LIMIT CYCLES FOR DISCONTINUOUS PLANAR PIECEWISE LINEAR DIFFERENTIAL SYSTEMS SEPARATED BY AN ALGEBRAIC CURVE

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ABSTRACT. We study how can change the maximum number of limit cycles of the discontinuous piecewise linear differential systems with only two pieces in function of the degree of the discontinuity algebraic curve between the two linear differential systems. These discontinuous differential systems appear frequently in applied sciences.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

The dynamics of the piecewise linear differential systems started to be studied around 1930, mainly in the book of Andronov et al [1]. Many researchers from different fields considered this kind of differential systems, because they are widely used to model phenomena appearing in mechanics, electronics, economy, neuroscience, ..., see for more details and applications the books of Bernardo et al [3] and of Simpson [33], the surveys of Makarenkov and Lamb [29], and Teixeira [36], the articles [19, 20, 21, 31, 35], and the references which appear in all these works.

A periodic orbit of a differential system which is isolated in the set of all periodic orbits of the system is a *limit cycle*. One of the main problems in the dynamics of the differential systems in the plane is to control the existence and the number of their limit cycles. This problem restricted to polynomial differential systems is the famous 16th Hilbert's problem, see more details in [14, 18, 22].

This last years many authors have studied the limit cycles of discontinuous piecewise linear differential systems in $\mathbb{R}^2$. Of course, the simplest piecewise linear differential systems in $\mathbb{R}^2$ are the ones having only two pieces separated by a curve, and when this curve is a straight line. Thus the limit cycles of these last class of discontinuous piecewise linear differential systems has been intensively studied, see [2, 4, 6, 7, 9, 10, 11, 12, 13, 15, 16, 17, 23, 24, 25, 26, 27, 28, 32] and the references quoted in these papers. Up to know the results of all these papers only provide examples that the discontinuous piecewise linear differential systems in $\mathbb{R}^2$ separated by a straight line can have 3 crossing limit cycles (they are defined in what follows), but it is an open question to know if 3 is the maximum number of crossing limit cycles that such discontinuous differential systems can have.

In the papers [5, 30] the authors considered discontinuous piecewise linear differential systems with only two pieces, but now the curve of discontinuity between the two pieces is not a straight line, and they shown that the number of limit cycles can increase arbitrarily with the number of oscillations of that curve. We note that the discontinuity curves considered in these two papers is not algebraic.

The main objective of this paper is to study how can change the maximum number of limit cycles of the discontinuous piecewise linear differential systems with only two pieces in function of the degree of the discontinuity curve when this curve is algebraic. Due to the big amount of computations necessary for studying this maximum number of limit cycles we restrict our analysis as follows: first to the class of discontinuous piecewise linear differential systems separated by the algebraic curve $y = x^n$ where $n > 1$ is an arbitrary positive integer; second to the class of discontinuous piecewise linear differential systems formed by two linear differential systems having only centers; and third we only consider crossing limit cycles, they are defined in what follows.

We consider discontinuous piecewise differential equations in $\mathbb{R}^2$ of the form

$$
(1) \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} X^-(x,y) & \text{if } (x,y) \in L = \{(x,y) : y \leq x^n\}, \\ X^+(x,y) & \text{if } (x,y) \in U = \{(x,y) : y \geq x^n\}, \end{cases}
$$

where the vector fields X^- and X^+ are linear. The dynamics over the curve of discontinuity $y = x^n$ is defined following the Filippov's convention, see [8]. Thus, on the curve of discontinuity $y = x^n$ we distinguish generically two different kind of points: the *crossing points* $\{p = (x, x^n) : X^-(p)X^+(p) > 0\}$, and the *sliding points* $\{p = (x, x^n) : X^-(p)X^+(p) < 0\}$.

We assume that each one of the linear differential systems of the discontinuous piecewise linear differential systems (1) has a center. We want to find upper bound for the maximum number of crossing limit cycles that such discontinuous piecewise linear differential systems can exhibit. A *crossing limit cycle* for such discontinuous differential systems is formed by two pieces of orbits, one piece of orbit of the linear differential system contained in $y < x^n$ and the other piece of orbit of the linear differential system contained in $y > x^n$, furthermore both pieces connect in two crossing points over the curve $y = x^n$. We must mention that when $n = 1$, i.e. when the discontinuous curve is a straight line, and both linear differential systems have a center the discontinuous differential system (1) has no crossing limit cycles, see for details [28]. This proves the result of the next Theorem 2 for the particular case $n = 1$.

An equilibrium p of a linear differential system of a discontinuous piecewise linear differential systems in $\mathbb{R}^2$ separated by the algebraic curve $y = x^n$ is called *real* when p belongs to the closure of the component of $\mathbb{R}^2 \setminus \{(x,y) : y = x^n\}$ where is defined the mentioned linear differential system, otherwise p is called *virtual*.

The following normal form for the discontinuous piecewise linear differential systems in $\mathbb{R}^2$ separated by the algebraic curve $y = x^n$ when both linear differential systems have a center will help us to prove our manin result, the theorem which follows the next proposition.

Proposition 1. *After a linear change of variables and a rescaling of the independent variable any discontinuous piecewise linear differential systems in $\mathbb{R}^2$ separated by the algebraic curve $f(x, y) = 0$ when both linear differential systems have a center can be written as*

(2)
$$\dot{x} = -bx - \frac{4b^2 + \omega^2}{4a}y + d, \quad \dot{y} = ax + by + c, \quad \text{in } L = \{(x, y) : y \leq x^n\},$$
$$\dot{x} = -y - \alpha, \qquad\qquad\qquad \dot{y} = x - \beta, \qquad \text{in } U = \{(x, y) : y \geq x^n\},$$
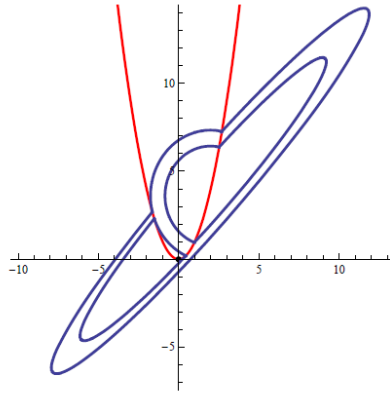
with $a > 0$ and $\omega > 0$.

FIGURE 1. The three limit cycles of a discontinuous piecewise linear differential system (2) for $n = 2$.

Theorem 2. *Let n be a positive integer.*

(a) *An upper bound for the maximum number of crossing limit cycles that a discontinuous piecewise linear differential system in $\mathbb{R}^2$ separated by the algebraic curve $y = x^n$ can have when both linear differential systems have a center, real or virtual, is d_n equal to*

$$\frac{2n^2 - n - 1}{2} \quad \text{if } n \text{ is odd}, \qquad \text{and} \quad \frac{2n^2 - n}{2} \quad \text{if } n \text{ is even}.$$

(b) *For $n = 2$ there are discontinuous piecewise linear differential systems which reach the upper bound of 3 limit cycles of statement (a), see the 3 limit cycles in Figure 1.*

(c) *For $n = 2, 3$ we provide the explicit polynomial of degree $2d_n$ whose real roots x provide the points (x, x^n) where the crossing limit cycles of the discontinuous piecewise linear differential system (2) intersect the curve $y = x^n$.*

Theorem 2 is proved in section 3.

We remark that in statement (a) of Theorem 2 we provide *upper bound* for the maximum number of crossing limit cycles that the discontinuous piecewise linear differential systems studied in that theorem can have. The major part of the

results mentioned previously on the limit cycles of discontinuous piecewise linear differential systems studied until now only provide *lower bounds* for that maximum number. The unique other three papers that we know providing upper bounds are when the curve of discontinuity is a straight line, and either one of the linear differential systems has an equilibrium on the the discontinuity straight line [7, 24], or one of the linear differential systems has a center [28].

It remains the following open question.

Open question. *Are the upper bounds given in the statement of Theorem 2 reached for values of $n > 2$?*

2. Normal form

Now we shall prove the normal form for the discontinuous piecewise linear differential systems that we study.

