surrounding a unique equilibrium point of piecewise linear differential systems with a unique straight line of discontinuity is at most two. This conjecture is analogous to Conjecture 1 of [Tonnelier, 2003]. Later on in [Huan & Yang, 2012] the authors provide numerical evidence on the existence of three limit cycles surrounding a unique equilibrium for the discontinuous piecewise linear differential systems with two linear zones separated by a straight line. In [Llibre & Ponce, 2012] it is proved the existence of such 3 limit cycles.

Up to now there are results that some classes of discontinuous piecewise linear differential systems with a unique straight line of discontinuity can have *at least k* limit cycles surrounding a unique equilibrium point. As far as we know there are no results that some classes of discontinuous piecewise linear differential systems with a unique straight line of discontinuity have *at most k* limit cycles surrounding a unique equilibrium point. Probably this is the first approach where such kind of results are stated.

We consider planar discontinuous piecewise linear differential systems with two zones separated by the straight line $\Sigma = \{x = 0\}$, i. e.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} M^+ \begin{pmatrix} x \\ y \end{pmatrix} + u^+ \text{ if } & x > 0, \\ M^- \begin{pmatrix} x \\ y \end{pmatrix} + u^- \text{ if } & x < 0, \end{cases} \tag{1}$$

where M^+ and M^- are 2×2 real matrices, and $(x, y)^T, u^+, u^- \in \mathbb{R}^2$. Here the dot denotes derivative with respect to the independent variable t, here called the time.

Select the following sets of hypotheses: $(Ha) = \{(Hak) : k = 1, 2, 3\}$, where

$(Ha1)$ s in $x > 0$ is a saddle for the system $(\dot{x}, \dot{y})^T = M^+(x, y)^T + u^+$,

$(Ha2)$ p in $x < 0$ is a center for the system $(\dot{x}, \dot{y})^T = M^-(x, y)^T + u^-$,

$(Ha3)$ the singular point p is a center for the system (1) such that its period annulus (formed by all the periodic orbits surrounding p) ends in a homoclinic loop of the saddle s,

$(Hb) = \{(Hbk) : k = 1, 2, 3\}$, where

$(Hb1)$ s in $x > 0$ is a saddle for the system $(\dot{x}, \dot{y})^T = M^+(x, y)^T + u^+$,

$(Hb2)$ p on $x = 0$ is a center for the system $(\dot{x}, \dot{y})^T = M^-(x, y)^T + u^-$,

$(Hb3)$ the singular point p is a center for the system (1) such that its period annulus ends in a homoclinic loop of the saddle s,

and $(Hc) = \{(Hck) : k = 1, 2, 3\}$, where

$(Hc1)$ s in $x > 0$ is a saddle for the system $(\dot{x}, \dot{y})^T = M^+(x, y)^T + u^+$,

$(Hc2)$ p in $x > 0$ is a virtual center for the system $(\dot{x}, \dot{y})^T = M^-(x, y)^T + u^-$,

$(Hc3)$ r is a fold–fold point which is a center for the system (1) such that its period annulus ends in a homoclinic loop of the saddle s.

Roughly speaking, a fold–fold singularity for system (1) is a point on $x = 0$ at which two curves of fold points meet, one from the solutions in $x \geq 0$ and the other from the solutions in $x \leq 0$.

An affine change of variables in the plane preserving the straight line $x = 0$ will be called in what follows a Σ*–preserving affine change of variables.*

Proposition 1. *The discontinuous piecewise linear differential systems (1) satisfying assumptions* (Ha) *after a* Σ*–preserving affine change of variables and a time–rescaling can be written as*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} A^+ \begin{pmatrix} x - c \\ y \end{pmatrix} & \text{if} \quad x > 0, \\ A^- \begin{pmatrix} x + d \\ y \end{pmatrix} & \text{if} \quad x < 0, \end{cases} \tag{2}$$

where

$$A^+ = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}, \qquad A^- = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and the parameters a, b, c, and d are positive (see Figure 1).

Proposition 2. *The discontinuous piecewise linear differential systems (1) satisfying assumptions* (Hb) *after an affine* Σ*–preserving change of variables and a time–rescaling can be written as system (2) with the parameters a, b and c positive and* $d = 0$ *(see Figure 2).*

Proposition 3. *The discontinuous piecewise linear differential systems (1) satisfying assumptions* (Hc) *after an affine* Σ*–preserving change of variables and a time–rescaling can be written as system (2) with the parameters a, b and c positive and d negative (see Figure 3).*

Propositions 1, 2 and 3 are proved in section 2.

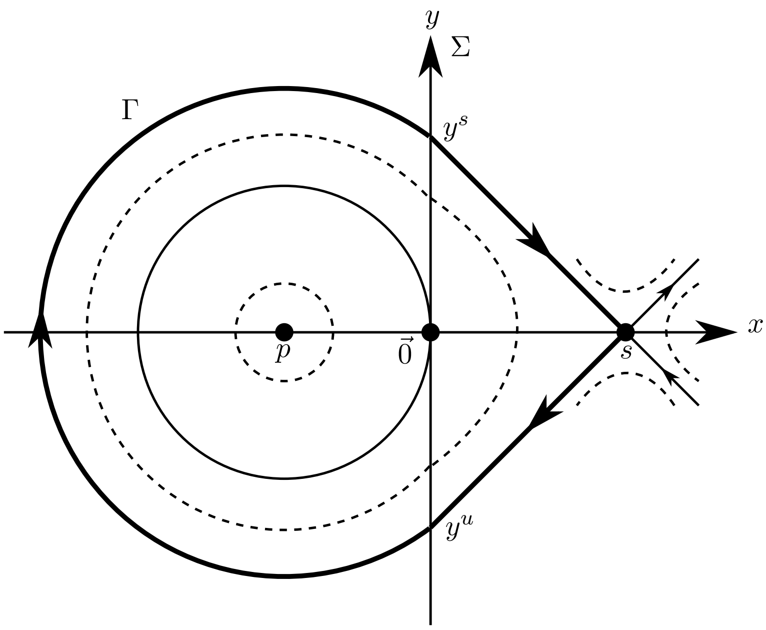

Fig. 1. Normal form of system (1) assuming the Hypotheses set (Ha). Here Γ is the homoclinic orbit; the points $(0, y^u)$ and $(0, y^s)$ are the intersections between Γ and Σ; the point p is both a center of the system $(\dot{x}, \dot{y})^T = M^-(x, y)^T + u^-$ and a center of the system (1) with its period annulus ending in Γ; the point s is a saddle of the system $(\dot{x}, \dot{y})^T = M^+(x, y)^T + u^+$; and the origin $\vec{0}$ is a fold–fold point of Σ.

