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LIMIT DISTRIBUTIONS OF EXTREME VALUES OF BOUNDED INDEPENDENT RANDOM FUNCTIONS

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ABSTRACT. We study the limit probabilities that extreme values of a sequence of independent normal random functions belong to extending intervals.

1. Introduction

Consider a sequence (ξ_i) of independent identically distributed random variables with the distribution function $F(x) = \mathsf{P}(\xi_i < x)$ and put $z_n = \max_{1\le i\le n} \xi_i$. Assume that for some constants $b_n > 0$ and a_n

$$b_n(z_n - a_n) \xrightarrow{D} \zeta \tag{1}$$

as $n \to \infty$ and let the distribution function $G(x) = \mathsf{P}(\zeta < x)$ of the random variable ζ be nondegenerate.

If relation (1) holds, then we say that the distribution function F belongs to the domain of attraction of the law G and we write $F \in D(G)$. According to the well-known extreme types theorem [1]–[3], a distribution function F may belong to the domain of attraction of one of the following three types of distributions:

$$\begin{gathered}\text{Type I:}\quad G_1(x) = \exp(-e^{-x}),\quad -\infty < x < \infty,\\ \text{Type II:}\quad G_2(x) = \begin{cases}0, & x \le 0,\\ \exp(-x^{-\alpha}), & \alpha > 0,\ x > 0,\end{cases}\\ \text{Type III:}\quad G_3(x) = \begin{cases}\exp(-(-x)^{\alpha}), & \alpha > 0,\ x \le 0,\\ 1, & x > 0.\end{cases}\end{gathered} \tag{2}$$

In an earlier series of papers (see, for example, [4, 5]) we studied the weak convergence of extreme values of a sequence of independent random elements in some Banach lattices; that is, we generalized relation (1) to the infinite-dimensional case.

In Banach spaces with an unconditional basis, one can develop a theory analogous to the classical theory of the weak convergence of extreme values [4] (perhaps this is the only case where such a generalization is available). For example, the limit laws are degenerate [5] for spaces of $L_p[0,1]$ type if some natural conditions are posed. We assumed in [5] that components of extreme values are asymptotically independent. From a certain point of view, this case seems to be the most important one.

When studying the weak convergence of extreme values of independent random elements in the space $C[0,1]$, we face a number of problems. One of the possible approaches

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to solving these problems is described in the paper [6] for independent Wiener processes. Another approach is considered in [7].

In this paper, we generalize the approach of [7] to the case of bounded elements in Banach lattices. We also obtain some results for abstract Banach lattices of $C(Q)$ type, but our primary attention is paid to the asymptotic behavior of probabilities that extreme values of independent random functions belong to extending intervals. This case is the most interesting one for possible applications.

2. Main results for Banach lattices of $C(Q)$ type

In what follows we use the following notation: B is a Banach lattice equipped with a norm $\|\cdot\|$, B_+ is the set of positive elements of the lattice B,

$$B_{(u)} = \{x \in B\colon \text{there exists } \lambda > 0 \text{ such that } |x| \le \lambda u\}$$

is the ideal generated by the element $u \in B_+$, and $\|x\|_u = \inf\{\lambda > 0\colon |x| \le \lambda u\}$ is the norm in $B_{(u)}$.

Let X be a random element assuming values in a Banach lattice B, let (X_n) be a sequence of independent copies of the random element X, and let

$$Z_n = \max_{1\le i\le n} X_i, \qquad W_n = \min_{1\le i\le n} X_i, \qquad n \ge 1,$$

be the corresponding extreme values. Assume that

$$X \in B_{(u)} \quad \text{almost surely.} \tag{3}$$

The fundamental extreme types theorem for the real axis [1] can be generalized to the case of random elements assuming values in a Banach lattice. Namely, the following result holds.

