LIMIT OF LATTICES IN A LIE GROUP

BY
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1. **Introduction.** Let G be a connected Lie group, Γ a discrete subgroup and G/Γ be the space of left cosets. Let μ be a right Haar measure of G. μ induces a measure $\bar{\mu}$ over G/Γ. If $\bar{\mu}(G/\Gamma)$ is finite, Γ is called a *lattice*. If G/Γ is compact, Γ, certainly being a lattice, is called a c-lattice. We use $A(G)$ to denote the group of all continuous automorphisms of G with compact open topology and $\mathscr{S}(G)$ ($\mathscr{S}_c(G)$) to denote the set of all lattices (c-lattices) of G. We give $\mathscr{S}(G)$ a topology induced from *limit of lattices* [2]. We know that $A(G)$ operates continuously on $\mathscr{S}(G)$. In [2], Chabauty conjectured that for a lattice Γ_0 of G, $A(G)\Gamma_0$ with topology induced from $\mathscr{S}(G)$ is homeomorphic to $A(G)/N(\Gamma_0)$ where $N(\Gamma_0)$ is the subgroup $\{\alpha : \alpha \in A(G), \alpha(\Gamma_0)=\Gamma_0\}$. The purpose of this paper is to study $\mathscr{S}(G)$ and Chabauty's conjecture.

1.1. Definition. A set $\{H_\lambda\}$ of subgroups of G is called *uniformly discrete* if $H_\lambda \cap V=\{e\}$ for a certain neighborhood V of the identity e and all λ.

1.2. Definition. A sequence $\{H_n\}$ of subgroups of G *converges to a subgroup H*, with notation $\lim H_n=H$, if given any compact subset K and neighborhood V of e, $H \cap K \subset VH_n$ and $H_n \cap K \subset VH$ holds for large n.

As a consequence of the definitions, we have the following

1.3. Lemma. *Let $\{H_n\}$ be a sequence of subgroups of G converging to a discrete subgroup H; then $\{H_n\}_{n\geq n_0}$, n_0 a certain integer, is uniformly discrete.*

Proof. Let V be a small compact neighborhood of e such that $V^{-1}V \cap H=\{e\}$. Since $e \notin V(H-\{e\})$ and $V(H-\{e\})$ is closed, there exists a neighborhood W of e disjoint with $V(H-\{e\})$. Choose a symmetric compact neighborhood K of e and a symmetric neighborhood L of e such that $K\subset V \cap W$ and $L^2 \subset K$. As $\lim H_n=H$, $H_n \cap K\subset LH$ holds for large n. Let n be sufficiently large and $x, y \in H_n \cap K$, then $x, y \in L$ for $K \cap L(H-\{e\})=\varnothing$. Hence $x^{-1}y \in H_n \cap L^2 \subset H_n \cap K$ and $H_n \cap K$ is a group. But K is a small neighborhood of e, $H_n \cap K=\{e\}$ for large n.

1.4. Lemma. (6) *Let $\{\Gamma_n\}$ be a sequence of subgroups of G converging to a c-lattice Γ of G; then there exists a compact subset K with $G=K\Gamma_n$ for large n.*

2. **A base for the topology of $\mathscr{S}(G)$.** Let K be a compact subset of G, V be a neighborhood of e and Γ_0 be a lattice of G. We define $W(K, V{:}\Gamma_0)=\{\Gamma : \Gamma \in \mathscr{S}(G), \Gamma \cap K\subset V\Gamma_0$ and $\Gamma_0 \cap K\subset V\Gamma\}$. It is easy to see that the family of all $W(K, V{:}\Gamma_0)$

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with K running through all compact subsets of G, V through all open neighborhoods of e and Γ_0 through $\mathscr{S}(G)$ forms a base for the topology of $\mathscr{S}(G)$ induced from limit of lattices.

2.1. Proposition. *$\mathscr{S}(G)$ is a Hausdorff space.*

Proof. Given any two lattices $\Gamma_1 \neq \Gamma_2$, we may assume that there is an element $\gamma_1 \in \Gamma_1 - \Gamma_2$. Let W be a relative compact, symmetric open neighborhood of e such that $\overline{W}^2\gamma_1 \cap \Gamma_2 = \varnothing$. Then $W(\{\gamma_1\}, W:\Gamma_1)$ and $W(\{\overline{W}\gamma_1, e\}, W:\Gamma_2)$ are disjoint.