We consider the more general affine perturbation of system (2) inside the class of discontinuous piecewise linear differential systems with two zones separated by the straight line $x = 0$, namely

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} A^+ \begin{pmatrix} x - c \\ y \end{pmatrix} + \varepsilon \left(B^+ \begin{pmatrix} x \\ y \end{pmatrix} + v^+ \right) & \text{if} \quad x > 0, \\ A^- \begin{pmatrix} x + d \\ y \end{pmatrix} + \varepsilon \left(B^- \begin{pmatrix} x \\ y \end{pmatrix} + v^- \right) & \text{if} \quad x < 0, \end{cases} \tag{3}$$

where ε is a small parameter and

$$B^+ = \begin{pmatrix} b_1^+ & b_2^+ \\ b_3^+ & b_4^+ \end{pmatrix}, \; B^- = \begin{pmatrix} b_1^- & b_2^- \\ b_3^- & b_4^- \end{pmatrix}, \; v^+ = \begin{pmatrix} v_1^+ \\ v_2^+ \end{pmatrix} \text{ and } v^- = \begin{pmatrix} v_1^- \\ v_2^- \end{pmatrix}.$$

Let N be the maximum number of limit cycles of perturbed system (3) which can bifurcate from the periodic solutions of the unperturbed system (2) when $|\varepsilon| \neq 0$ is sufficiently small.

Theorem A. *For every $a > 0$, $b > 0$, $c > 0$ and $d > 0$, we have that $N \geq 2$. Moreover we can find parameters $b_i^{\pm}$ for $i = 1, 2, 3, 4$ and $v_j^{\pm}$ for $j = 1, 2$ such that system (3), satisfying (Ha) when $\varepsilon = 0$, has at least 0, 1 or 2 limit cycles.*

Theorem A is proved in section 3.

In the next corollary we show that system (3) has at least 3 limit cycles for some values of the parameters for which the hypotheses of Theorem (A) hold.

Corollary 1. *We assume that $a = b = c = 1/2$, $d = 1/4$, $b_1^+ = 11712/5$, $b_1^- = -13312$, $v_1^+ = 4496$, and $b_2^{\pm} = b_3^{\pm} = b_4^{\pm} = v_1^- = v_2^{\pm} = 0$. Then for $|\varepsilon| \neq 0$ sufficiently small system (3) has at least 3 limit cycles. Numerically, we can see that these 3 limit cycles pass ε–close through the points $(0, y_i)$, where y_i*

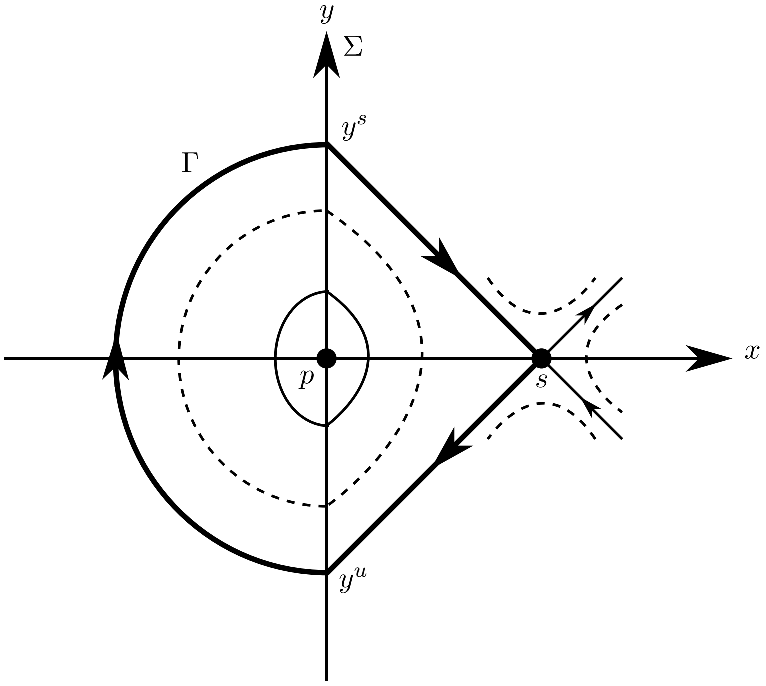

Fig. 2. Normal form of system (1) assuming the Hypotheses set (Hb). Here Γ is the homoclinic orbit; the points $(0, y^u)$ and $(0, y^s)$ are the intersections between Γ and Σ; the point $p = \vec{0}$ is both a center of the system $(\dot{x}, \dot{y})^T = M^-(x, y)^T + u^-$ and a center of the system (1) with its period annulus ending in Γ, moreover it is a fold–fold point of Σ; and the point s is a saddle of the system $(\dot{x}, \dot{y})^T = M^+(x, y)^T + u^+$.

for $i = 1, 2, 3$ *are the solutions of the following equation*

$$-5312y + 65(16y^2 + 1)\left(3\pi + 2\arctan\left(\frac{1 - 16y^2}{8y}\right)\right) - 183(4y^2 - 1)\log\left(\frac{1 + 2y}{1 - 2y}\right) = 0$$

for $y \in (0, 1/2)$. *Moreover* $y_1 \approx 0.312$, $y_2 \approx 0.439$ *and* $y_3 \approx 0.492$.

Corollary 1 is proved in section 3.