Proof of Proposition 1. Doing the linear change of variables which writes the matrix of the linear differential system in U, having a real or virtual center, in its real normal form, and rescaling the independent variable of the differential system (if necessary) we can write such a linear differential system as $\dot{x} = -y - \alpha$, $\dot{y} = x - \beta$.

Now the linear differential system in the region L is

$$\dot{x} = Ax + By + d, \qquad \dot{y} = ax + by + c,$$

assuming that it has a center. Since the eigenvalues of this system are

$$\frac{A + b \pm \sqrt{4aB + (A - b)^2}}{2},$$

in order to have a center we must have that $A + b = 0$ and $4aB + (A - b)^2 = -\omega^2$ for some $\omega > 0$ and $aB < 0$, or equivalently $A = -b$, $B = -(4b^2 + \omega^2)/(4a)$ and $a > 0$. This completes the proof of the proposition. $\qquad\square$

It is easy to check that the differential linear system in the region L has the first integral

$$H_L = 4(ax + by)^2 + 8a(cx - dy) + \omega^2 y^2,$$

while the first integral of the differential linear system in the region U is

$$H_U = (y + \alpha)^2 + (x - \beta)^2.$$

3. Proofs of Theorem 2

In order that the piecewise linear differential system (2) has a crossing limit cycle Γ, it must intersect the discontinuous curve $y = x^n$ in two points. Let (x_L, x_L^n) and (x_R, x_R^n) be two intersecting points. Then, taking into account that H_L and H_R are first integrals these two points satisfy the equations

(3)
$$\begin{aligned} e_L &:= H_L(x_L, x_L^n) - H_L(x_R, x_R^n) = 0, \\ e_U &:= H_U(x_L, x_L^n) - H_U(x_R, x_R^n) = 0. \end{aligned}$$

Note that e_L and e_U are two polynomials of degree $2n$ in the variables x_L and x_R. Clearly $x_L - x_R$ is a factor of both e_L and e_R. Some direct calculations show that $E_L = e_L/(x_L - x_R)$ and $E_R = e_R/(x_L - x_R)$ are given by

$$
\begin{aligned}
E_L = {} & \left(4b^2 + \omega^2\right)\left(x_R^{2n-1} + x_L x_R^{2n-2} + \ldots + x_L^{n-3} x_R^{n+2} + x_L^{n-2} x_R^{n+1}\right) \\
& + \left((4b^2 + \omega^2)x_L^{n-1} + 8ab\right)x_R^n + \left((4b^2 + \omega^2)x_L^n + 8abx_L - 8ad\right)x_R^{n-1} \\
& + \left((4b^2 + \omega^2)x_L^{n+1} + 8abx_L^2 - 8adx_L\right)x_R^{n-2} \\
& + \ldots + \left((4b^2 + \omega^2)x_L^{2n-3} + 8abx_L^{n-2} - 8adx_L^{n-3}\right)x_R^2 \\
& + \left((4b^2 + \omega^2)x_L^{2n-2} + 8abx_L^{n-1} - 8adx_L^{n-2} + 4a^2\right)x_R \\
& + (4b^2 + \omega^2)x_L^{2n-1} + 8abx_L^n - 8adx_L^{n-1} + 4a^2 x_L + 8ac,
\end{aligned}
$$

and

$$
\begin{aligned}
E_U = {} & x_R^{2n-1} + x_L x_R^{2n-2} + \ldots + x_L^{n-3} x_R^{n+2} + x_L^{n-2} x_R^{n+1} + x_L^{n-1} x_R^n \\
& + (x_L^n + 2\alpha)\left(x_R^{n-1} + x_L x_R^{n-2} + \ldots + x_L^{n-3} x_R^2\right) \\
& + \left((x_L^n + 2\alpha)x_L^{n-2} + 1\right)x_R + (x_L^n + 2\alpha)x_L^{n-1} + x_L - 2\beta.
\end{aligned}
$$

Note that E_L and E_U are polynomials in the variables x_L and x_R of degree $2n - 1$.

To study the existence and the number of limit cycles of system (2), it is necessary to compute the common zeros (x_L, x_R) of E_L and E_R. For doing so we calculate the resultants $\mathrm{Resultant}[E_L, E_R, x_R]$ and $\mathrm{Resultant}[E_L, E_R, x_L]$ of E_L and E_R with respect to x_R and x_L, respectively. By the symmetry of E_L and E_R with respect to x_L and x_R, we know that $\mathrm{Resultant}[E_L, E_R, x_R]$ and $\mathrm{Resultant}[E_L, E_R, x_L]$ have the same expression. So we only need to compute $\mathrm{Resultant}[E_L, E_R, x_R]$.

In what follows we denoted by R_L (respectively R_R) for simplifying the notation the $\mathrm{Resultant}[E_L, E_R, x_R]$ (respectively $\mathrm{Resultant}[E_L, E_R, x_L]$) because this resultant is a polynomial in the variable x_L (respectively x_R).

The next result provides the degree of the resultant R_L, which is the most technical lemma in the proof of Theorem 2. We remark that in general the degree of the resultant of two arbitrary polynomials of degree $2n - 1$ is $(2n - 1)^2$, but since the polynomials E_L and E_R are special the degree in this case is lower, and to compute this degree is the more difficult part of this paper.

Lemma 3. *The maximum degree of the resultant R_L is*

$$2n^2 - n - 1 \text{ if } n \text{ is odd,} \qquad \text{and} \qquad 2n^2 - n \text{ if } n \text{ is even.}$$

Proof. Let $S_0 = 4b^2 + \omega^2$, $S_1 = -8ad$, $S_2 = 4a^2$, $S_3 = 8ac$ and $A = 8ab$. And for simplifying notations we replace x_L by x. Then by the definition of the resultant of two polynomials R_L is the determinant of a $(4n - 2) \times (4n - 2)$ matrix, more precisely

$$
R_L = \begin{vmatrix}
C_1 & 0_1 & \cdots & 0_{n-1} & \cdots & 0_{2n-2} & C_2 & 0_1 & \cdots & 0_{n-1} & \cdots & 0_{2n-2} \\
 & C_1 & & & & & & C_2 & & & & \\
 & & \ddots & & & & & & \ddots & & & \\
 & & & C_1 & & & & & & C_2 & & \\
 & & & & \ddots & & & & & & \ddots & \\
 & & & & & C_1 & & & & & & C_2
\end{vmatrix},
$$

---

where 0_k is a k–dimensional column vector for $k \in \{1,\ldots,n-2\}$, the entries which are empty are zeros, and C_1 and C_2 are $2n$-dimensional column vectors given respectively as

$$
C_1 = \begin{bmatrix} S_0 \\ S_0 x \\ S_0 x^2 \\ \vdots \\ S_0 x^{n-3} \\ S_0 x^{n-2} \\ S_0 x^{n-1} + A \\ S_0 x^n + Ax + S_1 \\ S_0 x^{n+1} + Ax^2 + S_1 x \\ \vdots \\ S_0 x^{2n-3} + Ax^{n-2} + S_1 x^{n-3} \\ S_0 x^{2n-2} + Ax^{n-1} + S_1 x^{n-2} + S_2 \\ S_0 x^{2n-1} + Ax^n + S_1 x^{n-1} + S_2 x + S_3 \end{bmatrix}, \quad
C_2 = \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^{n-3} \\ x^{n-2} \\ x^{n-1} \\ x^n + 2\alpha \\ x^{n+1} + 2\alpha x \\ \vdots \\ x^{2n-3} + 2\alpha x^{n-3} \\ x^{2n-2} + 2\alpha x^{n-2} + 1 \\ x^{2n-1} + 2\alpha x^{n-1} + x - 2\beta \end{bmatrix}.
$$

For computing the determinant R_L, we first apply the next algorithm to R_L:

- We add the $(2n-1+j)$-th column multiplied by $-(4b^2 + \omega^2)$ to the j-th column for $j = 1,\ldots,2n-1$.