2.2. Since G is separable, there exists a countable dense subset $\mathscr{M} = \{x_1, \ldots, x_n, \ldots\}$ of G. Let $\mathscr{F}(\mathscr{M})$ be the set of all finite subsets of $\mathscr{M}$. Let K be a compact subset, V a neighborhood of e and $A \in \mathscr{F}(\mathscr{M})$, we define $V(A, K, V) = \{\Gamma : \Gamma \in \mathscr{S}(G), \Gamma \cap K \subset VA \text{ and } A \subset V\Gamma\}$. It is clear that $V(A, K, V) \subset W(K, V^2;\Gamma)$ for all $\Gamma \in V(A, K, V)$. Now let $\{V_n\}$ be a sequence of relative compact, symmetric open neighborhoods of e such that $V_n \supset \overline{V}_{n+1}^2$ and $\bigcap_{n=1}^{\infty} V_n = \{e\}$, and $\{K_n\}$ be a sequence of compact subsets of G with $e \in K_n \subset K_{n+1}$ and $\bigcup_{n=1}^{\infty} K_n^0 = G$(2).

Proposition. *Let $W_{A,n} = W(\overline{V}_n K_n, V_{n+1}; \Gamma_{A,n})$ if $V(A, \overline{V}_n K_n, V_{n+2})$ is not empty and $\Gamma_{A,n}$ a fixed element in it, =empty otherwise. Then $\{W_{A,n} : A \in \mathscr{F}(\mathscr{M}), n = 1, 2, \ldots\}$ is a countable base for the topology of $\mathscr{S}(G)$.*

Proof. Let Γ_0 be any lattice of G and $\mathscr{V}$ be any neighborhood of Γ_0 in $\mathscr{S}(G)$. Then $\Gamma_0 \in W(\overline{V}_n K_n, V_{n+1}; \Gamma_0) \subset W(K_n, V_n; \Gamma_0) \subset \mathscr{V}$ for certain n. Let $\Gamma_0 \cap \overline{V}_n K_n = \{\gamma_1, \ldots, \gamma_m\}$. Since $\mathscr{M}$ is dense, there exists $A = \{x_{i_1}, \ldots, x_{i_m}\}$ such that $\gamma_j \in V_{n+2} x_{i_j}$, $1 \leq j \leq m$. Then $\Gamma_0 \in V(A, \overline{V}_n K_n, V_{n+2}) \subset W(\overline{V}_n K_n, V_{n+1} : \Gamma_{A,n}) = W_{A,n}$ and one verifies readily $W_{A,n} \subset \mathscr{V}$.

2.3. By Proposition 2.2, $\mathscr{S}(G)$ is separable and by Proposition 2.1, $\mathscr{S}(G)$ is Hausdorff. It is easy to show that $W(K, V_{n+1}; \Gamma)^- \subset W(K, V_n; \Gamma)$ which implies that $\mathscr{S}(G)$ is regular. Hence by Urysohn's metrization theorem, we get the following

Theorem. *$\mathscr{S}(G)$ is separable metric.*

Remark. It is still unknown whether $\mathscr{S}(G)$ is locally compact or not. We shall see later that this problem can always be reduced to the case having G semisimple without compact factor.

3. **A homeomorphism.** Let G be a connected Lie group, $\tilde{G}$ be a covering group of G and $\tilde{G} \xrightarrow{p} G$ be the covering map. We define $p^*: \mathscr{S}(G) \to \mathscr{S}(\tilde{G})$ by $p^*(\Gamma) = p^{-1}(\Gamma)$, $\Gamma \in \mathscr{S}(G)$.

3.1. Proposition. *p^* is a homeomorphism onto a closed subset of $\mathscr{S}(\tilde{G})$.*

Proof. It is clear that p^* is 1-1 and $p^*(\mathscr{S}(G)) = \{\tilde{\Gamma} : \tilde{\Gamma} \in \mathscr{S}(\tilde{G}) \text{ and } \tilde{\Gamma} \supset \ker(p)\}$ is closed.

(2) X^0 denotes the set of all interior points of X.

Let $\{\Gamma_n\}$ be a sequence of lattices of G converging to a lattice Γ. We want to verify $\lim p^{-1}(\Gamma_n)=p^{-1}(\Gamma)$. Let $\tilde{K}$ be a compact subset of $\tilde{G}$ and $\tilde{V}$ a neighborhood of e in $\tilde{G}$. Then $\Gamma_n \cap p(\tilde{K}) \subset p(\tilde{V})\Gamma$ and $\Gamma \cap p(\tilde{K}) \subset p(\tilde{V})\Gamma_n$ holds for large n. Apply p^{-1} to get the desired inclusion relations. Hence p^* is continuous. Similarly one shows that p^{*-1} is continuous.

