STATISTICA, anno LXVI, n. 2, 2006

LIMIT DISTRIBUTIONS FOR THE TERMS OF CENTRAL ORDER STATISTICS UNDER POWER NORMALIZATION

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1. INTRODUCTION

Let $\{X_n,\ n \geq 1\}$ be a sequence of independent and identically distributed random variables (r.v.'s) with common distributed function (d.f.) $F(x) = P(X_n \leq x)$. Let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ denote the order statistics of $X_1, X_2, \ldots, X_n$ and let G be a nondegenerate d.f. A sequence $\{X_{r:n}\}$ is called a sequence of order statistics with variable rank if $1 \leq r = r_n \leq n$ and $r_n \to \infty$, as $n \to \infty$. Here, we have the following two cases:

(1) If $\frac{r_n}{n} \to 0$ as $n \to \infty$ (or $\frac{r_n}{n} \to 1$) then $X_{r_n:n}$ is called the lower intermediate order statistics (or upper intermediate order statistics).

(2) If $\frac{r_n}{n} \to \lambda (0 < \lambda < 1)$, as $n \to \infty$, then $X_{r_n:n}$ is called central order statistics.

The limit theory of order statistics with variable rank was developed by (Balkema and de Haan, 1978a,b), (Chibisov, 1964), (Smirnov, 1952, 1967) and (Wu, 1966). (Smirnov, 1952) has shown that, for any non-decreasing variable rank $\{r_n\}$, there exist constant $a_n > 0$, b_n such that, as $n \to \infty$,

$$P(X_{r_n:n} \leq a_n x + b_n) \xrightarrow{w} G(x) \tag{1}$$

for some d.f. $G(x)$, if and only if, as $n \to \infty$

$$\frac{n\left[F(a_n x + b_n) - \frac{r_n}{n}\right]}{\sqrt{r_n\left(1 - \frac{r_n}{n}\right)}} \to v(x) \tag{2}$$

where ($\xrightarrow{W}$) denotes the weak convergence and $v(.)$ is a nondecreasing right continuous and extended real function satisfying

$$\lim_{x\to-\infty} v(x) = -\infty \ , \qquad \lim_{x\to\infty} v(x) = +\infty \ ,$$

and

$$G(x) \ = \Phi(\ v(x)),$$

Where Φ is the standard normal distribution function. The case of central ranks, where $\frac{r_n}{n} \to \lambda$, as $n \to \infty$ $(0 < \lambda < 1)$, has been studied by (Smirnov, 1952) (see also Shokry, 1983)). First, we note that it is possible for two sequences $\{r_n\}$ and $\{r_n^*\}$ with $\lim_{n\to\infty} \frac{r_n}{n} = \lim_{n\to\infty} \frac{r_n^*}{n}$ to lead to different nondegenerate limiting d.f., s for $X_{r_n:n}$ and $X^*_{r_n:n}$ Specifically, as shown in (Smirnov, 1952), (1) may be holds as well as the relation

$$P(X_{r_n^*:n} \le a_n^* x + b_n^*) \xrightarrow{w} G^*(x) \tag{3}$$

where, a_n, $a_n^* > 0$ and $\frac{r_n^*}{n} \to \lambda$, as $n \to \infty$ and $G(x)$ and $G^*(x)$ are nondegenerate d.f.'s of different types. However, this is not possible if

$$\sqrt{n}\left(\frac{r_n}{n} - \lambda\right) \to 0, \quad as \quad n \to \infty \tag{4}$$

i.e., $r_n \ = \ \lambda n \ + \ o(\ \sqrt{n})$ as the following result shows.

Lemma 1. Suppose that (2) and (3) hold, where $G(x)$ and $G^* \ (x)$ are nondegenerate d.f.'s and r_n, r_n^* both satisfy (4). Then $G(x)$ and $G^* \ (x)$ are of the same type, i.e., $G^* \ (x) \ = \ G(ax \ + \ b)$ for some $a > 0$, b. It turns out that, for some sequences $\{r_n\}$ satisfying (4), just four forms of limiting distributions $G(x)$ satisfying (3) are possible for $X_{r_n:n}$. For completeness, we state this here as a theorem and refer to (Smirnov, 1952) for the proof.