Theorem B. *For every* $a > 0$, $b > 0$, $c > 0$ *and* $d = 0$, *we have that* $N = 2$. *Moreover we can find parameters* $b_i^{\pm}$ *for* $i = 1, 2, 3, 4$ *and* $v_j^{\pm}$ *for* $j = 1, 2$ *such that system* (3), *satisfying* (Hb) *when* $\varepsilon = 0$, *has exactly* 0, 1, *or* 2 *limit cycles.*

Theorem B is proved in section 3.

Theorem C. *For every* $a > 0$, $b > 0$, $c > 0$ *and* $d < 0$, *we have that* $N \geq 2$. *Moreover we can find parameters* $b_i^{\pm}$ *for* $i = 1, 2, 3, 4$ *and* $v_j^{\pm}$ *for* $j = 1, 2$ *such that* (3), *satisfying* (Hc) *when* $\varepsilon = 0$, *has at least* 0, 1 *or* 2 *limit cycles.*

Theorem C is proved in section 3.

2. Proofs of Propositions 1, 2, and 3

First we shall prove the normal forms given in the statements of Propositions 1, 2, and 3 for the discontinuous piecewise linear differential systems with two zones separated by the straight line Σ satisfying the assumptions either (Hak), or (Hbk), or (Hck) for $k = 1, 2, 3$, respectively.

Proof. [Proof of Proposition 1] From Hypothesis $(Ha2)$, $p = (-d, e)$, with $d > 0$ is a center for the system

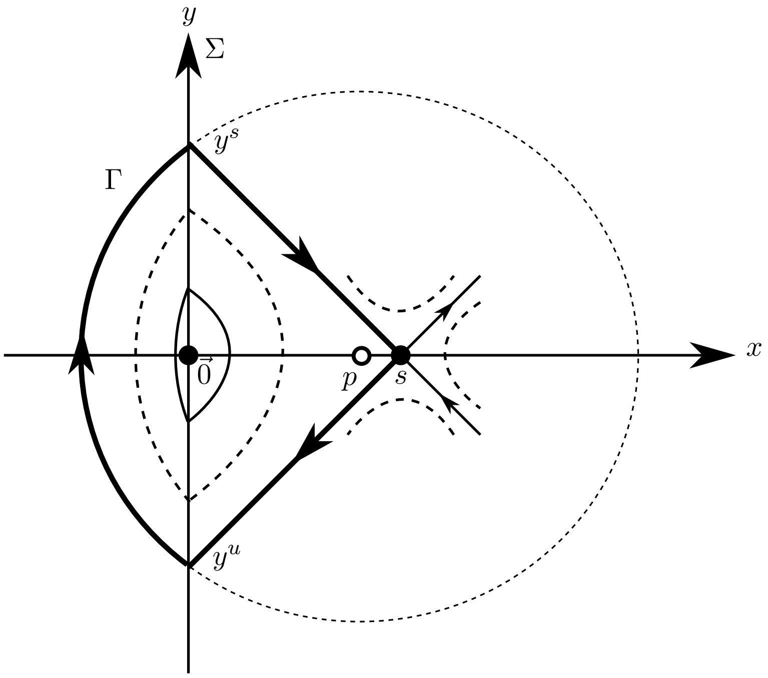

Fig. 3. Normal form of system (1) assuming the Hypotheses set (Hc). Here Γ is the homoclinic orbit; the points $(0, y^u)$ and $(0, y^s)$ are the intersections between Γ and Σ; the point p is a center of the system $(\dot{x}, \dot{y})^T = M^-(x, y)^T + u^-$; the point s is a saddle of the system $(\dot{x}, \dot{y})^T = M^+(x, y)^T + u^+$; and the origin $\vec{0}$ is a fold–fold point of Σ wich is a center of the system (1) with its period annulus ending in Γ.

$(\dot{x}, \dot{y})^T = M^-(x, y)^T + u^-$. So

$$M^- = \begin{pmatrix} m_1 & m_2 \\ m_3 & -m_1 \end{pmatrix},$$

with $m_1^2 + m_2 m_3 < 0$. By translating the system through the straight line $\{x = d\}$, which is a Σ–preserving change of variables, we can assume that $e = m_1\, d/m_2$.

Doing the Σ–preserving change of variables

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \sqrt{-m_1^2 - m_2 m_3} & 0 \\ m_1 & m_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

and rescaling the time by $\tau = \sqrt{-m_1^2 - m_2 m_3}\, t$, system (1) can be written as

$$\begin{pmatrix} \tilde{x}' \\ \tilde{y}' \end{pmatrix} = \begin{cases} A^+ \begin{pmatrix} \tilde{x} - c \\ \tilde{y} + \tilde{c} \end{pmatrix} & \text{if} \quad \tilde{x} > 0, \\ A^- \begin{pmatrix} \tilde{x} + d \\ \tilde{y} \end{pmatrix} & \text{if} \quad \tilde{x} < 0, \end{cases} \tag{4}$$

where

$$A^+ = \begin{pmatrix} \bar{a} & a \\ b & \bar{b} \end{pmatrix} \quad \text{and} \quad A^- = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

with a, b, c, d, $\bar{a}$, $\bar{b}$, $\bar{c} \in \mathbb{R}$, $c > 0$, and $d > 0$. Now the prime indicates derivative with respect to the new time τ.

From Hypothesis $(Ha1)$, $\tilde{s} = (c, -\tilde{c})$ is a saddle for the system $(\tilde{x}', \tilde{y}')^T = A^+(\tilde{x} - c, \tilde{y} + \tilde{c})^T$. So A^+ has two distincts real non–zero eigenvalues λ_1, λ_2 such that $\lambda_1 \lambda_2 < 0$. It is easy to see that this last condition is equivalent to

$$ab > \bar{a}\bar{b}. \tag{5}$$

As usual, $W^u(\tilde{s})$ and $W^s(\tilde{s})$ represent respectively the unstable and stable sets of the saddle $\tilde{s}$. We denote $y^{u,s} = \Sigma \cap W^{u,s}(\tilde{s})$. From Hypothesis $(Ha3)$, $\{y^u, y^s\} \subset W^u(\tilde{s}) \cap W^s(\tilde{s})$. Let $\Gamma = W^u(\tilde{s}) \cap W^s(\tilde{s})$ containing y^u and y^s. Hence Γ ia a homoclinic loop.

