LIMIT POINTS OF KLEINIAN GROUPS AND FINITE SIDED FUNDAMENTAL POLYHEDRA

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Let G be a discrete subgroup of $SL(2, C)/\{\pm 1\}$. Then G operates as a discontinuous group of isometries on hyperbolic 3-space, which we regard as the open unit ball $\mathbf{B}^3$ in Euclidean 3-space $\mathbf{E}^3$. G operates on $\mathbf{S}^2$, the boundary of $\mathbf{B}^3$, as a group of conformal homeomorphisms, but it need not be discontinuous there. The set of points of $\mathbf{S}^2$ at which G does not act discontinuously is the *limit set* $\Lambda(G)$.

If we fix a point 0 in $\mathbf{B}^3$, then the orbit of 0 under G accumulates precisely at $\Lambda(G)$. The approximation is, however, not uniform. We distinguish a class of limit points, called *points of aproximation*, which are approximated very well by translates of 0. The set of points of approximation includes all loxodromic (including hyperbolic) fixed points, and includes no parabolic fixed points. In § 1 we give several equivalent definitions of point of approximation, and derive some properties. We remark that these points were first discussed by Hedlund [7].

Starting with a suitable point 0 in $\mathbf{B}^3$, we can construct the Dirichlet fundamental polyhedron P_0 for G. It was shown by Greenberg [5] that even if G is finitely generated, P_0 need not have finitely many sides. Our next main result, given in § 2, is that if P_0 is finite-sided, then every point of $\Lambda(G)$ is either a point of approximation or a cusped parabolic fixed point (roughly speaking a parabolic fixed point is cusped if it represents the right number of punctures in $(\mathbf{S}^2-\Lambda(G))/G$).

The above theorem has several applications: one of these is a new proof of the following theorem of Ahlfors [1].

If P_0 has finitely many sides, then the Euclidean measure of $\Lambda(G)$ is either 0 *or* 4π.

Our next main result, given in § 3, is that the above necessary condition for P_0 to have finitely many sides is also sufficient. In fact, we prove that any convex fundamental polyhedron G has finitely many sides if and only if $\Lambda(G)$ consists entirely of points of

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approximation and cusped parabolic fixed points. As an application of this we give a new proof of the following theorem of Marden [11].

Every Dirichlet fundamental polyhedron is finite sided or none are.

§ 1

Let $\hat{\mathbf{E}}^3$ be the 1-point compactification of $\mathbf{E}^3$, the added point is of course called ∞. Then G acts on $\hat{\mathbf{E}}^3$ as a group of orientation preserving conformal homeomorphisms. In $\hat{\mathbf{E}}^3$, the unit ball $\mathbf{B}^3$, and the upper half-space

$$\mathbf{H}^3 = \{(z, x) \mid z \in C,\ x \in R,\ x > 0\}$$

are conformally equivalent. When convenient, we will regard G as acting on $\mathbf{H}^3$, and on C, its boundary.

In $\mathbf{E}^3$ we use $|x-y|$ for Euclidean distance, and in $\mathbf{B}^3$ or $\mathbf{H}^3$, we use $\varrho(x, y)$ for non-Euclidean distance.

The action of G on $\hat{\mathbf{E}}^3$ is most easily seen via isometric spheres. We assume that ∞ is not fixed by $g \in G$, and that $g(\mathbf{B}^3) = \mathbf{B}^3$. Then there are two 2-spheres S_g and S'_g, called the *isometric spheres* of g and g^{-1}, respectively, with the following properties: S_g and S'_g both have the same (Euclidean) radius R_g, and are both orthogonal to $\mathbf{S}^2$. The action of g is the composition of inversion in S_g, followed by reflection in the perpendicular bisector of the line segment joining the centers of S_g and S'_g, followed by a Euclidean rotation centered at the center of S'_g. The importance of this description is that g is the composition of inversion in S_g and a Euclidean isometry (which maps $g^{-1}(\infty)$ to $g(\infty)$).