Then

- We expand the resulting determinant of a $(4n-2)\times(4n-2)$ matrix through its first row.

- We expand the new resulting determinant of a $(4n-3)\times(4n-3)$ matrix through its first row.

- We expand this second new resulting determinant of a $(4n-4)\times(4n-4)$ matrix again through its first row.

- In total we repeat this process $n-1$ times, and at the end we have that R_L is the determinant of a $(3n-1)\times(3n-1)$ matrix.

After all these computations we achieve that $R_L = (-1)^{n-1} R_{L_1}$ with

$$
R_{L_1} = \begin{vmatrix} D_1 & 0_1 & \cdots & 0_{n-2} & 0_{n-1} & 0_n & \cdots & 0_{2n-2} & C_2 & 0_1 & \cdots & 0_{n-2} & 0_{n-1} \\ & D_1 & & & & & & & & C_2 & & & \\ & & \ddots & & & & & & & & \ddots & & \\ & & & D_1 & & & & & & & & C_2 & \\ & & & & D_1 & & & & & & & & C_2 \\ & & & & & D_1 & & & & & & & \\ & & & & & & \ddots & & & & & & \\ & & & & & & & D_1 & & & & & \end{vmatrix},
$$

where the entries without letters are zeros, and D_1 is the $n+1$–dimensional column vector given by

$$D_1 = \begin{bmatrix} A \\ Ax + B \\ Ax^2 + Bx \\ \vdots \\ Ax^{n-2} + Bx^{n-3} \\ Ax^{n-1} + Bx^{n-2} + C \\ Ax^n + Bx^{n-1} + Cx + D \end{bmatrix},$$

where $B = -8ad - 2\alpha\left(4b^2 + \omega^2\right)$, $C = 4a^2 - \left(4b^2 + \omega^2\right)$ and $D = 8ac + 2\beta\left(4b^2 + \omega^2\right)$.

We separate the rest of the proof in three cases, first $n > 3$, after $n = 2$ and finally $n = 3$.

We do the next computations with R_{L_1} when $n > 3$:

- We add the first row multiplied by $-x^\ell$ to the $(\ell+1)$-th row for $\ell = 1, \ldots, n$.
- In the new resulting determinant we add the second row multiplied by $-x^\ell$ to the $(\ell + 2)$-th row for $\ell = 1, \ldots, n - 1$.
- In this second new resulting determinant we add the third row multiplied by $-x^\ell$ to the $(\ell + 3)$-th row for $\ell = 1, \ldots, n - 2$.
- In this third new resulting determinant we add the fourth row multiplied by $-x^\ell$ to the $(\ell + 3)$-th row for $\ell = 1, \ldots, n - 3$.
- In total we repeat this process n times.

Now R_{L_1} is the determinant of a $(3n - 1) \times (3n - 1)$ matrix of the form

$$R_{L_1} = \begin{vmatrix} M_1 & M_2 \\ M_3 & M_4 \end{vmatrix}$$

where M_1, M_2, M_3 and M_4 are respectively the $(n+1) \times (n+1)$, $(n+1) \times (2n-2)$, $(2n-2) \times (n+1)$ and $(2n-2) \times (2n-2)$ matrices given by

$$M_1 = \begin{bmatrix} A & & & & & & \\ B & A & & & & & \\ & B & \ddots & & & & \\ & & \ddots & \ddots & & & \\ & & & \ddots & \ddots & & \\ C & & & & \ddots & A & \\ D & C & & & & B & A \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0_{n-2}^T & 1 & 0 & & & \\ & & 1 & & & \\ \vdots & & & \ddots & & \\ & & & & 1 & 0 \\ & & & & & 1 \\ 0_{n-2}^T & 2\alpha & & & & 0 \end{bmatrix},$$

$$
M_3 = \begin{bmatrix}
0_{2n-2} & f_n(x) & g_{n-1}(x) & h_{n-2}(x) & h_{n-3}(x) & \cdots & \cdots & h_1(x) \\
 & 0_{2n-3} & f_n(x) & g_{n-1}(x) & h_{n-2}(x) & \ddots & \ddots & \vdots \\
 & & 0_{2n-4} & \ddots & & \ddots & \ddots & \vdots \\
 & & & & \ddots & & \ddots & h_{n-3}(x) \\
 & & & & & \ddots & \ddots & h_{n-2}(x) \\
 & & & & & & \ddots & g_{n-1}(x) \\
 & & & & & & & f_n(x) \\
 & & & & & & & 0_{n-2}
\end{bmatrix},
$$

where

$$
f_n(x) = Ax^n + Bx^{n-1} + Cx + D,
$$
$$
g_{n-1}(x) = Ax^{n-1} + Bx^{n-2} + C,
$$
$$
h_j(x) = Ax^j + Bx^{j-1} \quad \text{for} \quad j = n-2, \ldots, 1,
$$

and M_4 is equal to

$$
\begin{bmatrix}
A & & & & & r_{n+1}(x) & r_n(x) & x^{n-1} & \cdots & \cdots & x^3 & x^2 \\
h_1(x) & A & & & & r_{n+2}(x) & r_{n+1}(x) & r_n(x) & \ddots & \ddots & x^4 & x^3 \\
\vdots & & \ddots & \ddots & & \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & r_{2n-3}(x) & & \ddots & & \ddots & \ddots & x^{n-1} & \vdots \\
\vdots & & \ddots & \ddots & A & q_{2n-2}(x) & & \ddots & & \ddots & \ddots & r_n(x) & x^{n-1} \\
h_{n-2}(x) & \ddots & \ddots & \ddots & h_1(x) & p_{2n-1}(x) & & \ddots & & \ddots & \ddots & r_{n+1}(x) & r_n(x) \\
g_{n-1}(x) & h_{n-2}(x) & \ddots & \ddots & \vdots & & & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
f_n(x) & g_{n-1}(x) & \ddots & \ddots & \vdots & & & & \ddots & \ddots & \ddots & \vdots & \vdots \\
 & f_n(x) & \ddots & \ddots & \vdots & & & & & \ddots & \ddots & r_{2n-3}(x) & \vdots \\
 & & \ddots & \ddots & h_{n-2}(x) & & & & & & \ddots & q_{2n-2}(x) & r_{2n-3}(x) \\
 & & & \ddots & g_{n-1}(x) & & & & & & & p_{2n-1}(x) & q_{2n-2}(x) \\
 & & & & f_n(x) & & & & & & & & p_{2n-1}(x)
\end{bmatrix},
$$

where

$$
p_{2n-1}(x) = x^{2n-1} + \left(2\alpha x^{n-2} + 1\right)x - 2\beta,
$$
$$
q_{2n-2}(x) = x^{2n-2} + 2\alpha x^{n-2} + 1,
$$
$$
r_j(x) = x^j + 2\alpha x^{j-n} \quad \text{for} \quad j = 2n-3, \ldots, n \quad \text{with} \quad j \geq 3.
$$

Since $\det M_1 \neq 0$ because $ab \neq 0$, doing some linear algebra we get that

$$
\det R_{L_1} = \det M_1 \det(M_4 - M_3 M_1^{-1} M_2) = (\det A)^{n+1} \det(M_4 - M_3 M_1^{-1} M_2).
$$

Direct calculations show that

$$
M_1^{-1} = \begin{bmatrix}
w_1 & & & & & & & & & \\
w_2 & w_1 & & & & & & & & \\
w_3 & w_2 & w_1 & & & & & & & \\
\vdots & & w_3 & w_2 & w_1 & & & & & \\
\vdots & \ddots & & \ddots & \ddots & \ddots & \ddots & & & \\
\vdots & & \ddots & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
w_{n-1} & \ddots & & \ddots & & \ddots & \ddots & \ddots & \ddots & \ddots \\
u_n & w_{n-1} & & \ddots & & \ddots & \ddots & \ddots & w_2 & w_1 \\
v_{n+1} & u_n & w_{n-1} & \cdots & & \cdots & \cdots & w_3 & w_2 & w_1
\end{bmatrix}
$$

where

$$
w_k = (-1)^{k-1} A^{-k} B^{k-1} \quad \text{for} \quad k = 1, 2, \ldots, n-1,
$$
$$
u_n = (-1)^{n-1} A^{-n} B^{n-1} - A^{-2} C,
$$
$$
v_{n+1} = (-1)^n A^{-(n+1)} B^n + 2 A^{-3} B C - A^{-2} D.
$$