Theorem 1. If the central rank sequence $\{r_n\}$ satisfies (4), then the only possible nondegenerate limit d.f.'s of G for which (2) holds are

$\Phi(v_{i;\beta}(x)), \ \ i = 1,2,3,4$ where

Types I): $v_{1;\beta}(x)=\begin{cases}-\infty, & x\le 0\\ cx^{\beta} & x>0,\ c,\beta\ge 0\end{cases}$

Type II): $v_{2;\beta}=\begin{cases}-c|x|^{\beta}, & x\le 0\\ \infty, & x>0,\ c,\beta>0\end{cases}$

Type III): $v_{3;\beta}=\begin{cases}-c_1|x|^{\beta}, & x\le 0, c_1>0\\ c_2x^{\beta}, & x>0,\ c_2,\beta>0\end{cases}$ (5)

Type IV): $v_4(x)=\begin{cases}-\infty, & x\le -1\\ 0, & -1<x\le 1\\ \infty, & x>1\end{cases}$

In contrast to the situation of the fixed rank cases (lower and upper extremes), we note that the third of these distributional types is continuous. The range of possible limit distribution is much larger and the situation becomes more complicated, if the restriction (4) is removed. Namely:

(i) When $\sqrt{n}\left(\frac{r_n}{n}-\lambda\right)\to t$, as, $n\to\infty$, $-\infty<t<\infty$, (Smirnov, 1952) has shown that the only possible limit d.f.'s of the normalized $X_{r_n:n}$ are $\Phi(v_{i;\beta}(x+c_\lambda t))$, $i=1,2,3,4$, *where* $c_\lambda=\frac{1}{\sqrt{\lambda(1-\lambda)}}$ and $v_{i;\beta}(x)$ are defined in (5). In this case, $F(x)$ belongs to (λ,t)-domain of attraction of that given type. If $t=0$ we say that $F(x)$ belongs to the normal λ-domain of attraction of that given type.

(ii) When $\sqrt{n}\left(\frac{r_n}{n}-\lambda\right)\to\pm\infty$ as $n\to\infty$, $-\infty<t<\infty$, (Wu, 1966) has shown that all possible limit d.f.'s $\frac{X_{r_n:n}-b_n}{a_n}, a_n>0$ belong to the normal type or log-normal type.

(iii) When $\sqrt{n}\left(\frac{r_n}{n}-\lambda\right)$ is bounded but does not tend to a limit, (Wu, 1966) has shown that the only possible type of limit d.f. of $\frac{X_{r_n:n}-b_n}{a_n}, a_n>0$ is the normal one.