We enumerate the elements of G as $\{g_n\}$, and let R_n be the radius of the isometric sphere of g_n. It was shown by Beardon and Nicholls [3], that for every positive ε,

$$\sum R_n^{4+\varepsilon} < \infty,$$

while

$$\sum R_n^4 < \infty$$

if G is discontinuous at some point of $\mathbf{S}^2$.

It is useful to compare R_g with $|g(0)|$ and $|g(\infty)|$ (0 is now the origin). As S_g and $\mathbf{S}^2$ are orthogonal,

$$R_g^2 + 1 = |g(\infty)|^2$$

and as $g(0)$ and $g(\infty)$ are inverse points with respect to $\mathbf{S}^2$, $|g(0)| \cdot |g(\infty)| = 1$. If $G = \{g_n\}$ is discrete then $|g_n(\infty)| \to 1$ and so

$$\tfrac{1}{2} R_n^2 \sim |g_n(\infty)| - 1 \sim 1 - |g_n(0)|.$$

as $n \to \infty$.

We can use the above description of g to derive the following result, the plane version of which is trivial. If g is a conformal isometry of $\mathbf{B}^3$ and if x and y are in $\mathbf{E}^3-\{\infty, g^{-1}(\infty)\}$ then

$$|g(x)-g(y)| = \frac{R_g^2\,|x-y|}{|x-g^{-1}(\infty)|\,|y-g^{-1}(\infty)|} \tag{1}$$

The proof is easy. If J denotes inversion in S_g we have that

$$|g(x)-g(y)| = |J(x)-J(y)|$$

and also that the triangles with (ordered) vertices $g^{-1}(\infty)$, x, y and $g^{-1}(\infty)$, $J(y)$, $J(x)$ are similar. These facts lead easily to (1).

Now let K be a compact subset of $\Omega(G)=\hat{\mathbf{E}}^3-\Lambda(G)$. It is easily seen from (1) that there are positive numbers k_1 and k_2 (depending on G and K) such that for all x and y in K and all but a finite number of n,

$$k_1 R_g^2 \leqslant |g_n(x)-g_n(y)| \leqslant k_2 R_g^2. \tag{2}$$

A limit point z is called a *point of approximation* of G if and only if there is a point x in $\Omega(G)$, a positive constant k and a sequence g_n of distinct elements of G with

$$|z-g_n(x)| < kR_g^2. \tag{3}$$

We remark that by (2) this holds for one x in $\Omega(G)$ if and only if it holds for all x in $\Omega(G)$. Further, the approximation (3) is uniform on compact subsets of $\Omega(G)$.

Another observation is that the rate of approximation by points in $\Omega(G)$ as expressed by (3) is the best possible. Indeed if we replace g, x and y in (1) by g_n^{-1}, z and 0 we find that

$$|z-g_n(\infty)| \geqslant k_3 R_n^2 \tag{4}$$

where k_3 is positive and depends only on G.

The identity (1) can be used to characterize points of approximation in another way. We put $y=z$ in (1) and deduce that z is a point of approximation if and only if for one (or all) x other than z, there is a positive number k and a sequence g_n of distinct elements of G with

$$|g_n(x)-g_n(z)| \geqslant k. \tag{5}$$

Again, if this holds for some x $(\neq z)$ it holds uniformly on compact subsets of $\hat{\mathbf{E}}^3-\{z\}$. In the other direction if (5) holds uniformly on a set A we find that z is not in the closure of A.

The conditions (3) and (5) are metrical: we now seek to describe points of approximation topologically. Observe first that if σ is a hyperbolic line in $\mathbf{B}^3$ with end points

x and z, say, then (5) holds for a class of g_n if and only if there is a compact subset K of $\mathbf{B}^3$ with

$$g_n(\sigma) \cap K \neq \varnothing \tag{6}$$

for the same class of g_n. We may, of course, take K to be $\{x \in \mathbf{B}^3 \colon \varrho(x, 0) \leqslant \varrho_0\}$ and write

$$T = \{x \in \mathbf{B}^3 \colon \varrho(x, \sigma) \leqslant \varrho_0\}.$$

We then see that (6) holds if and only if

$$g_n(x) \to z \tag{7}$$

in T for one (or all) x in K. A Stolz region at z is a cone in $\mathbf{B}^3$ of the form

$$\{x \in \mathbf{B}^3 \colon |z - x| \leqslant k_4(1 - |x|)\}$$

and near z, T contains and is contained in Stolz regions at z.