Proposition 1. *Let X be a random element assuming values in a separable Banach lattice B and let condition* (3) *hold. Assume that for some constants $b_n > 0$ and a_n,*

$$\mathsf{P}\left(Z_n \le \left(a_n + \frac{x}{b_n}\right) u\right) \longrightarrow G(x)$$

as $n \to \infty$, where $G(x)$ is a nondegenerate distribution function. Then, up to location shift and scale changes, the law $G(x)$ has one of the three extreme value distributions listed in (2).

In order that $G(x) = G_k(x)$, it is necessary and sufficient that the distribution function $F_u(x) = \mathsf{P}(X \le xu)$ belong to the domain of attraction of the law $G_k(x)$, $k \in \{1,2,3\}$.

Indeed, it is obvious that

$$\mathsf{P}\left(Z_n \le \left(a_n + \frac{x}{b_n}\right) u\right) = \left(F_u\left(a_n + \frac{x}{b_n}\right)\right)^n$$

and therefore Proposition 1 is reduced to the one-dimensional case.

Remark 1. Events of type $A = (X \le \lambda u)$, $u \in B_+$, $\lambda \in \mathbf{R}$, are measurable, since $\max(X, Y)$ is measurable for random elements X and Y. Indeed

$$A = (X - \lambda u \le 0) = (\max(X - \lambda u, 0) = 0) = \bigcap_{n\ge 1}\left(\|\max(X - \lambda u, 0)\| \le \frac{1}{n}\right).$$

It is a rather complicate problem to check whether or not $F_u \in D(G_k)$ and to determine the corresponding constants a_n and b_n. The problem becomes simpler for the normal distribution. We treat this case in what follows.

Below, X denotes a normal random element assuming values in a Banach lattice B, and $\mathfrak{S}X$ is the mean quadratic deviation of the random element X (the mean quadratic deviation can be defined by $\mathfrak{S}X = \sqrt{\pi/2}\mathsf{E}|X|$ for a normal random element in a Banach lattice). Since any Banach lattice ℓ is convex, condition (3) implies that $\mathfrak{S}X \in B_{(u)}$.

Put

$$\Psi_u(x) = \Phi^{-1}(F_u(x)), \qquad d_u = \lim_{x\to\infty}(\|\mathfrak{S}X\|_u \Psi_u(x) - x),$$

$$\theta_u(x) = \exp\left(x^2/2\right)(1 - F_u(\|\mathfrak{S}X\|_u x - d_u)), \tag{4}$$

where $\Phi^{-1}(x)$ is the inverse function of the standard normal distribution function $\Phi(x)$, and the distribution function $F_u(x)$ is defined in Proposition 1.

The following asymptotic equality is a generalization of relation (1) for the case of Banach lattices $B_{(u)}$: for all $x \in \mathbf{R}^1$

$$\lim_{n\to\infty} \mathsf{P}(Z_n \le \tau_n(x)u) = G_1(x), \tag{5}$$

where $\tau_n(x) = -d_u + \|\mathfrak{S}X\|_u(x/b_n + a_n)$, $G_1(x)$ is a type I distribution function defined in (2), and $a_n, b_n \in \mathbf{R}^1$.

Theorem 1. *Let X be a normal random element assuming values in a separable Banach lattice. Assume that X satisfies condition* (3) *and moreover $\|\mathfrak{S}X\|_u > 0$. Then equality* (5) *holds for*

$$b_n = \begin{cases} (2\ln(n))^{1/2}, & n > 1, \\ 1, & n = 1, \end{cases} \qquad a_n = b_n + \frac{\ln(\theta_u(b_n))}{b_n}, \tag{6}$$

where $\theta_u(x)$ and d_u are defined by equalities (4).

Remark 2. Under the assumptions of Theorem 1, equality (5) holds for $|Z_n|$ instead of Z_n with the same constants a_n and b_n.

It is known (see [3, pp. 28–29]) that extreme values Z_n and W_n are asymptotically independent if random variables X_i are independent. It turns out that a similar result holds for ideals of bounded random elements.