In order to improve the accuracy of the approximation of the distribution of the maximum for large values of *n*, (Weinstein, 1973) introduced a nonlinear normalization called the power normalization. (Pantcheva, 1985) and (Mohan and Ravi, 1992) studied the max domain of attraction of a d.f. under power normalization. The necessary and sufficient conditions for a d.f. to belong to max stable under power normalization is obtained by (Mohan and Ravi, 1992) and (Subramanya, 1994). In these papers the results of (Gnedenko, 1943) and (de Haan, 1971) concerning linear normalization are extended to p - max stable (max - stable under power normalization) laws. They showed that every d.f. attracted to l - max stable (max - stable under linear normalization) law is necessary attracted to some p - max stable and that p - max stable laws, in fact, attract more. This means that, the class of p - max stable laws contains many distributions more than l - max stable, i.e., in the case of power normalization the applications become more than the case of linear normalizations. Therefore, the advantage of normalizing with power functions, in the extreme case, is given by the fact that p - max stable laws attract more d.f.'s than linear-stable laws. However, in the central and the intermediate cases normalizing with power functions may also enlarges the class of the limit types, which creates more applications. A unified approach to the results of (Mohan and Ravi, 1992) and (Subramanya, 1994) has been obtained by (Christoph and Falk, 1996). Recently, (Barakat and Nigm, 2002) gave the sufficient conditions for the convergence of the extremes with random sample size, under power normalization, as well as the limit forms of the d.f.'s, where the normalizing constants do not contain the random sample size (stability of the power normalizing constants). The continuation of the restricted convergence of the power normalized extreme order statistics, on the half-line of real numbers to the whole-line, is proved under general conditions. Moreover, the continuation property of the power normalized extremes, with random sample indices, is proved in an important practical case (see Barakat and Nigm, 2003). (Barakat and *et al*, 2002), proved that the restricted convergence of the power normalized extremes on an arbitrary nondegenerate interval implies the weak convergence. Finally, (Barakat and *et al*, 2004) studied the weak convergence of the generally normalized extremes (extremes under nonlinear monotone normalization) of random number of independent (non-identically distributed) random variables. Now, the power normalization becomes an important contribution to the asymptotic theory of order statistics. The present paper investigates the classes of limit types having domains of normal λ - attraction for central terms of order statistics under power normalization.

2. MAIN RESULTS

Suppose that, as $n \to \infty$, $\frac{r_n}{n} \to \lambda$, $0 < \lambda < 1$ and for norming constants $a_n > 0$, and $b_n > 0$, we have

$$P\left(\left|\frac{X_{r_n:n}}{a_n}\right|^{\frac{1}{b_n}} S(X_{r_n:n})<x\right)=\Phi_{r_n:n}(a_n|x|^{b_n}\, S(x))\xrightarrow{w} H(x) \tag{6}$$

where $S(x)=-1$ if $x<0$, $=0$, if $x=0$ and $=1$ if $x>0$. We call $H(x)$ a central d.f. under power normalization or simply P-central d.f. if (6) holds. We can rewrite (1) and (3) for power normalization, as $n\to\infty$,

$$\Phi_{r_n:n}(a_n|x|^{b_n}\, S(x))\to H_1(x) \tag{7}$$

$$\Phi_{r_n^{'}:n}(a_n^{'}|x|^{b_n^{'}}\, S(x))\to H_2(x) \tag{8}$$

where a_n, $a_n^{'}$, b_n, $b_n^{'}>0$ are constants and $H_1(x)$, $H_2(x)$ are nondegenerate d.f.'s of different types. Also, both r_n and $r_n^{'}$ satisfy (4). The following theorem states the classes of limit laws (types) having domains of normal λ-attraction for central order statistics under power normalization (main result).

Theorem 2. If the central rank sequence $\{r_n\}$ satisfies (4), then the only possible nondegenerate limit d.f.'s of $H(x)=\Phi(u_{\alpha,i}^{*}(x))$, $i=1,2,...,7$, for which (6) holds are

Type I): $u_{1;\beta}^{*}(x)=\begin{cases} c(\log x)^{\beta}, & c>0,\ x\geq 1 \\ -\infty & x<1 \end{cases}$

Type II): $u_{2;\beta}^{*}(x)=\begin{cases} -\infty & x\leq 0 \\ -c(-\log x)^{\beta}, & c>0,\ 0<x\leq 1 \\ \infty & x>1 \end{cases}$

Type III): $u_{3;\beta}^{*}(x)=\begin{cases} -\infty & x\leq 0 \\ -c_1|\log x|^{\beta}, & c_1>0,\ 0<x<1 \\ c_2(\log x)^{\beta} & c_2>0,\ x\geq 1 \end{cases}$

Type IV): $u_{4;\beta}^{*}(x)=\begin{cases} -\infty, & x\leq \frac{1}{e} \\ 0, & \frac{1}{e}<x\leq e \\ \infty, & x\geq e \end{cases}$