The center at $(-d, 0)$ of the system $(\tilde{x}', \tilde{y}')^T = A^-(x+d, y)^T$ induces a symmetry on (4), namely: the solution of (4) starting in Σ for $y > 0$ has to return to Σ in $-y$ for $t < 0$, so the same occurs for $t > 0$ and for every y between y^u and y^s, because from $(Ha3)$ $(-d, 0)$ is a center of (4) such that its period annulus ends in the homoclinic loop Γ.

From the above symmetry $y^u = -y^s$, which is equivalent to

$$\bar{b} = (c\bar{a} - 2a\bar{c})/c. \tag{6}$$

Now, the origin $(0, 0)$ is a singularity for the system (4), because it is a point of tangency. From Hypothesis $(Ha3)$ the origin $(0, 0)$ must be a fold-fold point. Moreover, every orbit distinct to $(0, 0)$, inside the region delimited by Γ reaching the line Σ, have to cross Σ. Thus $(-d, 0)$ is a center with its period annulus ending in Γ. These conditions are satisfied if and only if $\bar{c} = \bar{a}c/a$ and $a > 0$. From (6) we conclude that $\bar{b} = -\bar{a}$

Computing the solution of (4) we conclude, from the above symmetry, that $\bar{a} = 0$. Furthermore, from (5) we have that $b > 0$.

Hence (4) has a center at $(-d, 0)$ such that its period annulus ends in the homoclinic loop Γ if and only if $\bar{a} = \bar{b} = \bar{c} = 0$ and $a, b, c > 0$. So we have conclude the proof. ∎

Proof. [Proof of Proposition 2] By translating the system through the straight line Σ, which is a Σ–preserving change of variables, we can assume that $p = (0, 0)$. From Hypothesis $(Hb2)$, $(0, 0)$ is a center for the system $(\dot{x}, \dot{y})^T = M^-(x, y)^T + u^-$. So

$$M^- = \begin{pmatrix} m_1 & m_2 \\ m_3 & -m_1 \end{pmatrix},$$

with $m_1^2 + m_2 m_3 < 0$. Doing the Σ–preserving change of variables

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \sqrt{-m_1^2 - m_2 m_3} & 0 \\ m_1 & m_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

and rescaling the time by $\tau = \sqrt{-m_1^2 - m_2 m_3}\, t$, system (1) can be written as

$$\begin{pmatrix} \tilde{x}' \\ \tilde{y}' \end{pmatrix} = \begin{cases} A^+ \begin{pmatrix} \tilde{x} - c \\ \tilde{y} + \bar{c} \end{pmatrix} & \text{if} \quad \tilde{x} > 0, \\ A^- \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} & \text{if} \quad \tilde{x} < 0, \end{cases} \tag{7}$$

where

$$A^+ = \begin{pmatrix} \bar{a} & a \\ b & \bar{b} \end{pmatrix} \quad \text{and} \quad A^- = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

with a, b, c, $\bar{a}$, $\bar{b}$, $\bar{c} \in \mathbb{R}$, and $c > 0$.

From here, the proof follows analogously to the proof of Proposition 1. ∎

Proof. [Proof of Proposition 3] From the Hypothesis $(Hc2)$, $p = (-d, e)$, with $d < 0$ is a center for the system $(\dot{x}, \dot{y})^T = M^-(x, y)^T + u^-$. So

$$M^- = \begin{pmatrix} m_1 & m_2 \\ m_3 & -m_1 \end{pmatrix},$$

with $m_1^2 + m_2 m_3 < 0$. By translating the system through the straight line $\{x = -d\}$, which is Σ–preserving change of variables, we can assume that $e = m_1\, d/m_2$.

Doing the Σ–preserving change of variables

$$\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}=\begin{pmatrix}\sqrt{-m_1^2-m_2m_3} & 0\\ m_1 & m_2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix},$$

and rescaling the time by $\tau=\sqrt{-m_1^2-m_2m_3}\,t$, system (1) can be written as

$$\begin{pmatrix}\tilde{x}'\\ \tilde{y}'\end{pmatrix}=\begin{cases}A^+\begin{pmatrix}\tilde{x}-c\\ \tilde{y}+\bar{c}\end{pmatrix} & \text{if} \quad \tilde{x}>0,\\ A^-\begin{pmatrix}\tilde{x}-d\\ \tilde{y}\end{pmatrix} & \text{if} \quad \tilde{x}<0,\end{cases} \tag{8}$$

where

$$A^+=\begin{pmatrix}\bar{a} & a\\ b & \bar{b}\end{pmatrix} \quad \text{and} \quad A^-=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},$$

with a, b, c, d, $\bar{a}$, $\bar{b}$, $\bar{c}\in\mathbb{R}$, $c>0$, and $d>0$.

From here the proof follows in a similar way to the proof of Proposition 1. ■

3. Proofs of Theorems A, B, C and Corollary 1

The idea of the proofs of Theorems A, B, and C is to compute the Taylor expansion at $\varepsilon=0$ up to order 1 in ε of the Poincaré map associated to (3) and then apply the Implicit Function Theorem.

Let I be a proper real interval and let $f_0,f_1,\ldots,f_n:I\to\mathbb{R}$. We say that $f_0,f_1,\ldots,f_n$ are *linearly independent* functions if and only if

$$\forall t\in I\ \sum_{i=0}^{n}\alpha_i f_i(t)=0\ \Rightarrow\ \alpha_0=\alpha_1=\cdots=\alpha_n=0.$$

Proposition 4. *If $f_0,f_1,\ldots,f_n$ are linearly independent then there exist $t_1,t_2\ldots,t_n\in I$ and $\alpha_0,\alpha_1,\ldots,\alpha_n\in\mathbb{R}$ such that for every $j\in\{1,2,\ldots,n\}$*

$$\sum_{i=0}^{n}\alpha_i f_i(t_j)=0.$$

For a proof of Proposition 4 see for instance [Llibre & Świrszcz, 2011].