Theorem 2. *Let B be a separable function Banach lattice, and let*

$$X = (X(t), t \in T)$$

be a normal random element in B satisfying condition (3). *Suppose that $\mathsf{E}X = 0$, $\mathfrak{S}X = (\sigma(t), t \in T)$, $\|\mathfrak{S}X\| > 0$, and there exists a number $\varepsilon > 0$ such that $\sigma(t) \ne 0$, $\sigma(s) \ne 0$, and, for all $t, s \in T$,*

$$\frac{\mathsf{E}X(t)X(s)}{\sigma(t)\sigma(s)} > -1 + \varepsilon. \tag{7}$$

Then

$$\lim_{n\to\infty} \mathsf{P}\big(-\tau_n(x)u \le W_n \le Z_n \le \tau_n(y)u\big) = G_1(x)G_1(y), \tag{8}$$

where $\tau_n(x)$ is defined by equality (5).

The simple example given below shows that Theorem 2 may fail in general if a condition of type (7) is not imposed on X. Nevertheless the asymptotic behavior can be obtained in abstract Banach lattices for the probability that random elements W_n and Z_n belong to symmetric extending intervals.

Theorem 3. *If the assumptions of Theorem* 1 *are satisfied, then*

$$\lim_{n\to\infty} \mathsf{P}\big(-\hat\tau_n(x)u \le W_n \le Z_n \le \hat\tau_n(x)u\big) = G_1(x),$$

where

$$\hat{\tau}_n(x) = -\hat{d}_u + \|\mathfrak{S}X\|_u \left(\hat{a}_n + \frac{x}{b_n}\right),$$

$$\hat{a}_n = b_n + \frac{\ln \hat{\theta}(b_n)}{b_n},$$

and the functions $\hat{\theta}(x)$ *and* $\hat{d}_u$ *are defined by* (4) *with*

$$\hat{F}_u(x) = \mathsf{P}(\|X\|_u < x)$$

instead of $F_u(x)$.

Remark 3. A Banach lattice $B_{(u)}$ is order isometric to the space $C(Q)$ for some compact Hausdorff space Q ([8, 9]). This implies that the general case of Theorems 1–3 follows from the particular case of bounded random functions defined on some parameter set T and $u = \{u(t) = 1, t \in T\}$.

3. Extreme values of normal random functions

Let $X = \{X(t), t \in T\}$ be a normal random function defined on a parameter set T, let $X_n = \{X_n(t), t \in T\}$, $n \geq 1$, be independent copies of the random variable X, and let

$$Z_n = \left\{Z_n(t) = \max_{1 \leq k \leq n} X_k(t), t \in T\right\}, \qquad W_n = \left\{W_n(t) = \min_{1 \leq k \leq n} X_k(t), t \in T\right\}.$$

By $R = \{R(t,s), t, s \in T\}$ and $\mathfrak{S}X = \{\sigma(t), t \in T\}$ we denote the correlation function and mean square deviation of the random function X, respectively, where

$$R(t,s) = \mathsf{E}(X(t) - \mathsf{E}X(t))\big(X(s) - \mathsf{E}(X(s))\big), \qquad \sigma(t) = (R(t,t))^{1/2}.$$

For $u \equiv 1$, notation (4) is equivalent to

$$F_{\sup}(x) = \mathsf{P}\left(\sup_{t \in T} X(t) < x\right), \qquad \Psi(x) = \Phi^{-1}\left(F_{\sup}(x)\right),$$

$$d = \lim_{x \to \infty} (\|\mathfrak{S}X\| \Psi(x) - x),$$

$$\theta(x) = \exp\big(x^2/2\big)(1 - F_{\sup}(\|\mathfrak{S}X\| x - d)) \tag{9}$$

(it is known that the limit

$$\lim_{x \to \infty} (\|\mathfrak{S}X\| \Psi(x) - x)$$

exists and is finite for a bounded normal random function; see [10, p. 139]).

In what follows we assume that X is a bounded random function; that is, $\|X\| < \infty$ almost surely and $\|\mathfrak{S}X\| > 0$ where $\|x\| = \sup_{t \in T} |x(t)|$ and b_n is defined by (6).