3.2. Let G, $\tilde{G}$, p have the same meaning as in 3.1, $\Gamma \in \mathscr{S}(G)$ and $\tilde{\Gamma}=p^{-1}(\Gamma)$.

PROPOSITION. *If* $A(\tilde{G})\tilde{\Gamma} \cong A(\tilde{G})/N(\tilde{\Gamma})$, *then* $A(G)\Gamma \cong A(G)/N(\Gamma)$.

Proof. We identify $\mathscr{S}(G)$ in $\mathscr{S}(\tilde{G})$ through p^* and identify $A(G) \subset A(\tilde{G})$ by $A(G) =\{\alpha : \alpha \in A(\tilde{G}),\ \alpha(\ker(p))=\ker(p)\}$. In order to verify this proposition, it suffices to verify that $A(G)\Gamma = A(G)\tilde{\Gamma}$ is locally compact. ker (p) is finitely generated abelian, say, generated by $\{z_1, \ldots, z_m\}$. Let W be a relative compact, symmetric neighborhood of e in $A(\tilde{G})$ such that for any $w \in W$, $wz_i \in Uz_i$, $1 \leqq i \leqq m$ and U a neighborhood of e in $\tilde{G}$ with $\bar{U} \cap \tilde{\Gamma}=\{e\}$. Claim that $A(G)\tilde{\Gamma} \cap W\tilde{\Gamma}$ is relative compact in $A(G)\tilde{\Gamma}$. Let $\{\alpha_n\}$ be a sequence of elements in $A(G)$ having $\alpha_n(\tilde{\Gamma}) \in W\tilde{\Gamma}$ for all n. Since $W\tilde{\Gamma}$ is relative compact and $A(\tilde{G})\tilde{\Gamma} \cong A(\tilde{G})/N(\tilde{\Gamma})$, there exists $\alpha \in \bar{W}$, $\beta_n \in N(\tilde{\Gamma})$ such that $\{\alpha_n\beta_n\}$ (more precisely a subsequence of this) converges to α. Then consider $\alpha^{-1}=\lim \beta_n^{-1}\alpha_n^{-1}$, $\beta_n^{-1}\alpha_n^{-1}(z_i) \in \tilde{\Gamma}$ for all n. But $\alpha^{-1}(z_i) \in \bar{U}z_i$ and $\bar{U} \cap \tilde{\Gamma} =\{e\}$, this implies $\beta_n(\ker(p))=\ker(p)$ for large n. Hence $\alpha \in A(G)$, and $A(G)\Gamma$ is locally compact at Γ. By action of $A(G)$, $A(G)\Gamma$ is locally compact. Thus $A(G)\Gamma \cong A(G)/N(\Gamma)$ follows easily.

REMARK. By Proposition 3.2, in order to study Chabauty's conjecture, one can consider only simply connected Lie groups.

4. **Representations and limit of lattices.** Let F be a free monoid on m generators $\gamma_1, \ldots, \gamma_m$ and $\{w_1(\gamma), \ldots, w_i(\gamma), \ldots\}$ be a set of words in F. If $w(\gamma)=\gamma_{j_1}\cdots\gamma_{j_s}$, we define $W: G^m \to G$ by $w(x)=x_{j_1}\cdots x_{j_s}$, where $x=(x_1, \ldots, x_m) \in G^m$.

4.1. LEMMA. *If* $G=GL(n, R)$, *then there exists a positive integer* n_0 *such that* $W_i(x)=e$ *for* $1 \leqq i \leqq n_0$ *implies* $W_j(x)=e$ *for all* j(3).