Furthermore,

$$
M_1^{-1} M_2 = \begin{bmatrix}
0_{n-2}^T & w_1 & & & & & & & \\
0_{n-2}^T & w_2 & w_1 & & & & & & \\
0_{n-2}^T & w_3 & w_2 & w_1 & & & & & \\
\vdots & \vdots & & w_3 & w_2 & w_1 & & & \\
\vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & & \\
\vdots & \vdots & & \ddots & & \ddots & \ddots & \ddots & \ddots \\
0_{n-2}^T & w_{n-1} & & \ddots & & \ddots & \ddots & \ddots & \ddots & \ddots \\
0_{n-2}^T & u_n & w_{n-1} & & \ddots & & \ddots & \ddots & w_2 & w_1 \\
0_{n-2}^T & v_{n+1} + 2\alpha w_1 & u_n & w_{n-1} & \cdots & \cdots & \cdots & w_3 & w_2
\end{bmatrix},
$$

and

$$
M_3 M_1^{-1} M_2 = \begin{bmatrix}
0_{n \times (n-2)} & M_{n \times n} \\
0_{(n-2) \times (n-2)} & 0_{(n-2) \times n}
\end{bmatrix}, \qquad M_{n \times n} = (F_{k\ell})_{k, \ell = 1, \ldots, n},
$$

with $0_{i \times j}$ the zero $i \times j$ matrix, and

$$
F_{k\ell} = \sum_{s=1}^{n+1-k} h_{k-1+s} w_{n+3-\ell-s},
$$

where for simplifying the notation we have used $w_j = 0$ for $j \le 0$, $h_n = f_n$, $h_{n-1} = g_{n-1}$, $w_n = u_n$ and $w_{n+1} = v_{n+1} + 2\alpha w_1$.

Since $\det A = 8ab \ne 0$, the computation of R_{L_1} has been reduced to calculate the determinant of the matrix $M_4 - M_3 M_1^{-1} M_2$. Note that M_4 and $M_3 M_1^{-1} M_2$ are $(2n-2) \times (2n-2)$ matrices. Now we work with these two matrices according to the following algorithm:

- we add the i-th row multiplied by $-x$ to the $(i+1)$-th row for $i = n-3, n-2, \ldots, 1$, successively.

At the end of this process the matrix M_4 has been reduced to the matrix M_4^* given by

$$
\left[
\begin{array}{ccccccccccccccc}
A & & & & & & & x^{n+1}+2\alpha x & x^n+2\alpha & x^{n-1} & \cdots & \cdots & \cdots & x^3 & x^2 \\
B & A & & & & & & & & 2\alpha & & & & & \\
& B & \ddots & & & & & & & & & & \ddots & & \\
& & \ddots & \ddots & & & & & & & & & & \ddots & \\
& & & \ddots & \ddots & & & & & & & & & & \ddots \\
& & & & \ddots & A & & 1 & & & & & 2\alpha & & \\
& & & & & B & & -2\beta & 1 & & & & & & 2\alpha \\
C & & & & & & & \widetilde{p}_{2n} & -2\beta & 1 & & & & & \\
D & C & & & & & & & \widetilde{p}_{2n} & -2\beta & \ddots & & & & \\
\widetilde{f}_{n+1} & D & \ddots & & & & & & & \widetilde{p}_{2n} & \ddots & \ddots & & & \\
& \widetilde{f}_{n+1} & \ddots & \ddots & & & & & & & \ddots & \ddots & \ddots & & \\
& & \ddots & \ddots & C & & & & & & & \ddots & \ddots & 1 & \\
& & & \ddots & D & C & & & & & & & \ddots & -2\beta & 1 \\
& & & & \widetilde{f}_{n+1} & D & & & & & & & & \widetilde{p}_{2n} & -2\beta \\
\end{array}
\right],
$$

where $\widetilde{f}_{n+1} = -xf_n(x)$ and $\widetilde{p}_{2n} = -xp_{2n-1}(x)$ are polynomials of degrees $n+1$ and $2n$ respectively. Applying the same algorithm to the matrix $M_3 M_1^{-1} M_2$, this has been reduced to the matrix M_3^* given by

$$
\left[
\begin{array}{cccccccccc}
& F_{11} & F_{12} & F_{13} & F_{14} & \cdots & \cdots & \cdots & F_{1,n-2} & F_{1,n-1} & F_{1,n} \\
& E_2(x) & E_1(x) & w_0 & Cw_1 & & & & & & \\
& E_3(x) & E_2(x) & E_1(x) & w_0 & Cw_1 & & & & & \\
\mathbf{0}_1 & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & & & & \\
& \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\
& \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
& \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
& E_{n-2}(x) & E_{n-3}(x) & E_{n-4}(x) & \cdots & \cdots & \cdots & E_2(x) & E_1(x) & w_0 & Cw_1 \\
& E_{n-1}(x) & E_{n-2}(x) & E_{n-3}(x) & \cdots & \cdots & \cdots & \cdots & E_2(x) & E_1(x) & w_0 \\
& F_n(x) & F_{n-1}(x) & F_{n-2}(x) & \cdots & \cdots & \cdots & \cdots & F_3(x) & F_2(x) & F_1(x) \\
& -xf_nw_{n+1} & -xf_nw_n & -xf_nw_{n-1} & \cdots & \cdots & \cdots & \cdots & -xf_nw_4 & -xf_nw_3 & -xf_nw_2 \\
\mathbf{0}_2 & & & \mathbf{0}_3 & & & & & & &
\end{array}
\right],
$$

where $\mathbf{0}_1 = 0_{(n+1)\times(n-2)}$, $\mathbf{0}_2 = 0_{(n-3)\times(n-2)}$, $\mathbf{0}_3 = 0_{(n-3)\times n}$, $w_0 = Dw_1 + Cw_2$, and

$$E_i(x) = -xf_n(x)w_i + Dw_{i+1} + Cw_{i+2} \quad \text{for } i = 1, \ldots, n-1,$$

$$F_i(x) = -xf_n(x)w_i + Dw_{i+1}, \quad \text{for } i = 1, \ldots, n,$$

$$F_{11} = -A^{-1}Bx^n + (A^{-2}BC - A^{-1}D + 2\alpha)x,$$
$$+ (-1)^n A^{-n-1}B^{n+1} + 2A^{-3}B^2C - 2A^{-2}BD + 2\alpha A^{-1}D,$$

$$F_{12} = x^n + (-1)^{n-1}A^{-n}B^n - A^{-2}BC + A^{-1}D,$$

$$F_{13} = x^{n-1} + (-1)^{n-2}A^{-(n-1)}B^{n-1} + A^{-1}C,$$

$$F_{1\ell} = x^{n+2-\ell} + (-1)^{n+1-\ell}A^{-(n+2-\ell)}B^{n+2-\ell}, \quad \text{for } \ell = 4, \ldots, n.$$