Proposition 2. *Let* $X = \{X(t), t \in T\}$ *be a bounded normal random function such that* $\|\mathfrak{S}X\| > 0$. *Then*

$$\lim_{n \to \infty} \mathsf{P}\left(b_n\left(\frac{\|Z_n\| + d}{\|\mathfrak{S}X\|} - a_n\right) \leq x\right) = G_1(x) \tag{10}$$

for all $x \in \mathbf{R}$ *where*

$$a_n = b_n + \frac{\ln(\theta(b_n))}{b_n} \tag{11}$$

and $\theta(x)$ *and* d *are defined by equalities* (9).

Lemma 1. *Let $X = \{X(t), t \in T\}$ be a bounded normal random function, let $d = 0$, and let $\|\mathfrak{S}X\| = 1$. Then*

$$\lim_{n\to\infty} \mathsf{P}\left(b_n\left(\sup_{t\in T} Z_n(t) - a_n\right) \le x\right) = G_1(x), \tag{12}$$

where the constants a_n are defined by (11).

The proof of Lemma 1 can be found in [7].

Proof of Proposition 2. The asymptotic relation

$$\lim_{x\to\infty} x^{-1}\left(\ln(1 - F_{\sup}(x)) + (x+d)^2/2\|\mathfrak{S}X\|^2\right) = 0 \tag{13}$$

is proved in [10, p. 139]. If $X(t)$ is a random function satisfying the assumptions of the proposition, then $\bar{X}(t) = (X(t)+d)\|\mathfrak{S}X\|^{-1}$ is a bounded normal random function such that $d(\bar{X}) = 0$ and $\|\mathfrak{S}\bar{X}\| = 1$ (see (13)). The definition of the function $\theta(x)$ implies that the random functions $X(t)$ and $\bar{X}(t)$ have the same distribution function $\theta(x)$. Applying Lemma 1 to the random function $\bar{X}(t)$, we get

$$\lim_{n\to\infty} \mathsf{P}\left(b_n\left(\frac{\sup_{t\in T}(Z_n + d)}{\|\mathfrak{S}X\|} - a_n\right) \le x\right) = G_1(x). \tag{14}$$

Furthermore, we apply the following simple bounds:

$$\begin{aligned} \mathsf{P}\left(\sup_{t\in T} Z_n(t) > u_n(x)\right) &\le \mathsf{P}(\|Z_n(t)\| > u_n(x)) \\ &\le \mathsf{P}\left(\sup_{t\in T} Z_n(t) > u_n(x)\right) + \mathsf{P}\left(\inf_{t\in T} Z_n(t) < -u_n(x)\right) \end{aligned} \tag{15}$$

and

$$\begin{aligned} \mathsf{P}\left(\inf_{t\in T} Z_n(t) < u_n(x)\right) &\le \mathsf{P}\left(\bigcap_{k=1}^{n}\left(\inf_{t\in T} X_k(t) < u_n(x)\right)\right) \\ &= \mathsf{P}\left(\inf_{t\in T} X(t) < u_n(x)\right)^n \underset{n\to\infty}{\longrightarrow} 0 \end{aligned} \tag{16}$$

as $u_n(x) \to \infty$. Relations (14)–(16) lead to equality (10). $\square$

According to Remark 3, Theorem 1 follows from the proof of Proposition 2 given above, while Theorems 2 and 3 follow from the following Propositions 3 and 4, respectively.

Proposition 3. *Let $X = \{X(t), t \in T\}$ be a bounded normal random function, let $\mathsf{E}X = 0$, $\|\mathfrak{S}X\| > 0$, and let there exist a number $\varepsilon > 0$ such that condition* (7) *holds for all $t, s \in T$ for which $\sigma(t) \ne 0$ and $\sigma(s) \ne 0$. Then for all $x, y \in \mathbf{R}$*

$$\lim_{n\to\infty} \mathsf{P}\left(-\tau_n(x) \le \inf_{t\in T} W_n(t) \le \sup_{t\in T} Z_n(t) \le \tau_n(y)\right) = G_1(x)\, G_2(y), \tag{17}$$

where $\tau_n(x) = -d + \|\mathfrak{S}X\|(a_n + x/b_n)$ and constants a_n are defined by equality (11).