Proof. Let $M_n(R)$ be the set of all real n-n matrices. We define $\bar{W}_i: M_n(R)^m \to M_n(R)$ by $\bar{W}_i(A)=A_{j_1}\cdots A_{j_s}-E$ where $A=(A_1, \ldots, A_m)$, $w_i(\gamma)=\gamma_{j_1}\cdots\gamma_{j_s}$ and E is the identity matrix. It is clear that $W_i(X)=E$ iff $\bar{W}_i(X)=0$ and $X \in G^m$. Let $p_{kl}: M_n(R) \to RE_{kl}$ be the projection which assigns to any matrix its (k, l) entry, and $W^i_{kl}=p_{kl} \circ \bar{W}_i$, $1 \leqq k, l \leqq n$ and $i=1, 2, \ldots$. Then $W^i_{kl} \in R[Y_1, \ldots, Y_{n^2m}]$, the ring of real polynomials on n^2m variables. Since $R[Y_1, \ldots Y_{n^2m}]$ is Noetherian, there exists a positive integer n_0 such that the ideal generated by $W^i_{k,l}$, $i \leqq n_0$ contains all $W^j_{s,t}$, and the Lemma follows immediately.

4.2. Let Γ be a discrete subgroup of G and $\mathscr{R}(\Gamma, G)$ be the space of all representations, i.e., homomorphisms of Γ into G with compact open topology.

(3) This was pointed out to the author by H. C. Wang.

THEOREM. *Let $\{\Gamma_n\}$ be a sequence of subgroups of G converging to a discrete subgroup Γ of G and T is a finitely generated subgroup of G contained in Γ; then there are $r_n \in \mathscr{R}(T, G)$ such that*

(i) $r_n(T) \subset \Gamma_n$ *and*

(ii) $\lim r_n = 1_T$ *where 1_T is the inclusion map of T into G.*

Proof. Let $\tilde{G} \xrightarrow{p} G$ be the universal covering map and $\tilde{G} \xrightarrow{q} GL(n, R)$ be a continuous homomorphism which is a local isomorphism. We set $\tilde{\Gamma} = p^{-1}(\Gamma)$, $\tilde{T} = p^{-1}(T)$, and $\tilde{\Gamma}_n = p^{-1}(\Gamma_n)$ for all n. By Lemma 1.3, Γ_n is discrete for large n, and by same argument used in 3.1, $\lim \tilde{\Gamma}_n = \tilde{\Gamma}$. Since T and ker (p) are finitely generated, $\tilde{T}$ is finitely generated. Hence there exists a finite generating subset $\{\tilde{\gamma}_1, \ldots, \tilde{\gamma}_m\}$ of $\tilde{T}$ with a set of fundamental relations $\{w_1(\tilde{\gamma}), \ldots, w_i(\tilde{\gamma}), \ldots\}$. By the preceding Lemma, there exists a positive integer n_0 such that $W_i(x) = e$, $1 \leqq i \leqq n_0$ and $x \in \tilde{G}^m$ implies $qW_j(x) = e$ for all j. Let $\mathscr{R} = \bigcap_{i=1}^{n_0} W_i^{-1}(e)$; then $\mathscr{R}$ is a real analytic subset of $\tilde{G}^m$, hence is locally connected. Let $\mathscr{V}$ be a connected neighborhood of $\tilde{\gamma} = (\tilde{\gamma}_1, \ldots \tilde{\gamma}_m)$ in $\mathscr{R}$. Since ker (q) is discrete, we must have that $W_i(x) = e$ for all $x \in \mathscr{V}$. Since $\lim \tilde{\Gamma}_n = \tilde{\Gamma}$, there are $\gamma_i^{(n)} \in \tilde{\Gamma}_n$, $1 \leqq i \leqq m$ such that $\lim_n \gamma_i^{(n)} = \tilde{\gamma}_i$, $1 \leqq i \leqq m$. From Lemma 1.3, we know that $\{\tilde{\Gamma}_n\}_{n>k}$, for certain positive integer k, is uniformly discrete. Since $W_i(\tilde{\gamma}^{(n)}) \in \tilde{\Gamma}_n$, $\lim_n W_i(\tilde{\gamma}^{(n)}) = W_i(\tilde{\gamma}) = e$, it follows that $\tilde{\gamma}^{(n)} = (\gamma_1^{(n)}, \ldots, \gamma_m^{(n)}) \in \mathscr{V}$. Hence there are representations $\tilde{r}_n : \tilde{T} \to \tilde{G}$ such that $\tilde{r}_n(\tilde{\gamma}) = \tilde{\gamma}^{(n)}$. It is clear that (i) and (ii) are satisfied for $\{\tilde{\Gamma}_n\}$, $\tilde{\Gamma}$, $\tilde{T}$ and $\{\tilde{r}_n\}$. But ker $(p) \subset \tilde{T}$, $\tilde{\Gamma}$ and $\tilde{\Gamma}_n$ for all n, we must have $\tilde{r}_n | \ker(p) = 1_{\ker(p)}$. Hence $\tilde{r}_n$ induces $r_n : T \to G$ which satisfies (i) and (ii).