Now $\det\left(M_4 - M_3 M_1^{-1} M_2\right) = \det\left(M_4^* - M_3^*\right)$, and this last determinant is the determinant of the matrix

$$\begin{bmatrix}
A & & & & & & G_1 & \sigma_2 & \sigma_3 & \sigma_4 & \cdots & \cdots & \sigma_{n-1} & \sigma_n \\
B & A & & & & & \widetilde{E}_2 & \widetilde{E}_1 & \delta & -cw_1 & & & & \\
& B & \ddots & & & & \widetilde{E}_3 & \widetilde{E}_2 & \widetilde{E}_1 & \delta & & \ddots & & \\
& & \ddots & \ddots & & & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & & \\
& & & \ddots & A & & \widetilde{E}_{n-3} & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \\
& & & & B & A & \widetilde{E}_{n-2} & \widetilde{E}_{n-3} & \widetilde{E}_{n-4} & \cdots & \cdots & \widetilde{E}_1 & \delta & -cw_1 \\
& & & & & B & \widetilde{E}_{n-1} & \widetilde{E}_{n-2} & \widetilde{E}_{n-3} & \widetilde{E}_{n-4} & \cdots & \cdots & \widetilde{E}_1 & \delta \\
C & & & & & & \widetilde{p}_{2n,1} & \widetilde{F}_{n-1} & \widetilde{F}_{n-2} & \widetilde{F}_{n-3} & \cdots & \cdots & \widetilde{F}_2 & \widetilde{F}_1 \\
D & C & & & & & xf_nw_{n+1} & \widetilde{p}_{2n,2} & \widetilde{f}_{n+1,1} & \widetilde{f}_{n+1,2} & xf_nw_{n-3} & \cdots & xf_nw_3 & xf_nw_2 \\
\widetilde{f}_{n+1} & D & \ddots & & & & & & \widetilde{p}_{2n} & -2\beta & 1 & & & \\
& \widetilde{f}_{n+1} & \ddots & \ddots & & & & & & \widetilde{p}_{2n} & & \ddots & \ddots & \\
& & \ddots & \ddots & C & & & & & & & \ddots & \ddots & 1 \\
& & & \ddots & D & C & & & & & & & -2\beta & 1 \\
& & & & \widetilde{f}_{n+1} & D & & & & & & & \widetilde{p}_{2n} & -2\beta
\end{bmatrix},$$

where the entries without anything are all zeros, and

$$G_1 = G_1(x) = x^{n+1} + A^{-1}Bx^n + (A^{-2}BC - A^{-1}D)x$$
$$- (-1)^n A^{-n-1}B^{n+1} - 2A^{-3}B^2C + 2A^{-2}BD - 2\alpha A^{-1}D,$$

$$\sigma_2 = 2\alpha + (-1)^n A^{-n}B^n + A^{-2}BC - A^{-1}D,$$

$$\sigma_3 = (-1)^{n-1}A^{-(n-1)}B^{n-1} - A^{-1}C,$$

$$\sigma_\ell = (-1)^{n+2-\ell}A^{-(n+2-\ell)}B^{n+2-\ell}, \quad \ell = 4, \ldots, n,$$

$$\delta = 2\alpha - Dw_1 - Cw_2,$$

$$\widetilde{E}_{n-2} = 1 - E_{n-2}, \quad \widetilde{E}_{n-1} = -2\beta - E_{n-1}, \quad \widetilde{E}_j = -E_j, \; j = 1, \ldots, n-3$$
$$\widetilde{F}_{n-1} = -2\beta - F_{n-1}, \quad \widetilde{F}_{n-2} = 1 - F_{n-2}, \quad \widetilde{F}_j = -F_j, \; j = 1, \ldots, n-3$$
$$\widetilde{f}_{n+1} = -xf_n, \quad \widetilde{f}_{n+1,1} = -2\beta + xf_nw_{n-1}, \quad \widetilde{f}_{n+1,2} = 1 + xf_nw_{n-2}$$
$$\widetilde{p}_{2n} = -xp_{2n-1}, \quad \widetilde{p}_{2n,1} = -xp_{2n-1} - F_n, \quad \widetilde{p}_{2n,2} = -xp_{2n-1} + xf_nw_n.$$

Denote this last $(2n-2)\times(2n-2)$ matrix as M_{2n-2}. Note that $A, B, C, D, \delta, -cw_1$, $-2\beta, \sigma_i$ are constants, $\widetilde{p}_{2n,1}, \widetilde{p}_{2n,2}$ and $\widetilde{p}_{2n}$ are polynomials of degree $2n$, all the other letters denote polynomials of degree $n + 1$.

Since by definition the determinant of a matrix (a_{ij}) of order m is

$$\sum_{\sigma(j_1,\ldots,j_m)} (-1)^{\tau(j_1,\ldots,j_m)} a_{1j_1} \ldots a_{mj_m},$$

where σ is a permutation of $\{1,\ldots,m\}$ and the summation is taken over all these possible permutations, and τ evaluated on a permutation of $\{1,\ldots,m\}$ is the minimum number of transpositions for passing the permutation to the identity, for calculating the degree of this determinant, we must first compute the highest degree terms $a_{1j_1} \ldots a_{mj_m}$ in the summation. In order to realize this aim we must choose the a_{ij_i}'s in such a way that the product has the highest possible degree. Applying this idea to the determinant of the matrix M_{2n-2}, in the row and column where $\widetilde{p}_{2n,1}, \widetilde{p}_{2n,2}, \widetilde{p}_{2n}$ is located, we must choose either one of these three polynomials or the two ones on the row and column where one of these three polynomials is located, because their product has the highest degree. The details are the following:

- For $\widetilde{p}_{2n,1}$ and $\widetilde{p}_{2n,2}$ we can choose respectively $G_1\widetilde{F}_1$ and $\widetilde{E}_1 x f_n w_1$. But we can choose only one of them, because $\widetilde{F}_1$ and $x f_n w_1$ are located on the same column. So in the product we choose $G_1\widetilde{F}_1$ and $\widetilde{p}_{2n,2}$. Of course we can also choose $\widetilde{p}_{2n,1}$ and $\widetilde{E}_1 x f_n w_1$ as the elements in the product.
- Next in order to get the highest degree in the product, we must choose the B or A in the second row, which is associated to $\widetilde{p}_{2n,1}$. We choose A.
- Then for the first $\widetilde{p}_{2n}$ from above, we choose $\widetilde{f}_{n+1}$ and $\widetilde{E}_1$, which are located respectively on the row and column where the first $\widetilde{p}_{2n}$ is located.
- For the second $\widetilde{p}_{2n}$, we can only choose itself, because there are no other two elements on the row and column where the second $\widetilde{p}_{2n}$ is located, whose product has degree larger than $2n$.
- Correspondingly in the fourth row we can choose either the B or the A, we choose A.
- To the third $\widetilde{p}_{2n}$, we choose $\widetilde{f}_{n+1}$ and $\widetilde{E}_1$, which are located respectively on the row and column where the third $\widetilde{p}_{2n}$ is located.
- In the row and column where the fourth $\widetilde{p}_{2n}$ are located, we can choose only $\widetilde{p}_{2n}$ itself as an element of the product.
- Then in the sixth row we can choose only either the B or the A, and for the fifth $\widetilde{p}_{2n}$ we must choose $\widetilde{f}_{n+1}$ and $\widetilde{E}_1$ as the elements of the product.
- According to this algorithm we get the highest degree terms in the summation in the following products
  - n polynomials of degree $n+1$ and $(n-2)/2$ polynomials of degree $2n$ provided that n is even,
  - $n-1$ polynomials of degree $n+1$ and $(n-1)/2$ polynomials of degree $2n$ provided that n is odd.

Summarizing we have proved that the highest degree of R_L is at most

$$n(n+1) + \frac{n-2}{2}2n = 2n^2 - n \qquad \text{if } n \text{ is even,}$$

$$(n-1)(n+1) + \frac{n-1}{2}2n = 2n^2 - n - 1 \quad \text{if } n \text{ is odd.}$$

Note that this last two numbers are both even, and recall that they are denoted $2d_n$.

The above proof provides the maximum degree of the polynomial R_L for $n > 3$. We now study the degree of R_L for $n = 2, 3$, and for these values of n we present the explicit expressions of R_L.