We say that an ordered set of complex–valued functions $F=(f_0,f_1,\ldots,f_n)$ defined on I is an *Extended Chebyshev* system or ET–system on I if and only if any nontrivial linear combination of the functions of F has at most n zeros counting multiplicities. We say that F is an *Extended Complete Chebyshev* system or an ECT–system on I if and only if for any k, $0\le k\le n$, $(f_0,f_1,\ldots,f_k)$ is an ET–system. For more details, see the book of Karlin and Studden [Karlin & Studden, 1966].

In order to prove that F is a ECT–system on I is sufficient and necessary to show that $W(f_0,f_1,\ldots,f_k)(t)\neq 0$ on I for $0\le K\le n$. Here $W(f_0,f_1,\ldots,f_k)(t)$ denotes the Wronskians of the functions $(f_0,f_1,\ldots,f_k)$ with respect to t. We recall the definition of the Wronskian.

$$W(f_0,\ldots,f_k)(t)=\begin{vmatrix}f_0(t) & \cdots & f_k(t)\\ f_0'(t) & \cdots & f_k'(t)\\ \vdots & \ddots & \vdots\\ f_0^{(k)}(t) & \cdots & f_k^{(k)}(t)\end{vmatrix}.$$

Now consider the functions

$$
\begin{aligned}
g_1(y) &= 1,\\
g_2(y) &= \frac{(ay^2 - b\,c^2)}{y}\log\left(\frac{\sqrt{b}\,c+\sqrt{a}\,y}{\sqrt{b}\,c-\sqrt{a}\,y}\right),\\
g_3^1(y) &= \frac{(d^2+y^2)}{y}\left(3\pi + 2\arctan\left(\frac{d^2-y^2}{2dy}\right)\right),\\
g_3^2(y) &= y, \quad \text{and}\\
g_3^3(y) &= \frac{(d^2+y^2)}{y}\left(\pi + 2\arctan\left(\frac{d^2-y^2}{2dy}\right)\right).
\end{aligned}
\tag{9}
$$

We define the sets of functions $G^i = \{g_1, g_2, g_3^i\}$ for $i = 1, 2, 3$. In the proofs of Theorems A, B, and C we shall use the following lemma.

Lemma 1. *Assume that $a > 0$, $b > 0$, $c > 0$.*

(a) *If $d > 0$ then G^1 is a set of functions linearly independent on the interval $(0, \sqrt{b}\,c/\sqrt{a})$.*
(b) *If $d = 0$ then G^2 is a ECT-system on the interval $(0, \sqrt{b}\,c/\sqrt{a})$.*
(c) *If $d < 0$ then G^3 is a set of functions linearly independent on the interval $(0, \sqrt{b}\,c/\sqrt{a})$.*

Proof. To prove statement (a) we compute the power series expansion in y up to order 2 of the functions g_i for $i = 1, 2$ and g_3^1. So

$$
\begin{aligned}
g_1 &= 1,\\
g_2 &= -2\sqrt{ab}c + \frac{4a^{3/2}}{3\sqrt{bc}}y^2 + \mathcal{O}(y^3), \quad \text{and}\\
g_3^1 &= 4d^2\pi\frac{1}{y} - 4d + 4\pi y - \frac{8}{3d}y^2 + \mathcal{O}(y^3).
\end{aligned}
\tag{10}
$$

Let C be the square matrix formed by the coefficients of y^i for $i = 0, 1, 2$ of (10), namely

$$
C = \begin{pmatrix} 1 & 0 & 0\\ -2\sqrt{ab}c & 0 & \dfrac{4a^{3/2}}{3\sqrt{bc}}\\ -4d & 4\pi & -\dfrac{8}{3d}\end{pmatrix}.
$$

Since $\det(C) = -(16\sqrt{a^2}\pi)/(3\sqrt{b}c) \neq 0$, we have that G^1 is a set of functions linearly independent for $y > 0$ near 0, which implies that G^1 is a set of functions linearly independent in $(0, \sqrt{b}\,c/\sqrt{a})$ for every $a, b, c, d > 0$. This concludes the proof of statement (a).

The proof of statement (c) follows analogously by considering the power series expansion in y up to order 2 of the functions g_i for $i = 1, 2$ and g_3^3.

To prove statement (b) we compute the Wronskians $W_1(y) = W(g_1)(y)$, $W_2(y) = W(g_1, g_3^2)(y)$ and $W_3(y) = W(g_1, g_3^2, g_2)(y)$. So

$$
\begin{aligned}
&W_1(y) = 1, \quad W_2(y) = 1, \quad \text{and}\\
&W_3(y) = \frac{4a}{(b\,c^2 - a\,y^2)^4}\left(5\sqrt{ab}\,cy + (b\,c^2 + 5ay^2)\log\left(\frac{\sqrt{b}\,c+\sqrt{a}\,y}{\sqrt{b}\,c-\sqrt{a}\,y}\right)\right).
\end{aligned}
$$

Clearly $W_1(y) \neq 0$ and $W_2(y) \neq 0$ for $y > 0$. Now, since

$$\left.\frac{\sqrt{b}\,c+\sqrt{a}\,y}{\sqrt{b}\,c-\sqrt{a}\,y}\right|_{y=1} = 1, \quad \text{and} \quad \frac{d}{dy}\left(\frac{\sqrt{b}\,c+\sqrt{a}\,y}{\sqrt{b}\,c-\sqrt{a}\,y}\right) = \frac{2\sqrt{abc}}{\left(\sqrt{bc}-\sqrt{ay}\right)^2} > 0,$$

for $y \in (0, \sqrt{b}c/\sqrt{a})$, it follows that

$$\log\left(\frac{\sqrt{b}\,c+\sqrt{a}\,y}{\sqrt{b}\,c-\sqrt{a}\,y}\right) > 0, \tag{11}$$

for $y \in (0, \sqrt{b}c/\sqrt{a})$. Now let

$$P(y) = 2\sqrt{abc}\,y + \left(ay^2 - bc^2\right)\log\left(\frac{\sqrt{b}\,c+\sqrt{a}\,y}{\sqrt{b}\,c-\sqrt{a}\,y}\right).$$