5. **Stability of subgroups.** Let M be a compactly generated, closed, normal subgroup of G with G/M semisimple and having no compact factor, and $M \cap \Gamma$ are c-lattices of M for all $\Gamma \in \mathscr{S}(G)$.

PROPOSITION. *If $\{\Gamma_n\}$ is a sequence of lattices of G converging to a lattice Γ_0, and $\{r_n\}$ is a sequence of representations of $T_0 \cap M$ such that* (i) $r_n(\Gamma_0 \cap M) \subset \Gamma_n$ *for all n and* (ii) $\lim r_n = 1_{\Gamma_0 \cap M}$, *then $r_n(\Gamma_0 \cap M) \subset M$ for large n.*

Proof. Let ρ be an irreducible representation of G/M over a complex vector space with ker (ρ) = center of G/M. Such a ρ always exists. Let $\pi : G \to G/M$ be the projection. Since $\pi(\Gamma_0)$ is a subgroup of G/M with property (S), $C\rho(\pi(\Gamma_0))$, the vector space generated by $\rho(\pi(\Gamma_0))$, contains $\rho(\pi(G))$ (1). As $C\rho(\pi(\Gamma_0))$ is of finite dimension, and $\Gamma_0 \cap M$, being a c-lattice of M, is finitely generated, there exists a finitely generated subgroup T of Γ_0 containing $\Gamma_0 \cap M$ and $C\rho(\pi(T)) = C\rho(G/M)$. By Proposition 4.2, there are representations r_n' of T such that (i) $r_n'(T) \subset \Gamma_n$, (ii) $\lim_n r_n' = 1_T$ and (iii) $r_n' | \Gamma_0 \cap M = r_n$ for large n, and (iv) $C\rho(\pi r_n'(T)) = C\rho(G/M)$ for large n. Assume that n is sufficiently large. Choose a neighborhood U_1 of e in G/M with $\lim_m L_m(U_1) = e$ (for definition see [9, p. 210]) and U_1 containing no central element $\neq e$, $U = \pi^{-1}(U_1)$. $\Gamma_0 \cap M$, a c-lattice, has a finite generating subset $Q = \{\beta_1, \ldots, \beta_s\}$. Hence $\beta_i^{-1} r_n(\beta_i) \in U$ for $1 \leqq i \leqq s$, $\lim_m L_m(\pi r_n(Q)) \subset \lim_m L_m(U_1) = e$.

But $\pi r_n(M \cap \Gamma_0) \subset \pi(\Gamma_n)$ is discrete. By a slight modification of [9, 3.2], $\pi r_n(M \cap \Gamma_0)$ being nilpotent and normal in $\pi r_n'(T)$, is central. Since Q generates $M \cap \Gamma_0$ and $\pi r_n(Q) = e$, we get $r_n(M \cap \Gamma_0) \subset M$ for large n.

6. In this section we deal with some properties of Haar measure. Let H be a locally compact and σ-compact group, Γ a lattice of H, K a closed subgroup containing Γ, and $H \xrightarrow{\theta} H/K$ be the projection.

6.1. Lemma. *If θ has a local cross section, then Γ is a lattice of K.*

Proof. Let V be a relative compact, open neighborhood of eK in G/K and s be a local cross section defined over $\bar{V}$. Given a right Haar measure μ of H, we define $\mu_v(B) = \mu(s(V)B)$ for any Borel subset B in K. It is easy to verify that μ_v is a right Haar measure of K. Let F be a fundamental domain in K with respect to Γ. Then $s(V)F$ is a Γ-packing in G, i.e. $F^{-1}s(v)^{-1}s(v)F \cap \Gamma = \{e\}$. Hence $\mu_v(F) = \mu(s(v)F) \leqq \bar{\mu}(G/\Gamma) < \infty$, and Γ is a lattice of K.