Type V): $u^{*}_{5;\beta}(x)=\begin{cases} c(\log(-x))^{\beta}, & x\le -1 \\ \infty \quad , & x>1 \end{cases}$

Type VI): $u^{*}_{6;\beta}(x)=\begin{cases} -\infty, & x\le -1 \\ -c(-\log(-x))^{\beta}, & -1< x < 0 \\ \infty \quad , & x\ge 0 \end{cases}$

$$\text{Type VII): } u^{*}_{7;\beta}(x)=\begin{cases} c_1(\log(-x))^{\beta}, & x\le -1 \\ -c_2\left|\log(-x)\right|^{\beta} & c_2>0\,, \quad -1<x<0 \\ \infty & x\ge 0 \end{cases} \tag{9}$$

The proof of Theorem 2., follows from the following lemmas

Lemma 2. Let $\frac{r_n}{n}$, $0<\lambda<1$ as $n\to\infty$. Then (6) will be satisfied, for norming constants $a_n>0$ and $b_n>0$, if and only if,

$$\frac{F(a_n|x|^{b}n\ S(x))-\lambda_{r_n}}{\tau_{r_n}}\sqrt{n}\xrightarrow{w}u(x) \tag{10}$$

where

$$\lambda_{r_n}=\frac{r_n}{n+1},\quad v_{r_b}=1-\lambda_{r_n},\quad \tau_{r_n}=\sqrt{\frac{\lambda_{r_n}v_{r_n}}{n+1}}, \tag{11}$$

and the non-decreasing function $u(x)$ is uniquely determined from $\Phi(x)$ by the equation

$$\Phi(x)=\frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{u}(x)\exp\left\{-\frac{t^2}{2}\right\}dt \tag{12}$$

Proof. (See Leadbetter *et al.*, 1983, Theorems 2.5.1 and 2.5.2).

In the desire to investigate as completely as possible the set of limit laws for sequences of central terms under power normalization, we are led naturally to the concepts of type distribution law in the sense of P. Levy-Khinchin (see Lemma 2.2 and Lemma 2.3).

Lemma 3. Let Fn be a sequence of d.f.'s and T_1 a nondegenerate d.f. Let $a_n>0$ and $b_n>0$, be constants such that $F_n(a_n|x|^{b}n\ S(x))\xrightarrow{w}T_1(x)$. Then, for some nondegenerate d.f. T_2 and constants $\alpha_n > 0$ and $\beta_n > 0$, $F_n(\alpha_n |x|^{\beta_n} S(x)) \xrightarrow{w} T_2(x)$, if and only if,

$$\left(\frac{\alpha_n}{a_n}\right)^{\frac{1}{b_n}} \to A > 0 \ \ and \ \ \frac{\beta_n}{b_n} \to B > 0,$$

for some $A > 0$ and $B > 0$, and then

$$T_2(x) = T_1(A|x|^B S(x)).$$

Proof. (See Barakat and Nigm, 2002).

Lemma 4. Each law of type $F(x)$ can belong to only one domain of normal λ-attraction of some proper law $\Phi(x)$.

Proof. From (7), (8) and Lemma 2, we have, as $n \to \infty$

$$\frac{nF(a_n |x|^{b_n} S(x)) - r_n}{\sqrt{r_n\left(1 - \frac{r_n}{n}\right)}} \to u(x), \tag{13}$$

where $H(x) = \Phi(u(x))$.
By (4), we then have

$$\sqrt{n}\,\frac{nF(a_n |x|^{b_n} S(x)) - \lambda_n}{\sqrt{\lambda(1-\lambda)}} \to u(x)$$

Again by (4) with r_n replaced by $r_n^{/}$, we must therefore have that (13) holds with $r_n^{/}$ replacing r_n, and hence by Lemma 2, that

$$\Phi_{r_n^{/}:n}(a_n |x|^{b_n} S(x)) \to \Phi(u(x)) = H(x).$$

But if $H_n(x)$ is the d.f. of $X_{r_n^{/}:n}$, this says that,

$$H_n(a_n |x|^{b_n} S(x)) \to H(x),$$

whereas also by (8)

$$H_n(a_n^{'} |x|^{b_n^{'}} S(x)) \to H_1(x)$$

Thus by Lemma 3, $H(x)$ and $H_1(x)$ are of the same type, as required.