So

$$W_3(y) = \frac{2bc^2}{bc^2y^3 - ay^5}P(y).$$

Moreover $P(0) = 8\sqrt{a^3}/(3\sqrt{b}c) > 0$ and

$$P'(y) = 4abc^2 y \log\left(\frac{\sqrt{b}\,c+\sqrt{a}\,y}{\sqrt{b}\,c-\sqrt{a}\,y}\right) > 0.$$

Since $bc^2y^3 - ay^5 > 0$ for every $y \in (0, \sqrt{b}c/\sqrt{a})$, we conclude that $W_3(y) > 0$ for $y \in (0, \sqrt{b}c/\sqrt{a})$.

Hence the lemma is proved. ■

Proof. [Proof of Theorem A] The solutions of the discontinuous piecewise linear differential systems (2) and (3) can be easily computed, because they are piecewise linear differential systems.

Let $\varphi^+(t,x,y,\varepsilon) = \left(\varphi_1^+(t,x,y,\varepsilon), \varphi_2^+(t,x,y,\varepsilon)\right)$ be the solution of (3) for $x > 0$ such that $\varphi^+(0,x,y,\varepsilon) = (x,y)$. Similarly, let $\varphi^-(t,x,y,\varepsilon) = \left(\varphi_1^-(t,x,y,\varepsilon),\ \varphi_2^-(t,x,y,\varepsilon)\right)$ be the solution of (3) for $x < 0$ such that $\varphi^-(0,x,y,\varepsilon) = (x,y)$.

For $y > 0$ let $t^+(y,\varepsilon)$ be the smallest positive time such that $\varphi_1^+(t^+(y,\varepsilon),0,y,\varepsilon) = 0$, and let $t^-(y,\varepsilon)$ be the biggest negative time such that $\varphi_1^-(t^-(y,\varepsilon),0,y,\varepsilon) = 0$. Clearly, there exists a limit cycle passing through y if and only if $\varphi_2^+(t^+(y,\varepsilon),0,y,\varepsilon) = \varphi_2^-(t^-(y,\varepsilon),0,y,\varepsilon)$. So we must study the zeros of the function

$$f(y,\varepsilon) = \varphi_2^+(t^+(y,\varepsilon),0,y,\varepsilon) - \varphi_2^-(t^-(y,\varepsilon),0,y,\varepsilon). \tag{12}$$

Using some algebraic manipulator as Mathematica or Maple, it is easy to see that

$$
\begin{aligned}
t^+(y,\varepsilon) = {} & \frac{1}{\sqrt{ab}}\log\left(\frac{\sqrt{b}\,c+\sqrt{a}\,y}{\sqrt{b}\,c-\sqrt{a}\,y}\right)+\frac{\varepsilon}{2a^2b^2\left(-bc^2y+ay^3\right)}\left(\log\left(\frac{\sqrt{b}\,c+\sqrt{a}\,y}{\sqrt{b}\,c-\sqrt{a}\,y}\right)\right.\\
& \left(\sqrt{ab}b^2c^3(b_1^+ + b_4^+) + c^2\left(\sqrt{ab}b^2b_2^+ab\sqrt{ab}b_3^+\right)y - ab\sqrt{ab}c(b_1^+ + b_4^+)y^2\right.\\
& \left.-\left(ab\sqrt{ab}b_2^+ + \sqrt{ab}a^2b_3^+\right)y^3\right) - 2aby\left(a\left(cb_3^+ + 2v_2^+\right)y\right.\\
& \left.\left.+\,bc\left(c\left(b_1^+ + b_4^+\right) + 2v_1^+b_2^+y\right)\right)\right) + \mathcal{O}(\varepsilon^2), \quad \text{and}\\
t^-(y,\varepsilon) = {} & -\arctan\left(\frac{d^2-y^2}{2dy}\right) - \frac{3\pi}{2} + \frac{\varepsilon}{2y\left(d^2+y^2\right)}\left(\left(-\arctan\left(\frac{d^2-y^2}{2dy}\right)+\frac{\pi}{2}\right)\right.\\
& \cdot\left(d^2+y^2\right)\left(\left(b_1^- + b_4^-\right)d - \left(b_2^- - b_3^-\right)y\right) - 2d^3\pi\left(b_1^- + b_4^-\right)\\
& +2\left(b_1^- - \pi(b_2^- - b_3^-) + b_4^-\right)d^2y + 2\left(\pi(-b_2^- + b_3^-)y - 2v_2^-\right)y^2\\
& \left.+2\left(\left(-b_2^- + b_3^- + \pi(b_1^- + b_4^-)\right)y - 2v_1^-\right)dy\right) + \mathcal{O}(\varepsilon^2).
\end{aligned}
\tag{13}
$$

Substituting (13) in (12) and expanding (12) in Taylor series at $\varepsilon = 0$ up to order 1 in ε we get $f(y,\varepsilon) = \varepsilon f_1(y) + \mathcal{O}(\varepsilon^2)$, where

$$
f_1(y) = k_1 g_1(y) + k_2 g_2(y) + k_3 g_3^1(y), \tag{14}
$$

and the functions g_i for $i = 1,2$ and g_3^1 are the ones defined in (9). Here, the coefficients k_m for $m = 1,2,3$ depend linearly on the parameters $b_i^\pm$ for $i = 1,2,3,4$ and $v_j^\pm$ for $j = 1,2$, namely

$$
\begin{aligned}
k_1 &= \frac{-c\left(b_1^+ + b_4^+\right) - ad\left(b_1^- + b_4^-\right) - 2v_1^+ + 2av_1^-}{a},\\
k_2 &= -\frac{b_1^+ + b_4^+}{2\sqrt{a^3b}}, \quad \text{and}\\
k_3 &= -\frac{b_1^- + b_4^-}{4}.
\end{aligned}
$$