We collect together the above results.

THEOREM 1. *The following statements are equivalent.*

(i) *z is a point of approximation.*

(ii) *For some (or all) x in $\Omega(G)$ there is a positive number k and a sequence of distinct elements g_n in G such that $|z - g_n(x)| < k \cdot R_g^2$.*

(iii) *For some x other than z, there is a positive number k and a sequence of distinct elements g_n in G such that $|g_n(x) - g_n(z)| \geqslant k$.*

(iv) *There exists a sequence g_n of distinct elements of G such that $|g_n(x) - g_n(z)|$ is bounded away from zero uniformly on compact subsets of $\hat{\mathbf{E}}^3 - \{z\}$.*

(v) *If σ is any hyperbolic line in $\mathbf{B}^3$ ending at z then there is a relatively compact subset K of $\mathbf{B}^3$ and a sequence of distinct elements g_n in G such that $g_n(\sigma) \cap K \neq \varnothing$.*

(vi) *For some (or all) x in $\mathbf{B}^3$ there is a Stolz region T at z and a sequence of distinct elements g_n in G such that $g_n(x) \to z$ in T.*

If h is now a Möbius transformation which maps $\mathbf{B}^3$ onto $\mathbf{H}^3$, then hGh^{-1} acts on $\mathbf{H}^3$ and C and so may be regarded as a group of matrices. The points of approximation of hGh^{-1} are the images under h of the points of approximation of G and Theorems (1)(v) shows that this definition is conjugation invariant and so is independent of h.

In the special case when $\Lambda(G)$ is a proper subset of $\mathbf{S}^2$ we can choose h so that $\infty \notin \Lambda(hGh^{-1})$. In this case we let σ be the vertical line through z on C and we conclude that z is a point of approximation if and only if there is a positive constant k with

$$|g(\infty) - g(z)| \geqslant k$$

for infinitely many g in hGh^{-1}.

We now let $hGh^{-1} = \{g_n\}$ where

$$g_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}, \quad a_n d_n - b_n c_n = 1$$

and we have proved the following result.

PROPOSITION 1. *In the above situation z is a point of approximation of hGh^{-1} if and only if there is a positive number k such that*

$$|z + d_n/c_n| \leqslant k|c_n|^{-2}$$

for infinitely many g_n in hGh^{-1}.

PROPOSITION 2. *If z is a fixed point of the loxodromic element $g \in G$, then z is a point of approximation.*

Proof. We can assume without loss of generality that z is the attractive fixed point. Then for every $x \in \Omega(G)$, $g^{-h}(x)$ converges to the other fixed point.

The parabolic case is somewhat more complicated. We normalize G so that it acts on $\mathbf{H}^3$ and so that $z \to z+1 \in G$. Let J be the stability subgroup of ∞; i.e., $J = \{g \in G \,|\, g(\infty) = \infty\}$.

We recall that in general, if we have a discrete group G acting on, say $\mathbf{H}^3$, and a subgroup $J \subset G$, then the set $A \subset \mathbf{H}^3$ is *precisely invariant under* J if for every $g \in G$ either

(i) $g \in J$ and $g(A) = A$, or

(ii) $g \notin J$ and $g(A) \cap A = \varnothing$.

It is well known (see, for example, Leutbecher [9] or Kra [8, p. 58]) that if $z \to z+1 \in G$, then for every $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ which is in G but not in J, $|c| \geqslant 1$. As an immediate consequence of this, we obtain

LEMMA 1. *Let $z \to z+1$ be an element of the discrete group G acting on $\mathbf{H}^3$. Then*

$$A = \{(z, x) \in \mathbf{H}^3 \,|\, x > 1\}$$

is precisely invariant under J, the stability subgroup of ∞.

We conclude that no orbit can approach ∞ in a Stoltz region at ∞ and so we have proven

PROPOSITION 3. *If z is the fixed point of a parabolic element of G, then z is not a point of approximation.*

§ 2

In this section we explore the relationship between points of approximation and finite-sided fundamental polyhedra.