Proof of Theorem 2. We show in order that a proper law $\Phi(x)$ can have domain of normal λ-attraction it is necessary that for each positive integer $\nu > 1$ the function $u(x)$ corresponding by (12) to $\Phi(x)$ satisfies the equation

$$u(x) = \sqrt{\nu u}(\alpha_v |x|^{\beta} v \ S(x)), \tag{14}$$

where $\alpha_v \succ 0$ and $\beta_v \succ 0$ are real constants.

Suppose that, as $n \to \infty$

$$\Phi_{r_n:n}(a_n |x|^{b_n} S(x)) \to \Phi(x). \tag{15}$$

Then, as $n \to \infty$

$$\sqrt{n}\frac{F(a_n |x|^{b_n} S(x)) - \lambda}{\sqrt{\lambda(1-\lambda)}} \to u(x), \tag{16}$$

must be satisfied.

Consider the subsequences $\{u_{mv} : m \succ 1\}$, then as $m \to \infty$, (16) becomes

$$\sqrt{m}\frac{F(a_{mv} |x|^{b_{mv}} S(x)) - \lambda}{\sqrt{\lambda(1-\lambda)}} \to \frac{u(x)}{\sqrt{v}} = u_1(x), \tag{17}$$

Setting $r_m^{'} = \left[\frac{r_{mv}}{\nu}\right]$ then we get;

$$\frac{r_m^{'}}{m} - \lambda = \frac{r_{mv}}{mv} - \lambda + o\left(\frac{1}{\sqrt{m}}\right) \tag{18}$$

Relying on Lemma 2, (17) and (18) we conclude as $m \to \infty$,

$$\Phi_{r_m^{'}:m}(a_{mv} |x|^b mv \ S(x)) \to \Phi_1(x) \tag{19}$$

for each integer ν and $\Phi_1(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u_1(x)} \exp\left\{-\frac{t^2}{2}\right\} dt$.

Comparing (15) with (19) and applying Lemma 3, we convince ourselves that the limit laws $\Phi(x)$ and $\Phi_1(x)$ belong to the same type. Consequently, there are constants $\alpha_\nu \succ 0$ and $\beta_\nu \succ 0$ such that

$$\Phi_1(x) = \Phi(\alpha_\nu |x|^{\beta_\nu} S(x))$$

Hence, $u_1(x) = u(\alpha_v |x|^{\beta_\nu} S(x))$, which proves (14).

Turning to the investigation of the fundamental equation (14) we distinguish three cases when $S(x) = 1$:

A) For all integers $\nu > 1$, we have $\beta_v \succ 1$

B) There exists an integer $\nu > 1$, such that $\beta_v = 1$.

C) There exists an integer $\nu > 1$, such that $\beta_v < 1$.

Case A). In this case for $x \succ x_0 = \alpha_v^{\frac{1}{1-\beta v}}$,

$$\alpha_\nu |x|^{\beta_\nu} \geq x,$$

and for $x < x_0$,

$$\alpha_\nu |x|^{\beta_\nu} \leq x.$$

Since $u(x)$ is a non-decreasing function

$$u(\alpha_v |x|^{\beta} v\ S(x)) \geq u(x)), \quad x \geq x_0 \tag{20}$$

and

$$u(\alpha_v |x|^{\beta} v\ S(x)) \leq u(x)), \quad x < x_0 \tag{21}$$

From (14) and (20) it follows that $u(x) < 0$ or $u(x) = \infty$ when $x > x_0$.