Proof of Proposition 3. Since the random function X is symmetric, equality (12) implies that

$$\lim_{n\to\infty} \mathsf{P}\left(\min_{1\le i\le n} \inf_{t\in T} X_i(t) > -\tau_n(x)\right) = G_1(x),$$

$$\lim_{n\to\infty} \mathsf{P}\left(\max_{1\le i\le n} \sup_{t\in T} X_i(t) < \tau_n(x)\right) = G_1(x).$$

These equalities are equivalent to the following asymptotic relations:

$$\mathsf{P}\left(\inf_{t\in T} X(t) \le -\tau_n(x)\right) \sim \frac{e^{-x}}{n},$$

$$\mathsf{P}\left(\sup_{t\in T} X(t) \ge \tau_n(y)\right) \sim \frac{e^{-y}}{n} \tag{18}$$

as $n\to\infty$. Furthermore,

$$\begin{aligned}\mathsf{P}_n(x,y) &= \mathsf{P}\left(-\tau_n(x) \le \inf_{t\in T} W_n(t) \le \sup_{t\in T} Z_n(t) \le \tau_n(y)\right)\\ &= \mathsf{P}\left(-\tau_n(x) \le \inf_{t\in T} X(t) \le \sup_{t\in T} X(t) \le \tau_n(y)\right)^n\\ &= \left(1-\mathsf{P}\left(\left(\inf_{t\in T} X(t) < -\tau_n(x)\right)\cup\left(\sup_{t\in T} X(t) > \tau_n(y)\right)\right)\right)^n\\ &= \left(1-\mathsf{P}\left(\inf_{t\in T} X(t) < -\tau_n(x)\right) - \mathsf{P}\left(\sup_{t\in T} X(t) > \tau_n(y)\right)\right.\\ &\qquad \left. + \mathsf{P}\left(\left(\inf_{t\in T} X(t) < -\tau_n(x)\right)\cap\left(\sup_{t\in T} X(t) > \tau_n(y)\right)\right)\right)^n.\end{aligned} \tag{19}$$

It is easy to see that condition (7) implies that

$$\begin{aligned}\sup_{t,s\in T} \mathsf{E}\,|X(t)-X(s)|^2 &\le \sup_{t,s\in T}\left(\sigma^2(t)+\sigma^2(s)+2(1-\varepsilon)\sigma(t)\sigma(s)\right)\\ &\le (4-2\varepsilon)\,\|\mathfrak{S}X\|^2.\end{aligned} \tag{20}$$

To estimate the latter term in (19), we apply estimates (13) and (20):

$$\begin{aligned}&\mathsf{P}\left(\left(\inf_{t\in T} X(t) < -\tau_n(x)\right)\cap\left(\sup_{t\in T} X(t) > \tau_n(y)\right)\right)\\ &\quad\le \mathsf{P}\left(\sup_{t,s\in T}(X(t)-X(s)) > \tau_n(x)+\tau_n(y)\right)\\ &\quad\le \exp\left(-\frac{(\tau_n(x)+\tau_n(y))^2 + \mathrm{O}(\tau_n(x)+\tau_n(y))}{2(4-2\varepsilon)\,\|\mathfrak{S}X\|^2}\right)\\ &\quad\le \exp\left(-\frac{8\ln n\,\|\mathfrak{S}X\|^2+\mathrm{O}(\ln n)^{1/2}}{(8-4\varepsilon)\,\|\mathfrak{S}X\|^2}\right) = O\left(\frac{1}{n^{1+\delta}}\right), \qquad \delta>0.\end{aligned} \tag{21}$$

Now it follows from bounds (18), (19), and (21) that

$$\mathsf{P}_n(x,y) = \left(1-\frac{e^{-x}+e^{-y}}{n} + o\left(\frac{1}{n}\right)\right)^n \to \exp\left(-\left(e^{-x}+e^{-y}\right)\right) = G_1(x)G_1(y)$$

as $n\to\infty$, whence relation (17) follows (cf. similar reasoning in [3, p. 28]). $\square$

The following example shows that one cannot omit condition (7) in Proposition 3 and in Theorem 3.