Now let y^* be a simple zero of (14). We note that the function $f(y,\varepsilon)$ satisfies the hypothesis of the Implicit Function Theorem in a neighborhood of the point $(y,\varepsilon) = (y^*,0)$ that is

$$
f(y^*,0) = 0 \quad \text{and} \quad \frac{\partial}{\partial z} f(z,\varepsilon)\Big|_{(y^*,0)} \neq 0.
$$

Therefore, from the Implicit Function Theorem, there exists a unique analytic function $\xi(\varepsilon)$ defined in a neighborhood of $\varepsilon = 0$ such that

$$
f(\xi(\varepsilon),\varepsilon) = 0 \quad \text{and} \quad \xi(0) = y^*.
$$

So, from the definition of the function (12) each simple zero y^* of (14) is associated with a hyperbolic limit cycle of (3). Hence applying Lemma 1(a) and Proposition 4 we conclude the proof of the theorem. ■

In the proof of Theorem B we shall need the following lemma.

Lemma 2. *Assume that $a > 0$, $b > 0$, $c > 0$ and $d = 0$. If $b_1^+ = -b_4^+$, $b_1^- = -b_4^-$, and $v_1^+ = av_1^-$, then system (2) does not admit limit cycles.*

Proof. From the assumptions the smallest positive time $t^+(y,\varepsilon)$ such that $\varphi_1^+(t^+(y,\varepsilon),0,y,\varepsilon)=0$, and the biggest negative time $t^-(y,\varepsilon)$ such that $\varphi_1^-(t^-(y,\varepsilon),0,y,\varepsilon)=0$ can be explicitly computed using some algebraic manipulator as Mathematica or Maple, namely

$$\begin{aligned} t^+(y,\varepsilon) = {} & \frac{1}{\sqrt{ab}}\log\left(\frac{\sqrt{b}\,c+\sqrt{a}\,y}{\sqrt{b}\,c-\sqrt{a}\,y}\right) - \varepsilon\left(\frac{ab_3^+ + bb_2^+}{2\sqrt{a^3b^3}}\log\left(\frac{\sqrt{b}\,c+\sqrt{a}\,y}{\sqrt{b}\,c-\sqrt{a}\,y}\right)\right. \\ & \left.-\frac{bcb_2^+y + a\left(2bcv_1^- + cb_3^+y + 2v_2^-y\right)}{ab\left(bc^2-ay^2\right)}\right) + \varepsilon^2\left(\log\left(\frac{\sqrt{b}\,c+\sqrt{a}\,y}{\sqrt{b}\,c-\sqrt{a}\,y}\right)\right. \\ & \cdot\frac{3(bb_2^+)^2 + 3(ab_3^+)^2 + 2ab\left(b_2^+b_3^+ - 2(b_1^+)^2\right)}{8\sqrt{a^5b^5}} + \frac{1}{\left(bc^2-ay^2\right)^2} \\ & \cdot\left(\frac{ab_3^+\left(5cb_3^+ + 8v_2^-\right)y^3}{4b^2} - \frac{3bc^3(b_2^+)^2y}{4a^2} - \frac{2bc^3b_2^+v_2^-}{a}\right. \\ & +\frac{y}{4b}\left(8c(v_2^+)^2 - 3c^3(b_3^+)^2 + 8av_1^-y\left(cb_3^+v_2^+\right)\right. \\ & \left.+2\left(3cb_2^+b_3^+ - 2c(b_1^+)^2 + 4b_2^+v_2^+ - 4ab_1^+v_1^-\right)y^2\right) \\ & +\frac{c}{4a}\left(2c^2y\left(2(b_1^+)^2 - b_2^+b_3^+\right) + 8a^2(v_1^-)^2y + 5(b_2^+)^2y^2\right. \\ & \left.\left.\left.+8av_1^-\left(cv_2^+ + cb_1^+y + 2b_2^+y^2\right)\right)\right)\right) + \mathcal{O}(\varepsilon^3), \quad \text{and} \end{aligned} \tag{15}$$

$$\begin{aligned} t^-(y,\varepsilon) = {} & \pi + \varepsilon\left(\frac{b_3^- - b_2^-}{2}\pi + \frac{2v_2^-}{y}\right) \\ & +\varepsilon^2\left(\frac{\pi}{8}\left(4(b_1^-)^2 + 3(b_2^-)^2 - 2b_2^-b_3^- + 3(b_3^-)^2\right)\right. \\ & \left.+\frac{2}{y^2}\left(v_2^-y\left(b_3^- - b_2^-\right) + v_1^-\left(b_1^-y - v_2^-\right)\right)\right) + \mathcal{O}(\varepsilon^3). \end{aligned}$$

Expanding (12) in Taylor series at $\varepsilon=0$ up to order 2 in ε we get

$$f(y,\varepsilon) = \varepsilon^2\frac{2v_1^-\left(b_2^+ - ab_2^-\right)}{a} + \mathcal{O}(\varepsilon^3).$$

Now suppose that, for $|\varepsilon|\neq 0$ sufficiently small, there exists a continuous branch $\bar{y}_\varepsilon$ of solutions of the equation $f(y,\varepsilon)=0$ such that $\bar{y}_\varepsilon\to\bar{y}$ when $\varepsilon\to 0$ for some $0<\bar{y}<\sqrt{b}\,c/\sqrt{a}$. It implies that $v_1^-=0$ or $b_2^+=ab_2^-$. However, in both cases we can verify that $f(y,\varepsilon)\equiv 0$, i.e. the equilibrium point of system (3) is a center. This implies the non–existence of limit cycles for the system (3). ■

Proof. [Proof of Theorem B] Following the proof of Theorem A we have to estimate (12). Again, using some algebraic manipulator as Mathematica or Maple, it is easy to see that $t^+(y,\varepsilon)$ is given by (13) and

$$t^-(y,\varepsilon) = -\pi + \varepsilon\left(\frac{b_2^- + b_3^-}{2}\pi + \frac{2v_2^-}{y}\right) + \mathcal{O}(\varepsilon^2). \tag{16}$$