Also, from (14) and (21) it follows that $u(x) > 0$ or $u(x) = -\infty$ when $x \leq x_0$.

Suppose that for some $x = \zeta$, $\zeta \succ 0$, $u(\zeta) \leq 0$; then for each $x \succ x_0$ and sufficiently large N we have

$$z_N = \alpha_v^{\frac{\beta_v^n - 1}{\beta_v - 1}} |\zeta|^{\beta_v^N} \qquad \mathrm{S}(\zeta) = \alpha_v^{\frac{1}{\beta_v - 1}} (\alpha_v |\zeta|^{\beta_v - 1})^{\frac{\beta_v^n}{\beta_v - 1}} \quad \mathrm{S}(\zeta) \to \infty$$

As $N \to \infty$, and

$$u(z_N) \geq u(x).$$

From (14), we have

$$v^{\frac{N}{2}} u(z_N) = v^{\frac{N-1}{2}} u(z_{N-1}) = \ldots = vu(z_3) = v^{\frac{1}{2}} u(z_1) = u(z_0)$$

and $u(x) \le u(z_N) \le 0$, for each $x > x_0$.

This contradicts the condition $u(\infty) = \infty$. Hence when $x > x_0$, $S(x) = 1,\ u(x) = +\infty$. Proceeding analogously, we show that, when $x < x_0$, $S(x) = 1,\ u(x) = -\infty$. The law $\Phi(x)$ corresponding to such a function $u(x)$ is evidently improper. Consequently if $\Phi(x)$ is a proper law satisfying (14) then for each v, $\beta_v \le 1$.

Case B). In cases $\alpha_v = 1$ the Equation (14) reduces to the form

$$v^{\frac{1}{2}} u(x) = u(x).$$

The only increasing functions satisfying this equation and corresponding to proper laws are;

$$u(x) = \begin{cases} -\infty, & \text{when} \quad x \le A \\ 0, & \mathrm{A} \prec \mathrm{x} \prec \mathrm{B} \\ +\infty & x \succ B \end{cases}$$

Then,

$$\Phi(x) = \begin{cases} 0, & \text{when} \quad x \le A \\ \frac{1}{2}, & \mathrm{A} \prec \mathrm{x} \prec \mathrm{B} \\ 1 & x \succ B \end{cases}$$

Also, in the case $S(x) = 0$, we have $v^{\frac{1}{2}} u(0) = u(0)$, which leads to $u(0) = 0$ or ∞ or $-\infty$.

But in the case, $\alpha_v \ne 1$. Then (14) has the form

$$u(x) = \sqrt{v}\, u(\alpha_v |x| S(x)). \tag{22}$$

These cases of course, are excluded by the condition,

$$u(\infty) = \infty. \tag{23}$$

Case C). Now suppose that $0 \prec \beta_v \prec 1$ for some v. Then, for $x \succ x_0$,

$$\alpha_v |x|^{\beta} v \leq x$$

and for $x \prec x_0$,

$$\alpha_v |x|^{\beta} v \geq x .$$

Since $u(x)$ is a non-decreasing function

$$u(\alpha_v |x|^{\beta} v \; S(x)) \leq u(x), \quad x \geq x_0 \tag{24}$$

and

$$u(\alpha_v |x|^{\beta} v \; S(x)) \geq u(x), \quad x < x_0 \tag{25}$$

Taking account (14), hence we conclude when $x > x_0$

$u(x) \geq 0$ or $u(x) = -\infty$ and when $x \leq x_0$, $u(x) \leq 0$ or $u(x) = +\infty$.