Example. Let $X(t)=\xi(1-2t)$, $t\in[0,1]$, and let ξ be a standard normal random variable. Then

$$\sigma^2(t) = (1-2t)^2, \qquad \|\mathfrak{S}X\| = 1, \qquad d=0,$$

since the process $X(t)$ is continuous. Now we show that equality (17) does not hold for $x=y$. Indeed, $\tau_n(x) = a_n + x/b_n$ where the constants a_n and b_n are defined by (11).

Since

$$\sup_{t\in[0,1]} X(t) = |\xi|,$$

we have

$$\theta(x) \sim \sqrt{\frac{2}{\pi}} x^{-1}.$$

Then

$$\lim_{n\to\infty} \mathsf{P}\left(\sup_{t\in[0,1]} Z_n(t) \le \tau_n(x)\right) = \lim_{n\to\infty} \big(\mathsf{P}(|\xi| \le \tau_n(x))\big)^n = G_1(x),$$

whence

$$\begin{aligned} &\mathsf{P}\left(-\tau_n(x) \le \inf_{t\in[0,1]} W_n(t) \le \sup_{t\in[0,1]} Z_n(t) \le \tau_n(x)\right) \\ &\quad = \left(\mathsf{P}\left(-\tau_n(x) \le \inf_{t\in[0,1]} X(t) \le \sup_{t\in[0,1]} X(t) \le \tau_n(x)\right)\right)^n \\ &\quad = \big(\mathsf{P}(-\tau_n(x) \le -|\xi| \le |\xi| \le \tau_n(x))\big)^n = \big(\mathsf{P}(|\xi| \le \tau_n(x))\big)^n \to G_1(x) \end{aligned}$$

as $n \to \infty$ and, as a result, equality (17) does not hold. It is clear that condition (7) also is not valid for the process $X(t)$ in Theorem 2, since $\mathsf{E}X(0)X(1) = -1$ and $\sigma(0) = \sigma(1) = 1$.

Proposition 4. *If $X = \{X(t), t \in T\}$ is a bounded normal random function such that $\|\mathfrak{S}X\| > 0$, then*

$$\lim_{n\leftarrow\infty} \mathsf{P}\big(-\hat\tau_n(x) \le W_n \le Z_n \le \hat\tau_n(x)\big) = G_1(x), \tag{22}$$

where

$$\hat\tau_n(x) = -\hat d + \|\mathfrak{S}X\| \left(\hat a_n + \frac{x}{b_n}\right), \qquad \hat a_n = b_n + \frac{\ln \hat\theta(b_n)}{b_n},$$

and the functions $\hat\theta(x)$ and $\hat d$ are defined by equalities (9) *with*

$$\hat F_{\sup}(x) = \mathsf{P}\left(\sup_{t\in T} |X(t)| < x\right)$$

instead of $F_{\sup}(x)$.

Proof of Proposition 4. It is clear that equality (22) is equivalent to

$$\lim_{n\leftarrow\infty} \mathsf{P}\left(b_n\left(\frac{\left\|\sup_{1\le k\le n} |X_k(t)|\right\| + \hat d}{\|\mathfrak{S}X\|} - \hat a_n\right) \le x\right) = G_1(x). \tag{23}$$

Consider another normal random function $\hat X = \big(X(s), s \in \hat T\big)$ where $\hat T = T \cup T^*$ and T^* is a copy of T for which we introduce a one-to-one correspondence $t \leftrightarrow t^*$ between T and T^*. The function $\hat X$ is defined as follows:

$$\hat X(s) = \begin{cases} X(t), & \text{for } s = t \in T, \\ -X(t), & \text{for } s = t^* \in T^*,\ t^* \leftrightarrow t. \end{cases}$$