Substituting $t^+(y,\varepsilon)$ and (16) in (12) and expanding (12) in Taylor series at $\varepsilon=0$ up to order 1 in ε we get $f(y,\varepsilon)=\varepsilon f_2(y)+\mathcal{O}(\varepsilon^2)$, where

$$f_2(y) = k_1g_1(y) + k_2g_2(y) + k_3g_3^2(y), \tag{17}$$

and the functions g_i for $i = 1,2$ and g_3^2 are the ones defined in (9). Here, the coefficients k_m for $m = 1,2,3$ depend linearly on the parameters $b_i^\pm$ for $i = 1,2,3,4$ and $v_j^\pm$ for $j = 1,2$, namely

$$\begin{aligned} k_1 &= \frac{-c\left(b_1^+ + b_4^+\right) - 2v_1^+ + 2av_1^-}{a}, \\ k_2 &= -\frac{b_1^+ + b_4^+}{2\sqrt{a^3 b}}, \quad \text{and} \\ k_3 &= -\frac{\pi(b_1^- + b_4^-)}{2}. \end{aligned}$$

Note that $f_2(y) \equiv 0$ if and only if $b_1^+ = -b_4^+$, $b_1^- = -b_4^-$, and $v_1^+ = av_1^-$. In this case, from Lemma 2 system (2) does not admit limit cycles. Thus we can assume that the function (17) is not identically zero. This implies that for $|\varepsilon| \neq 0$ sufficiently small the zeros of the function (12) on a given bounded interval, assuming the hypotheses of Theorem B, are completely controlled by the function (17).

From here, applying Lemma 1(b), the definition of the ECT–systems, and the Implicit Function Theorem the proof follows. ■

Proof. [Proof of Theorem C] Here $t^+(y,\varepsilon)$ is given by (13) and Here $t^+(y,\varepsilon)$ is given by (13) and

$$\begin{aligned} t^-(y,\varepsilon) = -\arctan\left(\frac{d^2-y^2}{2dy}\right) - \frac{\pi}{2} - \varepsilon\Bigg(&\frac{d\left(b_1^- + b_4^-\right) - y\left(b_2^- - b_3^-\right)}{2y} \\ &- \frac{d^2\left(b_1^- + b_4^-\right) - 2v_2^- y - d\left(2v_1^- + y\left(b_2^- - b_3^-\right)\right)}{d^2+y^2}\Bigg) + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{18}$$

This proof follows completely analogous to the proof of Theorem A by applying Lemma 1(c), Proposition 4 and the Implicit Function Theorem for the function

$$f_3(y) = k_1 g_1(y) + k_2 g_2(y) + k_3 g_3^3(y), \tag{19}$$

where the functions g_i for $i = 1,2$ and g_3^3 are the ones defined in (9). Here, the coefficients k_m for $m = 1,2,3$ depend linearly on the parameters $b_i^\pm$ for $i = 1,2,3,4$ and $v_j^\pm$ for $j = 1,2$. ■

Proof. [Proof of Corollary 1] For the given coefficients the function (14) from the proof of Theorem (13) becomes

$$f_1(y) = \frac{16}{5y}\left(-5312y + 65(16y^2+1)\left(3\pi + 2\arctan\left(\frac{1-16y^2}{8y}\right)\right) - 183(4y^2-1)\log\left(\frac{1+2y}{1-2y}\right)\right).$$

Evaluating this function in some points we can check that $f_1(1/10) > 0$, $f_1(4/10) < 0$, $f_1(45/100) > 0$ and $f_1(499/100) < 0$. So there exists $y_1 \in (1/10, 4/10)$, $y_2 \in (4/10, 45/100)$ and $y_3 \in (45/100, 499/1000)$ such that $f(y_i) = 0$ for $i = 1,2,3$. Since from the proof of Theorem (13) $f(y,\varepsilon) = \varepsilon f_1(y) + \mathcal{O}(\varepsilon^2)$, so for $|\varepsilon| \neq 0$ sufficiently small we also have that $f(1/10,\varepsilon) > 0$, $f(4/10,\varepsilon) < 0$, $f(45/100,\varepsilon) > 0$ and $f(499/100,\varepsilon) < 0$, which implies the existence of at least 3 limit cycles.

Now plotting the graphic of $f_1(y)$ and $f_1'(y)$ for $y \in (0, 1/2)$ (see Figure 4) and estimating the solutions of the equation $f_1(y) = 0$ we obtain the numerical conclusion of the corollary. ■

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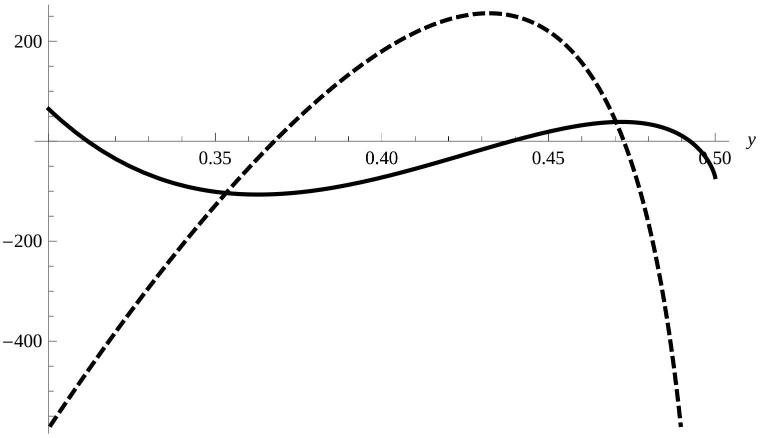

Fig. 4. The normal bold line is the graphic of the function $f_1(y)$ for $y \in (0, 1/2)$, and the dashed bold line is the graphic of the function $f_1'(y)$ for $y \in (0, 1/2)$. We can see that these functions have no common zeros for $y \in (0, 1/2)$, that is the zeros y_i for $i = 1, 2, 3$ of $f(y)$ are simple. It implies that they persist under small perturbation.

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