We need a definition of fundamental polyhedron when there are not necessarily finitely many sides. In this paper, we restrict ourselves to *convex* polyhedra.

A (convex) *polyhedron* P is an open subset of $\mathbf{B}^3$ (or of $\mathbf{H}^3$) defined as the intersection of countably many half-spaces Q_i with the following property. Each Q_i is bounded by a hyperplane S_i; the intersection of S_i with $\bar{P}$, the closure of P in $\mathbf{B}^3$ is called a *side* of P. We require that any compact subset of $\mathbf{B}^3$ meets only finitely many of the S_i: then the boundary of P in $\mathbf{B}^3$ consists only of sides.

The polyhedron P is a (convex) *fundamental polyhedron* for the discrete group G if

(a) no two points of P are equivalent under G.
(b) Every point of $\mathbf{B}^3$ is equivalent under G to some point of $\bar{P}$.
(c) The sides of P are pair-wise identified by elements of G.
(d) Every x in $\mathbf{B}^3$ has a neighbourhood that meets only finitely many translates of P.

We remark that there is a Fuchsian group and a polygon P which satisfies (a) and (b), but not (c). For Fuchsian groups (d) is a consequence of (a), (b) and (c).

PROPOSITION 4. *A point of approximation z of G cannot lie on the boundary of a convex fundamental polyhedron P_0 of G.*

Proof. As P_0 is convex we can select a hyperbolic line σ joining a point x in P_0 to the point of approximation z. Theorem 1 (v) is applicable and this is in direct contradiction with the defining property (d) of P_0.

One easily sees that the identification of sides of P induces an equivalence relation on $\bar{P}$, each equivalence class containing only finitely many points.

It is well known that there is at least one convex fundamental polyhedron for every discrete group. A particularly well known example is the *Dirichlet fundamental* polyhedron P_0 formed as follows: We start with say $0 \in \mathbf{B}^3$ where 0 is not fixed by any element of G. For each non-trivial $g \in G$, we form

$$Q_g = \{y \in \mathbf{B}^3 \,|\, \varrho(y, 0) \leqslant \varrho(y, g(0))\}.$$

One easily sees that Q_g is a half-space, and that $P_0 = \bigcap_g Q_g$ is a fundamental polyhedron for G.

For any polyhedron $P \subset \mathbf{B}^3$, $\bar{P}$ is the relative closure of P in $\mathbf{B}^3$; we let P^* be the intersection of $\mathbf{S}^2$ with the closure of P in $\hat{\mathbf{E}}^3$.

Our next definition is concerned with parabolic fixed points; they are limit points but they may have aspects similar to ordinary points. We assume that $z \in C$ is fixed point of some parabolic element of G, and let J be the stability subgroup of z. J is then a Kleinian group with exactly one fixed point; all such groups are known (see Ford [4], p. 139). In order to examine the possibilities, we assume that G acts on H^3, and that $z = \infty$.

A *cusped region* U is a subset of C with the following properties. U is precisely invariant under J, and U is the union of two disjoint non-empty open half-planes.

One easily sees that a cusped region U can exist only if J is a finite extension of a cyclic group, and in this case $U \cap \Lambda(G) = \emptyset$. We say that z is a *cusped* parabolic fixed point if either there is a non-empty cusped region U, or if J is not a finite extension of a cyclic group.

The existence of parabolic fixed points which are not cusped is given in Maskit [12].

THEOREM 2. *If there is a convex fundamental polyhedron P for G with finitely many sides, then every limit point of G is either a point of approximation or is a cusped parabolic fixed point.*

Proof. We start with the well known fact that every point of P^* is either in $\Omega(G)$ or is a cusped parabolic fixed point. Unfortunately, there is no ready reference for this fact, and so we outline a proof here.

The identifications of the sides of P induce an equivalence relation on $\bar{P}$, and on P^*. For each point $z \in P^*$, the set of points equivalent to z is called the (unordered) cycle at z. Since P has finitely many sides, the cycle contains finitely many points.