Then

$$\sup_{s\in\hat T} \hat X(s) = \sup_{t\in T} |X(t)|, \qquad \left\|\sup_{1\le k\le n} |X_k(t)|\right\| = \sup_{s\in\hat T} \hat Z_n(s)$$

almost surely; that is, equality (23) coincides with equality (12) involved in the proof of Proposition 2. ∎

4. Examples

Below we consider some corollaries of Theorems 1 and 2. First we consider a continuous normal stochastic process $X = \{X(t), t \in T = [0,h]\}$. Put

$$r(t,s) = \frac{R(t,s)}{\sigma(t)\sigma(s)}, \qquad \gamma^2(t) = R_{11}(t,t) = \left[\frac{\partial^2 R(t,s)}{\partial t \partial s}\right]_{t=s},$$
$$\mu(t) = \frac{R_{01}(t,t)}{\gamma(t)\sigma(t)}, \qquad R_{01}(t,s) = \left[\frac{\partial R(t,s)}{\partial s}\right],$$
$$\Lambda(t) = \int_0^t \frac{\gamma(s)}{\sigma(s)}(1-[\mu(s)]^2)^{1/2}\,ds, \qquad r_{11}(t,s) = \left[\frac{\partial^2 r(t,s)}{\partial t \partial s}\right].$$

This notation corresponds to that introduced in Chapter 13 of [11]. Recall that the function $\Lambda(t)$ is such that $\pi^{-1}\exp\bigl(-u^2/2\bigr)\Lambda(h) = \mathsf{E}N_u(h)$, where $N_u(h)$ is the number of crossings of a fixed level u by a trajectory of a normalized process $X(t)/\sigma(t)$ on the interval $[0,h]$ (see [11]).

We say that a process $X(t)$ satisfies condition (Ξ) if the function $R(t,s)$ has the continuous second order partial derivative $R_{11}(t,s)$, the joint normal distribution of $X(t)$ and its mean square derivative $X'(t)$ is nondegenerate for every $t \ge 0$, and

$$(\Xi) \qquad \begin{aligned} &\text{a)} \quad \mathsf{E}X(t) = 0,\ \sigma(t) > 0,\ |\mu(t)| < 1, \\ &\text{b)} \quad \sup_{t\in[0,h-s]} |r(t,t+s)| < 1,\ s > 0, \\ &\text{c)} \quad \sup_{|t-s|\le\delta} \left|1 - \frac{r_{11}(t,s)}{[r_{11}(t,t)r_{11}(s,s)]^{1/2}}\right| \underset{\delta\to 0}{\longrightarrow} 0\,. \end{aligned}$$

Note that if $X(t)$ is a stationary normal process with the spectral function $F(\lambda)$ such that $\lambda_2 = \int \lambda^2\,dF(\lambda) < \infty$, then condition (Ξ) holds and $\Lambda(t) = \lambda_2^{1/2} t$.

Corollary 1. *Let $T = [0,h]$ and let $X = \{X(t), t \in T\}$ be a normal stochastic process satisfying condition (Ξ). Then*

$$\lim_{n\to\infty} \mathsf{P}(Z_n \le \tau_n(x)\mathfrak{S}X) = G_1(x), \tag{24}$$

$$\lim_{n\to\infty} \mathsf{P}\bigl(-\tau_n(x)\mathfrak{S}X \le W_n \le Z_n \le \tau_n(y)\mathfrak{S}X\bigr) = G_1(x)G_1(y), \tag{25}$$

where $\tau_n(x) = b_n + (\ln(\Lambda(h)/2\pi)+x)/b_n$ and the constants b_n are defined in equalities (6).