6.2. Lemma. *Let G be a connected Lie group, H a closed normal subgroup and $G \xrightarrow{\pi} G/H$ be the projection. If $\mathscr{S}(G) \neq \varnothing$ and $\pi(\Gamma)$ is discrete for all $\Gamma \in \mathscr{S}(G)$, then there exist Haar measures μ_G, μ_H, $\mu_{G/H}$ such that $\bar{\mu}_G(G/\Gamma) = \bar{\mu}_{G/H}(G/\Gamma H)\bar{\mu}_H(H/\Gamma \cap H)$ for all $\Gamma \in \mathscr{S}(G)$.*

Proof. Since $\mathscr{S}(G) \neq \varnothing$ and $\pi(\Gamma)$ is discrete for $\Gamma \in \mathscr{S}(G)$, we must have that H and G are unimodular. We define μ_G by

$$(1) \qquad \int_G f(x)\, d\mu_G(x) = \int_{G/H} \int_H f(xh)\, d\mu_H(h)\, d\mu_{G/H}(xH)$$

where μ_H and $\mu_{G/H}$ are arbitrary. It is clear that μ_G, so defined, is a Haar measure. Given any $\Gamma \in \mathscr{S}(G)$, there are Borel subsets F_1, F_2 such that

(i) F_2 is a fundamental domain of $\Gamma \cap H$ in H,

(ii) $F_1^{-1}F_1 \cap H\Gamma = \{e\}$,

(iii) $F_1F_2\Gamma = G$.

It is clear that F_1F_2 is a fundamental domain of Γ in G, and F_1H is a fundamental domain of ΓH in G/H. Hence by formula (1), $\bar{\mu}_G(G/\Gamma) = \bar{\mu}_{G/H}(G/\Gamma H)\bar{\mu}_H(H/\Gamma \cap H)$ is immediate.

Remark. The lemma is true for any discrete subgroup Γ of G with $\pi(\Gamma)$ discrete.

7. **Some continuous maps.** Let G be a simply connected Lie group, R the radical of G, and C be the maximal compact normal subgroup of a semisimple part of G.

7.1. Lemma [4]. *Let $\Gamma \in \mathscr{S}(G)$, then $\Gamma \cap CR$ is a c-lattice of CR.*

7.2. Proposition. *Let $q: \mathscr{S}(G) \to \mathscr{S}(CR)$ be defined by $q(\Gamma) = \Gamma \cap CR$, $\Gamma \in \mathscr{S}(G)$. Then q is continuous.*

Proof. Immediate from Theorem 4.2.

7.3. By 6.2, we know that ΓCR is a lattice of G/CR for any $\Gamma \in \mathscr{S}(G)$. We define $\mathscr{S}(G) \xrightarrow{p} \mathscr{S}(G/CR)$ by $p(\Gamma) = \Gamma CR$, $\Gamma \in \mathscr{S}(G)$.

PROPOSITION. *p is continuous.*

Proof. Let $\{\Gamma_n\}$ be a sequence of lattices of G converging to $\Gamma \in \mathscr{S}(G)$. Since $\lim(\Gamma_n \cap CR) = \Gamma \cap CR$ in $\mathscr{S}(CR)$, and $\Gamma \cap CR$ is a c-lattice of CR, by Lemma 1.4, there exists a compact subset K_1 of CR with $K_1(\Gamma_n \cap CR) = CR$ for all n. Let K be any compact subset of G, V a neighborhood of e in G. Then $\Gamma_n CR \cap KCR = (\Gamma_n K_1^{-1} \cap K)CR \subset (\Gamma_n \cap KK_1)CR \subset V\Gamma CR$ and $\Gamma CR \cap KCR = (\Gamma K_1^{-1} \cap K)CR \subset (\Gamma \cap KK_1)CR \subset V\Gamma_n CR$ holds for large n. Hence p is continuous.

7.4. Since G is simply connected, G takes the form of a semidirect product $G = G_1 CR$, $G_1 \cong G/CR$. We define $f: \mathscr{S}(G) \to \mathscr{S}(G_1) \times \mathscr{S}(CR)$ by $f(\Gamma) = (p(\Gamma), q(\Gamma))$, $\Gamma \in \mathscr{S}(G)$.

LEMMA. *Let $\{\Gamma_\lambda\}$ be a set of lattices of G, if $\{p(\Gamma_\lambda)\}$ and $\{q(\Gamma_\lambda)\}$ are uniformly discrete in G_1, CR respectively, then $\{\Gamma_\lambda\}$ is uniformly discrete in G.*

Proof. Immediate.

REMARK. Let $(\Gamma', \Gamma'') \in \mathscr{S}(G_1) \times \mathscr{S}(CR)$. $f^{-1}(\Gamma', \Gamma'')$ is uniformly discrete. By 6.2, $\bar{\mu}(G/\Gamma)$ is constant for each $\Gamma \in f^{-1}(\Gamma', \Gamma'')$. Hence by a theorem of Chabauty [2], $f^{-1}(\Gamma', \Gamma'')$ is compact.