We now consider z in P^* and conjugate so that $z = \infty$ and the elements of G act on $\mathbf{H}^3$. We choose $g_1, ..., g_r$ in G so that the cycle of ∞ on P^* is $\{g_0(\infty), g_1^{-1}(\infty), ..., g_r^{-1}(\infty)\}$ where, for convenience, g_0 is the identity.

Now let J be the stabilizer of ∞ in G and J_0 the subgroup of parabolic elements (and g_0) that fix ∞ (J_0 may be trivial). If $\infty \in g(P^*)$ where $g \in G$ we can construct a geodesic σ from a point in $g(P)$ to ∞. This implies that for some i, $0 \leqslant i \leqslant r$, $g_i g^{-1}(\sigma)$ is a geodesic ending at ∞ and so $g_i g^{-1} \in J$. We conclude that

$$J \in J \cup Jg_1 \cup ... \cup Jg_r.$$

By Propositions 2 and 4, J can contain only elliptic and parabolic elements and we see from [4, p. 140–141] that in this case there are elliptic elements $e_1, ..., e_s$ such that

$$J = J_0 \cup J_0 e_1 \dots \cup J_0 e_s.$$

We conclude that g lies in one of a finite number of cosets $J_0 h_i$, $h_i \in G$.

If J_0 is trivial, then a neighbourhood of ∞ in $\mathbf{H}^3 \cup C$ meets only a finite number of images of P and so $\infty \in \Omega(G)$.

If J_0 is not a cyclic group, then by definition, ∞ is a cusped parabolic point.

Finally if J_0 is cyclic the images of P lie under one of the finite number of euclidean curved sides of P or the $h_i(P)$ or are translations under J_0 of these images and so a cusped region exists in this case.

We now assume without loss of generality, that $0 \in P$. Let $z \in S^2$, and let σ be the line from 0 to z. If σ intersects only finitely many translates of sides of P, then for some $g \in G$, $g(z) \in P^*$, and so by the above remark either $z \in \Omega(G)$ or z is a cusped parabolic fixed point. Observe that this situation must arise if $z \in \Omega(G)$, for the euclidean diameter of translates of P must converge to 0.

The only possibility left is that σ passes through infinitely many translates of some side M and in this case $z \in \Lambda(G)$. Then there is a sequence $\{g_n\}$ of distinct elements of G, and there is a sequence of points $\{y_n\}$ on M, so that $g_n(\sigma) \cap M = \{y_n\}$. We can assume that $y_n \to y$. If $y \in \mathbf{B}^3$, then by Theorem 1 (v) z is a point of approximation. If, as we now assume $y \notin \mathbf{B}^3$, then by the remarks above, y is a cusped parabolic fixed point. We again change normalization so that $y = \infty$, and we let J be the stability subgroup of ∞.

If J is not a finite extension of a cyclic group, then there is a compact set $K \subset C$, so that for every $z' \in C$, there is a $j \in J$ with $j(z') \in K$. Hence, we can choose a sequence $\{j_n\}$ of elements of J so that $j_n \circ g_n(z) \in K$, and $j_n \circ g_n(0) \to \infty$. Observe that this latter condition implies that infinitely many of the $\{j_n \circ g_n\}$ are distinct.

If J is a finite extension of a cyclic group, then we can assume that $z \to z+1 \in J$, the cusped region is $U = \{z \mid |\mathrm{Im}\, z| \geqslant t\}$, and that no translates of z lies in U. Exactly as above, we can find a sequence $\{j_n\}$ of elements of J so that

$$|\mathrm{Im}\,(j_n \circ g_n(z))| \leqslant t, \ |\mathrm{Re}\,(j_n \circ g_n(z))| \leqslant \tfrac{1}{2}.$$

This concludes the proof of Theorem 2 as we have now verified Theorem 1 (iii).

We remark first that as a corollary to the proof, we have the following well known statement.

COROLLARY 1. *Let P be a convex finite sided polyhedron for G. Let P^{*0} be the relative interior of P^*. Then no two points of P^{*0} are equivalent under G, and every point of $\Omega(G) \cap S^2$ is equivalent under G to some point in the closure of P^{*0}.*

For the following applications we recall that G is *elementary* if $\Lambda(G)$ is a finite set.