Proof of Corollary 1. Let $X = \{X(t), t \in T\}$ be a normal stochastic process satisfying condition (Ξ). Then

$$\lim_{x\to\infty} \exp\left(x^2/2\right) \mathsf{P}\left(\sup_{t\in[0,h]} X(t)/\sigma(t) > x\right) = \Lambda(h)/2\pi$$

(see [7]). Since the process $X(t)/\sigma(t)$ is continuous, $d_u = 0$ for $u = \mathfrak{S}X$ ([10, p. 147]). It remains to apply Theorems 1 and 2 to the process $X(t)$ with

$$u = \mathfrak{S}X, \qquad d_u = 0, \qquad \|\mathfrak{S}X\|_u = 1, \qquad \theta(x) \sim \Lambda(h)/2\pi, \quad x \to \infty. \quad \square$$

We study the case of a normal sequence $X(t)$, $t = 1,2,\dots$, in the following result. Put $r_{ij} = R(i,j)/\sigma(i)\sigma(j)$ and assume that

$$\sigma_0 = \sigma(1) = \sigma(2) = \cdots = \sigma(m) > \max_{t>m} \sigma(t), \tag{26}$$

$$\max_{1\le i<j\le m} |r_{ij}| < 1. \tag{27}$$

Corollary 2. *Let $X = \{X(t), t \in T = N\}$ be a centered bounded normal sequence satisfying conditions* (26) *and* (27). *Then equalities* (5) *and* (8) *hold for*

$$\tau_n(x) = b_n + \frac{\ln(m) - (\ln(4\pi) + \ln\ln(n))/2 + x}{b_n},$$

where the constants b_n are defined by equalities (6).

Proof of Corollary 2. Without loss of generality we suppose that $\|\mathfrak{S}X\| = 1$. Then condition (26) and equality (13) imply that

$$\mathsf{P}\left(\sup_{t\in N} X(t) > x\right) \sim \mathsf{P}\left(\sup_{1\le t\le m} X(t) > x\right) \tag{28}$$

as $x \to \infty$. Using the same reasoning as that in [7], we derive from asymptotic relation (28) that

$$\mathsf{P}\left(\sup_{t\in N} X(t) > x\right) \sim m(2\pi)^{-1/2}x^{-1}\exp\left(-x^2/2\right)$$

as $x \to \infty$. This relation and equality (13) imply that $d = 0$. It remains to apply Theorems 1 and 2 for $u \equiv 1$, $d = 0$, and $\theta(x) \sim m(2\pi)^{-1/2}x^{-1}$. □

Denote by $|\cdot|_m$ the Euclidean norm in $\mathbf{R}^m$. Let μ be the Lebesgue measure in $\mathbf{R}^m$ and let T be a measurable bounded closed set in $\mathbf{R}^m$.

Corollary 3. *Let $X = \{X(t), t \in T\}$ be a stationary centered normal field whose correlation function $R(t,s) = R(t-s)$ is continuous and such that*

$$R(t) = 1 - |t|_m^\alpha + o(|t|_m^\alpha) \tag{29}$$

for some α, $0 < \alpha \le 2$. We also assume for $t \ne 0$ that

$$|R(t)| < 1. \tag{30}$$

Then equalities (24) *and* (25) *hold with*

$$\mathfrak{S}X \equiv 1, \qquad \tau_n(x) = b_n + \frac{(m/\alpha - 1/2)(\ln 2 + \ln\ln(n)) + \ln(\mu(T)H_\alpha 2\pi)^{1/2} + x}{b_n},$$

where H_α is some constant and the b_n, $n \ge 1$, are defined by (6) *(note that $H_2 = \pi^{-m/2}$; see* [12], [13], *and* [10, pp. 203–207]*).*

Proof of Corollary 3. If bounds (29) and (30) hold for a random field $X(t)$, then

$$\mathsf{P}\left(\sup_{t\in T} X(t) > x\right) \sim (2\pi)^{-1/2}\mu(T)H_\alpha x^{(2m/\alpha)-1}\exp\left(-x^2/2\right)$$

as $x \to \infty$ (see [12], [13]). Since the field $X(t)$ is continuous, we put $u \equiv \mathfrak{S}X \equiv 1$, $d = 0$, and

$$\theta(x) \sim (2\pi)^{-1/2}\mu(T)H_\alpha x^{(2m/\alpha)-1}$$

in Theorems 1 and 2 and obtain equalities (24) and (25) with the corresponding constants a_n and b_n. □

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