For $n = 2$ we work in a similar way as we did for $n > 3$, and we get that

$$R_L = -(8ab)^3 \det (M_4^* - M_3^*),$$

where

$$M_4^* = \begin{bmatrix} p_3(x) & q_2(x) \\ -xp_3(x) & -2\beta \end{bmatrix},$$

$$M_3^* = \begin{bmatrix} -\eta_1 f_2 + (\eta_2 - A^{-2}D)g_1 + (2\alpha+1)A^{-1}g_1 & A^{-1}f_2 - \eta_1 g_1 \\ \eta_1 x f_2 + (\eta_2 - A^{-2}D)D + (2\alpha+1)A^{-1}D & -A^{-1}xf_2 - \eta_1 D \end{bmatrix},$$

with $\eta_1 = A^{-2}(B+C)$ and $\eta_2 = A^{-3}(B+C)^2$. Since $q_2(x) - (A^{-1}f_2(x) - \eta_1 g_1(x)) = A^{-2}(B+C)^2 - A^{-1}D + 2\alpha + 1$, it follows that R_L is of degree 6 with the monomial x^6 of higher degree, which comes from the product of the monomials x^3 of higher degree in the polynomials $p_3(x)$ and $A^{-1}xf_2(x)$.

For $n = 3$ similar computations yield

$$R_L = (8ab)^4 \det (M_4^* - M_3^*)$$

with

$$M_4^* = \begin{bmatrix} A & q_4(x) & r_3(x) & x^2 \\ B & -2\beta & 1 & 2\alpha \\ C & -xp_5(x) & -2\beta & 1 \\ D & 0 & -xp_5(x) & -2\beta \end{bmatrix},$$

$$M_3^* = \begin{bmatrix} 0 & -A^{-2}Bf_3 + u_3 g_2 + \gamma_4 h_1 & A^{-1}f_3 - A^{-2}Bg_2 + u_3 h_1 & A^{-1}g_2 - A^{-2}Bh_1 \\ 0 & A^{-2}Bxf_3 + u_3 D + \gamma_4 C & -A^{-1}xf_3 - A^{-2}BD + u_3 C & A^{-1}D - A^{-2}BC \\ 0 & -u_3 xf_3 + \gamma_4 D & A^{-2}Bxf_3 + u_3 D & -A^{-1}xf_3 - A^{-2}BD \\ 0 & -\gamma_4 xf_3 & -u_3 xf_3 & A^{-2}Bxf_3 \end{bmatrix},$$

where $\gamma_4 = -A^{-4}B^3 + 2A^{-3}BC - A^{-2}D + 2\alpha A^{-1}$. Note that $A^{-1}g_2 - A^{-2}Bh_1 = x^2 - A^{-2}B^2 + A^{-1}C$. Some additional calculations show that polynomial entries m_{i1}^* for $i = 1, \ldots, 4$, m_{13}^* and m_{14}^* of the matrix $M_4^* - M_3^* := (m_{ij}^*(x))_{4\times4}$ have degree zero, and that the degree of the polynomials m_{32}^* and m_{43}^* is 6. All the other entries have degree 4. In the determinant of the matrix $M_4^* - M_3^*$, the product containing the two entries of degree 6 provides only degree 12. Taking one degree 6 entry and two entries of degree 4 we get the maximum degree 14.

This completes the proof of the lemma. □

Proof of statement (a) Theorem 2. From (3) we have that for each solution (x_L, x_R) with $x_L \neq x_R$ of the polynomial systems $E_L = 0$ and $E_R = 0$, we have a crossing limit cycle of the discontinuous piecewise linear differential system (2) which intersect the curve $y = x^n$ at the points (x_L, x_L^n) and (x_R, x_R^n).

Recall that we denote by $2d_n$ the degree of the resultant R_L computed in Lemma 3. Of course, the expression of $2d_n$ changes according the parity of n.

If (x_L, x_R) with $x_L \neq x_R$ is a solution of the polynomial system $E_L = 0$ and $E_R = 0$, the value x_L is a root of the resultant R_L, and the value x_R is a root of the resultant R_R. But both resultants R_L and R_R have the same roots, because these two polynomials are the same, we can pass from one to the other interchanging the variable x_L by the variable x_R. So the possible values for x_L and for x_R are the same. Consequently we only have at most $2d_n$ points (x_L, x_L^n) and (x_R, x_R^n), all together, which can be points where the crossing limit cycles intersect the curve $y = x^n$. Hence we cannot have more then d_n crossing limit cycles. This completes the proof of statement (a) of the theorem. $\qquad\square$

Proof of statement (b) Theorem 2. We provide an example showing that the upper bound of 3 limit cycles given in Theorem 2 for the maximum number of limit cycles that a discontinuous piecewise linear differential system in $\mathbb{R}^2$ separated by the parabola $y = x^2$ when both linear differential systems have a center is reached. Consider the linear differential system

(4)
$$\dot{x} = -\frac{189x}{100} + \frac{38221y}{20500} - \frac{720042289}{205000000},$$
$$\dot{y} = -\frac{41x}{20} + \frac{189y}{100} - \frac{3296401}{1000000},$$

in the region L, and the linear differential system

(5)
$$\dot{x} = \frac{18}{5} - y,$$
$$\dot{y} = x - \frac{99}{50},$$

in the region U. Both linear differential systems form a discontinuous piecewise linear differential system in $\mathbb{R}^2$ separated by the parabola $y = x^2$ having two centers. This discontinuous piecewise linear differential system has the three limit cycles of the Figure 1.

We note that the value of the coordinate x of the six points (x, x^2) of the parabola which belong to the limit cycles are the six real roots of the polynomial R_L equal to

$10769750976621134140624876981 - 29736678501017804062959688750\,x +$
$251541921321105654630871975\,x^2 + 259450288777014130734750000000\,x^3 -$
$556663302223881267192750000\,x^4 - 50659681524188388750000000000\,x^5 +$
$145407565859062500000000000\,x^6.$

The discontinuous piecewise linear differential system (4)-(5) proves the statement (b) of Theorem 2. $\qquad\square$

Proof of statement (c) Theorem 2. For $n = 2$ we have that $d_n = 3$, and consequently by Theorem 2 the discontinuous piecewise linear differential system (2) will have at most 3 limit cycles. The concrete expression of the resultant R_L modulo the coefficient of the highest degree monomial is

$$R_L^* = x^6 + C_1 x^5 + C_2 x^4 + C_3 x^3 + C_4 x^2 + C_5 x + C_6,$$

as before we have replaced the variable x_L by the variable x for simplicity, and

$$C_1 = 2A\eta_1,$$
$$C_2 = A^{-1}D + 2A^2\eta_1^2 + \eta_3,$$
$$C_3 = 4D\eta_1 - 4\beta,$$
$$C_4 = A^{-2}D^2 - A^{-1}D\eta_3 + D\eta_2 + \eta_3^2 + A\eta_1(D\eta_1 - 4\beta) + A^2\eta_1^2\eta_3,$$
$$C_5 = 2A^{-1}D(D\eta_1 - \beta) - 2\beta\eta_3 - 2AD\eta_1\eta_2 - 2A^2(\beta - D\eta_1)\eta_1^2,$$
$$C_6 = A^{-3}D^3 - 2A^{-2}D^2\eta_3 + A^{-1}D(\eta_3 + 1)\eta_3 - D\eta_1(6\beta + D\eta_1 - \eta_1\eta_3)$$
$$\qquad + 4\beta^2 - A(D\eta_2^2 - 2\beta\eta_1\eta_3) + A^3\eta_1^3(2\beta + D\eta_1),$$

where $\eta_3 = 2\alpha + 1$.