COROLLARY 2. *Let G be non-elementary. Then the set of points of approximation has positive Hausdorff dimension.*

Proof. It was remarked by Myrberg [13] that every non-elementary discrete group G contains a Schottky subgroup G_1, defined by say $2n$ *circles*. G_1 is then a discrete group of the second kind, with a finite-sided fundamental polyhedron. It was shown by Beardon [2] that for every such G_1, $\Lambda(G_1)$ has positive Hausdorff dimension. Since G_1 is purely loxodromic, $\Lambda(G_1)$ contains only points of approximation for G_1, and so for G.

COROLLARY 3. *Let G have a finite-sided fundamental polyhedron, then the points of approximation of $\Lambda(G)$ are uniformly approximable, i.e., there is a constant $k>0$ so that, for every point of approximation z, there is a sequence $\{g_n\}$ of distinct elements of G with*

$$|z-g_n(\infty)| \leqslant kR_n^2.$$

Proof. Let $p_1, \ldots, p_r$ be the parabolic vertices on $\bar{P}$. In the notation of the proof of Theorem 1 we find that if $y_n \to y$, $y=p_j$, then $j_n \circ g_n(0)$ remains outside some neighbourhood of the set $\{j_n \circ g_n(z)\}$. If we consider G as now acting in $\mathbf{B}^3$ this means that (retaining the same notation despite conjugation),

$$|j_h \circ g_n(z) - j_n \circ g_n(0)| \geqslant k$$

The result now follows by (1) and (2).

A corollary of the above is the following theorem of Ahlfors [1].

COROLLARY 4. *Let G have a finite sided fundamental polyhedron. Then the 2-dimensional measure of $\Lambda(G)$ is either zero or 4π.*

Proof. The proof is essentially immediate from Corollary 3, and the fact remarked above, that if G is of the second kind, then

$$\sum_{g \in G} R_g^4 < \infty.$$

Exactly the same considerations yield

COROLLARY 5. *If G has a finite-sided fundamental polyhedron, and if*

$$\sum_{g \in G} R_g^{2t} < \infty,$$

then the t-dimensional measure of $\Lambda(G)$ is zero.

§ 3

In this section we prove the converse of Theorem 2. Specifically, our goal is to prove.

THEOREM 3. *Let P be a convex fundamental polyhedron for the discrete group G, where every point of $\Lambda(G)$ either is a point of approximation or is a cusped parabolic fixed point. Then P has finitely many sides.*

Proof. Throughout we assume that P is a convex fundamental polyhedron for the discrete group G which, for the moment, is assumed to act on $\mathbf{B}^3$. If P has infinitely many sides, these accumulate at some point z on $\bar{P}$. We begin by showing that $z \in \Lambda(G)$.

LEMMA 2. *Let M_1, M_1', M_2, M_2' be sides of P where there are pairing transformations $g_1, g_2 \in G$ with $g_i(M_i)=M_i'$. Then, $g_1=g_2$ if and only if $M_1=M_2$.*

Proof. Let S_1, S_1' be the hyperplanes on which M_1, M_1', respectively, lie, and let Q_1, Q_1' be the half spaces which are bounded by S_1, S_1', respectively, and which contain P. If M_2 does not lie on S_1, then $M_2 \subset Q_1$, and $g_1(M_2) \cap Q_1' = \emptyset$. We conclude that $g_1(M_2)$ can be a side of P only if $M_2 \subset S_1$; i.e., $M_2 = M_1$.

This lemma shows that infinitely many distinct images of P accumulate at z. As P is convex and locally finite the euclidean diameter of the images of P under G converge to zero, thus $z \in \Lambda(G)$.

Proposition 4 together with the hypotheses of the theorem now imply that z is necessarily a cusped parabolic fix-point. We complete the proof by showing that this is inconsistent with the assumption that infinitely many sides of P accumulate at z.

We shall assume that G acts on $\mathbf{H}^3$ and that $z=\infty$. Now let J be the stabilizer of ∞ and J_0 the subgroup of parabolic elements of J. We may assume that J_0 contains $z \to z+1$: J_0 is either cyclic or of rank 2.