Suppose that for some $x = \zeta$, $\zeta \succ 0$, and $u(x) = \infty$ for some $x \prec x_0$. *a* $\zeta \prec x$ can always be found such that $u(\zeta) \prec 0$. But for sufficiently large N when $\beta_v \prec 1$

$$z_N = \alpha_v |z_{N-1}|^{\beta_v} S(z_{N-1}) = \ldots = \alpha_v^{\frac{1-\beta_v^N}{1-\beta_v}} |z_1|^{\beta_v^N} S(z_1) \to \alpha_v^{\frac{1}{1-\beta_v}} S(z_1),$$

and consequently,

$$u(z_N) \geq u(x) = \infty .$$

Hence,

$$u(z_N) = \infty .$$

But

$$v^{\frac{N}{2}} u(z_N) = v^{\frac{N-1}{2}} u(z_{N-1}) = \ldots = v \;\; u(z_2) = v^{\frac{1}{2}} u(z_1) = u(\zeta) \prec 0,$$

which contradicts the above equation.

Therefore, $u(x) \prec \infty$ for each $x \prec x_0$. Analogously, we show that $u(x) \succ -\infty$ when $x \succ x_0$. Thus when $x \leq x_0$ and $u(x) > 0$ when $x \geq x_0$. If further, $u(\zeta) = -\infty$, for some $\zeta \prec x_0$, then $u(x) = -\infty$ for each $x \prec x_0$. In fact, if in this case for some $x \prec x_0$, $u(x) = c$, $-\infty \prec c \prec 0$, then, as we have already seen for sufficiently large N

$$u(z_N) \geq u(x) = c,$$

from $u(\zeta) \geq cv^{\frac{n}{2}} \succ -\infty$ which contradicts the assumed equality $u(\zeta) = -\infty$.

If the distribution law which corresponds to the function $u(x)$ is proper, then only three combinations are possible.

a) $u(x) = -\infty$ when $x \prec x_0$ and $0 \leq u(x) \prec \infty$ when $x \succ x_0$

b) $u(x) = \infty$ when $x > x_0$ and $-\infty \prec u(x) \leq 0$ when $x \prec x_0$

c) $-\infty \prec u(x) \leq 0$ when $x \leq x_0$ and $0 \leq u(x) \prec \infty$ when $x \succ x_0$

Moreover, the number $x_0 = \alpha_v^{\frac{1}{1-\beta_v}}$, does not depend on ν and from (14), we have

$$u(x) = \sqrt{v}\, u(|x|^{\beta_v} S(x)), \tag{26}$$

The solution of this equation has the following forms:

$$\text{(i)} \quad u(x) = \begin{cases} c(\log x)^{\beta}, & c \succ 0,\ \beta > 0,\ x \geq 1, \\ -\infty & x \prec 1, \end{cases}$$

$$\text{(ii)} \quad u(x) = \begin{cases} -\infty & x \leq 1, \\ -c(-\log x)^{\beta}, & c \succ 0,\ \beta \succ 0,\ 0 \prec x \leq 1, \\ \infty, & x \succ 1, \end{cases}$$

$$\text{(iii)} \quad u(x) = \begin{cases} -\infty & x \leq 0, \\ -c_1 |\log x|^{\beta}, & c_1 \succ 0,\ \ 0 \prec x < 1, \\ c_2(\log x)^{\beta}, & c_2 \succ 0,\ \ \beta \succ 0,\ \ x \geq 1. \end{cases}$$

The solutions (i), (ii) and (iii) when $S(x) = 1$. Also, we can deduce the similar solutions when $S(x) = -1$. Thus, we have a complete proof of theorem 2.

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ACKNOWLEDGEMENT

The author would like to thank the anonymous reviewers as well as the editorial board for several helpful comments and suggestions.

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RIASSUNTO

Distribuzioni limite per i termini delle statistiche d'ordine centrali sotto normalizzazione di potenza

In questo lavoro vengono determinate le distribuzioni limite per sequenze di termini centrali sotto normalizzazione non casuale di potenza. Vengono studiate le classi di limiti aventi dominio di attrazione -λ.

SUMMARY

Limit distributions for the terms of central order statistics under power normalization

In this paper the limiting distributions for sequences of central terms under power nonrandom normalization are obtained. The classes of the limit types having domain of λ- attraction are investigated.