7.5. PROPOSITION. *Let $\{\Gamma_n\}$ be a sequence of lattices of G with $\{f(\Gamma_n)\}$ converging to (Γ', Γ''). Then there exists a subsequence $\{\Gamma_{i(n)}\}$ converging to a lattice Γ of G with $f(\Gamma) = (\Gamma', \Gamma'')$.*

Proof. By Lemma 1.3 and Lemma 7.4, $\{\Gamma_n\}$ is uniformly discrete. Hence by a theorem of Chabauty [2], there is a subsequence $\{\Gamma_{i(n)}\}$ converging to a discrete subgroup Γ of G. Since CR is closed, and $\lim_n(\Gamma_{i(n)} \cap CR) = \Gamma''$, we must have $\Gamma'' \subset \Gamma$. By the same argument as used in 7.3, $\lim_n p(\Gamma_{i(n)}) = p(\Gamma) = \Gamma'$. By the Remark at 6.2, Γ is a lattice.

7.6. COROLLARY. *Let K be any compact subset of $\mathscr{S}(G_1) \times \mathscr{S}(CR)$, then $f^{-1}(K)$ is compact in $\mathscr{S}(G)$.*

7.7. COROLLARY. *If $\mathscr{S}(G_1)$ is locally compact, then $\mathscr{S}(G)$ is locally compact.*

8. **A map v from $\mathscr{S}(G)$ to the set of real numbers.** Let μ be a fixed Haar measure of G. Consider the map v defined by $v(\Gamma) = \bar{\mu}(G/\Gamma)$ for $\Gamma \in \mathscr{S}(G)$. In general v is not continuous, for an example see [6]. But the following always holds.

8.1. THEOREM [6]. *$v|\mathscr{S}_c(G)$ is continuous and $\mathscr{S}_c(G)$ is locally compact* (4).

8.2. COROLLARY. *Let G be a simply connected nilpotent Lie group. If G has a lattice Γ_0, then $\mathscr{S}(G) \cong A(G)/N(\Gamma_0)$.*

(4) In [6], that $v|\mathscr{S}_c(G)$ is continuous was proved. Since $\mathscr{S}_c(G)$ is open in $\mathscr{S}(G)$, by a theorem of Chabauty [2], $\mathscr{S}_c(G)$ is locally compact.

Proof. By a theorem of Mal'cev [7], $\mathscr{S}_c(G)=\mathscr{S}(G)$ and $A(G)\Gamma_0=\mathscr{S}(G)$. Since $A(G)$ and $\mathscr{S}(G)$ are locally compact and $A(G)$ is σ-compact, we have then $\mathscr{S}(G) \cong A(G)/N(\Gamma_0)$.

8.3. It is known that there is a continuous representation of $A(G)$ into the positive real numbers such that $v(\alpha(\Gamma))=r(\alpha)v(\Gamma)$ for all $\Gamma \in \mathscr{S}(G)$ and $\alpha \in A(G)$.

PROPOSITION. *$v|A(G)\Gamma$ is continuous.*

Proof. Consider $f: \mathscr{S}(G) \to \mathscr{S}(G_1) \times \mathscr{S}(CR)$. It is clear that $f(A(G)\Gamma) \subset A(G_1)\Gamma' \times \mathscr{S}(CR)$, $f(\Gamma)=(\Gamma', \Gamma'')$. By 6.2, we know that $v(T)=v(p(T))v(q(T))$, for any $T \in \mathscr{S}(G)$. Since f, $v|\mathscr{S}(CR)$, $v|A(G_1)\Gamma'$ are continuous(5), we have $v|A(G)\Gamma$ is continuous.

REMARK. By Proposition 3.1, Proposition 8.3 is true for all connected Lie groups.

9. Let Γ be a finitely generated lattice of G and $\mathscr{R}(\Gamma, G)$ be the space of all representations of Γ into G. Given any $\alpha \in A(G)$, $r \in \mathscr{R}(\Gamma, G)$ we define $\alpha(r)=\alpha\cdot r$. It is easy to see that $A(G)$ operates continuously on $\mathscr{R}(\Gamma, G)$.