We will need the following remark about convex polyhedra.

LEMMA 3. *Let (z_i, x_i), $i=1, \ldots, n$, be a finite set of points of P. Let B be the Euclidean convex hull of the points $z_1, \ldots, z_n$. Then*

(i) *there is a $t>0$ so that $\{(z, x) \in \mathbf{H}^3 \mid z \in B, x>t\} \subset \overline{P}$, and*

(ii) *no two distinct points of B are equivalent under J.*

Proof. Since J keeps each horosphere $x=$constant invariant, conclusion (ii) follows from conclusion (i).

Since $\overline{P}$ is convex and $\infty \in P^*$, if $(z, x_0) \in \overline{P}$, then so does (z, x) for every $x>x_0$. Conclusion (i) now follows from the fact that if τ is the non-Euclidean line from (z_1, x_1) to (z_2, x_2), then the projection of τ onto the z-plane is the Euclidean line from z_1 to z_2.

This leads easily to

LEMMA 4. *Let $z_n \to \infty$ in $\overline{P}$ with $z_n=(u_n+iv_n, x_n)$.*

(i) *If J_0 is cyclic, then $v_n^2+x_n^2$ is unbounded.*

(ii) *If J_0 is of rank 2, then $u_n^2+v_n^2$ is bounded, x_n^2 is unbounded.*

Proof. If the conclusion of (i) fails then, by Lemma 3, P contains a subset of the form $[u', +\infty) \times [v', v''] \times [x', +\infty)$ $(v'<v'')$ and this contains points equivalent under J_0. The proof of (ii) is similar.

We immediately deduce that if z_n is a sequence of distinct points in $\Lambda(G) \cap P^*$ then $z_n \not\rightarrow \infty$. Indeed in (i) we have $x_n = 0$ and $z_n \notin U$ so $|v_n| \leqslant V^*$ whereas in (ii) $x_n = 0$. The hypothesis of the Theorem together with Proposition 4 now implies that P^* contains only finitely many limit points, in particular the cycle of ∞ is finite.

If infinitely many sides M_n of P meet ∞ we can select g_n in G where $g_n(P)$ abuts P along M_n. By Lemma 2, these g_n are distinct. It is evident that P can abut at most one other translate of $g(P)$ under J_0 and so we conclude that the g_n lie in infinitely many distinct cosets $J_0 g$. This implies that the set $\{g_n^{-1}(\infty)\}$ is an infinite subset of $\overline{P}$ contrary to our previous remark. We have proved

LEMMA 5. *Only finitely many sides of P pass through ∞.*

We have assumed there is an infinite sequence of sides M_n of P accumulating at ∞. The previous lemma implies that we may assume that none of these contain ∞. We select z_n on M_n with $z_n \to \infty$ and choose distinct g_n so that $g_n(P)$ abuts P along M_n.

As $\infty \notin M_n$ we conclude that $g_n(\infty) \in C$ and we can find a sequence j_n in J_0 with $j_n \circ g_n(\infty)$ lying in a compact subset K of C. By Lemma 4 we observe that $j_n(z_n) \to \infty$. If τ_n is the geodesic in $j_n \circ g_n(P)$ joining $j_n(z_n)$ to $j_n \circ g_n(\infty)$ we find that the τ_n meet a compact subset of $\mathbf{H}^3$ contrary to the assumption that the tesselation is locally finite. The proof is now complete.

We remark in closing that we have used the fact that we are dealing with 3-dimensional hyperbolic space in a crucial manner only in the precise definition of cusped parabolic fixed point. In dimension 2, it is well-known, and one easily proves using Lemma 1, that every parabolic fixed point is cusped. It is also well-known (see Greenberg [6] or Marden [10]) that a Fuchsian group has a finite sided fundamental polygon if and only if it is finitely generated. Combining these with the trivial fact that a Fuchsian group has a finite sided fundamental polygon if and only if as a Kleinian group it has a finite sided fundamental polyhedron, we obtain

COROLLARY 6. *A Fuchsian group G is finitely-generated if and only if $\Lambda(G)$ consists entirely of points of approximation and parabolic fixed points.*

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