For $n = 3$ we have that $d_n = 7$, therefore by Theorem 2 the discontinuous piecewise linear differential system (2) will have at most 7 limit cycles. The resultant $R_L =$ Resultant$[E_L(x_L, x_R), E_R(x_L, x_R), x_R]$ has a huge expression directly using the parameters of the system. Here we choose the previous notations A, B, C, D, and still replace x_L by x for simplicity, we get the next formula R_L^* of R_L modulo a multiplicative constant.

$$R_L^* = C_0x^{14} + C_1x^{13} + C_2x^{12} + C_3x^{11} + C_4x^{10} + C_5x^9 + C_6x^8 + C_7x^7 + C_8x^6$$
$$\qquad + C_9x^5 + C_{10}x^4 + C_{11}x^3 + C_{12}x^2 + C_{13}x + C_{14},$$

where

$$C_0 = -3B,$$
$$C_1 = -6A^{-1}B^2,$$
$$C_2 = -4A^{-2}B^3 - 2A^{-1}B(C + 2D) - 3D + 4A\alpha,$$
$$C_3 = -2A^{-2}B^2(C + 5D) - 12A^{-1}BD,$$
$$C_4 = -2A^{-3}B^3(3C - 5D) - 18A^{-2}B^2D - A^{-2}BD(8C + D)$$
$$\qquad -2A^{-1}(3B^2\alpha + 2CD + D^2) - 7B + 6C\alpha + 4D\alpha - 4A\beta,$$
$$C_5 = 4A^{-4}B^4(C - D) - 2A^{-3}B^2(4BD - 5C^2 + 10CD + D^2)$$
$$\qquad +2A^{-2}B(4B^2\alpha - 5CD + 7D^2) - 2A^{-1}(5B^2 + 6BC\alpha - 3D^2)$$
$$\qquad +2B\beta + 2D\alpha + 8A\alpha^2,$$
$$C_6 = A^{-4}B^3(15C^2 - 18CD - D^2) + 2A^{-3}B(BCD - 13BD^2 + 2C^3 - 4C^2D - CD^2)$$
$$\qquad -A^{-2}(4B^3 - 4B^2C\alpha - 8B^2D\alpha - 18BD^2 - 6CD^2 - 3C^2D)$$
$$\qquad +2A^{-1}(3B^2\beta - BC - 3BD + 3C^2\alpha + 4CD\alpha + D^2\alpha) - 7D + 6C\beta + 4D\beta + 8A\alpha,$$
$$C_7 = 4A^{-5}B^4C(C - D) + 2A^{-4}B^2(8BCD + 8C^3 - 10C^2D - CD^2 - 10D^2)$$
$$\qquad +2A^{-3}B(B^3 - 4B^2D\alpha - 9BD^2 + C^2D - 12CD^2 - D^3)$$
$$\qquad -2A^{-2}(B^3\beta + 5B^2D + 9B^2D\alpha - 2BC^2\alpha - 2BCD\alpha - 2BD^2\alpha + 4CD^2 + 2D^3)$$
$$\qquad -4A^{-1}(BC\beta + BD\beta + 5BD - 3CD\alpha - D^2\alpha) + 8D\alpha^2) - 16A\alpha\beta,$$
$$C_8 = 2A^{-5}B^3(2BD + 5C^2)(C - D) + 4A^{-4}B^4(C - D)\alpha - 4A^{-4}B^3D^2$$
$$\qquad +2A^{-4}B^2(15C^2D - 20CD^2 - D^3) + A^{-4}BC^2(4C^2 - D^2 - 6CD)$$
$$\qquad -8A^{-3}B^4\beta + 2A^{-3}B^3(5C - 2D - 8D\alpha) - 2A^{-3}B^2(5C^2 + 2CD - D^2)\alpha$$
$$\qquad -2A^{-3}BD^2(8C + 7D) - 2A^{-3}CD(C^2 + CD + D^2) - 4A^{-2}B^3\alpha^2$$
$$\qquad +2A^{-2}B^2(3C\beta - 11D - 2D\beta) - A^{-2}B(8CD - 26CD\alpha + D^2 + 8D^2\alpha)$$
$$\qquad +2A^{-2}(2C^3\alpha + 2C^2D\alpha + 3CD^2\alpha - 2D^3) - 2A^{-1}B^2\alpha$$
$$\qquad +2A^{-1}B(2C\alpha^2 + 4D\alpha^2 + 3D\beta) - 2A^{-1}(2C^2\beta + 3CD + 6CD\beta + D^2 + D^2\alpha)$$
$$\qquad -5B - 4B\alpha\beta + 8C\alpha + 4D\alpha + 4D\alpha^2 + 8A(\alpha^3 - \beta),$$