9.1. LEMMA. *If $A(G)1_\Gamma$ is a neighborhood of 1_Γ in $\mathscr{R}(\Gamma, G)$, then $A(G)\Gamma$ is open $\mathscr{S}(G)$.*

Proof. Suppose false. Then there is a sequence $\{\Gamma_n\}$ of lattices converging to Γ with $\Gamma_n \notin A(G)\Gamma$ for all n. By Theorem 4.2, there are representations r_n of Γ such that (i) $r_n(\Gamma) \subset \Gamma_n$ and (ii) $\lim r_n = 1_\Gamma$. By assumption $r_n \in A(G)(1_\Gamma)$ for large n. Let $\alpha_n \in A(G)$ such that $\alpha_n(1_\Gamma)=r_n$. Then $\alpha_n(\Gamma) \subset \Gamma_n$ and $\lim \alpha_n(\Gamma)=\Gamma$. By Proposition 8.3, $\bar{\mu}(G/\Gamma)=\lim \bar{\mu}(G/\alpha_n(\Gamma)) \geqq \lim\sup_n \bar{\mu}(G/\Gamma_n)$. But in [2], Chabauty showed that $\bar{\mu}(G/\Gamma) \leqq \lim\inf_n \bar{\mu}(G/\Gamma_n)$. It follows that $\bar{\mu}(G/\alpha_n(\Gamma))=\bar{\mu}(G/\Gamma_n)$ for large n. Hence $\Gamma_n=\alpha_n(\Gamma)$ for large n which leads to a contradiction.

9.2. LEMMA. *If $A(G)(1_\Gamma)$ is a neighborhood of 1_Γ in $\mathscr{R}(\Gamma, G)$, then $A(G)\Gamma \cong A(G)/N(\Gamma)$.*

Proof. $A(G)\Gamma$ is open in $\mathscr{S}(G)$. Since $\mathscr{S}(G)$ is metric, $A(G)\Gamma$ is locally closed in $\mathscr{S}(G)$. By Lemma 1.3, and Proposition 8.3, there is a neighborhood $\mathscr{V}$ of Γ in $\mathscr{S}(G)$ such that $\bar{\mathscr{V}} \subset A(G)\Gamma$, $\bar{\mu}(G/T) < n_0$ for all $T \in \mathscr{V}$ and certain positive number n_0 and $\mathscr{V}$ is uniformly discrete. By a theorem of Chabauty [2], $\bar{\mathscr{V}}$ is compact. Hence $A(G)\Gamma$ is locally compact and $A(G)\Gamma \cong A(G)/N(\Gamma)$.

9.3. Let $\hat{G}$ be the Lie algebra of G and G operates on $\hat{G}$ by means of the adjoint group, $\{\gamma_1, \ldots, \gamma_n\}$ a finite generating subset of Γ and $\{w_1(\gamma), \ldots, w_i(\gamma), \ldots\}$ be a set of fundamental relations. Then $\mathscr{R}(\Gamma, G)$ can be identified with $\bigcap_i W_i^{-1}(e) \subset G^m$ where W_i's are defined in 4. In [9], A. Weil proved that $L=\bigcap_i L_i$, L_i = the kernel of the tangent mapping to W_i at $(\gamma_1, \ldots, \gamma_m)$, is the space $Z^1(\Gamma, \hat{G})$ of all cocycles of Γ in $\hat{G}$. Define $g: A(G) \to G^m$ by $g(\alpha)=(\alpha(\gamma_1), \ldots, \alpha(\gamma_m))$, $\alpha \in A(G)$. Let M be

(5) This follows from the fact that $A(G_1)$ is semisimple and $A(G_1)/A(G_1)_0$ is finite.

the image of tangent mapping of g at $e \in A(G)$. By Proposition 4, in [4], $M = \Phi Z^1(G, \hat{G})$ where $\Phi: Z^1(G, \hat{G}) \to Z^1(\Gamma, \hat{G})$ is the restriction map.

9.4. THEOREM. *If* $H^1(G, \hat{G}) \xrightarrow{\Phi^*} H^1(\Gamma, \hat{G})$ *is surjective, then the Chabauty's conjecture for* Γ *is true.*

Proof. By the above remark, $M = L$, hence by Lemma 1 in (9), $A(G)(1_\Gamma)$ is a neighborhood of 1_Γ in $\mathscr{R}(\Gamma, G)$. Thus by Lemma 9.2, $A(G)\Gamma \cong A(G)/N(\Gamma)$.

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