$$\begin{aligned}
C_9 =\ & 8A^{-5}B^2C\left(C^2+2BD\right)(C-D) + 2A^{-4}B^4(C+D) + 4A^{-4}B^3C(C-3D)\alpha \\
& + 2A^{-4}B^2D^2(5C-11D) + 4A^{-4}BCD(3C^2-5CD-D^2) \\
& -4A^{-3}B^3(2C\beta+D+2D\beta) + 2A^{-3}B^2(3C^2-4CD-4CD\alpha-D^2-8D^2\alpha) \\
& -4A^{-3}B(C^3\alpha-2C^2D\alpha-2CD^2\alpha+3D^3) - 4A^{-3}CD^2(2C+D) - 4A^{-2}B^3\alpha \\
& + 12A^{-2}B^2D\beta + 2A^{-2}B(6C^2\beta-3CD-10CD\beta-7D^2+D^2\beta) \\
& + 2A^{-2}(10C^2D\alpha-4CD^2\alpha+3D^3\alpha) - 4A^{-1}B^2 + 12A^{-1}BC\alpha \\
& -2A^{-1}(4C^2\alpha^2-8CD\alpha^2+2CD\beta+5D^2+4D^2\beta) \\
& + 4(B\beta+D\alpha-2C\alpha\beta-2D\alpha\beta) + 8A(\alpha^2+\beta^2), \\[4pt]
C_{10} =\ & -2A^{-5}B(C-D)\left(B^4-2B^3C\alpha-5B^2D^2+10BC^2D-C^4\right) \\
& + 2A^{-4}B^4(2C\beta+D-4D\beta) + A^{-4}B^3(4C^2+8CD\alpha-D^2-16D^2\alpha) \\
& -2A^{-4}B^2(2C^3\alpha+2C^2D\alpha-CD^2\alpha+3D^3) + A^{-4}B(6C^2D^2-20CD^3-D^4) \\
& -A^{-4}C^2D(C^2+2D^2) + 2A^{-3}B^4\alpha - 4A^{-3}B^3(C\alpha^2+2D\beta) \\
& + 2A^{-3}B^2(7C^2\beta+5CD-8CD\beta-8D^2-9D^2\alpha+2D^2\beta) \\
& + 2A^{-3}B(C^3-C^2D-CD^2+4D^3\alpha-C^2D\alpha) \\
& + 2A^{-3}(C^4\alpha+3C^2D^2\alpha-4CD^3-D^4) - A^{-2}B^3(1-4\alpha\beta) \\
& -4A^{-2}B^2(C+D)\alpha + A^{-2}B(4C^2\alpha^2+8CD\alpha^2-4CD\beta-21D^2-4D^2\beta) \\
& -A^{-2}D(2C^2+12C^2\beta+8CD-22CD\alpha+2CD\beta-D^2+4D^2\alpha) + 8A^{-1}B^2\beta \\
& -2A^{-1}B(8C\alpha\beta+D-3D\alpha) + 4A^{-1}(C^2\alpha+2CD\alpha-5CD\alpha^2+4D^2\alpha^2+2D^2\beta) \\
& -4B(\alpha^2+2\beta^2) - 8C\beta + 8C\alpha^3 - 5D - 20D\alpha\beta - 4D\beta - 4A(2\alpha\beta-1)\alpha, \\[4pt]
C_{11} =\ & 4A^{-5}B(C-D)\left(B^4\beta-B^3(C-D\alpha)+B^2C^2\alpha+5BCD^2+2C^3D\right) \\
& -8A^{-4}B^4D\beta + 2A^{-4}B^3(3C^2\beta+7CD-2CD\beta+D^2\beta-3D^2-4D^2\alpha) \\
& + 2A^{-4}B^2(2C^3-C^2D-3C^2D\alpha-CD^2-4CD^2\alpha+D^3\alpha) \\
& -2A^{-4}B(4C^4\alpha-2C^3D\alpha-2C^2D^2\alpha+3CD^3+3D^4) - 2A^{-4}CD^2(2C^2+D^2) \\
& -8A^{-3}B^4\alpha\beta - 8A^{-3}B^3D\alpha^2 - 2A^{-3}B^2D(4C\beta+7D) \\
& + 2A^{-3}B(6C^3\beta+2C^2D-2D^3\alpha-10C^2D\beta-6CD^2+2CD^2\alpha) \\
& + 2A^{-3}(6C^3D\alpha-2C^2D^2\alpha+4CD^3\alpha-D^4) + 4A^{-2}B^3\beta \\
& + 2A^{-2}B^2(C+8C\alpha\beta-D-2D\alpha+4D\alpha\beta) + 2A^{-2}BD(8C\alpha^2+4D\alpha^2+7D\beta) \\
& -2A^{-2}(4C^3\alpha^2-4C^2D\alpha^2+2C^2D\beta+3CD^2+4CD^2\beta+2D^3-3D^3\alpha) \\
& -16A^{-1}B^2\beta^2 - 8A^{-1}B(C\beta+D+4D\alpha\beta) + 8A^{-1}(CD-2CD\beta+D^2-2D^2\alpha)\alpha \\
& + 16B\beta\alpha^2 + 8(C\beta^2+2D\alpha^3) - 16A\alpha\beta, \\[4pt]
C_{12} =\ & 2A^{-5}B(C-D)\left(B^3(2C\beta-D)-B^2C(C-4D\alpha)+B(2C^3\alpha+3D^3)+6C^2D^2\right) \\
& -A^{-4}B^5 + 4A^{-4}B^4C\alpha - 4A^{-4}B^3(C^2\alpha^2+2CD\beta+D^2\beta) \\
& + A^{-4}B^2(6C^3\beta+16C^2D-8CD^2+2CD^2\beta-4CD^2\alpha-D^3-8D^3\alpha) \\
& -A^{-4}BD(18C^3\alpha+C^2D-8C^2D\alpha-4CD^2\alpha+3D^3) \\
& + A^{-4}D^2(2C^3D\alpha-6C^2D^2-D^4) - 2A^{-3}B^4\beta + 4A^{-3}B^3(C-C\alpha\beta-D\alpha) \\
& -2A^{-3}B^2(7C^2\alpha-3D^2\beta) + 4A^{-3}B(3C^3\alpha^2+C^2D\beta-2CD^2-4CD^2\beta-2D^3+D^3\beta) \\
& -2A^{-3}(2C^3D\beta+C^2D^2-10C^2D^2\alpha+2C^2D^2\beta+4CD^3\alpha-3D^4\alpha) \\
& -4A^{-2}B^3\beta^2 + A^{-2}B^2(4C\beta-5D+4D\beta) + 2A^{-2}B(4C^2\alpha\beta-CD+12CD\alpha) \\
& + A^{-2}(4C^2D\alpha-20C^2D\alpha^2+8CD^2\alpha^2+2CD^2\beta-3D^3) \\
& + 2A^{-1}B^2\alpha - 2A^{-1}B(4C\alpha^2-D\beta+4D\beta^2) \\
& + 2A^{-1}(4C^2\alpha^3-2C^2\beta-CD-4CD\beta-6CD\alpha\beta-2D^2\alpha-8D^2\alpha\beta) \\
& -B + 4B\alpha\beta + 2C\alpha + 8C\alpha^2\beta + 8D\alpha^2 + 4D\beta^2 + 4A(2\alpha\beta-1)\beta, \\[4pt]
C_{13} =\ & 4A^{-5}B(C-D)\left(2B^3D\beta-B^2(C^2\beta+CD-2D^2\alpha)+BC^2D\alpha+2CD^3\right) \\
& + 2A^{-4}B^5\beta - 2A^{-4}B^4(2C\beta-D)\alpha - 2A^{-4}B^3D(2C\alpha^2+3D\beta) \\
& -2A^{-4}B^2D(C^2\beta-6CD+4D^2+3D^2\alpha-2D^2\beta) \\
& + 2A^{-4}BC(4C^3\beta+2C^2D-CD^2-D^3) \\
& -2A^{-4}B(2C^3D\beta+9C^2D^2\alpha+C^2D^2\beta-4CD^3\alpha-2D^4\alpha) \\
& + 2A^{-4}CD^3(C\alpha-2D) + 4A^{-3}B^4\beta^2 - 4A^{-3}B^3(2C-D\alpha)\beta \\
& + 4A^{-3}B^2C(3C\beta-2D)\alpha + 4A^{-3}BD(3C^2\alpha^2+CD\beta-2D^2) \\
& -2A^{-3}D(2C^3\beta+C^2D+4C^2D\beta+2CD^2-8CD^2\alpha-D^3+2D^3\alpha)
\end{aligned}$$

$$-2A^{-2}B^2(4C\beta^2 - 5D\beta + 4D\beta^2) + 2A^{-2}B(2C^2\beta + CD - 14CD\beta\alpha - D^2 + 3D^2\alpha)$$
$$+4A^{-2}(2C^3\alpha\beta + C^2D\alpha - 2C^2D\alpha\beta - 5CD^2\alpha^2 + 2D^3\alpha^2 + D^3\beta)$$
$$-4A^{-1}B^2\alpha\beta + 4A^{-1}B(2C\alpha^2\beta - D\alpha^2 - D\beta^2)$$
$$+4A^{-1}(2C^2\beta^2 - CD\beta + 2CD\alpha^3 - D^2 - 2D^2\beta - D^2\alpha\beta)$$
$$+2(B\beta - 4B\alpha\beta^2 + D\alpha - 4D\alpha^2\beta) + 8A\beta^2),$$

$$C_{14} = 2A^{-5}B(C-D)\left(B^2D(4C\beta + D) - BC(2C^2\beta + CD - 2D^2\alpha) + D^4\right)$$
$$-4A^{-4}B^5\beta^2 - A^{-4}B^4(2C\beta + D + 8D\alpha\beta) + 2A^{-4}B^3(2C^2\beta + CD - 2D^2\alpha)\alpha$$
$$-2A^{-4}B^2D^2(9C\beta + D - 4D\beta)$$
$$+A^{-4}BD(10C^3\beta + 5C^2D - 4C^2D\beta - 10CD^2\alpha - 2CD^2 + 4CD^2\beta + D^4 + 4D^3\alpha)$$
$$-A^{-4}D^2(2C^3\beta + C^2D - 2CD^2\alpha + D^3)$$
$$+2A^{-3}B^3(10C\beta^2 + D\beta) + 2A^{-3}B^2(4C^2\beta + 2CD + 12CD\alpha\beta - D^2\alpha)$$
$$-2A^{-3}B(6C^3\alpha\beta + 3C^2D\alpha - 6CD^2\alpha^2 - 5D^3\beta)$$
$$-2A^{-3}D^2(2C^2\beta + CD + 2CD\beta + D^2 - 3D^2\alpha + D^2\beta)$$
$$-20A^{-2}B^2D\beta^2 - 2A^{-2}B(6C^2\beta^2 + 7CD\beta + 4CD\beta^2 + 2D^2 + 14D^2\alpha\beta)$$
$$+2A^{-2}D(6C^2\alpha\beta - 2C^2\beta - CD + 3CD\alpha + 2D^2\alpha - 6D^2\alpha^2)$$
$$+8A^{-1}B^2\alpha\beta^2 + 2A^{-1}B(2C\beta + D + 8D\alpha\beta)\alpha$$
$$-2A^{-1}(4C^2\alpha^2\beta + 2CD\alpha^2 - 6CD\beta^2 - 4D^2\alpha^3 - 3D^2\beta - 4D^2\beta^2)$$
$$-4B\beta^2 - 2C\beta - 16C\alpha\beta^2 - D - 12D\alpha\beta - 8A\beta^3,$$

where $u_3 = A^{-3}B^2 - A^{-2}C$ and $\eta_4 = -A^{-4}B^3 + 2A^{-3}BC - A^{-2}D + 2A^{-1}\alpha$. This completes the proof of statement (c) of Theorem 2. $\qquad\square$

Acknowledgements

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